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METRICAL DIOPHANTINE APPROXIMATION FOR CONTINUED FRACTION LIKE MAPS OF THE INTERVAL

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Abstract. We study the metrical properties of a class of continued fraction-like mappings of the unit interval, each of which is defined as the fractional part of a Möbius transformation taking the endpoints of the interval to zero and infinity.

1. Introduction

Working with an ergodic transformation T of the interval that is given as a piecewise Möbius transformation, one comes to expect an associated continued fraction theory. The continued fraction expansion of a number x is a concrete realization of the symbolic representation of the number as a sequence of locations within intervals of monotonicity of the iterates T^n. The piecewise Möbius nature of the map allows for a representation of the endpoints of the intervals of monotonicity nesting about x as particular "fractions", $\frac{p_n}{q_n}$, called the convergents of x.

To offer a few familiar examples, the classical continued fraction theory may be seen as arising from a study of the Gauss map,

$$G(x) = \langle \frac{1}{x} \rangle = \frac{1}{x} - [\frac{1}{x}],$$

where $[a]$ denotes the integral part of a and consequently $G(x)$ is the fractional part of $\frac{1}{x}$ [3, 9]. There are two other well-known continued fraction theories, one arising from the Renyi or backward continued fraction map $R(x) = \langle \frac{1}{1-x} \rangle$ [1, 17] and the other from the nearest integer map $H(x) = \frac{1}{x} - [\frac{1}{x} - \frac{1}{2}]$ [16]. G and R are defined on the unit interval, while R is a self-map of $[-\frac{1}{2}, \frac{1}{2}]$. All are piecewise Möbius transformations and have actual rational fractions as convergents.

Several related families of mappings have been studied, with attention to their dynamical properties as well as to the resulting continued fraction theories [6, 10, 15, 22]. This fits into a broader series of connections which have been observed between continued fractions, diophantine approximation, ergodic theory and hyperbolic geometry, perhaps originating with Artin and more recently seen popping up all over [1, 2, 7, 8, 10]. Most commonly the dynamical system is introduced as a tool for studying the continued fraction theory. Although there has been much progress, only in a few cases, for example see [20, 21], is it possible to achieve the incredibly refined results acheived in the classical "metrical" theory of continued fractions [3, 5, 13, 18].

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We shall focus on the endomorphisms of the unit interval of the form $T(x) = \langle A(x) \rangle$, where A is a Möbius transformation taking one of the endpoints $\{0, 1\}$ to 0 and the other to ∞. Not all such maps are ergodic or even recurrent. Among those that are, there is a realization of the natural automorphic extension of T [9] acting on a subset of $\mathbb{R}^2$ for which the density $\frac{1}{(x-y)^2}$ defines an invariant measure. Except for two maps for which the invariant measure is infinite, the others are all Bernoulli with respect to the normalized measure [11]. Earlier work by Rudolfer and Wilkinson looked at the dynamic properties of a subclass of these mappings and showed them to be weak Bernoulli [22]. Observe that both the Gauss and Rényi maps are among the ones being considered so far, although both are usually defined by a Möbius transformation taking one of the endpoints to 1 instead of 0.

Our goal is to develop the basic continued fraction theory for the transformations with finite invariant measure and then to use the natural extension, in the manner pioneered by Bosma, Jager and Wiedijk [5] in the classical cases, to elucidate some fairly refined properties of the approximating convergents and consequently of the associated interval structure for the mappings. An important tool for our work, which distills a technique from [5], is Theorem 3. This result allows us to conclude convergence for ergodic sums in two variables almost everywhere in one of the variables.

For each of the mappings we shall derive what is classically known as the Khintchine constant, which gives the limiting geometric mean of a generic digit expansion. We shall also derive the Khintchine-Lévy constants. These describe the generic rate of growth of the denominators of convergents and can be used to determine something like the generic rate of shrinkage for nesting intervals of mononticity. Similar techniques allow the computation, for almost all x, of the values

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{|x - \frac{p_i}{q_i}|}{|x - \frac{p_{i-1}}{q_{i-1}}|} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right|$$

for all of the mappings. Several of these values relate simply and directly to the entropy of the mappings. The Euler dilogarithm makes a few notable appearances. Along the way we deduce a variety of diophantine, metrical and other properties of the continued fractions.

While the classical theory of the Gauss map is contained as a special case of our approach, the Rényi map is one of those with infinite invariant measure and therefore its convergents are not distributed in a way that can be understood using our methods.

2. u-continued fractions and dynamics

2.1. Matrices and maps: Basic definitions.

A 2×2 matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

with real entries and determinant $ad - bc = \pm 1$ acts on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ as a Möbius transformation by

$$C(z) = \frac{az+b}{cz+d}\ ,\ C(\infty) = \frac{a}{c} \quad \text{and} \quad C(-\frac{d}{c}) = \infty. \tag{2}$$

Such matrices preserve the extended real line $\mathbb{R} \cup \{\infty\}$ and either preserve or interchange the lower and upper half planes, depending on whether the determinant is positive or negative. Matrix multiplication goes over, in this setting, to the composition of the corresponding Möbius transformations [2].

We shall be considering the family of Möbius transformations parameterized by $\Psi \subset \mathbb{R} \times \{0,1\}$, where Ψ is defined as the union of

$$\Psi_0 = ((-\infty,-1)\cup(0,\infty))\times\{0\} \quad \text{and} \quad \Psi_1 = ((-\infty,0)\cup(1,\infty))\times\{1\}.$$

Define the matrices

$$A_{(k,0)} = \begin{pmatrix} \frac{k}{\sqrt{|k|}} & \frac{-k}{\sqrt{|k|}} \\ \frac{-1}{\sqrt{|k|}} & 0 \end{pmatrix} \quad \text{and} \quad A_{(k,1)} = \begin{pmatrix} \frac{k}{\sqrt{|k|}} & 0 \\ \frac{-1}{\sqrt{|k|}} & \frac{1}{\sqrt{|k|}} \end{pmatrix}. \tag{3}$$

Written as Möbius transformations, these have the form

$$A_{(k,0)}(z) = \frac{k(1-z)}{z} \quad \text{and} \quad A_{(k,1)}(z) = \frac{kz}{1-z}.$$

The second coordinate of the parameterization indicates the endpoint of the interval $[0,1]$ that is mapped to ∞ by the transformation. The remaining endpoint always maps to zero. We use the notation $u=(k,m)\in\Psi$.

For any real number t, $[t]$ shall denote the greatest integer less than or equal to t, otherwise known as the *integer part* of t. For technical reasons, define the integer part of ∞ to be zero. The *fractional part* of t, written $\langle t\rangle$, is $t-[t]$. Also define $\langle\infty\rangle = 0$. Let $I=[0,1]$, the closed unit interval. Then for $u\in\Psi$ the interval transformation $T_u : I \to I$ is the piecewise Möbius transformation

$$T_u(x) = \langle A_u(x)\rangle. \tag{4}$$

T_u will be called *Gauss-like* when $u\in\Psi_0$ and *Renyi-like* when $u\in\Psi_1$. $T_{(1,0)}$ and $T_{(1,1)}$ are respectively the classical Gauss and Renyi transformations, also known as the continued fraction and backward continued fraction maps [1, 10]. These transformations provide a dynamical system approach for studying properties of the regular and backwards continued fractions. Another important distinction will often be made depending on whether the parameter u satisfies either $(-1)^m k > 0$ or $(-1)^m k < 0$. As we shall see, this latter classification is more significant when considering the dynamical or "metrical" behavior of the transformations.

2.2. **Intervals of monotonicity and finite continued fractions.** Given $u\in\Psi$, the set of integers

$$V_u = \{l\in\mathbb{Z} \mid [A_u(x)] = l \text{ for some } x\in(0,1)\} \tag{5}$$

are the *u-digits*. When $k>0$, V_u is the set of non-negative integers and when $k<0$, it is the set of negative integers. V_u should be thought of as the set of integers a so that $A_u(x)-a\in(0,1)$ for some $x\in(0,1)$.

For each sequence of n integers $a_1,\dots,a_n\in V_u$, the open interval (cylinder set) at level n is the image of the interval $(0,1)$ under a composition of transformations, given specifically as

$$\triangle^{(n)}_{a_1\dots a_n} = A_u^{-1}B^{a_1}A_u^{-1}B^{a_2}\dots A_u^{-1}B^{a_n}(0,1), \quad \text{where} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

By an easy induction argument, each such interval is contained in the unit interval, and for positive integers n we have

$$\triangle^{(n+1)}_{a_1\dots a_{n+1}} \subset \triangle^{(n)}_{a_1\dots a_n}.$$

Observe that for $a \in V_u$ the Möbius transformation $A_u^{-1}B^a$ maps $(0,1)$ homeomorphically onto $\triangle_a^{(1)}$. Thus for any $x \in \triangle_a^{(1)}$ there is a unique $y \in (0,1)$ with $x = A_u^{-1}B^a(y)$. Then

$$T_u(x) = \left\langle A_u A_u^{-1} B^a(y) \right\rangle = \langle y + a \rangle = y.$$

It follows that T_u restricted to $\triangle_a^{(1)}$ is the bijection $B^{-a}A_u$ mapping the interval onto $(0,1)$, which therefore naturally extends to a homeomorphism of $\overline{\Delta}_a^{(1)}$ onto I.

Suppressing reference to the particular u or a_i we shall write

$$C_n = B^{-a_n}A_u \dots B^{-a_1}A_u. \tag{7}$$

The notation T^n will always refer to the nth iterate of the map T. The obvious extension of the above by induction gives

Proposition 1. *For each $n \geq 1$ and $a_1, \dots, a_n \in V_u$, the restriction of T_u^n to $\triangle_{a_1 \dots a_n}^{(n)}$ is the Möbius transformation C_n mapping $\triangle_{a_1 \dots a_n}^{(n)}$ onto $(0,1)$ that extends to a homeomorphism of $\overline{\Delta}_{a_1 \dots a_n}^{(n)}$ to $[0,1]$. Similarly, $T : \triangle_{a_1 \dots a_n}^{(n)} \to \triangle_{a_2 \dots a_n}^{(n-1)}$ is the Möbius transformation $B^{-a_1}A_u$, which extends to a homeomorphism of the closures.*

It is clear from the above that the intervals $\triangle_{a_1 \dots a_n}^{(n)}$ are the maximal open intervals on which the transformation T_u^n is a homeomorphism. They are sometimes referred to in the literature as *intervals of monotonicity.* Given a finite sequence of integers $a_1, a_2, \dots, a_n \in V_u$, define the finite *u-continued fraction* expansion

$$[a_1, a_2, \dots, a_n]_u = A_u^{-1}B^{a_1}A_u^{-1}B^{a_2} \dots A_u^{-1}B^{a_n}A_u^{-1}(\infty) \in [0,1]. \tag{8}$$

It follows immediately from the proposition that T_u acts as a shift on finite u-continued fraction expansions by $T[a_1, a_2, \dots, a_n]_u = [a_2, \dots, a_n]_u$. If we write

$$C_n^{-1}A_u^{-1} = \begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix}, \tag{9}$$

then the expansion may be associated with a "fraction" as follows: $[a_1, a_2, \dots, a_n]_u = C_n^{-1}A_u^{-1}(\infty) = \frac{p_n}{q_n}$. Again, although the explicit reference has been supressed, it is important to keep in mind that the matrix (9) does depend on the digits $a_1, a_2, \dots, a_n$ and the parameter u. For technical reasons we define $p_0 = 0$ and $q_0 = 1$. Also be aware that with this approach p_n and q_n are not always nonnegative.

Since C_n maps $\overline{\Delta}_{a_1 \dots a_n}^{(n)}$ homeomorphically onto I, $C_n^{-1}(0)$ and $C_n^{-1}(1)$ are the endpoints of the interval $\triangle_{a_1 \dots a_n}^{(n)}$. If u is Gauss-like, then $A_u^{-1}(\infty) = 0$ and if u is Renyi-like, then $A_u^{-1}(\infty) = 1$. Therefore we have the following values for the endpoints:

$$C_n^{-1}(0) = \begin{cases} [a_1, a_2, \dots, a_n]_u = \frac{p_n}{q_n} & \text{if } u \text{ is Gauss-like,} \\ [a_1, a_2, \dots, a_n - 1]_u & \text{if } u \text{ is Renyi-like.} \end{cases}$$

$$C_n^{-1}(1) = \begin{cases} [a_1, a_2, \dots, a_n + 1]_u & \text{if } u \text{ is Gauss-like,} \\ [a_1, a_2, \dots, a_n]_u = \frac{p_n}{q_n} & \text{if } u \text{ is Renyi-like.} \end{cases}$$

Note that $[a_1, a_2, \dots, a_n \pm 1]_u$ is defined as in (8) above, even though it is possible that $a_n \pm 1 \notin V_u$.

2.3. **The partition by intervals of montonicity and infinite continued fractions.** For $n \geq 1$ define $\mathbb{Q}_u^{(n)} = \{x \in I \mid T_u^m(x) = 0 \text{ for some } m \leq n\}$, which we shall call the preimages of zero of order n. Let $\mathbb{Q}_u = \bigcup_{n=1}^{\infty} \mathbb{Q}_u^{(n)}$ be the full set of preimages of 0. Given $n \geq 1$, define the partition P_n of (0,1) consisting of the non-empty cylinder sets of level n. Actually, this is only a partition modulo a countable set of points, and more precisely we have

Proposition 2. *For each integer* $n \geq 1$

$$[0,1] = \left(\bigcup_{a_1, a_2, \ldots, a_n \in V_u} \Delta^{(n)}_{a_1, a_2, \ldots, a_n} \right) \cup \mathbb{Q}_u^{(n)}.$$

Proof. We argue by induction. First suppose $n = 1$. The transformation A_u maps [0,1] to either $[0, \infty) \cup \{\infty\}$ or $(-\infty, 0] \cup \{\infty\}$ in $\mathbb{R} \cup \{\infty\}$. $A_u[0,1]$ therefore consists of ∞, some integers and intervals of the form $(k, k+1)$ for some $k \in \mathbb{Z}$. Take the fractional part to get T_u, thereby sending the integers and ∞ to 0, while each interval $(k, k+1)$ is mapped to $(0,1)$. In other words, each of the open intervals in the complement of $\mathbb{Z}$ in $A_u[0,1]$ is the image under A_u of a cylinder $\Delta_k^{(n)}$ and all such cylinders must be accounted for. The result thus holds for $n = 1$.

Suppose the proposition holds for some positive integer n. By the induction hypothesis we can write

$$[0,1] = \left(\bigcup_{a_2, \ldots, a_{n+1} \in V_u} \Delta^{(n)}_{a_2, \ldots, a_{n+1}} \right) \cup \mathbb{Q}_u^{(n)},$$

where we have taken the liberty of indexing the u-digits from 2 to $n+1$. Recall that by Proposition 1, the restriction of T_u to $\Delta_{a_1}^{(1)}$ has a homeomorphic inverse $T_u^{-1} = A_u^{-1} B^{a_1}$ mapping $(0,1)$ onto $\Delta_{a_1}^{(1)}$. Then, beginning with what we already know for $n = 1$,

$$\begin{aligned}
[0,1] &= \Big(\bigcup_{a_1 \in V_u} \Delta_{a_1}^{(1)} \Big) \cup \mathbb{Q}_u^{(1)} = \Big(\bigcup_{a_1 \in V_u} A_u^{-1} B^{a_1}(0,1) \Big) \cup \mathbb{Q}_u^{(1)} \\
&= \Big(\bigcup_{a_1 \in V_u} A_u^{-1} B^{a_1} \left(\left(\bigcup_{a_2, \ldots, a_{n+1} \in V_u} \Delta^{(n)}_{a_2, \ldots, a_{n+1}} \right) \cup \mathbb{Q}_u^{(n)} \setminus \{0,1\} \right) \Big) \cup \mathbb{Q}_u^{(1)} \\
&= \left(\bigcup_{a_1, a_2, \ldots, a_{n+1} \in V_u} \Delta^{(n)}_{a_1, a_2, \ldots, a_{n+1}} \right) \cup T_u^{-1}(\mathbb{Q}_u^{(n)} \setminus \{0,1\}) \cup \mathbb{Q}_u^{(1)},
\end{aligned}$$

proving the proposition. □

By analogy to the classical case, $u = (1,0)$, we shall refer to $\mathbb{Q}_u$ as the *u-rational* numbers and to its complement in I as the *u-irrational* numbers. In the classical case this notation agrees with the commonly used dichotomy of the reals. This is no longer true for the backward continued fractions.

It was shown in [11] that all of the maps T_u we consider have well-defined derivatives on $(0,1)$ and are eventually expanding. In particular, the second derivative is always bounded below by a constant larger than one. A simple consequence of this is that there is a value $\lambda_u > 1$ and a constant $K_u \geq 0$ so that for any $n > 1$ the length of an interval $\triangle^{(n)}$ at level n satisfies

$$| \triangle^{(n)} | < K_u \lambda_u^{-n}. \tag{10}$$

Therefore given an infinite sequence of u-digits $\{a_n\}_{n=1}^{\infty}$, there is a sequence of nested closed intervals $\{\overline{\triangle}_{a_1\dots a_n}^{(n)}\}_{n=1}^{\infty}$ with lengths converging to zero. Thus, since one endpoint of $\overline{\triangle}_{a_1\dots a_n}^{(n)}$ is $\frac{p_n}{q_n}$, it makes sense to define the value of the infinite u-continued fraction

$$[a_1, a_2, \dots]_u = \lim_{n\to\infty} [a_1, a_2, \dots, a_n]_u = \lim_{n\to\infty} \frac{p_n}{q_n} \in [0, 1]. \tag{11}$$

If x has an infinite u-expansion, then

$$x = [a_1, a_2, \dots]_u = \bigcap_{k=1}^{\infty} \overline{\Delta}_{a_1\dots a_k} \tag{12}$$

and consequently

$$T_u(x) = \bigcap_{k=1}^{\infty} T_u \overline{\Delta}_{a_1\dots a_k} = \bigcap_{k=2}^{\infty} \overline{\Delta}_{a_2\dots a_k} = [a_2, a_3, \dots]_u. \tag{13}$$

This shows that T_u will always act as a shift on u-continued fraction expansions.

Given a u-irrational number $x \in (0, 1)$, there is an infinite sequence of u-digits $\{a_i\}_{i=1}^{\infty}$ so that x lies in each of the intervals $\triangle_{a_1\dots a_n}^{(n)}$. Since one of the endpoints of $\triangle_{a_1\dots a_n}^{(n)}$ is $\frac{p_n}{q_n}$, $x = \lim_{n\to\infty} \frac{p_n}{q_n}$.

From the above it also follows that a u-irrational number lies in the interior of a unique interval at each level and consequently the u-expansion is unique.

What has been proved above is now collected together as a proposition.

Proposition 3. *Each u-irrational x has a unique, infinite u-expansion. The continued fraction expansion of the u-irrational x has the form $[a_1, \dots, a_n, \dots]$ if and only if $x \in \triangle_{a_1\dots a_n}^{(n)}$ for each integer $n > 0$, and similarly $x = [a_1, a_2, \dots]_u$ if and only if $T_u^{n-1}(x) \in \triangle_{a_n}$ for each integer $n > 0$. Furthermore, T_u acts as a shift map on u-continued fraction expansions.*

The fractions $\frac{p_n}{q_n}$ are called the *u-convergents* of x.

There are basic properties of the u-continued fractions and their approximations that remain to be elucidated. At this point it is surprisingly easy to derive some metrical results about the u-continued fractions which parallel the classical case. With the techniques developed later, in Section 3, it will become possible to prove more of the basic non-metrical properties of the approximations.

3. Invariant measures and basic metrical results

In an earlier paper by one of the authors the transformations $T_u(x)$ were studied from a purely dynamic point of view [11]. In particular, it was shown that the transformations $T_u(x)$ have invariant probability measures μ_u, equivalent to Lebesgue measure and defined by the densities

$$\mu_u(x) = \begin{cases} \frac{c_u}{x+k} & \text{if } u \text{ is Gauss-like,} \\ \frac{c_u}{x+k-1} & \text{if } u \text{ is Renyi-like,} \end{cases}$$

where $c_u = \operatorname{sgn}(k) \left(\log \left| \frac{k+1-m}{k-m} \right| \right)^{-1}$. Furthermore, T_u is ergodic with respect to μ_u.

The following proposition is proved in much the same way as its classical analogues are in [3] and [9]. The case where $m = 1$ and k is a positive integer was proved in [10].

Proposition 4. *For all $u \in \Psi$ and for almost all u-irrational $x = [a_1, a_2, \ldots]_u \in [0,1]$,*

$$\lim_{n\to\infty} \frac{1}{n}\left(a_1 + \ldots + a_n\right) = \pm\infty,$$

where the sign agrees with the sign of k and

$$\lim_{n\to\infty} \sqrt[n]{\alpha_1 \ldots \alpha_n} = \prod_{i=2}^{\infty} \left(\frac{(i+|k|)^2}{(i+|k|)^2 - 1} \right)^{\frac{\operatorname{sgn}(k)\log i}{\log\left|\frac{k+1-m}{k-m}\right|}},$$

where $\alpha_i = a_i + 1$ if $a_i \geq 0$ and $\alpha_i = |a_i|$ if $a_i < 0$.

Proof. We suppose u is Gauss-like with $k > 0$. The other cases are similar. Define the functions $\alpha_n : [0,1) \to \mathbb{N}$, where $\alpha_n(x) = \alpha_n$ for u-irrational $x = [a_1, a_2, \ldots]_u$ and α_n is as defined above. Also, define the truncation of α_1 for $L > 1$ by $b_L(x) = \alpha_1(x)$ if $x \in \bigcup_{|i|\leq L} \Delta_i^{(1)}$ and $b_L(x) = L$ otherwise. Then for almost all x

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \alpha_i = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \alpha_1(T^i x) \geq \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} b_L(T^i x).$$

Applying the Birkoff Ergodic Theorem [3, 9] to the function b_L, we see that the right-hand sum above is equal to

$$\int_0^1 b_L(x)\mu_u(x)\,dx \geq C_u \int_0^1 b_L(x)\,dx \geq C_u \sum_{i=0}^{L} (i+1)|\Delta_i^{(1)}| \tag{14}$$

for some constant $C_u > 0$. Then by definition (6) the length of cylinder at level one is $|\Delta_i^{(1)}| = |A_u^{-1}B^i(0) - A_u^{-1}B^i(1)| = \left|\frac{k}{(i+k)(i+1+k)}\right|$ and consequently for all $L > 0$, the right-hand side of equation (14) is bounded below by $C_u \sum_{i=0}^{L} \left|\frac{(i+1)k}{(i+k)(i+1+k)}\right|$. This sum diverges as L goes to infinity. That proves the first assertion of the proposition.

In order to prove the second assertion of the proposition, take the logarithm of the left-hand side and apply the Ergodic Theorem. This yields

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log \alpha_n = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log \alpha_1(T^k x) = \int_0^1 \log \alpha_1(x)\mu_u(x)\,dx.$$

Arguing as above, it is clear that the integral can be bounded above by a convergent sum of the form $K_u \sum_{i=0}^{\infty} \log(i+1)|\Delta_i^{(1)}| = K_u \sum_{i=1}^{\infty} \left|\frac{k\log(i+1)}{(i+k)(i+1+k)}\right|$ for some $K_u > 0$, which justifies the use of the Ergodic Theorem. Compute

$$\int_0^1 \log \alpha_1(x)\mu_u(x)\,dx = c_u \sum_{i=0}^{\infty} \log(i+1) \int_{\Delta_i^{(1)}} \frac{dx}{x+k}$$
$$= c_u \sum_{i=0}^{\infty} \log(i+1) \int_{A_u^{-1}B^i(1)}^{A_u^{-1}B^i(0)} \frac{dx}{x+k} = c_u \sum_{i=0}^{\infty} \log(i+1) \int_{\frac{k}{i+1+k}}^{\frac{k}{i+k}} \frac{dx}{x+k}$$
$$= c_u \sum_{i=0}^{\infty} \log(i+1) \left[\log\left(\frac{k}{i+k}+k\right) - \log\left(\frac{k}{i+1+k}+k\right)\right]$$
$$= c_u \sum_{i=1}^{\infty} \log i \left(\log \frac{(i+k)^2}{(i+k-1)(i+k+1)}\right).$$

Turning the sum into the product of the exponents and simplifying gives

$$\prod_{i=2}^{\infty} \left(\frac{(i+k)^2}{(i+k)^2-1}\right)^{c_u \log i}.$$

It is surprising that in the other cases the computations simplify in a similar way. □

More could be done with these methods, but the natural automorphic extension, developed in the next section, is a far more powerful tool.

4. The natural extension and its relation to diophantine approximation

4.1. **Basic definitions and properties.** For each $u = (k, m) \in \Psi$ define the intervals

$$J_u = \begin{cases} (-\infty, -k] & \text{if } m = 0, k > 0, \\ [-k, \infty) & \text{if } m = 0, k < -1, \\ (-\infty, 1-k] & \text{if } m = 1, k > 1, \\ [1-k, \infty) & \text{if } m = 1, k < 0. \end{cases} \tag{15}$$

For a Borel subets D of $I \times J_u$ let

$$\rho(D) = \int\int_D \frac{dxdy}{(x-y)^2} \tag{16}$$

and define the probability measure $\rho_u = c_u\rho$, where c_u is defined as in Section 3 and satisfies $1/c_u = \rho(I \times J_u)$. Observe that we also have the formula

$$c_u = \begin{cases} \left(\log(\frac{|k|+1}{|k|})\right)^{-1} & \text{if } (-1)^m k > 0, \\ \left(\log(\frac{|k|}{|k|-1})\right)^{-1} & \text{if } (-1)^m k < 0. \end{cases} \tag{17}$$

The function $\rho_u(x, y) = \frac{c_u}{(x-y)^2}$ is referred to as the density for the measure ρ_u. Since ρ_u and Lebesgue measure on $\mathbb{R}^2$ have the same sets of measure zero, "almost everywhere" statements regarding ρ_u or one of its projections to I or J_u are true with respect to Lebesgue measure on these sets.

We use S to denote the σ-algebra of Borel sets for any subset of $\mathbb{R}^n$. The results from [11] regarding the natural extension of T_u will be stated as a theorem for later use.

Theorem 1. a) *There are sets* $I_u \subset I$ *and* $J_u^* \subset J_u$, *which differ from* I *and* J_u *on a countable set, so that the map* $\tilde{T}_u : I_u \times J_u^* \to I_u \times J_u^*$ *defined by* $\tilde{T}_u(x,y) = (T_u(x), A_u(y) - [A_u(x)])$ *is an automorphism with invariant measure* ρ_u.

b) $\tilde{T}_u$ *is a Bernoulli automorphism of the probability space* $(I_u \times J_u^*, \rho_u, S)$.

The following remarks highlight some elementary but important properties of the map $\tilde{T}_u$. In the abstract setting, it is known that the endomorphism T_u possesses a natural automorphic extension [9]. The map $\tilde{T}_u$ is a particular, concrete realization of that natural automorphic extension, but by a common abuse of notation we shall refer to it as *the natural automorphic extension* of T_u.

Remark 1. As it is defined above $\tilde{T}_u$ maps $(0,1) \times (J_u \cup \{\infty\})$ to $[0,1) \times (J_u \cup \{\infty\})$ and agrees with the above Bernoulli automorphism off a set of measure zero. In most cases we shall work with this larger domain.

Remark 2. It follows from the definition of $\tilde{T}_u$ and Proposition 1 that for $x \in \triangle^{(n)}_{a_1 \ldots a_n}$ and $y \in J_u \cup \{\infty\}$

$$\tilde{T}_u^n(x,y) = (C_n(x), C_n(y)). \tag{18}$$

In particular, by restricting the first coordinate of the domain in this way $\tilde{T}_u$ becomes a fixed Möbius transformation acting on each of the coordinates. Furthermore, if $x = [a_1, a_2, \ldots]_u$ is a fixed u-irrational, then for each integer $n > 0$ the u-digits $a_1, \ldots, a_n$ determine a matrix C_n, as in equation (7), and for any $y \in J_u \cup \{\infty\}$ equation (18) holds.

Remark 3. From a measure theoretic point of view the dynamical systems $(\tilde{T}_u, I_u \times J_u^*, \rho_u)$ are not all distinct. Set $\alpha(x,y) = (1-x, 1-y)$. Then conjugacy by the order two measure preserving transformation α induces an isomorphism between the dynamical system with $u = (k,1)$ and the one with $u = (-k, 0)$. First, note that the set of points (x,y) with $x \notin \mathbb{Q}_u$ has full measure and for such x, $A_u(x) \notin \mathbb{Z}$. Then on this set of full measure compute

$$\begin{aligned}\alpha \circ \tilde{T}_{(k,1)} \circ \alpha(x,y) &= (-\frac{k}{x} + k + 1 + [\frac{k}{x} - k], -\frac{k}{y} + k + 1 + [\frac{k}{x} - k]) \\ &= (-\frac{k}{x} + k + 1 - [-\frac{k}{x} + k + 1], -\frac{k}{y} + k + 1 - [-\frac{k}{x} + k + 1]) \\ &= (-\frac{k}{x} + k - [-\frac{k}{x} + k], -\frac{k}{y} + k - [-\frac{k}{x} + k]) = \tilde{T}_{(-k,0)}(x,y),\end{aligned}$$

where we have used the fact that, for any $a \notin \mathbb{Z}$, $[a] = -[1-a]$.

Remark 4. It should be mentioned that the explicit realization of the natural extension given by Nakada [15] is another avenue for extending the family T_u and might provide an alternative to the approach taken here. It has proved very useful in a variety of other settings; for example see [6, 5, 12]. Schweiger has also incorporated Nakada's approach into his theory of fibered systems [20, 21] to achieve metrical results for a variety of continued fraction theories in one dimension and higher.

4.2. **Diophantine approximation.** Fix $u \in \Psi$. Given $x \in (0,1)$ set $(x_0, y_0) = \tilde{T}_u(x, \infty)$ and define $\tilde{T}_u^n(x_0, y_0) = (x_n, y_n)$. If x is a u-irrational, (x_n, y_n) is defined for all integers $n \geq 0$. One important measure of the degree to which a u-convergent $\frac{p_n}{q_n}$ approximates x is given in classical form by the value

$$\theta_n = |q_n||q_n x - p_n|. \tag{19}$$

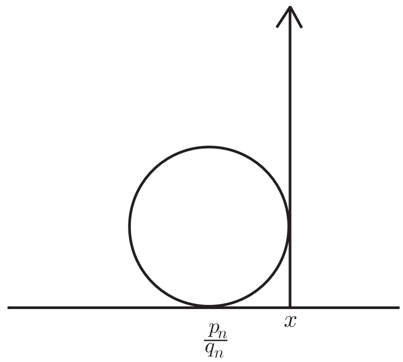

FIGURE 1. The Ford circle of radius $\frac{1}{2\tau_n q_n^2}$ at $\frac{p_n}{q_n}$ tangent to $\overline{x\infty}$.

In terms of the natural extension we have

Theorem 2. $\theta_n = \frac{1}{|x_n - y_n|}$.

Proof. Let C be a Möbius transformation given by the real 2×2 matrix of determinant $\delta = \pm 1$ as in formula (1). Möbius transformations map circles on the Riemann sphere to circles and therefore the image of the line $y = \tau$ for $\tau > 0$ is a circle S through the point $C(\infty) = a/c$ [2].

Since the extended real line remains invariant under C and meets $y = \tau$ only at ∞, S is tangent to the real axis at a/c. This circle will lie in the upper or lower half-plane depending on whether $\delta = 1$ or $\delta = -1$, respectively. Computing the image of a particularly nice point on the line $y = \tau$ under C gives $C(-\frac{d}{c} + i\tau) = \frac{a}{c} \pm \frac{i}{\tau c^2}$, where the sign depends on whether $\delta = \pm 1$. We conclude that S has radius $\frac{1}{2\tau c^2}$.

Let C be the matrix C_{n+1} determined by the u-irrational $x = [a_1, a_2, \ldots]_u$ as in Propositiion 1 and suppose that C_{n+1} has determinant $\delta = 1$. Then C_{n+1} takes the vertical ray $\overline{x\infty}$ in the upper half-plane beginning at x to the semi-circle $C_{n+1}(\overline{x\infty}) = \widehat{x_n y_n}$ orthogonal to $\mathbb{R}$ at the points $x_n = C_{n+1}(x)$ and $y_n = C_{n+1}(\infty)$. The semi-circle $\widehat{x_n y_n}$ is tangent to the line $y = \tau_n$ where $\tau_n = \frac{|x_n - y_n|}{2}$.

By definition

$$C_{n+1}^{-1}(\infty) = C_n^{-1} A_u^{-1} B^{a_{n+1}}(\infty) = C_n^{-1} A_u^{-1}(\infty) = \frac{p_n}{q_n},$$

the n^{th} u-convergent to x. By the general considerations in the first paragraph above, C_{n+1}^{-1} takes the line $y = \tau_n$ to the circle in the upper half-plane of radius $\frac{1}{2\tau_n q_n^2}$ which is tangent to $\mathbb{R}$ at the point p_n/q_n. Tangencies are preserved by Mobius transformations, so the ray $\overline{x\infty}$ is tangent to this image circle. See Figure 1. It follows that

$$|x - \frac{p_n}{q_n}| = \frac{1}{2\tau_n q_n^2} \quad \text{and} \quad \theta_n = \frac{1}{2\tau_n} = \frac{1}{|x_n - y_n|}. \tag{20}$$

The case of $\delta = -1$ is similar except that the Möbius transformations C_n alternately preserve and interchange the upper and lower half-planes. Although some of the geometric configurations will then be in the lower half-plane, that will have no effect on the result. $\square$

Now, by computing the supremum of $\frac{1}{|x-y|}$ for $(x,y) \in I \times J_u$ and noting that $x_n \neq 0, 1$, we get the following version of the classical result of Dirichlet for the rough approximation of a u-irrational number by u-rational numbers.

Corollary 1. *For each u-irrational number $x \in (0,1)$ the convergents $\frac{p_n}{q_n}$ (infinite in number) satisfy*

$$\left| x - \frac{p_n}{q_n} \right| < \begin{cases} \frac{1}{|k| q_n^2} & \text{if } \ (-1)^m k > 0, \\ \frac{1}{(|k|-1) q_n^2} & \text{if } \ (-1)^m k < 0. \end{cases}$$

5. The dynamics of $\tilde{T}_u$ and applications to diophantine approximation

5.1. **Distribution under the automorphic extension.** One consequence of Theorem 1 is that $\tilde{T}_u$ is ergodic. In particular, as a consequence of Birkoff's Ergodic Theorem [9], for almost all $(x,y) \in I \times J_u$ and any Borel set D or $f \in L^1(\rho_u)$,

$$\lim_{n\to\infty} \frac{1}{n} \#\{j \leq n \ \mid \ \tilde{T}_u^n(x,y) \in D\} = \rho_u(D) \ \text{ and}$$
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\tilde{T}_u^n(x,y)) = \int f d\rho_u. \tag{21}$$

What is more surprising is that with some additional hypotheses on D and f the identities (21) hold for almost all $x \in I$ for any $y \in J_u$. The remainder of this section will be devoted to the proof of this fact, which is Theorem 3.

Lemma 1. *Given a u-irrational $x \in (0,1)$ and $\epsilon > 0$ there is a value N so that for $n > N$ and $y, y' \in J_u \cup \{\infty\}$, $|\tilde{T}_u^n(x,y) - \tilde{T}_u^n(x,y')| < \epsilon$.*

Proof. Define the Möbius transformations

$$M_{(k,0)} = \begin{pmatrix} 0 & \frac{-k}{\sqrt{|k|}} \\ \frac{1}{\sqrt{|k|}} & 0 \end{pmatrix} \ \text{ and } \ M_{(k,1)} = \begin{pmatrix} \frac{1}{\sqrt{|k|}} & \frac{k-1}{\sqrt{|k|}} \\ \frac{1}{\sqrt{|k|}} & \frac{-1}{\sqrt{|k|}} \end{pmatrix}. \tag{22}$$

M_u is an order two homeomorphism which interchanges the intervals I and $J_u \cup \{\infty\}$. One easily sees by computation that for $u \in \Psi$ and $n \in V_u$, $M_u B^{-n} A_u M_u^{-1} = (-1)^m \text{sgn}(k)(B^{-n}A_u)^{-1}$. Note that multiplication of a matrix by a scalar, as we have above, is term by term and has no effect on the resulting Möbius transformation.

Fix a u-irrational x with u-expansion $[a_1, \ldots]_u$. Then the transformations C_n are determined and we can define the associated transformation

$$\hat{C}_n = B^{-a_1} A_u \ldots B^{-a_n} A_u.$$

By Proposition 1 $\hat{C}_n$ maps $\overline{\Delta}^{(n)}_{a_n \ldots a_1}$ onto I. Then it follows from the previous observation that $M_u C_n M_u^{-1} = ((-1)^m \text{sgn}(k))^n \hat{C}_n^{-1}$. Applying Remark 2,

$$\tilde{T}_u^n \left(\{x\} \times (J_u \cup \{\infty\})\right) = \{C_n(x)\} \times C_n(J_u \cup \{\infty\}).$$

The second coordinate can be rewritten as

$$C_n(J_u \cup \{\infty\}) = M_u \hat{C}_n^{-1} M_u(J_u \cup \{\infty\}) = M_u \hat{C}_n^{-1}(I) = M_u(\overline{\Delta}^{(n)}_{a_n \ldots a_1}).$$

On $(0,1)$ we have $|(M_uA_u^{-1}B^a)'| = 1$ for any $a \in V_u$. By equation (10), N can be chosen so that for $n > N$, $|\overline{\Delta}^{(n-1)}_{a_{n-1}\dots a_1}| < \epsilon$. Then the interval $C_n(J_u \cup \{\infty\}) = M_u(\overline{\Delta}^{(n)}_{a_n\dots a_1}) = M_uA_u^{-1}B^{a_n}(\overline{\Delta}^{(n-1)}_{a_{n-1}\dots a_1})$ has length less than ϵ. For $y, y' \in J_u \cup \{\infty\}$ the second coordinates of the points $\tilde{T}^n_u(x,y)$ and $\tilde{T}^n_u(x,y')$ are $C_n(y)$ and $C_n(y')$, both of which belong to the interval $M_u(\overline{\Delta}^{(n)}_{a_n\dots a_1})$. The lemma follows. □

Theorem 3. *For $u \in \Psi$ and for almost all $x \in [0,1]$ the sequence of points $\tilde{T}^n_u(x,y)$ for positive integers n is distributed in the interior of the region $I \times J_u$ according to the density function $\rho_u(w,z)$ for all $y \in J_u \cup \{\infty\}$. In other words, for any Borel set $\mathbf{B} \subset I \times J_u$ which has boundary of measure zero and for almost all x,*

$$\lim_{n\to\infty} \frac{1}{n}\#\{j \le n \mid \tilde{T}^n_u(x,y) \in \mathbf{B}\} = \rho_u(\mathbf{B})$$

for all $y \in J_u \cup \{\infty\}$. Furthermore, for any uniformly continuous function $f \in L^1(\rho_u)$ and for almost all $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(\tilde{T}^n_u(x,y)) = \int f d\rho_u$$

for all $y \in J_u \cup \{\infty\}$.

The proof is modeled on arguments in [5] and [12].

Proof. Let $\mathbf{B}$ be a Borel subset of $I \times J_u$ whose boundary has measure zero. Consider the ϵ-collar $\mathbf{C}_\epsilon$ of $\mathbf{B}$, consisting of all points in the plane within ϵ of the boundary of $\mathbf{B}$. Define $\mathbf{B}^+_\epsilon = \mathbf{B} \cup \mathbf{C}_\epsilon$ and $\mathbf{B}^-_\epsilon = \mathbf{B} - \mathbf{C}_\epsilon$. Then we have $\mathbf{B}^-_\epsilon \subset \mathbf{B} \subset \mathbf{B}^+_\epsilon$.

As a consequence of Lemma 1, given $x \in (0,1)$ u-irrational and $\epsilon > 0$, there exists $N^- > 0$ so that for $n > N^-$, if $\tilde{T}^n_u(x,y) \in \mathbf{B}^-_\epsilon$ for some $y \in J_u \cup \{\infty\}$, then $\tilde{T}^n_u(x,y) \in \mathbf{B}$ for all y. Similarly, N^+ can be chosen so that for $n > N^+$, if $\tilde{T}^n_u(x,y) \in \mathbf{B}$ for some $y \in J_u \cup \{\infty\}$, then $\tilde{T}^n_u(x,y) \in \mathbf{B}^+_\epsilon$ for all y.

Fix $y^* \in J_u \cup \{\infty\}$. By the above, for any $(x,y) \in I \times J_u$ for which the first and last limits exist,

$$\begin{aligned} (23) \qquad & \lim_{n\to\infty} \frac{1}{n}\#\{j \le n \mid \tilde{T}^j_u(x,y) \in \mathbf{B}^-_\epsilon\} \\ \le\ & \liminf_{n\to\infty} \frac{1}{n}\#\{j \le n \mid \tilde{T}^j_u(x,y^*) \in \mathbf{B}\} \\ \le\ & \limsup_{n\to\infty} \frac{1}{n}\#\{j \le n \mid \tilde{T}^j_u(x,y^*) \in \mathbf{B}\} \\ \le\ & \lim_{n\to\infty} \frac{1}{n}\#\{j \le n \mid \tilde{T}^j_u(x,y) \in \mathbf{B}^+_\epsilon\}. \end{aligned}$$

By the ergodicity of the natural extension and Birkoff's Ergodic Theorem the first and last limits exist for almost all $(x,y) \in I \times J_u$ and are, respectively, $\rho_u(\mathbf{B}^-_\epsilon)$ and $\rho_u(\mathbf{B}^+_\epsilon)$. Since $\mathbf{B}$ has boundary of measure zero, these two values can be made arbitrarily close by choosing ϵ small enough. Since $y^* \in J_u \cup \{\infty\}$ is arbitrary it follows that for almost all x

$$\lim_{n\to\infty} \frac{1}{n}\#\{j \le n \mid \tilde{T}^j_u(x,y^*) \in \mathbf{B}\} = \rho_u(\mathbf{B}),$$

proving that $\tilde{T}^n_u(x,y)$ is distributed according to the density ρ_u.

Turning now to the second assertion of the theorem, suppose that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(\tilde{T}_u^n(x,y))$ converges to a value L. Then for $y' \neq y$

$$\left|\frac{1}{n}\sum_{i=1}^{n} f(\tilde{T}_u^n(x,y')) - L\right| \tag{24}$$
$$< \left|\frac{1}{n}\sum_{i=1}^{n} f(\tilde{T}_u^n(x,y')) - f(\tilde{T}_u^n(x,y))\right| + \left|\frac{1}{n}\sum_{i=1}^{n} f(\tilde{T}_u^n(x,y)) - L\right|.$$

The first sum to the right of the inequality can be made arbitrarily small when n is large, as a consequence of Lemma 1 and the uniform continuity of f. The second term goes to zero by hypothesis. Thus the limit with y' replacing y also converges to L. Since f is integrable, the ergodicity of $\tilde{T}_u$ and the Ergodic Theorem give that for almost all x the limit L exists and has the value $\int f d\rho_u$. □

5.2. **Application 1: The Khintchine-Lévy Theorem.** As a first application of Theorem 3 we prove the following generalization of the classical result of Khintchine and Lévy [18].

Theorem 4. *For $u \in \Psi$ and almost all $x \in I$*

$$\lim_{n\to\infty} \frac{\log|q_n|}{n} = \begin{cases} \log\sqrt{|k|} - \left(\log\frac{|k|+1}{|k|}\right)^{-1} \mathcal{L}_2(-\frac{1}{|k|}) & \text{if } k(-1)^m > 0, \\ \log\sqrt{|k|} + \left(\log\frac{|k|}{|k|-1}\right)^{-1} \mathcal{L}_2(\frac{1}{|k|}) & \text{if } k(-1)^m < 0, \end{cases}$$

where $\mathcal{L}_2(z) = \int_z^0 \frac{\log(1-t)}{t}\, dt$ is the Euler dilogarithm.

The following lemma is useful for relating the limit in Theorem 4 to an ergodic sum.

Lemma 2. *For $u \in \Psi$ and $x = [a_1, a_2, \ldots]_u$ u-irrational, the Möbius transformations C_n are defined and*

$$C_n(\infty) = \begin{cases} \frac{-\operatorname{sgn}(k)\sqrt{|k|}q_n}{q_{n-1}} & \text{if } u \text{ is Gauss-like}, \\ 1 - \operatorname{sgn}(k)\frac{\sqrt{|k|}q_n}{q_{n-1}} & \text{if } u \text{ is Renyi-like}. \end{cases}$$

Proof. By taking inverses in formula (9) we get

$$C_n(\infty) = A_u^{-1}\begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix}^{-1}(\infty) = \begin{cases} \frac{kq_n}{kq_n - s_n} & \text{if } u \text{ is Gauss-like}, \\ \frac{s_n}{s_n - kq_n} & \text{if } u \text{ is Renyi-like}. \end{cases} \tag{25}$$

In order to simplify these expressions consider the identity

$$\begin{pmatrix} p_n & r_n \\ q_n & s_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & r_{n-1} \\ q_{n-1} & s_{n-1} \end{pmatrix} B^{a_n} A_u^{-1}. \tag{26}$$

When u is Gauss-like, the right-hand side of (26) becomes

$$\operatorname{sgn}(k)\begin{pmatrix} \frac{1}{\sqrt{|k|}}(a_n p_{n-1} + r_{n-1}) & \frac{k}{\sqrt{|k|}}(p_{n-1}(a_n+1) + r_{n-1}) \\ \frac{1}{\sqrt{|k|}}(a_n q_{n-1} + s_{n-1}) & \frac{k}{\sqrt{|k|}}(q_{n-1}(a_n+1) + s_{n-1}) \end{pmatrix}.$$

Comparing lower left-hand entries and solving for s_{n-1} gives $s_{n-1} = \operatorname{sgn}(k)\sqrt{|k|}q_n - a_n q_{n-1}$. If we substitute this value for s_{n-1} into the expression

$$s_n = \operatorname{sgn}(k)\frac{k}{\sqrt{|k|}}(q_{n-1}(a_n + 1) + s_{n-1})$$

that results from equating the lower right-hand entries, we get $s_n = \sqrt{|k|}q_{n-1} + kq_n$. Substituting this value into formula (25) gives the result when u is Gauss-like. The same type of matrix computation when u is Renyi-like gives $s_n = -\sqrt{|k|}q_{n-1} + kq_n$, which proves the lemma. □

While Lemma 2 is needed for the proof of Theorem 4, there are several other interesting consequences that can be drawn from the lemma and its proof. The following two propositions are generalizations of well-known classical results about continued fractions [18].

Proposition 5. *For all $u \in \Psi$ and $x = [a_1, a_2, \ldots]_u$ u-irrational,*
a) $|p_n q_{n-1} - q_n p_{n-1}| = \frac{1}{\sqrt{|k|}}$,
b) $\triangle^{(n)}_{a_1 \ldots a_n}$ *has the second endpoint* $\frac{\sqrt{|k|}p_n \pm p_{n-1}}{\sqrt{|k|}q_n \pm q_{n-1}}$ *with the plus sign if u is Gauss-like and the minus sign if u is Renyi-like.*

Proof. The values for r_n can be determined using the same methods of comparing entries, as in the proof of Lemma 2. The result is $r_n = kp_n \pm \sqrt{|k|}p_{n-1}$ where, as above, the sign is plus if u is Gauss-like and the sign is minus if u is Renyi-like. The first part then follows by observing that the determinant of the matrix (9) is $p_n s_n - q_n r_n = \sqrt{|k|}(p_n q_{n-1} - q_n p_{n-1})$, which has absolute value one.

The second endpoint of $\triangle^{(n)}_{a_1 \ldots a_n}$ is $C_n^{-1}(1) = C_n^{-1}A_u^{-1}(0) = \frac{r_n}{s_n}$ if u is Gauss-like and $C_n^{-1}(0) = C_n^{-1}A_u^{-1}(0) = \frac{r_n}{s_n}$ if u is Renyi-like. Thus the second part is a consequence of the determinations of r_n and s_n. □

Proposition 6. *If $x = [a_1, a_2, \ldots]_u$ is a u-irrational number with convergents $[a_1, a_2, \ldots, a_n]_u = \frac{p_n}{q_n}$, then*

$$[a_n, \ldots, a_1]_u = \begin{cases} \frac{\sqrt{|k|}q_{n-1}}{q_n} & \textit{if } u \textit{ is Gauss-like,} \\ 1 - \frac{\sqrt{|k|}q_{n-1}}{q_n} & \textit{if } u \textit{ is Renyi-like.} \end{cases}$$

Proof. By the properties of $\hat{C}_n$ and M_u from the proof of Lemma 1,

$$[a_n, \ldots, a_1]_u = \hat{C}_n^{-1}A_u^{-1}(\infty) = M_u C_n M_u^{-1} A_u^{-1}(\infty) = M_u C_n(\infty)$$

$$= \begin{cases} M_u(\frac{-\operatorname{sgn}(k)\sqrt{|k|}q_n}{q_{n-1}}) = \frac{\sqrt{|k|}q_{n-1}}{q_n} & \text{if } u \text{ is Gauss-like,} \\ M_u(1 - \operatorname{sgn}(k)\frac{\sqrt{|k|}q_n}{q_{n-1}}) = 1 - \frac{\sqrt{|k|}q_{n-1}}{q_n} & \text{if } u \text{ is Renyi-like.} \end{cases}$$

□

Proof of Theorem 4. Let $g : I \times J_u \to \mathbb{R}$ be a ρ_u-integrable, uniformly continuous function. As at the beginning of Section 4.2 let $(x_n, y_n) = \tilde{T}_u^{n+1}(x, \infty)$ for integers $n \geq 0$. Without loss of generality suppose x to be u-irrational with u-expansion

$[a_1, a_2, \ldots]_u$, in which case $y_n = C_{n+1}(\infty)$. It follows from Theorem 3 that for almost all $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g(x_j, y_j) = \int_{I\times J_u} g d\rho_u. \tag{27}$$

If u is Gauss-like, then let $g(x_j, y_j) = \log|y_j|$. From Lemma 2 we get $g(x_j, y_j) = \log|\frac{\sqrt{|k|}q_{j+1}}{q_j}|$. Unpacking the left-hand side of (27) gives

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} (\log\sqrt{|k|} + \log|q_{j+1}| - \log|q_j|), \tag{28}$$

$$\log\sqrt{|k|} + \lim_{n\to\infty} \frac{\log|q_n|}{n}.$$

If u is Renyi-like, then take $g(x_j, y_j) = \log|1-y_j| = \log|\frac{\sqrt{|k|}q_{j+1}}{q_j}|$, and the left-hand side of (27) is again given by (28).

By first integrating with respect to x and then making a simple substition, e.g. $y = -\frac{1}{x}$ if $m = 0$ and $k > 0$, the integral in (27) takes the form

$$\int_{I\times J_u} g d\rho_u = -c_u \int_0^{\frac{1}{|k|}} \frac{\log x}{1 + (-1)^m \mathrm{sgn}(k) x} dx. \tag{29}$$

Suppose $(-1)^m k > 0$. After integration by parts the right-hand side of (29) becomes

$$\begin{aligned} -c_u \int_0^{\frac{1}{|k|}} \frac{\log x}{1+x} dx &= -c_u \left(\log x \log(1+x) \Big|_0^{\frac{1}{|k|}} - \int_0^{\frac{1}{|k|}} \frac{\log(1+x)}{x} \quad dx \right) \\ &= -c_u \left(-\log|k| \log(\frac{|k|+1}{|k|}) - \int_0^{-\frac{1}{|k|}} \frac{\log(1-u)}{u} \quad du \right) \\ &= -\left(\log(\frac{|k|+1}{|k|}) \right)^{-1} \left(-\log|k| \log(\frac{|k|+1}{|k|}) + \mathcal{L}_2(-\frac{1}{|k|}) \right). \end{aligned}$$

In light of (28) the proof is complete.

If $(-1)^m k < 0$, then we also have $|k| > 1$. Integrate by parts to get

$$\begin{aligned} -c_u \int_0^{\frac{1}{|k|}} \frac{\log x}{1-x} dx &= -c_u \left(-\log x \log(1-x) \Big|_0^{\frac{1}{|k|}} + \int_0^{\frac{1}{|k|}} \frac{\log(1-x)}{x} \quad dx \right) \\ &= -\left(\log(\frac{|k|}{|k|-1}) \right)^{-1} \left(\log|k| \log(\frac{|k|-1}{|k|}) - \mathcal{L}_2(\frac{1}{|k|}) \right). \end{aligned}$$

Combined with (28) the final case of Theorem 4 is proved. □

Since by Section 2.2 and Proposition 5 we know the endpoints of an interval of monotonicity, it becomes possible to compute its length:

$$\left|\triangle^{(n)}_{a_1\ldots a_n}\right| = \left| \frac{p_n}{q_n} - \frac{\sqrt{|k|}p_n \pm p_{n-1}}{\sqrt{|k|}q_n \pm q_{n-1}} \right| = \left| \frac{1}{q_n(\sqrt{|k|}q_n \pm q_{n-1})} \right|,$$

where the sign depends on the type of u as in Proposition 5. This lead to the following corollary to Theorem 4, which has its classical analogue in [3].

Corollary 2. *For $u \in \Psi$ and almost all $x \in I$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \left|\triangle^{(n)}_{a_1\ldots a_n}\right| = \begin{cases} -\log k + 2\left(\log \frac{|k|+1}{|k|}\right)^{-1} \mathcal{L}_2(-\frac{1}{|k|}) & \text{if } (-1)^m k > 0, \\ -\log k - 2\left(\log \frac{|k|}{|k|-1}\right)^{-1} \mathcal{L}_2(\frac{1}{|k|}) & \text{if } (-1)^m k < 0. \end{cases}$$

5.3. **Application 2: The rate of approximation.** The natural extension can be used to prove the analogues of two other well-known results from the classical theory [3, 5].

Theorem 5. *For $u \in \Psi$ and almost all $x \in I$,*

1. $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{|x - \frac{p_i}{q_i}|}{|x - \frac{p_{i-1}}{q_{i-1}}|} = \begin{cases} -1 - \left(\log \frac{|k|+1}{|k|}\right)^{-1} \mathcal{L}_2(-\frac{1}{|k|}) & \text{if } (-1)^m k > 0, \\ 1 - \left(\log \frac{|k|}{|k|-1}\right)^{-1} \mathcal{L}_2(\frac{1}{|k|}) & \text{if } (-1)^m k < 0. \end{cases}$$

2. $$\lim_{n\to\infty} \frac{1}{n} \log \left|x - \frac{p_n}{q_n}\right| = \begin{cases} -\log k + 2\left(\log \frac{|k|+1}{|k|}\right)^{-1} \mathcal{L}_2(-\frac{1}{|k|}) & \text{if } (-1)^m k > 0, \\ -\log k - 2\left(\log \frac{|k|}{|k|-1}\right)^{-1} \mathcal{L}_2(\frac{1}{|k|}) & \text{if } (-1)^m k < 0. \end{cases}$$

Lemma 3. *For u-irrational $x \in (0,1)$ and for all integers $n \geq 1$,*

$$\left|x - \frac{p_i}{q_i}\right| \left|x - \frac{p_{i-1}}{q_{i-1}}\right|^{-1} = \begin{cases} \left|\frac{x_{i-1}}{y_{i-1}}\right| & \text{if } u \text{ is Gauss-like,} \\ \left|\frac{1-x_{i-1}}{1-y_{i-1}}\right| & \text{if } u \text{ is Renyi-like.} \end{cases}$$

Proof. We make use of the following well-known identity that holds for any Möbius transformation C and for $z, w \in \mathbb{C}$ with $C(z), C(w) \neq \infty$:

$$\left(\frac{C(z) - C(w)}{z - w}\right)^2 = C'(z)C'(w)\,. \tag{30}$$

Suppose u is Gauss-like. From the definitions of θ_i and (x_i, y_i), Theorem 2 and Remark 2, we get

$$\frac{|x - \frac{p_i}{q_i}|}{|x - \frac{p_{i-1}}{q_{i-1}}|} = \frac{\theta_i}{\theta_{i-1}} \left(\frac{q_{i-1}}{q_i}\right)^2 = \left|\frac{x_{i-1} - y_{i-1}}{x_i - y_i}\right| \frac{|k|}{(C_i(\infty))^2} = \left|\frac{x_{i-1} - y_{i-1}}{x_i - y_i}\right| \frac{|k|}{y_{i-1}^2}.$$

Using the relation $x_i = B^{-a_i} A_u(x_{i-1})$ and a similar one for y_i, we can rewrite the final expression above and then apply the identity (30) with $C = B^{-a_n} A_u$ to get

$$\begin{aligned} &\left|\frac{B^{-a_i} A_u(x_{i-1}) - B^{-a_i} A_u(y_{i-1})}{x_{i-1} - y_{i-1}}\right|^{-1} \frac{|k|}{y_{i-1}^2} \\ &= \left|(B^{-a_i} A_u)'(x_{i-1})(B^{-a_i} A_u)'(y_{i-1})\right|^{-\frac{1}{2}} \frac{|k|}{y_{i-1}^2}. \end{aligned}$$

Since $(B^{-a_i} A_u)'(x) = A_u'(x) = \frac{-k}{x^2}$, the above is equal to

$$\left(\frac{|x_{i-1} y_{i-1}|}{|k|}\right) \frac{|k|}{y_{i-1}^2} = \left|\frac{x_{i-1}}{y_{i-1}}\right|.$$

If u is Renyi-like, then $C_i(\infty)$ is replaced by $1 - C_i(\infty) = 1 - y_{i-1}$ and $(B^{-a_i} A_u)'(x) = A_u'(x) = \frac{k}{(1-x)^2}$, which also gives the desired result. □

Proof of Theorem 5. Define the function

$$f_u(x,y) = \begin{cases} \left|\frac{x}{y}\right| & \text{if } u \text{ is Gauss-like,} \\ \left|\frac{1-x}{1-y}\right| & \text{if } u \text{ is Renyi-like.} \end{cases}$$

Then by Theorem 3 and the previous Lemma 3, for almost all x

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{|x - \frac{p_i}{q_i}|}{|x - \frac{p_{i-1}}{q_{i-1}}|} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f_u(\tilde{T}_u^{n+1}(x,\infty)) = \int_{I\times J_u} f_u \quad d\rho_u.$$

When $m = 0$ and $k > 0$ the integral becomes

$$c_u \int_0^1 \int_{-\infty}^{-k} \frac{x}{-y(x-y)^2} \quad dydx = c_u \int_0^1 \int_{-\infty}^{-k-x} \left(-\frac{1}{x(v+x)} + \frac{1}{xv} - \frac{1}{v^2} \right) \quad dvdx$$
$$= c_u \int_0^1 \left(\frac{-\log k + \log(x+k)}{x} - \frac{1}{x+k} \right) \quad dx,$$

where we have substituted $v = y - x$. Now, substituting $t = -x/k$, the above becomes

$$c_u \left(\int_0^{-\frac{1}{k}} \frac{\log(1-t)}{t} \quad dt - \log\left(\frac{1+k}{k}\right) \right) = c_u \left(-\mathcal{L}_2(-\frac{1}{k}) - \log\left(\frac{1+k}{k}\right) \right)$$

which is what we expect for part 1. In the other cases much the same approach using simple substitutions and partial fractions gives the result.

Part 2 would be similarly straightforward except that $\log f_u$ is far from being absolutely continuous on $I \times J_u$. Let us revisit the proof of the second assertion of Theorem 3. Suppose u is Gauss-like, f is the function $\log f_u$ and $x = [a_1, a_2, \ldots]$ thereby defining the specific transformations C_n for positive integers n. Then the first term on the second line of the inequality (24) becomes

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(\tilde{T}_u^n(x,y')) - f(\tilde{T}_u^n(x,y)) \right|$$
$$= \left| \frac{1}{n} \sum_{i=1}^{n} \log\left|\frac{C_n(x)}{C_n(y')}\right| - \log\left|\frac{C_n(x)}{C_n(y)}\right| \quad \right| = \left| \frac{1}{n} \sum_{i=1}^{n} \log\left|\frac{C_n(y)}{C_n(y')}\right| \quad \right|.$$

It follows from Lemma 1 that $\lim_{n\to\infty} \log\left|\frac{C_n(y)}{C_n(y')}\right| = 0$ and consequently that the limit of the above sum is also zero. That proves the second assertion of Theorem 3 with the revised assumption that f is the function $\log f_u$, as defined above, with u Gauss-like. There is little modification required to prove it when u is Renyi-like.

To prove part 2 of Theorem 5 we work with the function $\log f_u$. By the above strengthening of Theorem 3 and Lemma 3, for almost all x,

$$\lim_{n\to\infty} \frac{1}{n} \log\left|x - \frac{p_n}{q_n}\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log\left(\frac{|x - \frac{p_i}{q_i}|}{|x - \frac{p_{i-1}}{q_{i-1}}|} \right) = \int_{I\times J_u} \log f_u \quad d\rho_u.$$

When $m = 1$ and $k > 1$ the integral becomes

$$c_u \int_0^1 \int_{-\infty}^{1-k} \log\left(\frac{1-x}{1-y}\right) \frac{1}{(x-y)^2} \quad dydx$$
$$= c_u \int_{-1}^{0} \int_{-\infty}^{-k} \log\left(\frac{-x}{-y}\right) \frac{1}{(x-y)^2} \quad dydx,$$

where the obvious substitution has been made. Then we seperate the integrand into two pieces and integrate each seperately to get

$$c_u \int_{-1}^{0} \left(\int_{-\infty}^{-k} \frac{\log(-x)}{(x-y)^2} \quad dy - \int_{-\infty}^{-k} \frac{\log(-y)}{(x-y)^2} \quad dy \right) \quad dx$$
$$= c_u \int_{-1}^{0} \left(\frac{\log(-x)}{(x+k)} - \frac{\log k}{(x+k)} + \frac{\log k}{x} - \frac{\log(x+k)}{x} \quad \right) dx.$$

With a little manipulation this becomes

$$c_u \left(\frac{1}{k} \int_{-1}^{0} \frac{\log(-\frac{x}{k})}{(\frac{x}{k}+1)} \quad dx - \frac{1}{k} \int_{-1}^{0} \frac{\log(1+\frac{x}{k})}{(\frac{x}{k})} \quad dx \right).$$

Substituting $v = -\frac{x}{k}$ in both integrals results in

$$-c_u \left(\int_{\frac{1}{k}}^{0} \frac{\log v}{1-v} \quad dv \right) - c_u \left(\int_{\frac{1}{k}}^{0} \frac{\log(1-v)}{v} \quad dv \right).$$

The first integral was computed in the proof of Theorem 4 and is

$$c_u \left(\log|k| \log(\frac{|k|-1}{|k|}) - \mathcal{L}_2(\frac{1}{|k|}) \right);$$

the second is exactly $-c_u \mathcal{L}_2(\frac{1}{k})$. A similar approach works in the other cases. □

Remark 5. With the function f_u as defined in the beginning of the proof above, Theorem 5 actually shows that for almost all $x \in (0,1)$, for any $y \in J_u \cup \{\infty\}$ the ergodic sums

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f_u(\tilde{T}_u^{n+1}(x,y)) \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log f_u(\tilde{T}_u^{n+1}(x,y))$$

converge to the values given respectively in parts 1 and 2 of the theorem.

5.4. **Application 3: Entropy ties it all together and $\frac{1}{n}\log\theta_n$.** Let $h(T)$ denote the measure theoretic entropy of the transformation T [9, 14]. Rohlin's entropy formula [14, 19]

$$h(T_u) = \int_0^1 \log|T_u'(x)| \, d\mu_u \tag{31}$$

can be applied to compute the entropy of the transformations T_u. Combined with Theorems 4 and 5 we have

Corollary 3. *For $u \in \Psi$ and for almost all x,*

$$h(T_u) = 2 \lim_{n\to\infty} \frac{\log|q_n|}{n} = -\lim_{n\to\infty} \frac{1}{n} \log\left|x - \frac{p_n}{q_n}\right| = \lim_{n\to\infty} \frac{1}{n} \log\left|\triangle^{(n)}_{a_1\ldots a_n}\right|.$$

Proof. Formula (31) is easily manipulated to give

$$h(T_u) = -2c_u \int_0^{\frac{1}{|k|}} \frac{\log x}{1 + (-1)^m \mathrm{sgn}(k)x} dx - \log |k|.$$

The first equality in the corollary follows from equations (27) and (29). The second equality follows by equating Theorem 4 with part 2 of Theorem 5, with the appropriate multiples inserted. □

A similar result appears in [10] for the cases $u = (1, n)$, where n is a positive integer.

Observe that from the second equality in Corollary 3 we get

$$0 = \lim_{n\to\infty} \left(\frac{1}{n} \log \left| x - \frac{p_n}{q_n} \right| + 2\frac{\log |q_n|}{n} \right) = \lim_{n\to\infty} \frac{1}{n} \left(\log \left(\frac{\theta_n}{q_n^2} \right) + 2 \log |q_n| \right),$$

which simplifies to prove

Corollary 4. *For $u \in \Psi$ and almost all $x \in I$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \theta_n = 0.$$

References

[1] R.L. Adler, L. Flatto, Geodesic flows, interval maps, and symbolic dynamics. *Bull. Amer. Math. Soc.* **25** (1991), 229–334. MR **92b:**58172

[2] T. Bedford, H. Keane, C. Series, eds., *Ergodic Theory, Symbolic Dynamics and Hyperbolic Spaces*, Oxford Univ. Press, 1991. MR **93e:**58002

[3] C. Billingsley, *Ergodic Theory and Information*. J. Wiley & Sons, New York-London, 1965. MR **33:**254

[4] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York-London, 1968. MR **38:**1718

[5] W. Bosma, H. Jager and F. Wiedijk, Some metrical observations on the approximation of continued factions. *Indag. Math.* **45** (1983), 281–299. MR **85f:**11059

[6] R. Burton, C. Kraaikamp and T. Schmidt, Natural extensions for the Rosen Fractions. *Trans. Amer. Math. Soc.* **352** (1999), 1277–1298. MR **2000j:**11123

[7] H. Cohn, Representation of Markoff's binary quadratic forms by geodesics on a perforated torus. *Acta Arith.* **18** (1971), 125–136. MR **44:**5277

[8] C. Series, The modular surface and continued fractions. *J. London Math. Soc.* **31** (1985), 69 – 80. MR **87c:**58094

[9] I. P. Cornfeld, S.V. Fomin, and Ya. G. Sinai, *Ergodic Theory*. Springer-Verlag, Berlin-Heidelberg-New York, 1982. MR **87f:**28019

[10] K. Gröchenig and A. Haas, Backward continued fractions, invariant measures, and mappings of the interval. *Ergodic Th. and Dyn. Sys.* **16** (1996), 1241–1274. MR **97m:**58114

[11] A. Haas, Invariant Measures and Natural Extensions. *Canad. Math. Bull.* **45** (2002), 97–108. MR **2002j:**37005

[12] H. Jager, The distribution of certain sequences connected with the continued fraction. *Indag. Math.* **89** (1) (1986), 61–69. MR **87g:**11092

[13] A. Ya. Khintchine, *Continued Fractions* (trans. from 3rd Russian ed. by P. Wymm). P. Noordhoff, Ltd., Groningen. MR **28:**5038

[14] R. Mañé, *Ergodic Theory and Differentiable Dynamics*. Springer-Verlag, Berlin, 1983. MR **88c:**58040

[15] H. Nakada, Metrical theory for a class of continued fraction transformations and their natural extensions. *Tokyo J. Math.* **4** (1981), 399–426. MR **83k:**10095

[16] G. J. Rieger, Mischung und Ergodizität bei Kettenbrüchen nach nächsten Ganzen. *J. Reine Angew. Math.* **310** (1979), 171–181. MR **81c:**10066

[17] A. Rényi, Valòs szàmok elöàllitàsàra szölgàlò algoritmusokròl. *M. T.A. Mat. ès Fiz. Oszt. Kžl.* **7** (1957), 265–293. MR **20:**4113

[18] A. Rockett and P. Szüsz, *Continued Fractions*. World Sci. Pub., Singapore, 1992. MR **93m:**11060

[19] V.A. Rohlin, Exact endomorphisms of a Lebesgue space. *A.M.S. Transl.* **39** (1964), 1–37. MR **37:**4234

[20] F. Schweiger, *Ergodic Theory of Fibered Systems and Metric Number Theory*. Clarendon Press, 1995. MR **97h:**11083

[21] F. Schweiger, *Multidimensional Continued Fractions*. Oxford Univ. Press, 2000.

[22] S.M. Rudolfer and K.M. Wilkinson, A number-theoretic class of weak Bernoulli transformations. *Math. Systems Theory* **7** (1973), 14–24. MR **48:**2100

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