TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 7, Pages 2557–2568
S 0002-9947(04)03585-8
Article electronically published on February 27, 2004

A JOIN THEOREM FOR THE COMPUTABLY ENUMERABLE DEGREES

CARL G. JOCKUSCH, JR., ANGSHENG LI, AND YUE YANG

Abstract. It is shown that for any computably enumerable (c.e.) degree $\mathbf{w}$, if $\mathbf{w} \neq \mathbf{0}$, then there is a c.e. degree $\mathbf{a}$ such that $(\mathbf{a} \vee \mathbf{w})' = \mathbf{a}'' = \mathbf{0}''$ (so $\mathbf{a}$ is low$_2$ and $\mathbf{a} \vee \mathbf{w}$ is high). It follows from this and previous work of P. Cholak, M. Groszek and T. Slaman that the low and low$_2$ c.e. degrees are not elementarily equivalent as partial orderings.

1. Introduction

This paper concerns properties of the join operation on the computably enumerable degrees. Part of our motivation comes from the study of the join operation on the Turing degrees by D. Posner and R. Robinson. In [9], Theorem 1, they showed that in the Turing degrees, for any nonzero degree $\mathbf{w} \leq \mathbf{0}'$, there exists a degree $\mathbf{a}$ such that $\mathbf{a} \vee \mathbf{w} = \mathbf{a}' = \mathbf{0}'$, where $\mathbf{a}'$ denotes the Turing jump of $\mathbf{a}$. In contrast, by S. B. Cooper [3] and C. E. M. Yates (unpublished), there exist nonzero noncuppable degrees in $\mathcal{R}$ — the class of all c.e. degrees. Hence there is no hope for such a join theorem to hold in $\mathcal{R}$. However, we can still salvage the idea by applying the jump operator one more time. The main theorem of this paper establishes the strongest possible analogue for the c.e. degrees of the Posner-Robinson join theorem [9].

Theorem 1.1 (Join theorem for c.e. degrees)**.** *For any c.e. degree* $\mathbf{w}$*, if* $\mathbf{w} \neq \mathbf{0}$*, then there is a c.e. degree* $\mathbf{a}$ *such that* $(\mathbf{a} \vee \mathbf{w})' = \mathbf{a}'' = \mathbf{0}''$*. Equivalently, for any nonzero c.e. degree* $\mathbf{w}$*, there is a low*$_2$ *c.e. degree* $\mathbf{a}$ *such that* $\mathbf{a} \vee \mathbf{w}$ *is high.*

Many interesting results follow from Theorem 1.1. The most significant one is that the jump classes low and low$_2$ are not elementarily equivalent. To see that, we need to recall the notion of deep degrees and almost deep degrees, and we make some remarks about definable ideals along the way.

M. Bickford and C. Mills [1] introduced the notion of *deep degree.* They called a c.e. degree $\mathbf{a}$ *deep* if for every c.e. degree $\mathbf{x}$, $\mathbf{x}' = (\mathbf{a} \vee \mathbf{x})'$, namely, joining with

Received by the editors June 11, 2002.
2000 *Mathematics Subject Classification.* Primary 03D25, 03D30; Secondary 03D35.
Key words and phrases. Join theorem, computably enumerable degree, definable ideals, Turing jump.
The first author was partially supported by NSF Grant DMS-98-03073. The second author was supported by EPSRC Research Grant no. GR/M 91419, "Turing Definability" (UK), by NSF Grant No. 69973048, by NSF Major Grant No. 19931020 (P. R. China), and by National Distinguished Young Investigator Award no. 60325206 (China). The third author was partially supported by the NSTB OAP programme and NUS Grant No. R-146-000-028-112 (Singapore). All three authors were partially supported by NSFC grant No. 60310213 "New Directions in Theory and Applications of Models of Computation" (China).

$\mathbf{a}$ preserves the jump of every c.e. degree. S. Lempp and T. Slaman [5] showed that the only deep degree is $\mathbf{0}$. Extending the notion of deep degree, P. Cholak, M. Groszek and T. Slaman [2], page 900, called a c.e. degree $\mathbf{a}$ *n-deep* if for every c.e. degree $\mathbf{x}$, $\mathbf{x}^{(n)} = (\mathbf{a} \vee \mathbf{x})^{(n)}$. Let $\mathbf{DP}_n$ be the set of all n-deep degrees. They stated without proof that, for all n, $\mathbf{0}$ is the only n-deep degree. This follows at once from our main result, Theorem 1.1. They also stated without proof that for every nonzero c.e. degree $\mathbf{w}$ there is a nonhigh c.e. degree $\mathbf{a}$ such that $\mathbf{a} \vee \mathbf{w}$ is high, and our main result strengthens this by showing that $\mathbf{a}$ can be chosen to be low$_2$.

In the same paper [2], Cholak, Groszek and Slaman introduced the notion of *almost deep degrees*, the c.e. degrees $\mathbf{a}$ such that for any low c.e. degree $\mathbf{x}$, $\mathbf{a} \vee \mathbf{x}$ is low. They obtained the following result.

Theorem 1.2 (Cholak, Groszek and Slaman [2])**.** *There is a nonzero almost deep degree.*

The motivation is to find definable ideals in the c.e. degrees. The collection of all almost deep degrees is a nontrivial ideal. However, it is not known to be definable, since it is defined in terms of low degrees. By a result of A. Nies, R. Shore and T. Slaman [8], all other jump classes ($\mathbf{L_n}$ ($n \geq 2$) and $\mathbf{H_n}$ ($n \geq 0$)) are definable. This raises the possibility that one could generalise the notion of almost deep degrees and get sequences of definable ideals. For example, one could consider the c.e. degrees $\mathbf{a}$ such that for every low$_n$ c.e. degree $\mathbf{x}$, $\mathbf{a} \vee \mathbf{x}$ is low$_n$: Define $\mathbf{PL}_n$ by

$$\mathbf{PL}_n = \{\mathbf{a} \in \mathcal{R} \mid (\forall \mathbf{x} \in \mathcal{R})[\mathbf{x}^{(n)} = \mathbf{0}^{(n)} \leftrightarrow (\mathbf{a} \vee \mathbf{x})^{(n)} = \mathbf{0}^{(n)}]\}.$$

Or one could consider the c.e. degrees $\mathbf{a}$ such that for any c.e. degree $\mathbf{x}$, $\mathbf{a} \vee \mathbf{x}$ is high$_n$ implies $\mathbf{x}$ is high$_n$: Define $\mathbf{PH}_n$ by

$$\mathbf{PH}_n = \{\mathbf{a} \in \mathcal{R} \mid (\forall \mathbf{x} \in \mathcal{R})[\mathbf{x}^{(n)} = \mathbf{0}^{(n+1)} \leftrightarrow (\mathbf{a} \vee \mathbf{x})^{(n+1)} = \mathbf{0}^{(n+1)}]\}.$$

By the definability of jump classes other than low$_1$, for each $n \geq 1$, both $\mathbf{PL}_{n+1}$ and $\mathbf{PH}_n$ are definable ideals of $\mathcal{R}$. Cholak, Groszek and Slaman [2] stated without proof (as remarked above) that $\mathbf{PH}_1 = \{\mathbf{0}\}$. They also stated that it is conceivable that $\mathbf{PL}_2 \neq \{\mathbf{0}\}$, and the same may hold for $\mathbf{PL}_n$, $n \geq 3$. By Theorem 1.1, we now know that all such ideals are trivial:

Corollary 1.3. *For each $n \geq 1$,*

$$\mathbf{DP}_n = \mathbf{PL}_{n+1} = \mathbf{PH}_n = \{\mathbf{0}\}.$$

Although Theorem 1.1 rules out some approaches to producing nontrivial definable ideals, it sheds some light on the elementary equivalence problems of jump classes. By making use of splitting properties, it is known that for all pairs (m, n), if $m \leq 2$ and $n > 2$, then $\mathbf{Th}(\mathbf{L}_m) \neq \mathbf{Th}(\mathbf{L}_n)$ (see the discussion in Li [7]). Recently, after showing that $\mathbf{Th}(\mathbf{H}_1) \neq \mathbf{Th}(\mathbf{H}_n)$ ([7]), Li [7], [6] raises the question again:

(1) Are there any $m \neq n$ such that $\mathbf{Th}(\mathbf{H}_m) = \mathbf{Th}(\mathbf{H}_n)$?
(2) Are there any $m \neq n$ such that $\mathbf{Th}(\mathbf{L}_m) = \mathbf{Th}(\mathbf{L}_n)$?
(3) In particular, are the low$_1$ and low$_2$ c.e. degrees elementarily equivalent?

By Theorem 1.1 and Theorem 1.2, we can answer (3), whereas the other two remain open.

Corollary 1.4. *The low$_1$ and low$_2$ c.e. degrees are not elementarily equivalent.*

In fact, it follows by the same argument that for all $n \geq 2$ the low$_1$ and low$_n$ c.e. degrees are not elementarily equivalent.

The rest of the paper is devoted to the proof of Theorem 1.1 and it is organised as follows. In section 2, we describe the requirements and the strategies to satisfy the requirements; in section 3, we describe the priority tree of strategies and describe the full construction; and finally in section 4, we verify that all requirements are satisfied.

Our notation and terminology are standard and generally follow Soare [10]. We assume that the reader is familiar with tree constructions. We say that a number is *fresh* at a given point in the construction if it is the least natural number greater than any number mentioned so far.

2. Requirements and strategies

2.1. **The requirements.** Fix a c.e. set W. Our construction will be nonuniform. We construct c.e. sets A and B_e together with Turing functionals Γ, Δ_e and a computable partial function $\Omega_{e,i}$ such that, for each $e, i \in \omega$, either $A'' \leq_{\mathrm{T}} (A \oplus W)'$ via the functional Γ, or $B_e' \leq_{\mathrm{T}} B_e \oplus W$ via Δ_e, or (the characteristic function of) W is computed by $\Omega_{e,i}$. Notice that if $B_e' \leq_{\mathrm{T}} B_e \oplus W$, then we can simply take A to be B_e to establish Theorem 1.1. Also, if $\mathbf{a}'' \leq (\mathbf{a} \vee \mathbf{w})'$ for c.e. degrees $\mathbf{a}$ and $\mathbf{w}$, then $\mathbf{0}'' \leq \mathbf{a}'' \leq (\mathbf{a} \vee \mathbf{w})' \leq \mathbf{0}''$ so $\mathbf{a}'' = (\mathbf{a} \vee \mathbf{w})' = \mathbf{0}''$. Thus it suffices to satisfy the following requirements:

$\mathcal{R}_e$: $\mathrm{Tot}^A(e) = \lim_y \Gamma(A, W; e, y)$ or $(\exists B_e, \Delta_e)[\Delta_e(B_e, W)$ total and $(\forall i)\mathcal{S}_{e,i}]$,

$\mathcal{S}_{e,i}$: $W = \Omega_{e,i}$ or $\Delta_e(B_e, W; i) = B_e'(i)$,

where $e, i \in \omega$, $\mathrm{Tot}^A = \{e \mid \Phi_e(A) \text{ is total}\}$, $B_e' = \{i \mid \Psi_i(B_e; i)\downarrow\}$, $\{\Phi_e \mid e \in \omega\}$ and $\{\Psi_i \mid i \in \omega\}$ are fixed effective enumerations of all Turing functionals.

We will use a tree construction, but we first describe how individual requirements are met.

2.2. **The basic module for $\mathcal{R}_e$.** Let α be a node labelled $\mathcal{R}_e$. Roughly speaking, $\mathcal{R}_e$ has two basic responsibilities. First, $\mathcal{R}_e$ defines the functional $\Gamma(A, W; e, y)$ for more and more values of y and it also corrects the wrong values of $\Gamma(A, W; e, z)$ which were defined by nodes to the right of α; this is similar to making $A \oplus W$ high. Second, $\mathcal{R}_e$ preserves more and more $\Phi_e(A; x)$ computations; this is similar to making A low$_2$. Before we make the $\mathcal{R}_e$-strategy more precise, let us make some conventions on building Γ, called Γ-*rules*.

Whenever we define $\Gamma(A, W; e, y)$, we define its use $\gamma(e, y)$ as fresh, and *locate* it at some node of the priority tree. If $\gamma(e, y)$ is enumerated into A, then $\Gamma(A, W; e, y)$ is set to be undefined automatically. If $W \upharpoonright (\gamma(e, y) + 1)$ changes, then unless we explicitly set $\Gamma(A, W; e, y)$ to be undefined, $\Gamma(A, W; e, y)$ is redefined with the same value and the same use automatically. We do the same if $A \upharpoonright \gamma(e, y)$ changes. The Γ-rules will ensure that actions of other requirements and irrelevant W-changes do not make $\Gamma(A, W)$ nontotal.

Returning to the $\mathcal{R}_e$-strategy, we define the *length function* by

$$l(e) = \max\{x \mid (\forall y < x)[\Phi_e(A; y)\downarrow]\}.$$

We say that s is $\mathcal{R}_e$-*expansionary* if $l(e)[s] > l(e)[v]$ for all $v < s$. At non-$\mathcal{R}_e$-expansionary stages, we will define $\Gamma(A, W; e, y) = 0$ for more and more y. This ensures that if there are only finitely many $\mathcal{R}_e$-expansionary stages, then

$(\lambda y)\Gamma(A, W; e, y)$ is total and $\lim_y \Gamma(A, W; e, y) = 0$, and consequently, $\mathcal{R}_e$ is satisfied.

Suppose that there are infinitely many $\mathcal{R}_e$-expansionary stages. Then the $\mathcal{S}_{e,i}$-substrategies of $\mathcal{R}_e$ will try to correct the values of $\Gamma(A, W; e, y)$, at the risk of destroying some computation $\Phi_e(A; x)$ which should be preserved. The hope is to build a c.e. set B_e, and a Turing functional Δ_e to try to satisfy $\mathcal{S}_{e,i}$ for all $i \in \omega$.

2.3. **The basic module for $\mathcal{S}_{e,i}$.** An $\mathcal{S}_{e,i}$-strategy usually works at $\mathcal{R}_e$-expansionary stages. The goals for $\mathcal{S}_{e,i}$ are: Keep $\Delta_e(B_e, W; i) = B'_e(i)$, which is similar to making $B_e \oplus W$ complete; and preserve $\Psi_i(B_e; i)$, which is similar to making B_e low. We define Δ_e as follows. If $\Delta_e(B_e, W; i)$ is currently undefined, we define it to be $B'_e[i](s)$ with fresh use $\delta_e(i)$. Δ_e will have its Δ-*rules* similar to those for Γ, in order to ensure that actions of other requirements and irrelevant W-changes will not make $\Delta_e(B_e, W)$ nontotal. Whenever $\Psi_i(B_e; i)$ becomes convergent, $\mathcal{S}_{e,i}$ has the ability to put $\delta_e(i)$ into B_e to correct $\Delta_e(B_e, W; i)$. However, $\mathcal{S}_{e,i}$ will not do this immediately but rather "open a gap", in a fashion to be described. If W changes below $\delta_e(i) + 1$ during the gap, then $\Delta_e(B_e, W; i)$ can be corrected and the markers $\delta_e(j)$ can be moved to numbers bigger than $\psi_i(i)$ without putting their current positions into B_e. This provides a finitary win for $\mathcal{S}_{e,i}$. If such W-changes do not occur, then $\mathcal{S}_{e,i}$ puts $\delta_e(i)$ into B_e to correct $\Delta_e(B_e, W; i)$, and in this case, progress is made towards showing that W is computable.

2.4. **Description of one gap/cogap strategy.** We now see how a single $\mathcal{R}_e$ can be combined with its $\mathcal{S}_{e,i}$-substrategies. (Thus, we treat e as fixed and i as variable in this discussion.) $\mathcal{S}_{e,i}$ will have an auxiliary set C_i consisting of the numbers x such that $\Phi_e(A; x)$ is defined and preserved (meaning: able to survive the $\Gamma(A, W; e, y)$ correction). Let us call C_i the *clearing set* of $\mathcal{S}_{e,i}$. $\mathcal{S}_{e,i}$ also builds the computable partial function $\Omega_{e,i}$ hoping to establish the computability of W. At every stage, the domain of $\Omega_{e,i}$ will be a finite initial segment of the natural numbers.

$\mathcal{S}_{e,i}$ will be a *gap/cogap* strategy, which has the following parameters:

- $p(i)$: the largest number in the domain of $\Omega_{e,i}$. If the domain of $\Omega_{e,i}$ is empty, let $p(i) = -1$;
- $r(i)$: the A-restraint imposed by $\mathcal{S}_{e,i}$ during A-cogaps of $\mathcal{S}_{e,i}$.

The $\mathcal{S}_{e,i}$-strategy proceeds as follows:

(1) (Opening an A-gap) Wait for an $\mathcal{R}_e$-expansionary stage, say v, at which the A-gap is currently closed; $\Psi_i(B_e; i) \downarrow$ and either $\Delta_e(B_e, W; i) = 0$ or $\delta_e(i) \leq \psi_i(i)$. Then open an A-gap as follows:
  - Drop its restraint on A by setting $r(i) = -1$.
  - For all y, if $\Gamma(A, W; e, y) = 0$ and $\Gamma(A, W; e, y)$ is located at some node corresponding to an $S_{e,j}$ for $j > i$ and $\gamma(e, y) > R(i) = \max\{r(i') \mid i' \leq i\}$, then enumerate $\gamma(e, y)$ into A. For each $j > i$, set $C_j = \emptyset$.
  - For all $z \leq \delta_e(i)[s]$ with $\Omega_{e,i}(z)$ not already defined, define $\Omega_{e,i}(z) = W(z)[s]$. (We threaten to make $\Omega_{e,i} = W$.)
  - Let y be the least z such that $\Gamma(A, W; e, z) \uparrow$, define $\Gamma(A, W; e, y) = 1$ with $\gamma(e, y)$ fresh, and locate it at the node corresponding to $\mathcal{S}_{e,i}$.

  The last two actions correct the values of $\Gamma(A, W; e, y)$ which are wrongly defined to be zero at some node to the right, except those "controlled" by higher priority nodes.

(2) (Closing the A-gap) A gap opened at stage v is closed at the next $\mathcal{R}_e$-expansionary stage, say $s > v$, if any. We close the A-gap as follows:

Case 2a. (Successful Closure) $W_v \upharpoonright (\delta_e(i)+1) \neq W_s \upharpoonright (\delta_e(i)+1)$.

Then: For all $j \geq i$, if $\Delta_e(B_e, W; j)$ is defined, then set it to be undefined and stop. (By doing so, we lift all $\delta_e(j)$ beyond $\psi_i(i)$ hence preserve $\Psi_i(B_e; i)\downarrow$, and we are able to redefine $\Delta_e(B_e, W; i) = 1$ correctly. So $\mathcal{S}_{e,i}$ is satisfied finitarily.)

Case 2b. (Unsuccessful Closure) Otherwise, then:

- Let x be the least number which is not in C_i, define the A-restraint $r(i) = \varphi_e(x)$, and enumerate $\delta_e(j)$ into B_e for all $j \geq i$. (We set $r(i) = \varphi_e(x)$ in order to preserve $\Phi_e(A; x)$ until a W-change (if any) gives an opportunity to add x to C_i as described in the next paragraph. Note that $\Phi_e(A; x)$ is currently defined because we are at an $\mathcal{R}_e$-expansionary stage, and all numbers y previously added to C_i were such that $\Phi_e(A; y)$ was defined when y was added to C_i. Note also that for all y with $\gamma(e, y)\downarrow$, either $\gamma(e, y) \leq R(i)$ or $\gamma(e, y) > p(i)$, where $p(i)$ is the greatest element of the domain of $\Omega_{e,i}$.)

(3) (Building C_i) At any stage t where $r(i) \downarrow\neq -1$ (not necessarily an $\mathcal{R}_e$-expansionary stage) if $W_s \upharpoonright p(i) \neq W_t \upharpoonright p(i)$, where $p(i)$ is the largest number in the domain of $\Omega_{e,i}$, then:

- For any e, y, if $\Gamma(A, W; e, y)$ is defined and located at some node to the right of the current one, and $\gamma(e, y) \geq p(i)$, then set $\Gamma(A, W; e, y)$ to be undefined.
- Enumerate the least number x which is not in C_i into C_i, and restart $\Omega_{e,i}$ completely. Note that, because of the restraint imposed at the last stage when the A-gap was (unsuccessfully) closed, $\Phi_e(A, x)\downarrow$. Furthermore, all defined marker positions $\gamma(e, y)$ are either $\leq R(i)$ or are located at some node for $\mathcal{S}_{e,j}$ for $j < i$. Thus, if no requirement $\mathcal{S}_{e,j}$ for $j < i$ acts at any future stage, the computation $\Phi_e(A, x)$ is permanent, and does not require any restraint to be preserved.
- Set $r(i) = -1$.

We now analyse the possible outcomes for the strategy $\mathcal{S}_{e,i}$, under the assumption that $\mathcal{R}_e$ has infinitely many expansionary stages and $\mathcal{S}_{e,i'}$ never acts for all $i' < i$. It is easy to see that if $\mathcal{S}_{e,i}$ acts only finitely many times, then $\Delta_e(B_e, W; i) = B'_e(i)$ holds eventually and permanently. So we consider only the following two cases:

Case 1. Step (3) occurs infinitely many times.

By the strategy, at the stage at which x is enumerated into C_i, $\Phi_e(A; x)$ is defined and cleared from all γ-markers located at some node to the right of that for $\mathcal{S}_{e,i}$. Hence we are able to ensure that $\Phi_e(A)$ is total. On the other hand, Step (1) ensures that for almost every y, $\Gamma(A, W; e, y)$ is correctly defined to be equal to 1. Also $(\lambda y)\Gamma(A, W; e, y)$ is total, and hence $\lim_y \Gamma(A, W; e, y) = 1$. $\mathcal{R}_e$ is satisfied through the first clause.

Case 2. Otherwise, that is, after some stage W never changes below $p(i)$ during any A-cogap. By assumption, $\mathcal{S}_{e,i}$ never closes any A-gap successfully. Hence W never changes below $\delta_e(i)$, which is larger than $p(i)$, during any A-gap. Since $p(i)[s]$ is nondecreasing in s and unbounded over the construction, and $W_s \upharpoonright p(i)[s] = W \upharpoonright p(i)[s]$, W is computable.

2.5. **Coordination among $\mathcal{R}$-strategies.** In general, we have to protect computations of the form $\Phi_e(A;x)$ of the $\mathcal{R}_e$-strategy from injuries by other $\mathcal{R}'$-strategies. If $\mathcal{R}'$ has higher priority, the usual "believable computation" trick works. If $\mathcal{R}'$ has lower priority than $\mathcal{R}$, then it can be done by the *slow-down* method (see for example Cooper and Li [4]). Let ξ be a node extending a node $\alpha\hat{\ }\langle g_i\rangle$ working on the $\mathcal{S}_{e,i}$-strategy. We say that ξ is *ready to define* $\Gamma(A,W;d,y)$, if for $l=\max\{b(\xi),d,y,m\}$, $l\in C_i$, where $b(\xi)$ is a natural number coding the node ξ and m is the number of times that ξ has defined $\Gamma(A,W;d,y)$. If ξ is not ready, then it simply waits. This ensures that for each x there are at most finitely many times that x is removed from C_i because the convergence of $\Phi_e(A;x)$ is injured by the enumeration of some number less than $\varphi_e(x)$ by some node ξ extending $\alpha\hat{\ }\langle g_i\rangle$. Hence if the $\mathcal{S}_{e,i}$-strategy puts numbers into C_i infinitely often, then $C_i=\omega$ at the end of the construction, i.e. every $x\in\omega$ eventually remains in C_i. And since $C_i=\omega$, the delay in defining $\Gamma(d,y)$ caused by $\mathcal{S}_{e,i}$ will not last forever.

3. Priority tree and construction

3.1. **Priority tree.** We first define the priority tree T, which is $(\omega+1)$–branching. The tree T consists of all finite sequences of elements of the set $\{g_0,g1,\dots,1\}$. For each node α on T, α has the following outgoing edges ordered from left to right:

$$g_0<_L g_1<_L\cdots<_L 1.$$

Fix a priority ranking of the requirements $\mathcal{R}_e$ by

$$\mathcal{R}_0<\mathcal{R}_1<\mathcal{R}_2<\cdots.$$

For each node α on T, if the length of α is e, then we label α with the requirement $\mathcal{R}_e$, and we associate with each node $\alpha\hat{\ }\langle g_i\rangle$ the subrequirement $\mathcal{S}_{e,i}$, and we associate with $\alpha\hat{\ }\langle 1\rangle$ the outcome that there are only finitely many α-expansionary stages, as defined below.

3.2. **Parameters and conventions.** Suppose that α is a node on T. α has the following parameters:

(a) B_α: the c.e. set built by α;
(b) Δ_α: the Turing functional built by α;
(c) $b(\alpha)$: where $b:T\to\omega$ is a 1-1 computable function;
(d) $R(\alpha)$: the maximum A-restraint imposed by nodes to the left of α or extended by α.

For each $i\in\omega$ and $\alpha\in T$ of length e, we associate with the node $\alpha\hat{\ }\langle g_i\rangle$ the following parameters:

(a) C_i^α: the clearing set of α for $\mathcal{S}_{e,i}$;
(b) $r^\alpha(i)$: the A-restraint on node α imposed by the $\mathcal{S}_{e,i}$-strategy as described before;
(c) Ω_i^α: the computable partial function built at the node $\alpha\hat{\ }\langle g_i\rangle$;
(d) $p^\alpha(i)$: the largest number in the domain of Ω_i^α (it is -1 if Ω_i^α is empty or undefined).

Definition 3.1. Given an $\mathcal{R}_e$-strategy α and a stage s:

(i) We say that $\Phi_e(A;x)=y$ is α-*believable*, if for all m,n, if $\Gamma(A,W;m,n)$ is defined and located at some node to the right of α, then either $\gamma(m,n)>\varphi_e(x)$ or $\gamma(m,n)\le R(\alpha)$.

(ii) We define the *length function* $l(\alpha)$ by

$$l(\alpha) = \max\{x \mid (\forall y < x)[\Phi_e(A; y) \downarrow \text{ via an } \alpha\text{-believable computation}]\}.$$

(iii) We say that a stage s is *α-expansionary* if α is accessible at stage s and $l(\alpha)[s] > l(\alpha)[v]$ for all $v < s$ at which α was accessible.

Definition 3.2. Given a node α, and $e, y \in \omega$, let $m(\alpha, e, y)$ be the number of times that α has defined $\Gamma(A, W; e, y)$, and define $d(\alpha, e, y) = \max\{b(\alpha), e, y, m(\alpha, e, y)\}$.

We say that α is *ready to define* $\Gamma(A, W; e, y)$ if for each $\mathcal{R}$-strategy α' and each $i \in \omega$ and if the node $\alpha'\hat{\ }\langle g_i\rangle \subseteq \alpha$, then $d(\alpha, x, y) \in C_i^{\alpha'}$, the clearing set of α' for strategy $\mathcal{S}_{e',i}$, where $\mathcal{R}_{e'}$ is the requirement associated with α'.

Suppose that α is an $\mathcal{R}$-strategy. If α is *initialised*, then all of its nodes $\alpha\hat{\ }\langle g_i\rangle$ are initialised, and both B_α and Δ_α are set to be $\emptyset$.

Suppose that $\alpha\hat{\ }\langle g_i\rangle$ for some $i \in \omega$ is a node coming out of α and associated with $\mathcal{S}_{e,i}$. If it is *initialised*, then both C_i^α and Ω_i^α are set to be $\emptyset$, any gap is set to be closed and $r^\alpha(i)$ is set to be undefined.

Without loss of generality, we suppose that the c.e. set W is enumerated by $\{W_s\}_{s\in\omega}$ such that, for all n, both $W_{2n+1} = W_{2n}$ and $|W_{2n+2} - W_{2n+1}| = 1$ hold. Finally we require that the Γ-rules and Δ-rules described in section 2 be followed automatically.

3.3. **Stage-by-stage construction.** The construction proceeds in stages as follows:

Stage $s = 0$. Set $A = \Gamma = \emptyset$.

Stage $s = 2n + 1$. We first describe a node α being *accessible at stage s* inductively on substages t of stage s. First we allow the root node $\emptyset$ to be accessible at substage $t = 0$.

Substage t. Let α be accessible at substage t of stage s and let α be labelled $\mathcal{R}_e$ for some $e \in \omega$. If $t = s$, then initialise all ξ with $\alpha <_{\mathrm{L}} \xi$ and go to stage $s + 1$; if $t < s$, then proceed as follows.

Case 1. s is not α-expansionary.

Let $\alpha\hat{\ }\langle 1\rangle$ be accessible. Let y be the least z such that $\Gamma(A, W; e, z)$ is undefined.

Case 1a. α is ready to define $\Gamma(A, W; e, y)$. Then

- for any m, n, if $\Gamma(A, W; m, n)$ is defined and located at some ξ with $\alpha\hat{\ }\langle 1\rangle <_{\mathrm{L}} \xi$ and $\gamma(m, n) > R(\alpha\hat{\ }\langle 1\rangle)$, then enumerate $\gamma(m, n)$ into A, and initialise all ξ with $\alpha <_{\mathrm{L}} \xi$;
- define $\Gamma(A, W; e, y) = 0$ with $\gamma(e, y)$ fresh, and locate it at $\alpha\hat{\ }\langle 1\rangle$;
- go to substage $t + 1$.

Case 1b. Otherwise, then initialise all ξ with $\alpha <_{\mathrm{L}} \xi$, and go to substage $t + 1$.

Case 2. s is α-expansionary.

Choose the least i such that the strategy $\mathcal{S}_{e,i}$ is either ready to open or ready to close an A-gap at stage s, or such that $\Delta_\alpha(B_\alpha, W; i)[s]$ is undefined, and act accordingly as described below. (Such an i exists because $\Delta_\alpha(B_\alpha, W; j)[s]$ is undefined for all sufficiently large j.) Initialise all ξ with $\alpha\hat{\ }\langle g_i\rangle <_{\mathrm{L}} \xi$. If $\Delta_\alpha(B_\alpha, W; i)[s]$ is undefined, define it to be $B'_\alpha(i)[s]$, and go to stage $s + 1$. Otherwise, open or close a gap as described below.

We say that the $\mathcal{S}_{e,i}$-strategy is *ready to open an A-gap at stage s*, if the gap is closed at the beginning of stage s, $\Psi_i(B_\alpha; i)[s] \downarrow$, and either $\Delta_\alpha(B_\alpha, W; i) = 0[s]$ or for some $j \geq i$, $\delta_\alpha(j) < \psi_i(i)$. If this holds, we *open an A-gap* as follows:

- set $r^\alpha(i) = -1$;
- define $p^\alpha(i) = \delta_\alpha(i)$ and for each $z \leq p^\alpha(i)$ define $\Omega_i^\alpha(z) = W(z)[s]$ if it is currently undefined;
- for any m, n, if $\Gamma(A, W; m, n) = 0$ via an axiom located at some η with $\alpha\hat{\ }\langle g_i\rangle <_{\mathrm{L}} \eta$ and $\gamma(m,n) > R(\alpha\hat{\ }\langle g_i\rangle)$, then enumerate $\gamma(m,n)$ into A;
- (Building Γ) let y be the least z such that $\Gamma(A, W; e, z)\uparrow$. If $\mathcal{S}_{e,i}$ is ready to define $\Gamma(A, W; e, y)$, then define $\Gamma(A, W; e, y) = 1$ with $\gamma(e, y)$ fresh, and locate it at $\alpha\hat{\ }\langle g_i\rangle$;
- (Updating C_j^β) for each node $\beta\hat{\ }\langle g_j\rangle$ on T properly extended by $\alpha\hat{\ }\langle g_i\rangle$ and working on, say, $\mathcal{S}_{e',j}$, if there exists $x \in C_j^\beta$ with $\Phi_{e'}(A, x)\uparrow$, let x_0 be the least such x and remove from C_j^β all $y \geq x_0$ currently in C_j^β.
- go to substage $t + 1$.

We say that the $\mathcal{S}_{e,i}$-strategy is *ready to close an A-gap at stage s* if a gap is currently open. Let s^- be the stage at which the current gap was opened; we *close the A-gap* in a manner depending on the two cases below.

Case 2a. (Successful Closure) $W_{s^-} \upharpoonright (p^\alpha(i) + 1) \neq W_s \upharpoonright (p^\alpha(i) + 1)$. Then:

- for every $j \geq i$, set $\Delta_\alpha(B_\alpha, W; j)$ to be undefined, and we say that $\delta_\alpha(j)$ is *lifted at stage s*;
- cancel the partial function Ω_i^α.
- initialise all ξ with $\alpha\hat{\ }\langle g_i\rangle <_{\mathrm{L}} \xi$ and go to stage $s + 1$.

Case 2b. (Unsuccessful Closure) Otherwise, then

- let x be the least number $\notin C_i^\alpha$ and define $r^\alpha(i) = \varphi_e(A; x)$ (note that $\Phi_e(A; x)\downarrow$ via an α-believable computation as stage s is α-expansionary);
- enumerate $\delta_\alpha(j)$ into B_α for every $j \geq i$ with $\delta_\alpha(j)\downarrow$ and say that $\delta_\alpha(j)$ is *lifted at stage s*;
- initialise all ξ with $\alpha\hat{\ }\langle g_i\rangle <_{\mathrm{L}} \xi$ and go to stage $s + 1$.

Stage $s = 2n+2$. Let w_s be the number $w \in W_s - W_{s-1}$. For each $\mathcal{S}_{e,i}$-strategy associated with a node, say $\alpha\hat{\ }\langle g_i\rangle$, such that $r^\alpha(i)\downarrow\neq -1$ and $p^\alpha(i) \geq w_s$, from left to right (in fact, the order is not essential as there are no conflicts) execute the following *cogap permitting action of* $\alpha\hat{\ }\langle g_i\rangle$ provided that $\mathcal{S}_{e,i}$ is not in a gap:

(1) set $r^\alpha(i) = -1$;
(2) for all y, if $\Gamma(A, W; e, y)$ is defined and located at some node to the right of $\alpha\hat{\ }\langle g_i\rangle$ and $\gamma(e, y) > w_s$, then set $\Gamma(A, W; e, y)$ to be undefined;
(3) enumerate the least number x which is not in C_i^α into C_i^α, and cancel Ω_i^α. Note that $\Phi_e(A; x)\downarrow$ via an α-believable computation because of the restraint $r^\alpha(i)$ imposed the last time that $\mathcal{S}_{e,i}$ unsuccessfully closed an A-gap.

This completes the description of the construction.

4. The verification

Let the *true path TP* be the leftmost path through T consisting of nodes which are accessible infinitely often.

Lemma 4.1 (Existence of the true path)**.** *Suppose that there is no c.e. set B such that $B' \leq_{\mathrm{T}} B \oplus W$. Then the true path TP is an infinite path through the tree T.*

Proof. We show by induction on e that there is a leftmost node α_e of length e which is accessible infinitely often. It then follows easily that α_{e+1} extends α_e for all e, and clearly $TP = \{\alpha_e : e \in \omega\}$. Clearly we may let α_0 be the empty node. Now suppose inductively that α_e exists and let it be denoted α. Let s_0 be a stage after which no node to the left of α is accessible. We consider two cases.

Case 1. There are only finitely many α-expansionary stages.

Clearly $\alpha\hat{}\langle 1\rangle$ is accessible infinitely many times, and is initialised only finitely many times and so is the leftmost node of length $e+1$ which is accessible infinitely often.

Case 2. There are infinitely many α-expansionary stages.

Claim. *If for all $i \in \omega$ the node $\beta = \alpha\hat{}\langle g_i\rangle$ is accessible only finitely often, then for all i, $\Delta_\alpha(B_\alpha, W; i)$ is defined and equal to $B'_\alpha(i)$.*

Proof of the Claim. First note that, by induction on i, $\Delta_\alpha(B_\alpha, W; i)[s]$ is defined for all sufficiently large stages at which α is accessible. Now fix i. By hypothesis there is a stage s_1 after which no node $\alpha\hat{}\langle g_{i'}\rangle$ is accessible for any $i' \leq i$. We may also assume that s_1 is so large that $\Delta_\alpha(B_\alpha, W; j)[s]$ is defined for all $j \leq i$ and $s \geq s_1$ such that α is accessible at s. First, suppose that $i \in B'_\alpha[s_2]$ for some stage $s_2 > s_1$ at which α is accessible. Then $\Delta_\alpha(B_\alpha, W; i)[s_2] = 1$ and $\delta_\alpha(j) \geq \psi_i(i)[s_2]$ for all $j \geq i$ such that $\delta_\alpha(j)\downarrow$, since otherwise $\alpha\hat{}\langle g_{i'}\rangle$ would be accessible at s_2 for some $i' \leq i$, contrary to the choice of s_1. It follows that both the computations $i \in B'_\alpha[s_2]$ and $\Delta_\alpha(B_\alpha, W; i)[s_2] = 1$ are permanent. To see that the first is permanent, note that no marker position $\delta_\alpha(i')$ for $i' \leq i$ can enter B_α after s_2 since no node $\alpha\hat{}\langle g_{i'}\rangle$ for $i' \leq i$ is accessible after s_2. The marker positions $\delta_\alpha(j)$ for $j \geq i$ are greater than the use $\psi_i(i)$ of the computation showing $i \in B'_\alpha[s_2]$ and remain greater than this use. To see that the second computation $\Delta_\alpha(B_\alpha, W; i)[s_2] = 1$ is never injured after stage s_2, let s_3 be the stage at which it was originally defined with the same use as at s_2. At stage s_3 all nodes $\alpha\hat{}\langle j\rangle$ for $j > i$ were initialised, and marker positions $\delta_\alpha(j)$ for $j > i$ are chosen greater than its use and remain greater than its use. Since no node $\alpha\hat{}\langle g_{i'}\rangle$ for $i' < i$ is accessible between stages s_3 and s_2 (or else $\mathcal{S}_{e,i}$ would be initialised and the use of $\Delta_\alpha(B_\alpha, W; i)$ would change between s_3 and s_2), no marker position $\delta_\alpha(i')$ for $i' < i$ enters B_α between s_3 and s_2. Finally, W–changes do not affect the computation by the Δ-rules and the fact that no node $\alpha\hat{}\langle g_{i'}\rangle$ for $i' < i$ is accessible after s_2. Thus, if $i \in B'_\alpha[s_2]$ for some stage $s_2 > s_1$ at which α is accessible, $\Delta_\alpha(B_\alpha, W; i) = 1 = B'(i)$.

Now suppose there is no such stage s_2. Since α is accessible at infinitely many stages, it follows that $B'_\alpha(i) = 0$. Recall that $\Delta_\alpha(B_\alpha, W; i)[s]$ is defined for all $s \geq s_2$. Since there are only finitely many stages s at which α is accessible and $i \in B'_\alpha[s]$, there are only finitely many stages s at which α is accessible and $\Delta_\alpha(B_\alpha, W; i)[s] = 1$. Also, there are only finitely many stages at which $\Delta_\alpha(B_\alpha, W; i)[s]$ is set equal to 0 since there are only finitely many stages at which $\alpha\hat{}\langle g_i\rangle$ is accessible. Let s_4 be the last stage at which $\Delta_\alpha(B_\alpha, W; i)$ is set equal to 0. Then no marker position $\delta_\alpha(i')$ for $i' < i$ enters B_α after s_4 for any $i' < i$, since otherwise $\alpha\hat{}\langle g_i\rangle$ would be initialised after s_4. The marker positions $\delta_\alpha(j)$ for $j > i$ become undefined at s_4 and, when defined at any stage after s_4, take values larger than the use of $\Delta_\alpha(B_\alpha, W; i)$ at s_4. The marker position $\delta_\alpha(i)$ does not enter B_α after s_4 because otherwise $\Delta_\alpha(B_\alpha, W; i)[s]$ would again be set equal to 0 after s_4, contrary to the choice of s_4.

Thus there is an (least) i such that the strategy $\mathcal{S}_{e,i}$ associated with $\alpha\hat{\ }\langle g_i\rangle$ acts infinitely often. In particular, the A-gap will be opened infinitely often. Consequently the node $\alpha\hat{\ }\langle g_i\rangle$ is accessible infinitely often. □

Lemma 4.2. *For any node $\alpha\hat{\ }\langle g_i\rangle$ on TP, if there is a stage after which there is no cogap permitting action for $\alpha\hat{\ }\langle g_i\rangle$, then W is computable.*

Proof. Let $\alpha\hat{\ }\langle g_i\rangle$ be a string on TP such that after some stage, say s, there is no cogap permitting for $\alpha\hat{\ }\langle g_i\rangle$. We may assume also that $\alpha\hat{\ }\langle g_i\rangle$ is never initialised after s. Observe that the associated $\mathcal{S}_{e,i}$-strategy must open and close the A-gap infinitely often, and each closure after s must be an unsuccessful one. Thus Ω_i^α is never cancelled after stage s, and $p^\alpha(i)[t] = \delta_\alpha(i)[t]$ goes to infinity as t goes to infinity. For each z, once $\Omega_i^\alpha(z)$ is defined after s, $W \upharpoonright z$ never changes: It cannot change during an A-gap since otherwise the closure would be successful; it does not change during an A-cogap by the choice of s. Therefore $\Omega_i^\alpha = W$. □

Lemma 4.3. *Suppose that there is no c.e. set B such that $B' \leq_{\mathrm{T}} B \oplus W$ and W is not computable. Then if $\alpha\hat{\ }\langle g_i\rangle$ is on TP, $C_i^\alpha = \omega$; more precisely, for any $x \in \omega$, there is a stage s_x after which $x \in C_i^\alpha$. Hence $\Phi_e(A)$ is total, where α is associated with $\mathcal{R}_e$.*

Proof. Consider $\alpha\hat{\ }\langle g_i\rangle$ on TP. Let s_0 be a stage after which no node to the left of $\alpha\hat{\ }\langle g_i\rangle$ is accessible.

We first prove that for each x, there are only finitely many stages at which x is removed from C_i^α. This is proved by induction on x. Fix x, and consider a stage s after which no $x' < x$ is removed from C_i^α. Since $\alpha\hat{\ }\langle g_i\rangle \in TP$ we may assume also that s is so large that $\alpha\hat{\ }\langle g_i\rangle$ is not initialised after s. Suppose that x is removed from C_i^α at stage $t > s$. Then some $\gamma(m,n) < \varphi_e(x)$ is enumerated into A at stage t. Let $\gamma(m,n)$ be located at η. If η is to the left of $\alpha\hat{\ }\langle g_i\rangle$ or η is extended by $\alpha\hat{\ }\langle g_i\rangle$ or $\eta = \alpha\hat{\ }\langle g_i\rangle$, then $\alpha\hat{\ }\langle g_i\rangle$ is initialised at t, as some node to the left of η acts at t. Thus η is to the right of $\alpha\hat{\ }\langle g_i\rangle$, or η extends $\alpha\hat{\ }\langle g_i\rangle$. We next prove that there are only finitely many stages t such that η (as described above) extends $\alpha\hat{\ }\langle g_i\rangle$. Suppose that η extends $\alpha\hat{\ }\langle g_i\rangle$ and causes x to be removed from C_i^α via the enumeration of $\gamma(m,n)$ (located at η) in A. Then $b(\eta) < x$ since otherwise η must wait until $x \in C_i^\alpha$ to be ready to define $\gamma(m,n)$ and then must choose $\gamma(m,n) > \varphi_e(x)$. Similarly, $m < x$ and $n < x$. Thus, there are only finitely many choices for η, m, and n. For fixed η, m, and n there are only finitely many stages at which the enumeration of $\gamma(m,n)$ located at η into A can cause x to be removed from C_i^α, since once $\gamma(m,n)$ has been defined by η more than x times, it must be chosen after x has entered C_i^α and hence larger than $\varphi_e(x)$. This concludes the proof that there are only finitely many stages t at which x is removed from C_i^α by the enumeration of $\gamma(m,n)$ into A, where $\gamma(m,n)$ is located at some η extending $\alpha\hat{\ }\langle g_i\rangle$. Let s_1 be a stage at which $\mathcal{S}_{e,i}$ opens an A-gap and which is larger than all such t. At s_1 all markers $\gamma(m,n)$ located to the right of $\alpha\hat{\ }\langle g_i\rangle$ become undefined or take values greater than $p^\alpha(i)$. At any future stage at which $p^\alpha(i)$ increases, an A-gap is opened by $\alpha\hat{\ }\langle g_i\rangle$, and so again all such markers are undefined or greater than $p^\alpha(i)$. Thus, at all stages $s_2 > s_1$, all markers $\gamma(m,n)$ located to the right of $\alpha\hat{\ }\langle g_i\rangle$ are greater than $p^\alpha(i)$. If there is no stage $s_2 > s_1$ at which x is added to C_i^α, then clearly x is removed from C_i^α at at most one stage $s_2 > s_1$. Suppose now that x is added to C_i^α at stage $s_2 > s_1$. Then $\Phi_e(A;x)$ as defined at s_2, as remarked in the construction. At s_2, all $\gamma(m,n)$ located to the right of $\alpha\hat{\ }\langle g_i\rangle$ with $\gamma(m,n)$

greater than or equal to $p^\alpha(i)$ become undefined. As remarked above, these $\gamma(m,n)$ are the only numbers which can injure the computation of $\Phi_e(A;x)$. Hence x is never removed from C_i^α after stage s_2. This completes the proof that x is removed from C_i^α at most finitely often.

It is now easy to see that, for all x, $x \in C_i^\alpha$ from some stage on. Assume this is false, and let x_0 be the least x for which this is false. Choose s_0 so large so that, for all $s \geq s_0$ no $x \leq x_0$ is added to or removed from C_i^α at stage s. Thus, at the beginning of each stage $s \geq s_0$, x is the least number not in C_i^α. Let s_1 be a stage $s > s_1$ at which cogap permitting action occurs for $\alpha\hat{\ }\langle g_i\rangle$. Such a stage s_1 exists by Lemma 4.2. Then x_0 is added to C_i^α at s_1, which gives us the desired contradiction.

It remains to show that $\Phi_e(A;x)$ is total. When x enters C_i^α for the last time, $\Phi_e(A;x)$ is defined. It remains defined forever thereafter with the same computation, since otherwise it would be removed from C_i^α. □

Lemma 4.4. *Suppose that $W \not\leq_{\mathrm{T}} \emptyset$, and that for any c.e. set B, $B' \not\leq_{\mathrm{T}} B \oplus W$. Suppose that α is a node on TP and $i \in \omega$. Then:*

- (i) *If $\alpha\hat{\ }\langle 1\rangle \in TP$, then*
  - (a) $(\lambda y)\Gamma(A,W;e,y)$ *is total,*
  - (b) $\lim_y \Gamma(A,W;e,y) = 0$, *and*
  - (c) $\Phi_e(A)$ *is not total.*
- (ii) *If $\alpha\hat{\ }\langle g_i\rangle \in TP$, then*
  - (a) $(\lambda y)\Gamma(A,W;e,y)$ *is total,*
  - (b) $\lim_y \Gamma(A,W;e,y) = 1$, *and*
  - (c) $\Phi_e(A)$ *is total.*

Proof. For (i). Let s_0 be minimal after which $\alpha\hat{\ }\langle 1\rangle$ is never initialised. Let $r_0 = \max\{r(\xi)[s_0] \mid \xi <_{\mathrm{L}} \alpha\hat{\ }\langle 1\rangle\}$. By the construction, for any y, if $\Gamma(A,W;e,y)$ is defined and located at some node to the right of $\alpha\hat{\ }\langle 1\rangle$ and $\gamma(e,y) > r_0$, then $\gamma(e,y) \in A$. Thus for almost every y, $\Gamma(A,W;e,y) = 0$ is defined by α and located at $\alpha\hat{\ }\langle 1\rangle$ eventually. Also, by the choice of s_0, once $\Gamma(A,W;e,y) = 0$ is defined after s_0 and located at $\alpha\hat{\ }\langle 1\rangle$, it will never be destroyed, and hence it will be kept permanently. Since we always define $\Gamma(A,W;e,y)$ for the least y for which $\Gamma(A,W;e,y)$ is undefined, $(\lambda y)\Gamma(A,W;e,y)$ is total. Hence, both (a) and (b) hold. (c) follows from the fact that there are only finitely many α-expansionary stages and that, for α on the true path, if $\Phi_e(A;x)$ is defined, it is eventually defined by an α-believable computation.

For (ii). (a) and (b) follow from the proof for (i) above. (c) follows from Lemma 4.3. □

This completes the proof of Theorem 1.1.

References

[1] M. Bickford and C. F. Mills, Lowness properties of r.e. degrees, Unpublished preprint, 1982.

[2] Peter Cholak, Marcia Groszek, and Theodore Slaman, An almost deep degree, *J. Symbolic Logic*, 66(2):881–901, 2001. MR **2002d:**03070

[3] S. Barry Cooper, On a theorem of C. E. M. Yates, Handwritten notes, 1974.

[4] S. Barry Cooper and Angsheng Li, Splitting and nonsplitting, II: A low_2 c.e. degree above which $\mathbf{0}'$ is not splittable, *J. Symbolic Logic*, 67:1391–1430, 2002. MR **2003k:**03055

[5] Steffen Lempp and Theodore A. Slaman, A limit on relative genericity in the recursively enumerable sets, *J. Symbolic Logic*, 54:376–395, 1989. MR **90f:**03077

[6] Angsheng Li, Definable relations on the computably enumerable degrees, in *Computability and Models*, pages 267–288. edited by S. B. Cooper and S. S. Goncharov, Kluwer Academic / Plenum Publishers, New York, 2003.

[7] Angsheng Li, Elementary differences among jump hierarchies, preprint.

[8] André Nies, Richard A. Shore, and Theodore A. Slaman, Interpretability and definability in the recursively enumerable degrees, *Proc. London Math. Soc. (3)*, 77(2):241–291, 1998. MR **99m:**03083

[9] David B. Posner and Robert W. Robinson, Degrees joining to $\mathbf{0}'$, *J. Symbolic Logic*, 46(4):714–722, 1981. MR **83c:**03040

[10] Robert I. Soare, *Recursively Enumerable Sets and Degrees*, Springer–Verlag, Heidelberg, 1987. MR **88m:**03003

Department of Mathematics, University of Illinois, 1409 W. Green St., Urbana, Illinois 61801

E-mail address: `jockusch@math.uiuc.edu`

Institute of Software, Chinese Academy of Sciences, P. O. Box 8718, Beijing, 100080, People's Republic of China

E-mail address: `angsheng@gcl.iscas.ac.cn`

Department of Mathematics, Faculty of Science, National University of Singapore, Lower Kent Ridge Road, Singapore 119260

E-mail address: `matyangy@leonis.nus.edu.sg`