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A NEW VARIATIONAL CHARACTERIZATION OF n-DIMENSIONAL SPACE FORMS

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ABSTRACT. A Riemannian manifold (M^n, g) is associated with a *Schouten* $(0,2)$-tensor C_g which is a naturally defined Codazzi tensor in case (M^n, g) is a locally conformally flat Riemannian manifold. In this paper, we study the Riemannian functional $\mathcal{F}_k[g] = \int_M \sigma_k(C_g) dvol_g$ defined on $\mathcal{M}_1 = \{g \in \mathcal{M} | Vol(g) = 1\}$, where $\mathcal{M}$ is the space of smooth Riemannian metrics on a compact smooth manifold M and $\{\sigma_k(C_g),\ 1 \le k \le n\}$ is the elementary symmetric functions of the eigenvalues of C_g with respect to g. We prove that if $n \ge 5$ and a conformally flat metric g is a critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] \ge 0$, then g must have constant sectional curvature. This is a generalization of Gursky and Viaclovsky's very recent theorem that the critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] \ge 0$ characterized the three-dimensional space forms.

1. Introduction

Let M^n be an n-dimensional compact and smooth manifold. Denote by $\mathcal{M}$ and $\mathcal{G}$ the space of smooth Riemannian metrics and the diffeomorphism group of M, respectively. We call a functional $\mathcal{F} : \mathcal{M} \to R$ Riemannian if $\mathcal{F}$ is invariant under the action of $\mathcal{G}$, i.e., $\mathcal{F}(\varphi^* g) = \mathcal{F}(g)$ for each $\varphi \in \mathcal{G}$ and $g \in \mathcal{M}$.

By letting $S_2(M)$ denote the bundle of symmetric $(0,2)$-tensors on M^n, we say that $\mathcal{F} : \mathcal{M} \to R$ has a gradient at $g \in \mathcal{M}$ if

$$\frac{d}{dt}\mathcal{F}(g + th)|_{t=0} = \int_M \langle h, \nabla \mathcal{F} \rangle_g dVol_g \tag{1.1}$$

for some $\nabla \mathcal{F} \in \Gamma(S_2(M))$ and all $h \in \Gamma(S_2(M))$. The theory of Riemannian functionals has a long history; for details and references we refer to [1] and [4], among many others.

Following [4] and [10], we consider the functional

$$\mathcal{F}_k[g] = \int_{M^n} \sigma_k(C_g) dVol_g, \tag{1.2}$$

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where $\sigma_k(C_g)$ is the k-th elementary symmetric function of the eigenvalues of the *Schouten* tensor $C_g := \mathrm{Ric} - \frac{r}{2(n-1)}g$ with respect to g. Here Ric and r denote the Ricci tensor and the scalar curvature of g, respectively. We note that in classical terms the Schouten tensor is $\frac{1}{n-2}C_g$.

Recall that an n-dimensional Riemannian manifold (M^n, g) is said to be locally conformally flat if it admits a coordinate covering $\{U_\alpha, \varphi_\alpha\}$ such that the map $f_\alpha : (U_\alpha, g_\alpha) \to (S^n, g_0)$ is a conformal map, where g_0 is the standard metric on S^n. Since a 2-dimensional Riemannian manifold is always locally conformally flat, we will assume $n \geq 3$ throughout this paper.

In [10] Viaclovsky studied $\mathcal{F}_k|_{[g]_1}$, where g is a fixed smooth metric on M^n and $[g]_1$ denotes the space of smooth metrics which are conformal to g and have unit volume, and he proved that a metric $\tilde{g}$ is critical for $\mathcal{F}_k|_{[g]_1}$ if and only if $\sigma_k(C_{\tilde{g}}) = \mathit{constant}$ provided $k = 1, 2$; or $k \geq 3$ and (M^n, g) is locally conformally flat. Let $\mathcal{M}_1 = \{g \in \mathcal{M} | \mathrm{Vol}(g) = 1\}$. In [4] the authors considered $\mathcal{F}_2[g]$ on $\mathcal{M}_1$ and proved the following important result.

Theorem A ([4])**.** *Let M be compact and three-dimensional. Then a metric g with $\mathcal{F}_2[g] \geq 0$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if g has constant sectional curvature.*

Our main purpose in this paper is to generalize Theorem A to a higher-dimensional situation. This is achieved when we restrict ourselves to the case that the critical metric is locally conformally flat. Precisely, our main result is the following

Theorem B. *Let M^n be compact with dimension $n \geq 5$. Then a conformally flat metric g with $\mathcal{F}_2[g] \geq 0$ is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if g has constant sectional curvature.*

Remark 1.1. Similar to the $n = 3$ case in Theorem A, the condition $\mathcal{F}_2[g] \geq 0$ in Theorem B remains necessary: Let $E = Ric - \frac{r}{n}g$ denote the trace-free Ricci tensor; then

$$\sigma_2(C_g) = -\frac{1}{2}|E|^2 + \frac{(n-2)^2}{8n(n-1)}r^2. \tag{1.3}$$

If g has constant sectional curvature, then $E = 0$ and $\sigma_2(C_g) = (n-2)^2 r^2 / [8n(n-1)] \geq 0$. However, there do exist critical metrics with $\mathcal{F}_2 < 0$; see Remark 7.1 below.

In this paper, all manifolds are supposed to be smooth, connected and orientable for compact ones.

2. Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold. We choose a local orthonormal vector field $\{e_1, \cdots, e_n\}$ adapted to the Riemannian metric of (M^n, g) with $\{\omega_1, \cdots, \omega_n\}$ its dual coframe. Then the connection forms $\{\omega_{ij}\}$ of (M^n, g) are characterized by the structure equations

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0, \tag{2.1}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.2}$$

where R_{ijkl} are the components of the Riemannian curvature tensor of (M^n, g). Let W_{ijkl} denote the components of the Weyl curvature tensor of (M^n, g), i.e. (see [1]),

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(C_{ik}g_{jl} - C_{il}g_{jk} + C_{jl}g_{ik} - C_{jk}g_{il}), \tag{2.3}$$

where C is the so-called *Schouten* tensor and it is a symmetric $(0,2)$-tensor defined by

$$C = Ric - \frac{r}{2(n-1)}g \tag{2.4}$$

with Ric and r denoting the Ricci curvature tensor and scalar curvature of g, respectively. In the sequel we often write C as C_g in emphasizing its dependence on the metric g.

Let R_{ij} be the components of Ric; then $C_{ij} = R_{ij} - \frac{r}{2(n-1)}g_{ij}$. Denote by ∇ the covariant derivative on (M^n, g) and write, e.g., $R_{ij,k} = \nabla_k R_{ij}$, $R_{ij,kl} = \nabla_l \nabla_k R_{ij}$, $C_{ij,k} = \nabla_k C_{ij}$, $C_{ij,kl} = \nabla_l \nabla_k C_{ij}$, and so on. Then we have the following Ricci identities (cf. [5]):

$$R_{ij,kl} - R_{ij,lk} = \sum_m R_{mj}R_{mikl} + \sum_m R_{im}R_{mjkl}, \tag{2.5}$$

$$C_{ij,kl} - C_{ij,lk} = \sum_m C_{mj}R_{mikl} + \sum_m C_{im}R_{mjkl}. \tag{2.6}$$

Let B denote the Cotten tensor, i.e., $B_{ijk} = C_{ij,k} - C_{ik,j}$. Now, we can state the following well-known facts:

(i) If $n = 3$, we always have $W_{ijkl} \equiv 0$. (M^3, g) is locally conformally flat if and only if $B_{ijk} \equiv 0$.

(ii) If $n \geq 4$, (M^n, g) is locally conformally flat if and only if $W_{ijkl} \equiv 0$.

(iii) $\sum_i W_{ijkl,i} = \frac{n-3}{n-2}B_{jkl}$, and so $B_{ijk} \equiv 0$ provided (M^n, g) is locally conformally flat and $n \geq 4$.

Hence, if (M^n, g) is a locally conformally flat manifold with $n \geq 3$, then we have

$$R_{ijkl} = \frac{1}{n-2}\left(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}\right) - \frac{r}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right), \tag{2.7}$$

$$C_{ij,k} = C_{ik,j}; \tag{2.8}$$

the latter means that C_{ij} is a Codazzi tensor.

Let $\tilde{C}_i^j = C_{ik}g^{kj}$. Then we define a family of invariant functions $\sigma_k(C_g)$ of (M^n, g) by

$$\det(\tilde{C} + tI) := \det(\tilde{C}_i^j + t\delta_i^j) = \sum_{k=0}^{n} \sigma_k(C_g)t^{n-k}, \tag{2.9}$$

i.e., $\sigma_k(C_g)$ are the k-th elementary symmetric functions of the eigenvalues of the tensor C_g with respect to g for $1 \leq k \leq n$, and $\sigma_0(C_g) = 1$. Here the $(1,1)$-tensor $\tilde{C}$ is considered as an endomorphism on TM and I denotes both the unit matrix and the identity endomorphism on TM.

Notice that by considering the elementary symmetric functions on R^n,

$$S_k(x_1, \cdots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \quad 1 \leq k \leq n; \quad S_0 := 1, \tag{2.10}$$

we then have

$$\sigma_k(C_g) = S_k(\lambda_1, \cdots, \lambda_n), \qquad 0 \le k \le n, \tag{2.11}$$

where the λ_i's are the eigenvalues of C_g with respect to g.

By the $(1,1)$-tensor $\tilde{C}$, we can define a series of new endomorphism fields $\{T_k\}_{0\le k\le n-1}$ on TM, the so-called *Newtonian* transformations, by

$$\det(\tilde{C} + tI)\cdot(\tilde{C} + tI)^{-1} = \sum_{k=0}^{n-1} T_k(C)t^{n-k-1}, \tag{2.12}$$

in the subset Ω_1 of $M \times R$ with $\Omega_1 := \{(p,t)|\det(\tilde{C}(p) + tI(p)) \ne 0\}$.

From (2.9) and (2.12), one can derive the formula

$$T_k = \sigma_k I - \sigma_{k-1}\tilde{C} + \cdots + (-1)^k \tilde{C}^k, \qquad k = 0, 1, \cdots, n-1, \tag{2.13}$$

where $\tilde{C}^k = \tilde{C}\cdot\tilde{C}\cdots\tilde{C}$($k$-tuples), $\sigma_k = \sigma_k(C_g)$.

3. Euler-Lagrange equation of $\mathcal{F}_k|_{\mathcal{M}_1}$

We will need the following properties of the Newtonian transformations $\{T_k\}_{0\le k\le n-1}$ as defined in section two.

Lemma 3.1 (see [4], [8], [9] and [11])**.** *For any $n \times n$ matrix $C = (C_i^j)$, we define $\sigma_k(C)$ and $T_k(C)$ as the k-th elementary symmetric polynomials of eigenvalues and the Newtonians of C, respectively. Then we have*

(1) $(k+1)\sigma_{k+1} = \mathrm{tr}(C \cdot T_k), \qquad k = 0, 1, \cdots, n-1.$

(2) $(T_k)_i^j = (k!)^{-1} \sum\limits_{i_1,\cdots,i_k;j_1,\cdots,j_k} \delta^{j_1\cdots j_k j}_{i_1\cdots i_k i} C^{i_1}_{j_1}\cdots C^{i_k}_{j_k},$

where $\delta^{j_1\cdots j_m}_{i_1\cdots i_m}$ is the usual generalized Kronecker symbol, i.e., $\delta^{j_1\cdots j_m}_{i_1\cdots i_m}$ equals $+1$ (resp. -1) if $(j_1\cdots j_m)$ is an even (resp. odd) permutation of $(i_1\cdots i_m)$ and in other cases it equals zero.

(3) *If the matrix $C = C(t)$ depends smoothly on a real variable $t \in R$, then the corresponding $\sigma_k = \sigma_k(C(t))$, $T_k = T_k(C(t))$ satisfy*

$$\frac{d}{dt}\sigma_{k+1} = \mathrm{tr}\left(\frac{dC}{dt}\cdot T_k\right), \qquad k = 0, 1, \cdots, n-1.$$

(4) *If C is a Codazzi tensor of $(1,1)$-type on a Riemannian manifold (M^n, g), then the corresponding Newtonians have vanishing divergence, i.e.,*

$$\sum_j \left[(T_k)_i^j\right]_{,j} = 0, \qquad \forall i,\ k.$$

(5) *If $C = C_g$ is given by (2.4) on a Riemannian manifold (M^n, g), then the Newtonian $T_1(C_g)$ has vanishing divergence.*

Remark 3.1. Property (5), a fact observed in [4] for $n = 3$, follows directly from the second Bianchi identity.

Proposition 3.1. *At any $g \in \mathcal{M}$, the functional $\mathcal{F}_k[g] = \int_M \sigma_k(C_g) dVol_g$ defined on $\mathcal{M}$ has a gradient $\nabla \mathcal{F}_k \in \Gamma(S_2(M))$ with the local expression $\nabla \mathcal{F}_k = \sum_{i,j} (\nabla \mathcal{F}_k)_{ij} \omega_i \otimes \omega_j$, where*

$$
\begin{aligned}
(\nabla \mathcal{F}_k)_{ij} = & -\frac{1}{2}\Delta_g (T_{k-1})_{ij} - \sum_{m,l} (T_{k-1})^{ml} W_{milj} + \frac{r}{n(n-1)} (T_{k-1})_{ij} \\
& + \sum_l \left[(T_{k-1})_i^l \right]_{,lj} + \frac{2}{n-2} \sum_l (T_{k-1})_j^l E_{il} \\
& - \frac{n}{2(n-1)(n-2)} \mathrm{tr}_g T_{k-1} \cdot E_{ij} + \frac{n-2k-2}{2(n-2)} \sigma_k g_{ij} \\
& + \frac{1}{2(n-1)} \Delta_g (\mathrm{tr}_g T_{k-1}) \cdot g_{ij} - \frac{1}{2(n-1)} \left(\mathrm{tr}_g T_{k-1} \right)_{,ij} \\
& - \frac{1}{2} \sum_{m,l} \left[(T_{k-1})^{ml} \right]_{,ml} g_{ij},
\end{aligned}
\tag{3.1}
$$

and where the quantities are all defined by the metric tensor g, $T_{k-1} = T_{k-1}(C_g)$, $\sigma_k = \sigma_k(C_g)$, $E_{ij} = R_{ij} - \frac{r}{n} g_{ij}$. As usual we use the metric tensor g to raise and lower indices.

Proof. This is a direct calculation, for the convenience of the readers, we conclude it here. We choose a local smooth frame field $\{e_i\}$ with dual $\{\omega_i\}$ on M^n. Let $g \in \mathcal{M}$ be an arbitrary fixed metric with local expression $g = \sum_{i,j} g_{ij} \omega_i \otimes \omega_j$. Set $(g^{ij}) = (g_{ij})^{-1}$.

Consider a smooth variation $\tilde{g}(t)$ of g with $\tilde{g}(0) = g$ and $\tilde{g}(t) = \sum_{i,j} \tilde{g}_{ij}(t) \omega_i \otimes \omega_j$. Let $h_{ij} := \delta \tilde{g}_{ij}$, where and later in this section $\delta := \frac{d}{dt}|_{t=0}$. As has been explained in Proposition 3.1, we will use g_{ij} or g^{ij} to lower or raise indices, e.g., $h^{ij} := g^{ik} h_{kl} g^{lj}$. The covariant derivation is with respect to the fixed metric g. Then we have the following formulas (cf. [1], [4], [6]):

$$\delta \tilde{g}_{ij} = h_{ij}, \quad \delta \tilde{g}^{ij} = -h^{ij}, \tag{3.2}$$

$$\delta \tilde{\Gamma}_{ijk} = \frac{1}{2}(h_{jk,i} + h_{ik,j} - h_{ij,k}) + \sum_l h_{kl} \Gamma_{ij}^l, \tag{3.3}$$

$$\delta \tilde{\Gamma}_{ij}^k = \frac{1}{2}(\nabla_i h_j^k + \nabla_j h_i^k - \nabla^k h_{ij}), \tag{3.4}$$

where $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$, $\Gamma_{ij}^l = \sum_k g^{kl} \Gamma_{ijk}$ denote the Christoffel symbols w.r.t. the metric g; analogously the $\tilde{*}$- notations correspond to the metric $\tilde{g}_{ij}$.

Then from the expression of the Riemannian curvature tensor

$$R^i{}_{jkl} = \partial_k \Gamma_{jl}^i - \partial_l \Gamma_{jk}^i + \sum_m (\Gamma_{mk}^i \Gamma_{jl}^m - \Gamma_{ml}^i \Gamma_{jk}^m), \quad R_{ijkl} = \sum_m g_{im} R^m{}_{jkl}, \tag{3.5}$$

by a direct calculation we have

$$\delta \tilde{R}_{ijkl} = \sum_m h_{im} R^m{}_{jkl} + \frac{1}{2}(h_{il,jk} + h_{ij,lk} - h_{jl,ik} - h_{ik,jl} - h_{ij,kl} + h_{jk,il}). \tag{3.6}$$

From $R_{jl} = \sum_{i,k} g^{ik} R_{ijkl}$, we have

$$\begin{aligned}\delta\tilde{R}_{jl} &= \frac{1}{2}\sum_k (h_l^k)_{,jk} + \frac{1}{2}\sum_k (h_j^k)_{,lk} - \frac{1}{2}\Delta_g h_{jl} - \frac{1}{2}\left(\mathrm{tr}_g h\right)_{,jl} \\ &= -\frac{1}{2}\Delta_g h_{jl} - \frac{1}{2}(\mathrm{tr}_g h)_{,jl} - \sum_{k,m} h^{mk} R_{jmlk} \\ &\quad + \frac{1}{2}\sum_m \left[h_{ml}R_j^m + h_{jm}R_l^m + (h_l^m)_{,mj} + (h_j^m)_{,ml}\right],\end{aligned} \tag{3.7}$$

where we have used the Ricci identity

$$\sum_k (h_l^k)_{,jk} = \sum_k (h_l^k)_{,kj} + \sum_{k,m} h^{mk} R_{mljk} + \sum_m h_{ml} R_j^m. \tag{3.8}$$

From $r = \sum_{i,j} g^{ij} R_{ij}$, $R_i^j = \sum_l g^{jl} R_{il}$ and (3.2), (3.7), we easily get

$$\delta\tilde{r} = -\sum_{i,j} h^{ij} R_{ij} + \sum_{i,j} g^{ij}\delta\tilde{R}_{ij} = -\sum_{i,j} h^{ij} R_{ij} + \sum_{i,j} (h^{ij})_{,ij} - \Delta_g(\mathrm{tr}_g h), \tag{3.9}$$

$$\begin{aligned}\delta\tilde{R}_i^j &= -\sum_l h^{jl} R_{il} - \frac{1}{2}\Delta_g h_i^j - \frac{1}{2}\sum_l g^{jl}(\mathrm{tr}_g h)_{,il} - \sum_{k,m,l} g^{jl} h^{mk} R_{imlk} \\ &\quad + \frac{1}{2}\sum_m \Big[h_{mi}R^{mj} + h_m^j R_i^m + \sum_l g^{jl}(h_i^m)_{,ml} + (h^{jm})_{,mi} + \sum_l g^{jl}(h_i^m)_{,ml}\Big].\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) we have

$$\begin{aligned}\delta\tilde{C}_i^j &= \delta\tilde{R}_i^j - \frac{\delta\tilde{r}}{2(n-1)}\delta_i^j \\ &= -\sum_l h^{jl} R_{il} - \frac{1}{2}\Delta_g h_i^j - \frac{1}{2}\sum_l g^{jl}(\mathrm{tr}_g h)_{,il} - \sum_{k,m,l} g^{jl} h^{mk} R_{imlk} \\ &\quad + \frac{1}{2}\sum_m \left[h_{mi}R^{mj} + h_m^j R_i^m + \sum_l g^{jl}(h_i^m)_{,ml} + (h^{mj})_{,mi} + \sum_l g^{jl}(h_i^m)_{,ml}\right] \\ &\quad - \frac{1}{2(n-1)}\delta_i^j\left[-\sum_{m,l} h^{ml} R_{ml} + \sum_{m,l} (h^{ml})_{,ml} - \Delta_g(\mathrm{tr}_g h)\right].\end{aligned} \tag{3.11}$$

Therefore, by using Lemma 3.1(3) and (3.11), we obtain

(3.12)

$$\begin{aligned}\delta\sigma_k(C_{\tilde{g}}) &= \sum_{i,j} (T_{k-1})^i_j\, \delta\tilde{C}^j_i \\ &= -\sum_{i,j,l}(T_{k-1})^i_j h^{jl} R_{il} - \sum_{i,j,m,l}(T_{k-1})^{ij} h^{ml} R_{imjl} \\ &\quad - \frac{1}{2}\sum_{i,j}\left[(T_{k-1})^i_j \Delta_g h^j_i + (T_{k-1})^{ij}(\mathrm{tr}_g h)_{,ij}\right] \\ &\quad + \frac{1}{2}\sum_{i,j,m}(T_{k-1})^i_j\left[h_{mi}R^{mj} + h^j_m R^m_i + (h^{mj})_{,mi} + \sum_l g^{jl}(h^m_i)_{,ml}\right] \\ &\quad - \frac{1}{2(n-1)}\mathrm{tr}(T_{k-1})\left[-\sum_{i,j} h^{ij}R_{ij} + \sum_{i,j}(h^{ij})_{,ij} - \Delta_g(\mathrm{tr}_g h)\right].\end{aligned}$$

Note that $\delta\sqrt{\det(\tilde{g}_{ij})} = \frac{1}{2}\mathrm{tr}_g h\sqrt{\det(g_{ij})}$ and therefore $\delta dVol_{\tilde{g}} = \frac{1}{2}\mathrm{tr}_g h \cdot dVol_g$. Then we can compute the variation of $\mathcal{F}_k[g]$:

(3.13)

$$\begin{aligned}\delta\mathcal{F}_k[\tilde{g}] &= \int_M \delta\sigma_k(C_{\tilde{g}})dVol_g + \int_M \sigma_k(C_g)\delta dVol_{\tilde{g}} \\ &= \int_M \left\{ -\sum_{i,j,l}(T_{k-1})^i_j h^{jl}R_{il} - \sum_{i,j,m,l}(T_{k-1})^{ij}h^{ml}R_{imjl}\right. \\ &\quad - \frac{1}{2}\sum_{i,j}\left[(T_{k-1})^i_j\Delta_g h^j_i + (T_{k-1})^{ij}(\mathrm{tr}_g h)_{,ij}\right] + \frac{1}{2}\sigma_k(C_g)\mathrm{tr}_g h \\ &\quad + \frac{1}{2}\sum_{i,j,m}(T_{k-1})^i_j\left[h_{mi}R^{mj} + h^j_m R^m_i + (h^{mj})_{,mi} + \sum_l g^{jl}(h^m_i)_{,ml}\right] \\ &\quad \left. -\frac{1}{2(n-1)}\mathrm{tr}(T_{k-1})\left[\sum_{i,j}(h^{ij})_{,ij} - \sum_{i,j}h^{ij}R_{ij} - \Delta_g(\mathrm{tr}_g h)\right]\right\} dVol_g.\end{aligned}$$

By using the Stokes' formula and (4) of Lemma 3.1, we can rewrite (3.13) as follows:

$$\begin{aligned}\delta\mathcal{F}_k[\tilde{g}] = \int_M \sum_{i,j} &\left\{ -\frac{1}{2}\Delta_g(T_{k-1})_{ij} - \sum_{m,l}(T_{k-1})^{ml}R_{milj} + \sum_l \left((T_{k-1})^l_i\right)_{,lj}\right. \\ &+ \frac{1}{2(n-1)}\mathrm{tr}(T_{k-1})R_{ij} - \frac{1}{2(n-1)}\left(\mathrm{tr}(T_{k-1})\right)_{,ij} + \frac{1}{2}\sigma_k(C_g)g_{ij} \\ &\left. -\frac{1}{2}\sum_{m,l}((T_{k-1})^{ml})_{,ml}g_{ij} + \frac{1}{2(n-1)}\Delta_g(\mathrm{tr}T_{k-1})g_{ij}\right\} h^{ij}dVol_g.\end{aligned} \tag{3.14}$$

From the decomposition (2.3), (2.4) of the Riemannian curvature tensor, Lemma 3.1(1) and the fact $T_{k-1}C = CT_{k-1}$, we get

$$\begin{aligned}-\sum_{m,l}(T_{k-1})^{ml}R_{milj} = &-\sum_{m,l}(T_{k-1})^{ml}W_{milj} - \frac{k}{n-2}\sigma_k(C_g)g_{ij}\\ &- \frac{1}{n-2}\mathrm{tr}(T_{k-1})C_{ij} + \frac{2}{n-2}\sum_m (T_{k-1})_i^m C_{mj}.\end{aligned} \tag{3.15}$$

Putting (3.15) into (3.14) and then using

$$R_{ij} = E_{ij} + \frac{r}{n}g_{ij}, \qquad C_{ij} = E_{ij} + \frac{n-2}{2n(n-1)}rg_{ij}, \tag{3.16}$$

we immediately obtain (3.1) by noting (1.1). □

From Proposition 3.1, we can deduce our main result of this section.

Theorem 3.1. *Suppose M^n is compact. Then a metric $g \in \mathcal{M}_1$ is a critical point for $\mathcal{F}_k|_{\mathcal{M}_1}$ if and only if it satisfies the following two equations:*

$$(n-2k)\sigma_k(C_g) - (n-2)\sum_{i,j}\left[(T_{k-1})^{ij}\right]_{,ij} = const := 2n\lambda; \tag{3.17}$$

(3.18)

$$\begin{aligned}&\Delta_g(T_{k-1})_{ij} + 2\sum_{m,l}(T_{k-1})^{ml}W_{imjl} - \frac{4}{n-2}\sum_l (T_{k-1})_j^l E_{il} - 2\sum_l \left[(T_{k-1})_i^l\right]_{,lj}\\ &- \frac{2r}{n(n-1)}(T_{k-1})_{ij} + \frac{n}{(n-1)(n-2)}\mathrm{tr}(T_{k-1})E_{ij} + \frac{4k}{n(n-2)}\sigma_k g_{ij}\\ &- \frac{1}{n-1}\Delta_g(\mathrm{tr}T_{k-1})g_{ij} + \frac{1}{n-1}(\mathrm{tr}T_{k-1})_{,ij} + \frac{2}{n}\sum_{m,l}\left[(T_{k-1})^{ml}\right]_{,ml} g_{ij} = 0.\end{aligned}$$

Proof. According to the principle of Lagrange's multiplier, $g \in \mathcal{M}_1$ is a critical point of $\mathcal{F}_k|_{\mathcal{M}_1}$ if and only if for some constant λ, it is a critical point of the auxiliary functional

$$\tilde{\mathcal{F}}_k:\ g \mapsto \int_M \sigma_k(C_g)dVol_g - 2\lambda\big[Vol(M,g) - 1\big],$$

defined on $\mathcal{M}$. From the proof of Proposition 3.1 we easily known that, at any fixed $g \in \mathcal{M}$, $(\nabla\tilde{\mathcal{F}}_k)_{ij} = (\nabla\mathcal{F}_k)_{ij} - \lambda g_{ij}$. This implies that $g \in \mathcal{M}_1$ is a critical point of $\tilde{\mathcal{F}}_k|_{\mathcal{M}}$ if and only if it satisfies

$$(\nabla\mathcal{F}_k)_{ij} = \lambda g_{ij}. \tag{3.19}$$

By contracting (3.19), using $\sum_{i,j} W_{imjl}g^{ij} = 0$ and Lemma 3.1(1), we can get (3.17). Inserting (3.17) into (3.19), we obtain (3.18). □

Corollary 3.1. *Suppose M^n is compact and $g \in \mathcal{M}_1$ is a critical point of $\mathcal{F}_k|_{\mathcal{M}_1}$ $(n \neq 2k)$. Then*

1. *when $k = 1, 2$, we have $\sigma_k(C_g) = const$ on M^n;*
2. *when $k \geq 3$ and (M^n, g) is a locally conformally flat manifold, we also have $\sigma_k(C_g) = const$ on M^n.*

Proof. Since $T_0(C_g) = I,\ T_1(C_g) = -Ric + \frac{r}{2}\mathrm{I}$, where I denotes the identity transformation on TM, i.e., with respect to a local frames field $\{e_i\}$,

$$T_0(C_g)_i^j = \delta_i^j, \qquad T_1(C_g)_i^j = -Ric_i^j + \frac{r}{2}\delta_i^j. \tag{3.20}$$

Now the conclusion follows from (3.17), (4)-(5) of Lemma 3.1 and the fact that C_g is a Codazzi tensor in case g is a locally conformally flat metric. □

Combining Theorem 3.1 with Lemma 3.1(4) and the proof of Corollary 3.1, we immediately have

Corollary 3.2. *Suppose M^n is compact and $g \in \mathcal{M}_1$ is a locally conformally flat metric. Then g is a critical point of $\mathcal{F}_k|\mathcal{M}_1$ $(n \neq 2k)$ if and only if it satisfies the conditions that $\sigma_k(C_g) = const$ and*

$$\begin{aligned}&\Delta_g(T_{k-1})_{ij} - \frac{1}{n-1}\Delta_g(\mathrm{tr}T_{k-1})g_{ij} + \frac{1}{n-1}(\mathrm{tr}T_{k-1})_{,ij} + \frac{4k}{n(n-2)}\sigma_k g_{ij}\\ &- \frac{4}{n-2}\sum_l (T_{k-1})_j^l E_{il} - \frac{2r}{n(n-1)}(T_{k-1})_{ij} + \frac{n}{(n-1)(n-2)}\mathrm{tr}(T_{k-1})E_{ij} = 0.\end{aligned}$$

Remark 3.2. The condition $n \neq 2k$ is natural due to the fact that, as has been observed by Viaclovsky in [10], if $n = 2k$ and M carries a locally conformally flat metric, then $\mathcal{F}_k[g]$ is an invariant and, in fact, is a multiple of the Euler characteristic of M for locally conformally flat metric g.

4. General properties for critical points of $\mathcal{F}_2|_{\mathcal{M}_1}$

From now on, we restrict our attention to $\mathcal{F}_2|_{\mathcal{M}_1}$ with $n \neq 4$. On the one hand, we have

Proposition 4.1. *Suppose M^n $(n \neq 4)$ is compact, and a metric $g \in \mathcal{M}_1$ is a critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$. Then, with respect to a local orthonormal frame field $\{e_i\}$ of g, the norm square of the trace-free Ricci tensor $E = Ric_g - \frac{r}{n}g$ satisfies*

$$\begin{aligned}\frac{1}{2}\Delta_g\sum_{i,j}(E_{ij})^2 =& \sum_{i,j,k}(E_{ij,k})^2 + \frac{n-2}{2(n-1)}\sum_{i,j}E_{ij}r_{,ij} + \frac{4}{n-2}\sum_{i,j,k}E_{ij}E_{jk}E_{ki}\\ &+ \frac{n^2-4n+8}{2n(n-1)}r\sum_{i,j}(E_{ij})^2 - 2\sum_{i,j,m,l}E_{ij}E_{ml}W_{milj}.\end{aligned} \tag{4.1}$$

Proof. Using the identities (3.16) and (3.20), we have

$$(T_1(C_g))_{ij} = -E_{ij} + \frac{n-2}{2n}r\delta_{ij}. \tag{4.2}$$

By use of (4.2) and the fact that T_1 has vanishing divergence, we have from (3.18)

$$\begin{aligned}\Delta_g E_{ij} =& \frac{n-2}{2(n-1)}r_{,ij} - \frac{n-2}{2n(n-1)}\Delta_g r\delta_{ij} - 2\sum_{m,l}E_{ml}W_{imjl} - \frac{n-2}{n^2(n-1)}r^2\delta_{ij}\\ &+ \frac{4}{n-2}\sum_l E_{il}E_{lj} + \frac{n^2-4n+8}{2n(n-1)}rE_{ij} + \frac{8}{n(n-2)}\sigma_2\delta_{ij}.\end{aligned} \tag{4.3}$$

Combining (4.3) with

$$\frac{1}{2}\Delta_g\sum_{i,j}(E_{ij})^2=\sum_{i,j,k}(E_{ij,k})^2+\sum_{i,j}E_{ij}\Delta_gE_{ij}, \tag{4.4}$$

we get (4.1).

On the other hand, we can prove

Proposition 4.2. *Let (M^n,g) be a locally conformally flat manifold. Then, with respect to an orthonormal frame field $\{e_i\}$ of g, we have*

$$\begin{aligned}&\frac{1}{2}\Delta_g\sum_{i,j}(E_{ij})^2\\&=\sum_{i,j,k}(E_{ij,k})^2+\frac{n-2}{2(n-1)}\sum_{i,j}E_{ij}r_{,ij}+\frac{n}{n-2}\sum_{i,j,l}E_{il}E_{lj}E_{ji}+\frac{r}{n-1}\sum_{i,j}(E_{ij})^2.\end{aligned} \tag{4.5}$$

Proof. By use of (3.16) and (2.6)-(2.8), we have

$$\begin{aligned}\Delta_gE_{ij}&=\Delta_gC_{ij}-\frac{n-2}{2n(n-1)}\Delta r\cdot g_{ij}\\&=\sum_l C_{ll,ij}+\sum_{m,l}(C_{mi}R_{mljl}+C_{ml}R_{mijl})-\frac{n-2}{2n(n-1)}\Delta r\cdot g_{ij}\\&=\frac{n-2}{2(n-1)}r_{,ij}-\frac{n-2}{2n(n-1)}\Delta_g r\delta_{ij}\\&\quad+\frac{n}{n-2}\sum_l E_{il}E_{lj}-\frac{1}{n-2}\sum_{m,l}(E_{ml})^2\delta_{ij}+\frac{r}{n-1}E_{ij}.\end{aligned} \tag{4.6}$$

Putting (4.6) into (4.4), we get (4.5).

From (3.16), a simple calculation gives

$$\sigma_2(C_g)=-\frac{1}{2}|E|^2+\frac{(n-2)^2}{8n(n-1)}r^2. \tag{4.7}$$

Comparing (4.3) with (4.6), and then making use of (4.7) and Corollary 3.2 in the case $k=2$, we immediately obtain the following

Corollary 4.1. *Suppose M^n $(n\neq 4)$ is compact. Then a locally conformally flat metric $g\in\mathcal{M}_1$ is a critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$ if and only if $\sigma_2(C_g)=const$ and the following algebraic identities hold:*

$$\sum_l E_i^lE_{lj}-\frac{(n-2)^2}{2n(n-1)}rE_{ij}-\frac{1}{n}|E|^2g_{ij}=0, \tag{4.8}$$

$$\mathrm{tr}_gE^3-\frac{(n-2)^2}{2n(n-1)}r|E|^2=0. \tag{4.9}$$

5. Locally conformally flat critical g of $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] > 0$

We first state an algebraic lemma.

Lemma 5.1 (see [7] or [5]). *For any real numbers $a_1, \cdots, a_n$ with $\sum_i a_i = 0$, there holds*

$$-\frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{i=1}^n a_i^2\right)^{3/2} \le \sum_{i=1}^n a_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{i=1}^n a_i^2\right)^{3/2}, \tag{5.1}$$

and equality holds in (5.1) if and only if at least $n-1$ of the a_i's are equal. In particular, for $\sum_{i=1}^n a_i^2 \neq 0$, if $\sum_{i=1}^n a_i^3 = -\frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{i=1}^n a_i^2\right)^{3/2}$, then the $n-1$ of the a_i's which are equal must be positive; if $\sum_{i=1}^n a_i^3 = \frac{n-2}{\sqrt{n(n-1)}}\left(\sum_{i=1}^n a_i^2\right)^{3/2}$, then the $n-1$ of the a_i's which are equal must be negative.

Now, we suppose that (M^n, g) is a compact locally conformally flat manifold and $g \in \mathcal{M}_1$ is a critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$. Then, according to Corollary 3.2, $\sigma_2(C_g) = const$. To prove Theorem B, we first consider the case $\mathcal{F}_2[g] > 0$, i.e., $\sigma_2(C_g) = const > 0$.

Proposition 5.1. *Suppose M^n $(n \neq 4)$ is compact, and $g \in \mathcal{M}_1$ is a locally conformally flat metric. If g is a critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\sigma_2(C_g) > 0$, then (M^n, g) is a space form.*

Proof. To prove the proposition, it suffices to show that $|E| \equiv 0$ on M^n. If it is not the case, then there exists a point $p \in M^n$ such that $|E|(p) \neq 0$. We will derive a contradiction.

Note that $\sigma_2(C_g) > 0$ and (4.7) imply that

$$\frac{n-2}{2\sqrt{n(n-1)}}|r| > |E| \ge 0, \quad \text{on} \quad M^n. \tag{5.2}$$

Since M^n is connected, from (5.2) we have only two possible cases: $r > 0$ on M^n or $r < 0$ on M^n.

If $r > 0$, then at p, by applying Lemma 5.1, we have from (4.9) and (5.2)

$$\begin{aligned} 0 = \mathrm{tr}_g E^3 - \frac{(n-2)^2}{2n(n-1)}r|E|^2 &\le \frac{n-2}{\sqrt{n(n-1)}}|E|^3 - \frac{(n-2)^2}{2n(n-1)}r|E|^2 \\ &< \frac{n-2}{\sqrt{n(n-1)}}|E|^3 - \frac{n-2}{\sqrt{n(n-1)}}|E|^3 = 0, \end{aligned}$$

which is a contradiction.

If $r < 0$, then at p, from (4.9) and (5.2) and applying Lemma 5.1, we have

$$\begin{aligned} 0 = \mathrm{tr}_g E^3 - \frac{(n-2)^2}{2n(n-1)}r|E|^2 &\ge -\frac{n-2}{\sqrt{n(n-1)}}|E|^3 - \frac{(n-2)^2}{2n(n-1)}r|E|^2 \\ &> -\frac{n-2}{\sqrt{n(n-1)}}|E|^3 + \frac{n-2}{\sqrt{n(n-1)}}|E|^3 = 0, \end{aligned}$$

which is also a contradiction. This completes the proof of Proposition 5.1.

6. Locally conformally flat critical metric g of $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g]=0$

In this section, we find that the analysis of [4] can be carried through by some modifications so as to extend the 3-dimensional results there to any dimension $n \geq 5$ for conformally flat critical points of $\mathcal{F}_2|_{\mathcal{M}_1}$.

We first recall a result about the elementary symmetric function $S_2(x_1, \cdots, x_n)$ defined by (2.10). Note that as a homogeneous function of degree two, S_2 has signature $(1, n-1)$ with index $n-1$. Thus the set $\{x \in R^n | S_2(x) > 0\}$ has exactly two components. Following [4], we let Γ_2^+ denote the component which contains the positive cone. For a symmetric linear transformation $C : V^n \to V^n$, where V^n is an n-dimensional inner product space, the notation $C \in \Gamma_2^+$ will mean that the eigenvalues $(\lambda_1, \cdots, \lambda_n)$ of C lies in Γ_2^+. We will need the following important properties of Γ_2^+.

Lemma 6.1 (cf. [4])**.** (1) *The set Γ_2^+ is an open convex cone with vertex at the origin.*

(2) *If $C \in \Gamma_2^+$, then the Newtonian $T_1(C)$ defined by* (2.12) *is positive definite.*

(3) $\Gamma_2^+ \subset \Gamma_1^+ := \{x \in R^n | S_1(x) > 0\}$.

Now we present a global characterization for the null criticals, i.e., metrics g which are critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] = 0$.

Theorem 6.1. *Let (M^n, g) $(n \geq 5)$ be compact and let $g \in \mathcal{M}_1$ be a conformally flat metric. Then g is a critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] = 0$ if and only if $r \leq 0$ and the eigenvalues of the tensor C_g are $\{\underbrace{0, \cdots, 0}_{n-1}, \frac{n-2}{2(n-1)} r\}$.*

Proof. The *if* part is a direct check with application of Corollary 4.1. We will only give a detailed proof for the *only if* part. We first prove our claim that $r \leq 0$.

Note that $\mathcal{F}_2[g] = 0$ implies $\sigma_2(C_g) = 0$ and

$$|E|^2 = \frac{(n-2)^2}{4n(n-1)} r^2. \tag{6.1}$$

Then by (4.2) and (4.3), we easily get

$$0 = |\nabla E|^2 - \frac{n-2}{2(n-1)} \sum_{i,j} (T_1(C_g))_{ij} r_{,ij} + \frac{n}{n-2} \mathrm{tr} E^3 + \frac{r}{n-1} |E|^2 - \frac{(n-2)^2}{4n(n-1)} |\nabla r|^2. \tag{6.2}$$

Let $p \in M^n$ be a point where r achieves its maximum. Then $\nabla r(p) = 0$ and $(r_{,ij}(p))$ is negative semi-definite. If $r(p) \geq 0$, then

$$\sigma_1(C_g)(p) = (\mathrm{tr}_g C_g)(p) = \frac{n-2}{2(n-1)} r(p) \geq 0.$$

Because $\sigma_2(C_g)(p) = 0$, from Lemma 6.1 we conclude that p must be on the boundary of the positive cone Γ_2^+ and $T_1(C_g)(p)$ is positive semi-definite. Now (6.2) implies

$$0 \geq |\nabla E|^2(p) + \frac{n}{n-2} \mathrm{tr} E^3(p) + \frac{r(p)}{n-1} |E|^2(p) = |\nabla E|^2(p) + \frac{n}{2(n-1)} r(p) |E|^2(p),$$

where we used (4.9) in the last step. Therefore $r(p) \cdot |E|(p) = 0$ which implies $r(p) = 0$ by (6.1), so $r \leq 0$ everywhere on M^n.

Now from (4.9), (6.1) and the fact that $r \leq 0$ on M^n, we get

$$\mathrm{tr}_g E^3 = -\frac{n-2}{\sqrt{n(n-1)}}|E|^3.$$

By Lemma 5.1 we see that the eigenvalues of E are of the form $\{a, \cdots, a, -(n-1)a\}$ for some function $a \geq 0$. Then we deduce from (3.16) that the eigenvalues $\{\lambda_i\}$ of C_g satisfy

$$\lambda_1 = \cdots = \lambda_{n-1} = a + \frac{n-2}{2n(n-1)}r, \ \lambda_n = -(n-1)a + \frac{n-2}{2n(n-1)}r.$$

Now we have $|E|^2 = n(n-1)a^2$, which in combination with (6.1) gives $a = -\frac{n-2}{2n(n-1)}r$. Therefore we find $\lambda_1 = \cdots = \lambda_{n-1} = 0$ and $\lambda_n = \frac{n-2}{2(n-1)}r$. This proves Theorem 6.1.

Theorem 6.2. *Let M^n $(n \geq 5)$ be compact and let $g \in \mathcal{M}_1$ be a conformally flat metric. If g is critical for $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\mathcal{F}_2[g] = 0$, then for each $p \in M^n$, either*

(i) *the sectional curvature vanishs at p, or*

(ii) *there exists a local coordinate system $\{x_1, \cdots, x_{n-1}, y\}$ around p mapping a neighborhood of p to a cube in R^n in which the metric g takes the form*

$$g = dx_1^2 + \cdots + dx_{n-1}^2 + f(x_1, \cdots, x_{n-1}, y)^2 dy^2 \tag{6.3}$$

with

$$f(x_1, \cdots, x_{n-1}, y) = a(y)\sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} b_i(y)x_i + c(y), \tag{6.4}$$

where $a(y)$, $\{b_i(y)\}$, $c(y)$ are some functions of y.

Proof. If $r = 0$ at p, then (i) holds. We may therefore assume, by Theorem 6.1, that $r < 0$ around p. Then in a neighborhood of p, $r \neq 0$ and TM has a decomposition as $TM = V_1 \oplus V_2$, where V_1 and V_2 are the eigenspaces of the tensor C_g with eigenvalues 0 and $\frac{n-2}{2(n-1)}r$, where $\dim V_1 = n-1$, $\dim V_2 = 1$.

Since C_g is a Codazzi tensor, according to A. Derziński [3], V_1 is an integrable distribution. V_2 is a 1-dimensional distribution, therefore it is also integrable. Thus we have a local coordinate system $\{x_1, \cdots, x_n\}$ mapping a neighborhood of $p \in M^n$ to a cube in R^n with $Span\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-1}}\} = V_1$ and $Span\{\frac{\partial}{\partial x_n}\} = V_2$.

Since eigenspaces corresponding to distinct eigenvalues are orthogonal, we have that the metric g locally takes the form

$$\begin{pmatrix} (g_{ij}(x))_{(n-1)\times(n-1)} & 0 \\ 0 & g_{nn}(x) \end{pmatrix}.$$

Claim 6.1. *Integral manifolds of V_1 are flat and totally geodesic submanifolds of M^n.*

Proof. Since V_1 is the eigenspace of a Codazzi tensor with the constant eigenvalue 0, by [3], the integral manifolds of V_1 are totally geodesic submanifolds of M^n. Let $\{e_i\}_{1\leq i\leq n-1}$ be an orthonormal frame field on V_1. Since (M^n, g) is conformally flat, from (2.3) we have

$$R(e_i, e_j, e_i, e_j) = \frac{1}{n-2}(C_{ii} + C_{jj}) = 0, \quad \forall i \neq j.$$

Therefore, from the Gauss equation, the integral manifolds of V_1 are of zero sectional curvature and thus are flat. $\square$

Let us consider a slice of an integral manifold $N := \{x_n = const\}$ of V_1. Let Π denote the second fundamental form of $N \hookrightarrow M^n$. Note that $\{x_1, \cdots, x_{n-1}\}$ forms a local coordinate system on N. Denote $\partial_a = \frac{\partial}{\partial x_a}$. In the sequel we will make use of the following convention on the range of indices: $1 \le i, j, k, l \le n-1$; $1 \le a, b, c, d \le n$.

Since $e_n = g_{nn}^{-1/2}\partial_n$ is the unit normal vector of $N \hookrightarrow M^n$, from Claim 6.1 we have

$$0 = \Pi(\partial_i, \partial_j) = -g(\nabla_{\partial_i} e_n, \partial_j) = -g_{nn}^{-1/2}\sum_a \Gamma_{ni}^a g_{ja} = -\frac{1}{2} g_{nn}^{-1/2}\partial_n g_{ij},$$

i.e., $\partial_n g_{ij} = 0,\ 1 \le i,\ j \le n-1$. Therefore, since the slice N is flat, by changing the coordinates $\{x_1, \cdots, x_{n-1}\}$ if necessary, we can assume that the metric has the local form: $g_{ij} = \delta_{ij}$ and $g_{na} = f^2(x_1, \cdots, x_n)\delta_{na}$ for some positive function $f(x) > 0$.

Claim 6.2.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_1^2}\delta_{ij}, \qquad 1 \le i,\ j \le n-1.$$

Proof. Since $\partial_n g_{ij} = 0 = g_{ni},\ \forall i, j$, by a simple calculation we have

$$\begin{cases} \Gamma_{nn}^i = -\frac{1}{2}\partial_i g_{nn} = -f\partial_i f, \\ \Gamma_{ni}^n = \Gamma_{in}^n = \frac{1}{2} g^{nn}\partial_i g_{nn} = \partial_i \log f, 1 \le i \le n-1, \\ \Gamma_{nn}^n = \frac{1}{2} g^{nn}\partial_n g_{nn} = \partial_n \log f, \\ \Gamma_{ab}^c = 0, \quad \text{for all other cases.} \end{cases} \tag{6.5}$$

According to (3.5), the components of the Ricci tensor are given by

$$R_{ab} = \sum_c R^c{}_{acb} = \sum_c (\partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c) + \sum_{c,d} (\Gamma_{cd}^c \Gamma_{ab}^d - \Gamma_{bd}^c \Gamma_{ac}^d). \tag{6.6}$$

From (6.5) and (6.6), we have

$$R_{ij} = -f^{-1}\frac{\partial^2 f}{\partial x_i x_j}, \quad R_{ni} = 0, \quad 1 \le i, j \le n-1; \quad R_{nn} = -\sum_i f\frac{\partial^2 f}{\partial x_i^2}.$$

Then we get the scalar curvature as follows:

$$r = \sum_{a,b} g^{ab} R_{ab} = \sum_i R_{ii} + f^{-2} R_{nn} = -2f^{-1}\sum_i \frac{\partial^2 f}{\partial x_i^2}.$$

From (2.4) and the above calculation, we have

$$\begin{cases} C_{ij} = R_{ij} = -f^{-1} f_{ij}, \quad i \ne j, \\ C_{ii} = R_{ii} - \frac{r}{2(n-1)} = -f^{-1} f_{ii} + \frac{1}{(n-1)f}\sum_j f_{jj}, \\ C_{nn} = R_{nn} - \frac{r}{2(n-1)} = \frac{2-n}{n-1} f \sum_i f_{ii}, \qquad C_{ni} = 0, \end{cases} \tag{6.7}$$

where we denote $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

Since all the eigenvalues of $(C_{ij})_{(n-1)\times(n-1)}$ are zero, we have $C_{ij} \equiv 0, \quad 1 \le i,\ j \le n-1$. From this and (6.7) we conclude that $f_{ij} = 0$ when $i \ne j$. Note that $C_{11} = \cdots = C_{n-1,n-1} = 0$ and (6.7) imply $R_{11} = \cdots = R_{n-1,n-1}$, which gives

$f_{11} = \cdots = f_{n-1,n-1}$. Then from (6.7) again we find $C_{nn} = -(n-2)ff_{ii}$ for any $1 \le i \le n-1$. This proves our Claim 6.2.

Claim 6.3. $f_{iii} = 0, \quad 1 \le i \le n-1.$

Proof. From the fact that C_g is a Codazzi tensor, we have

$$\nabla_i C_{nn} = \nabla_n C_{ni},\ 1 \le i \le n-1.$$

From the direct calculation

$$\nabla_i C_{nn} = \partial_i(C_{nn}) - 2\sum_a \Gamma^a_{ni} C_{na} = (n-2)f_i f_{ii} - (n-2)ff_{iii},$$

$$\nabla_n C_{ni} = \partial_n(C_{ni}) - \sum_a \Gamma^a_{nn} C_{ai} - \sum_a \Gamma^a_{ni} C_{na} = -\sum_a \Gamma^n_{ni} C_{nn} = (n-2)f_i f_{ii},$$

we find that $ff_{iii} = 0$, $1 \le i \le n-1$. Then Claim 6.3 follows.

From Claims 6.2 and 6.3, we have finished the proof of Theorem 6.2.

Proposition 6.1. *Let g be a metric on $R^n = \{x_1, \cdots, x_{n-1}, x_n\}$ of the form*

$$g = dx_1^2 + \cdots + dx_{n-1}^2 + f(x_1, \cdots, x_{n-1}, x_n)^2 dx_n^2$$

with

$$f(x_1, \cdots, x_{n-1}, x_n) = a(x_n)\sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} b_i(x_n)x_i + c(x_n). \tag{6.8}$$

Then g is locally conformally flat and satisfies $\sigma_2(C_g) = 0$. Furthermore, g is critical for $\mathcal{F}_2$ with respect to all compactly supported variations.

Proof. Simple calculations show that the metric g satisfies (6.5) and the following relations:

(i) $R_{ij} = -f^{-1}f_{ij}$; so $R_{11} = \cdots = R_{n-1,n-1},\ R_{ij} = 0,\ i \ne j;\ \ 1 \le i,j \le n-1.$
$R_{ni} = 0,\ \ 1 \le i \le n-1;\ \ R_{nn} = -(n-1)ff_{11};\ \ r = -2(n-1)f^{-1}f_{11}.$
(ii) $C_{ij} = 0;\ \ C_{ni} = 0;\ \ C_{nn} = -(n-2)ff_{11},\ \ 1 \le i,j \le n-1.$
(iii) C_g is a Codazzi tensor.

By formula (3.5) of the full curvature tensor R_{abcd}, a direct check proves that C and g satisfy $R_{abcd} = \frac{1}{n-2}(C_{ac}g_{bd} - C_{ad}g_{bc} + C_{bd}g_{ac} - C_{bc}g_{ad})$. This and (iii) verify that (R^n, g) is a non-compact locally conformally flat Riemannian manifold.

From (ii) above, we see that $\sigma_2(C_g) = 0$. Now we can also check that (4.8) is satisfied. Then, by making use of (4.6) and (4.8), we obtain from (3.1) that $\nabla\mathcal{F}_2 = 0$, which implies that g is critical for $\mathcal{F}_2$ with respect to all compactly supported variations. This proves Proposition 6.1.

Remark 6.1. Similar to the 3-dimensional situation in [4], Proposition 6.1 implies that, for higher dimension $n \ge 4$, there might exist abundance complete null critical metrics of non-constant sectional curvature on a non-compact manifold, e.g. R^n. Here we notice that, by appropriately choosing the functions $\{a(x_n),\ b_i(x_n),\ c(x_n)\}$ in (6.8), the Riemannian manifold (R^n, g) is complete, e.g., $f(x_1, \cdots, x_n) = 1 + x_1^2 + \cdots + x_n^2$.

To prove Theorem B for the null critical case, it suffices to show the scalar curvature $r \equiv 0$ on M^n. Since we have proved in Theorem 6.1 that $r \le 0$ on M^n, we now consider the set $M_- \equiv \{p \in M :\ r(p) < 0\}$. If $M_- = \varnothing$, then we are done.

So in the sequel we assume $M_- \neq \varnothing$. To derive a needed contradiction, we will adopt the method due to Gursky and Viaclovsky (cf. [4]) by undertaking a careful study of the leaves of the foliation of M_- defined by the distribution V_1 in Theorem 6.2. Notice that V_1 is integrable, so by the well-known Frobenius Integrability Theorem, we are guaranteed the existence of a unique maximal connected integral manifold through each point where the scalar curvature is negative.

We first present a result which is crucial for our proof of Theorem B in the null critical case.

Proposition 6.2. *Let $i : N \hookrightarrow M_-$ be a maximal connected integral manifold of V_1. Then N is isometric to R^{n-1} with the flat metric, and i is a proper imbedding.*

Proof. By definition, i is an injective immersion. From Claim 6.1, we know that N is a flat $(n-1)$-dimensional Riemannian manifold and i is a totally geodesic isometric immersion.

To prove the completeness of N, we first prove the following

Lemma 6.2. *Define a function S on N by $S(p) = [r(i(p))]^{-1}$, $\forall p \in N$. Then the Hessian of S is given by*

$$D^2_N(S) = -\frac{1}{2(n-1)} g_N. \tag{6.9}$$

Proof. We will use the notations and conclusions in Theorem 6.2. Given $p \in N$, we can find a local coordinate system $\{x_1, \cdots, x_{n-1}, x_n\}$ around $i(p)$ such that the immersion i is modeled by $(x_1, \cdots, x_{n-1}) \mapsto (x_1, \cdots, x_{n-1}, 0)$ and the components of the metric g is given by $g_{ij} = \delta_{ij},\ g_{ni} = 0,\ g_{nn} = f^2,\ 1 \le i, j \le n-1$. From the proof of Theorem 6.2, $r = -2(n-1)f^{-1}f_{11}$ and $f_{11} = \cdots = f_{n-1,n-1}$, so we have

$$S = -\frac{f}{2(n-1)f_{11}}.$$

From Claim 6.3 we have $f_{11}|_N = const.$ The flatness of N implies that $D^2_N(S)(\partial_i, \partial_j) = \partial_i\partial_j S$. Then from Claim 6.2, we have

$$D^2_N(S)(\partial_i, \partial_j) = \partial_i\partial_j\left(-\frac{f}{2(n-1)f_{11}}\right) = -\frac{f_{ij}}{2(n-1)f_{11}} = -\frac{1}{2(n-1)}\delta_{ij} = -\frac{1}{2(n-1)}g_{ij}.$$

This proves Lemma 6.2.

The following proof of Proposition 6.2 is almost identical with that in [4] except that in the second step we make use of Cheeger-Gromoll's splitting theorem. For the reader's convenience, we will keep it here.

We will now use Lemma 6.2 to show that N is necessarily complete. Let $\gamma : (a, b) \to N$ be any bounded geodesic segment in N. To show that N is complete, it is sufficient to show that we can extend γ to a longer segment in N. Since i is a totally geodesic immersion, $i \circ \gamma$ is a geodesic segment in M_-. From (6.9), S restricted to γ is a quadratic function of the arc length, so S is bounded on γ. Therefore $r \le -c < 0$ on the image of $i \circ \gamma$ and at the endpoints of γ, r is negative and thus the endpoints of $i \circ \gamma$ are in the interior of the open set $M_- \subset M$. Thus we may extend $i \circ \gamma$ to a longer geodesic segment in M_-. Applying Theorem 6.2 at the endpoints of $i \circ \gamma$, we see that N can be extended so that its image strictly contains the extension of $i \circ \gamma$. Since N is a maximal leaf, this proves that N is complete.

Next we show that N is necessarily isometric to R^{n-1}. From (6.9), we see that S is a globally concave function on N. If N were compact, then S would attain a minimum and the Hessian would be positive semidefinite at that point, a contradiction to (6.9). Therefore N is in fact a complete, non-compact and flat $(n-1)$-manifold. Then, by Cheeger and Gromoll's Splitting Theorem (cf. [2]), N must be either isometric to R^{n-1} or isometric to $R^l \times \overline{N}^{n-l-1}$ for some manifold $\overline{N}$ which contains no geodesic lines. In our case, if $\dim \overline{N} \geq 2$, then $\overline{N}$ is a flat manifold. Therefore in the latter case, $\overline{N}$ cannot be simply connected, which implies that there exists a closed geodesic on N. By restricting S to such a closed geodesic, it would attain a minimum, contradicting (6.9). Therefore N is isometric to R^{n-1}, as claimed.

Finally we show that $i : N \to M_-$ is a proper imbedding. Let K be a compact subset of M_-. Then $r \leq -c < 0$ on K. Since N was shown to be R^{n-1}, the local coordinate system $\{x_i\}_{1\leq i\leq n-1}$ becomes a global one. Then (6.9) shows that S is a function of a strictly concave quadratic polynomial in $\{x_i\}$. Therefore $r\circ i = S^{-1} \to 0$ as $\sum_{i=1}^{n-1} x_i^2 \to \infty$. Now $r \leq -c < 0$ on K implies that $i^{-1}(K)$ lies in a compact set and this proves that i is proper. Because the maximal integral manifold passing through a fixed point is unique, we see that i is in fact an imbedding. We have completed the proof of Proposition 6.2.

7. Completion of the proof of Theorem B

Because of Corollary 3.1, Proposition 5.1, Theorem 6.1 and Theorem 6.2, to prove Theorem B it is now sufficient to derive a contradiction in the case $M_- \neq \varnothing$ for the null critical case.

For any leaf $i : N \to M_-$ of V_1, we have that $i \circ \exp_N = \exp_M \circ i_*$ since i is totally geodesic. Therefore for any $p \in M_-$, $\exp_M$ restricted to $V_1(p)$ is a maximal connected integral manifold through p and, by Proposition 6.2, is a properly imbedded R^{n-1}. We now fix $p \in M_-$ and let $\beta : (-\varepsilon, \varepsilon) \to M_-$ be an integral curve of V_2 (in Theorem 6.2) passing through p. We identify $V_1(\beta(t))$ with R^{n-1}, and consider the normal exponential map Φ_t along β. Define

$$\Phi : \ R^{n-1} \times (-\varepsilon, \varepsilon) \to M_- \tag{7.1}$$

by $\Phi(x_1, \cdots, x_{n-1}, t) = \Phi_t(x_1, \cdots, x_{n-1}) : R^{n-1} \to M_-$. As observed above, for each t, the map $\Phi_t : R^{n-1} \to M_-$ gives a maximal integral manifold of V_1 and by Proposition 6.2, Φ_t is further a proper imbedding. Now, by completely the same argument as in the 3-dimensional case of [4], p. 272, we can show that for ε small enough, the map Φ is an imbedding.

Claim 7.1. *For $\Phi : \ R^{n-1} \times (-\varepsilon, \varepsilon) \to (M_-, g)$, we have*

$$\Phi^* g = dx_1^2 + \cdots + dx_{n-1}^2 + f(x_1, \cdots, x_{n-1}, t)^2 dt^2, \tag{7.2}$$

where

$$f(x_1, \cdots, x_{n-1}, t) = a(t) \sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} b_i(t) x_i + c(t) \tag{7.3}$$

with $a(t) > 0, \ c(t) > 0$.

Proof. The form (7.2) and (7.3) just follow from the proof of Theorem 6.2. Since we are writing the metric in the form (7.2), without loss of generality we may assume $f > 0$. Then for small t, we must have $c(t) > 0$. From the proof of Claim 6.2, we have that $r(x_1, \cdots, x_{n-1}, t) = -4(n-1)a(t)f^{-1}(x_1, \cdots, x_{n-1}, t)$ and $r(0, \cdots, 0, t) = -4(n-1)a(t)/c(t)$. Therefore for small t, we must have $a(t) > 0$. $\square$

Now we consider the image $U = \Phi\left(R^{n-1} \times (-\varepsilon, \varepsilon)\right)$ which is an open subset of M. Since the map Φ is an imbedding, the volume of U in the induced metric is

$$\begin{aligned} Vol_{\Phi^* g}(U) &= \int_{R^{n-1}\times(-\varepsilon,\varepsilon)} \sqrt{\det(g_{ij})}dx_1 \cdots x_{n-1}dt \\ &= \int_{R^{n-1}\times(-\varepsilon,\varepsilon)} \left(a(t)\sum_{i=1}^{n-1} x_i^2 + \sum_{i=1}^{n-1} b_i(t)x_i + c(t) \right) dx_1 \cdots dx_{n-1}dt \\ &= \infty. \end{aligned}$$

Since M is compact, any open subset of M should have finite volume. Thus we achieve the needed contradiction which shows that we must have $M_- = \varnothing$.

Therefore the null critical metric can only be flat and we complete the proof of Theorem B.

Remark 7.1. As stated in Remark 1.1, a constant curvature metric necessarily has $\mathcal{F}_2[g] \geq 0$. Theorem B shows that the converse is true for a critical metric. It should be noted that there indeed exist critical metrics such that $\mathcal{F}_2[g] < 0$. Take for example, if $n = 2m$, consider the Riemannian product $(M^n, g) = N^m(-c) \times S^m(c)$, where $c > 0$ and $N^m(-c)$ denotes a compact space form of constant sectional curvature $-c$, and $S^m(c)$ denotes the usual sphere of constant sectional curvature c. Now we choose c such that $Vol(M^n, g) = 1$. Then a simple calculation shows that (M^n, g) is a compact, locally conformally flat and non-Einstein manifold with scalar curvature identically zero. By using Corollary 4.1, we can easily prove that g is in fact a critical point of $\mathcal{F}_2|_{\mathcal{M}_1}$ with $\sigma_2(C_g) = -m(m-1)^2c^2 < 0$.

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References

[1] A. L. Besse, *Einstein Manifolds,* Springer-Verlag, Berlin, 1987. MR **88f:**53087

[2] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom., 6(1971/72): 119-128. MR **46:**2597

[3] A. Derdzinski, *Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor*, Math. Z., 172(1980): 273-280. MR **82e:**53053

[4] M. J. Gursky and J. A. Viaclovsky, *A new variational characterization of three-dimensional space forms*, Invent. Math., 145(2001): 251-278. MR **2002j:**53039

[5] H. Li, *Global rigidity theorems of hypersurfaces*, Ark. Math., 35(1997): 327-351. MR **98j:**53074

[6] A. Lichnerowicz, *Propagateurs et commutateurs on relativité générale*, Inst. Hautes Études Sci. Publ. Math. No.10(1961), 56. MR **28:**967

[7] M. Okumura, *Hypersurfaces and a pinching problem on the second fundamental tensor*, Amer. J. Math., 96(1974): 207-213. MR **50:**5701

[8] R.C. Reilly, *Variational properties of function of the mean curvatures for hypersurfaces in space forms*, J. Diff. Geom., 8(1973): 465-477. MR **49:**6102

[9] H. Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Sci. Math., 117(1993): 211-239. MR **94b:**53097

[10] J. A. Viaclovsky, *Conformal geometry, contact geometry and the calculus of variations*, Duke Math. J., 101(2000): 283-316. MR **2001b:**53038

[11] K. Voss, *Einige differentialgeometrische Kongruenzsätze für geschlossene Flächen und Hyperflächen*, Math. Ann., 131(1956): 180-218. MR **18:**229f

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