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SZEGÖ KERNELS AND FINITE GROUP ACTIONS

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ABSTRACT. In the context of almost complex quantization, a natural generalization of algebro-geometric linear series on a compact symplectic manifold has been proposed. Here we suppose given a compatible action of a finite group and consider the linear subseries associated to the irreducible representations of G, give conditions under which these are base-point-free and study properties of the associated projective morphisms. The results obtained are new even in the complex projective case.

1. INTRODUCTION

Let (M, ω) be a compact symplectic manifold of dimension $2n$, such that $[\omega] \in H^2(M, \mathbb{Z})$. Fix $J \in \mathcal{J}(M, \omega)$ (the contractible space of all almost complex structures on M compatible with ω), and let h and $g = \mathcal{R}(h)$ be the induced hermitian and riemannian structures. There exist an hermitian line bundle (A, h) on M and a unitary covariant derivative ∇_A on A, such that $-2\pi i\omega$ is the curvature of ∇_A.

In this set-up, the usual $\overline{\partial}$-complex from complex geometry can be replaced by a complex of pseudodifferential operators enjoying similar symbolic properties [BdM], [BdMG]; building on this foundational result, a theory of almost complex quantization has been developed and studied by several authors.

Namely, one can define spaces of quantized sections

$$H(M, A^{\otimes k}) \subseteq \mathcal{C}^\infty(M, A^{\otimes k}),$$

as the kernel of the first operator in the complex. A related approach is in terms of the asymptotic spectral properties of a suitable renormalized laplacian [GU], [BU1].

These linear series determine projective embeddings of M enjoying the same metric and symplectic asymptotic properties as in the integrable projective case [BU2], [Z], [SZ2], [T]. In the integrable case the theory reduces to the usual classical constructions of complex algebraic geometry.

Suppose now that G is a finite group with a symplectic action

$$\nu : G \times M \to M,$$

so that J may be chosen G-invariant. Then ν preserves g and h. Assume also that ν lifts to a linear action $\tilde{\nu} : G \times A \to A$, and that $\tilde{\nu}$ preserves h_A and ∇_A. Then $\tilde{\nu}$ preserves each of the spaces $H(M, A^{\otimes N})$. Let

$$\rho_i : G \to \mathrm{GL}(V_i) \quad (1 \leq i \leq c)$$

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be the irreducible representations of G; we shall assume that $i = 1$ corresponds to the trivial one-dimensional representation. For each N, we have a G-equivariant decomposition

$$H(M, A^{\otimes N}) = \bigoplus_{i=1}^{c} H(M, A^{\otimes N})_i,$$

where $H(M, A^{\otimes N})_i$ consists of a direct sum of copies of V_i. It is natural to ask whether the linear series $|H(M, A^{\otimes N})_i|$ are base-point-free and, if so, what about their asymptotic properties? In this note, we apply arguments from [BU2] and [Z], [SZ2] to these questions.

If $x \in M$, let $G_x = \{g \in G : g \cdot x = x\}$ be its stabiliser. Let $\chi_i : G \to \mathbb{C}$ be the character of the i-th irreducible representation. Let A_x be the fibre of A over $x \in M$. Clearly, G_x acts on A_x and thus we have a unitary character $\alpha_x : G_x \to S^1 \subset \mathbb{C}^*$. Let

$$\gamma_{i,N}(x) := (\alpha_x^N, \chi_i)_{G_x} = \sum_{g \in G_x} \alpha_x(g)^N \cdot \overline{\chi}_i(g) \quad (x \in M,\, 1 \le i \le c,\, N \in \mathbb{N}),$$

$(\,,\,)_{G_x}$ denoting the L^2-product with respect to the counting measure on G_x. Note that $\gamma_{i,N} = \gamma_{i,N+|G|}$ for every i and N, where $|G|$ denotes the order of G. Set

$$B_{i,N} := \{x \in M : \gamma_{i,N}(x) = 0\} = B_{i,N+|G|} \quad (1 \le i \le c).$$

Clearly, $x \in B_{i,N}$ implies $G_x \ne \{e\}$.

Our first goal is to determine the base locus of the spaces of sections $H(M, A^{\otimes k})_i$ for $k \gg 0$. In algebro-geometric terminology, the base locus of a vector subspace $W \subseteq \mathcal{C}^\infty(M, A^{\otimes N})$ is

$$\mathrm{Bs}(|W|) := \{x \in M :\, s(x) = 0 \,\forall\, s \in W\}.$$

To begin with, we shall prove:

Theorem 1.1. *Suppose $1 \le i \le c$, $0 \le r \le |G| - 1$, $x \in M$ and $\gamma_{i,r}(x) \ne 0$. Then for $N \gg 0$, $N \equiv r (\mathrm{mod}\ |G|)$ there exists a section $s \in H(M, A^{\otimes N})_i$ such that $s(x) \ne 0$.*

This has a number of consequences:

Corollary 1.1. *Suppose that the action of G on M is effective. Then*

$$\dim(H(M, A^{\otimes k})_i) > 0$$

for every $i = 1, \ldots, c$ and every $k \gg 0$.

In fact, it is proved in [P] that under the same hypothesis

$$\dim(H(M, A^{\otimes k})_i) = \frac{\dim(V_i)^2}{|G|} \cdot \frac{k^n}{n!} \cdot c_1(A)^n + o(k^n).$$

Proposition 1.1. *Suppose $1 \le i \le c$, $0 \le r \le |G| - 1$, and $\gamma_{i,r}(x) \ne 0$ for every $x \in M$. Then $H(M, A^{\otimes k})_i$ globally generates $A^{\otimes k}$ if $k \gg 0$ and $k \equiv r (\mathrm{mod}\ |G|)$, that is, for every $x \in M$ there is $s \in H(M, A^{\otimes k})_i$ such that $s(x) \ne 0$.*

Corollary 1.2. *If $k \gg 0$ and $i = 1, \ldots, c$, the subspace of G-invariant sections*

$$H(M, A^{\otimes k|G|})^G \subseteq H(M, A^{\otimes k|G|})$$

globally generates $A^{\otimes k|G|}$.

Corollary 1.3. *If M is a complex projective manifold and A is ample, for every $i=1,\ldots,c$ and $r=0,\ldots,|G|-1$ the base loci* $\mathrm{Bs}\big(|H^0(M,A^{\otimes(r+k|G|)})_i|\big)$ *stabilize for $k\gg 0$. Furthermore, for every $k\gg 0$,*

$$\mathrm{Bs}\left(|H^0(M,A^{\otimes(r+k|G|)})_i|\right)\subseteq B_{i,r}.$$

In the reverse direction, it is easily seen that if $G_x=G$ and there exists $s\in\mathcal{C}^\infty(M,A^{\otimes N})_i$ with $s(x)\neq 0$, then

$$(\alpha_x^N,\chi_i)_G\neq 0.$$

Therefore,

Corollary 1.4. *In the hypothesis of Corollary 1.3, suppose in addition that either $G_x=\{e\}$ or $G_x=G$ for every $x\in G$. Then*

$$\mathrm{Bs}\left(|H(M,A^{\otimes N})_i|\right)=B_{i,N}$$

for $i=1,\ldots,c$ and $N\gg 0$.

In the almost complex case, for any $i=1,\ldots,c$ and $r=0,\ldots,|G|-1$ we may still define the (i,r)-th equivariant asymptotic base locus of A as

$$\mathrm{Bs}(A,i,r)_\infty =: \{x\in M:\,\forall s>0\,\exists\, k>s,\ k\equiv r\ (\mathrm{mod}|G|)\\ \text{such that }\ x\in\mathrm{Bs}\left(|H(M,A^{\otimes k})_i|\right)\}.$$

The general case (symplectic, almost complex) of Corollary 1.3 is then

Corollary 1.5. *In the above situation,*

$$\mathrm{Bs}(A,i,r)_\infty\subseteq B_{i,r}.$$

If furthermore $K\subset M$ is any compact subset with $K\cap B_{i,r}=\emptyset$, then

$$K\cap\mathrm{Bs}\left(|H(M,A^{\otimes k})_i|\right)=\emptyset$$

for all $k\gg 0$ with $k\equiv r$ (mod $|G|$).

Next, if $\mathrm{Bs}\left(|H(M,A^{\otimes N})_i|\right)=\emptyset$, there are associated projective morphisms

$$\Phi_{i,r+k|G|}:M\to\mathbb{P}(H(M,A^{\otimes(r+k|G|)})_i^*),$$

and we now consider their asymptotic properties as $k\to+\infty$.

Theorem 1.2. *Suppose* $\mathrm{Bs}\left(|H(M,A^{\otimes N})_i|\right)=\emptyset$ *for some $1\leq i\leq c$ and $0\leq r\leq |G|-1$. Let $U\subseteq M$ be the open subset of M where the order $|G_x|$ is locally constant. Suppose $U'\subset U$ is open with $\overline{U'}\subset U$. Then $\Phi_{i,r+k|G|}$ is an immersion on U' for $k\gg 0$.*

Corollary 1.6. *$|H(M,A^{\otimes N})^G|$ is base-point-free and $\Phi_{1,N}$ is an immersion on compact subsets of U if $N\gg 0$ and $\sum_{g\in G_x}\alpha_x(g)^N\neq 0$ for every $x\in G$.*

In general $\Phi_{i,N}$ is not injective; for example it is constant on every orbit for any G if i corresponds to the trivial representation, or for any i if G is abelian. We may still ask, however, if in these cases points in different orbits have different images under $\Phi_{i,N}$.

Let $d_G:M\times M\to\mathbb{R}$ be the *orbit distance*:

$$d_G(x,y):=\min\{d(gx,y):g\in G\}\qquad(x,y\in M).$$

Clearly, $d_G(x,y)>0$ if and only if $x\notin G\cdot y$.

Proposition 1.2. *Assume that either G is abelian, or G is arbitrary and $i = 1$. Let $U \subseteq M$ be as in Theorem 1.1, $N \in \mathbb{N}$ and suppose that* $\mathrm{Bs}\left(|H(M, A^{\otimes N})_i|\right) = \emptyset$ *and that $\gamma_{i,N}$ is constant on W. Let $K \subseteq W$ be a compact subset. There exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, $x, y \in K$ and $d_G(x, y) > 0$, then*

$$\Phi_{i,N+k|G|}(x) \neq \Phi_{i,N+k|G|}(y).$$

Corollary 1.7. *If the action of G on M is free, then $\Phi_{i,N}$ is well defined and is an embedding $M/G \hookrightarrow \mathbb{P}\big(H(M, A^{\otimes k})^{G^*}\big)$ for any $i = 1, \ldots, c$ and $N \gg 0$.*

Similar statements hold for the asymptotic metric and almost complex properties, in the vein of Theorem 1.1 of [BU2].

2. Proofs

Proof of Theorem 1.1. We recall some notation from [BU2], [Z], [SZ2]. Let $A^* = A^{-1}$ be the dual line bundle with the induced hermitian stucture h_{A^*}, and let $A^* \supset \mathbb{S} \xrightarrow{\pi} M$ be the unit circle bundle, a strictly pseudoconvex domain. Given the connection, $\mathbb{S}$ has natural riemannian and almost CR structures. We shall identify functions and half-forms throughout.

As $\mathbb{S}$ is a principal S^1-bundle, $\mathcal{C}^\infty(\mathbb{S}) = \bigoplus_{N\in\mathbb{Z}} \mathcal{C}^\infty(\mathbb{S})_N$, where $\mathcal{C}^\infty(\mathbb{S})_N$ is the N-th isotype for the S^1-action. We shall identify $\mathcal{C}^\infty(M, A^{\otimes N})$ and $\mathcal{C}^\infty(\mathbb{S})_N$ in the standard manner. Set $H(\mathbb{S}) := \bigoplus_{N\in\mathbb{N}} H(\mathbb{S})_N$, where $H(\mathbb{S})_N \cong H(M, A^{\otimes N})$ under this identification; in the integrable projective case, $H(\mathbb{S})$ is the Hardy space of boundary values of holomorphic functions on A^*. Let $\Pi : L^2(\mathbb{S}) \to H(\mathbb{S})$ be the orthogonal projector and $\tilde{\Pi} \in \mathcal{D}'(\mathbb{S} \times \mathbb{S})$ its Schwartz kernel; decompose it as $\tilde{\Pi} = \bigoplus_{N\in\mathbb{N}} \tilde{\Pi}_N$, where $\tilde{\Pi}_N \in \mathcal{C}^\infty(\mathbb{S} \times \mathbb{S})$ is the N-th Fourier coefficent. We have $\tilde{\Pi}_N(x, y) = \sum_{i=0}^{d_N} s_i^N(x) \otimes \overline{s}_i^N(y)$, where $\{s_0^N, \ldots, s_{d_N}^N\}$ is an orthonormal basis of $H(\mathbb{S})_N$. Let $\tilde{\Phi}_{i,N} : \mathbb{S} \to H(M, A^{\otimes N})^*$ be the coherent state map, given by evaluation, which is a lifting of $\Phi_{i,N}$ when the latter is defined. Then $\tilde{\Pi}_N(p, q) = \big(\tilde{\Phi}_{i,N}(p), \tilde{\Phi}_{i,N}(q)\big)$ $(p, q \in \mathbb{S})$, where $(\cdot, \cdot)$ denotes the L^2-hermitian product on $H(M, A^{\otimes N})^*$.

The induced action of G on A^* preserves $\mathbb{S}$ and the riemannian and almost CR structures on $\mathbb{S}$, and the isomorphisms $H(\mathbb{S})_N \cong H(M, A^{\otimes N})$ are G-equivariant. For $N \gg 0$, we have G-equivariant decompositions $H(\mathbb{S})_N = \bigoplus_i H(\mathbb{S})_{i,N}$, where $H(\mathbb{S})_{i,N}$ is the factor consisting of a direct sum of copies of V_i, $1 \leq i \leq c$. Similarly, $H(\mathbb{S}) = \bigoplus_i H(\mathbb{S})_i$. We shall implicitly identify $H(\mathbb{S})_N$ and $H(\mathbb{S})_{i,N}$ with $H(M, A^{\otimes N})$ and $H(M, A^{\otimes N})_i$, respectively. For each i, let $\Pi_i : L^2(\mathbb{S}) \to H(\mathbb{S})_i$ denote the orthogonal projection and let $\tilde{\Pi}_i \in \mathcal{D}'(\mathbb{S}\times\mathbb{S})$ be its Schwartz kernel. For each i and N, let $\Pi_{i,N} : L^2(\mathbb{S}) \to H(\mathbb{S})_{i,N}$ be the orthogonal pojection and $\tilde{\Pi}_{i,N}$ its Schwartz kernel, the N-th Fourier coefficient of $\tilde{\Pi}_i$: if $\{s_0^{(i,N)}, \ldots, s_{d_{i,N}}^{(i,N)}\}$ is an orthonormal basis of $H(\mathbb{S})_{i,N}$, then

$$\tilde{\Pi}_{i,N}(p, q) = \sum_{j=0}^{d_{i,N}} s_j^{(i,N)}(p) \otimes \overline{s_j^{(i,N)}(q)} \qquad (p, q \in \mathbb{S}).$$

Clearly, $\tilde{\Pi} = \sum_{i=1}^{c} \tilde{\Pi}_i$. Notice that the Fourier components of the total and equivariant Szegö kernels, Π_N and $\Pi_{i,N}$, when restricted to the diagonal in $\mathbb{S} \times \mathbb{S}$, descend to well-defined smooth functions on the diagonal in $M \times M$, that is, with some abuse of language we may write $\Pi_N(p, p) = \Pi_N(x, x)$ and $\Pi_{i,N}(p, p) = \Pi_{i,N}(x, x)$ for any $p \in \mathbb{S}$ and $x \in M$ with $\pi(p) = x$. This will be done implicitly below.

By the projection formula, for each $i = 1, \dots, c$ we have

$$\tilde{\Pi}_{i,N} = \sum_{j=0}^{d_N} \Pi_i(s_j^N) \otimes \overline{s}_j^N = (\dim(V_i)/|G|) \cdot \sum_g \sum_j \overline{\chi}_i(g)\rho(g)(s_j^N) \otimes \overline{s}_j^N,$$

where $\rho : G \to \mathrm{GL}(H(\mathbb{S})_N)$ is the induced representation; explicitly, $\rho(g)\sigma = \sigma \circ g^{-1}$ ($g \in G$, $\sigma \in H(\mathbb{S})_N$), where we view g^{-1} as a contactomorphism of $\mathbb{S}$. Thus,

$$\begin{aligned}\tilde{\Pi}_{i,N}(p,q) &= (\dim(V_i)/|G|) \cdot \sum_g \sum_j \overline{\chi}_i(g) s_j^N(g^{-1}p)\overline{s}_j^N(q) \\ &= (\dim(V_i)/|G|) \cdot \sum_g \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,q).\end{aligned}$$

On the diagonal, $\tilde{\Pi}_{i,N}(p,p) = (\dim(V_i)/|G|) \cdot \sum_g \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p)$. Let d be the geodesic distance function on M and also its pull-back $d \circ \pi$ to $\mathbb{S}$. If $x \in M$ and $G \cdot x \neq \{x\}$, set $a_x := \min\{d(gx,x) : g \in G \setminus G_x\}$. Suppose $p \in \mathbb{S}$, $x = \pi(p)$. Then

$$\begin{aligned}\tilde{\pi}_{i,N}(p,p) \quad = \quad & (\dim(V_i)/|G|) \cdot \sum_{g \in G_x} \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p) \\ & +(\dim(V_i)/|G|) \cdot \sum_{g \notin G_x} \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p).\end{aligned}$$

By virtue of Lemma 4.5 of [BU2], the latter term is bounded in absolute value by $C\tilde{\Pi}_N(p,p)e^{-a_x^2 N/2} + O(N^{(n-1)/2})$, where C is independent of x and N. By (13) of [SZ1] and the definition of dual action, $\tilde{\Pi}_N(g^{-1}p,p) = \alpha_x(g)^N\tilde{\Pi}_N(p,p)$ if $g \in G_x$. Thus the former term is

$$\begin{aligned}& (\dim(V_i)/|G|) \cdot \Big[\sum_{g \in G_x} \overline{\chi}_i(g)\alpha_x(g)^N\Big]\tilde{\Pi}_N(p,p) \\ &= (\dim(V_i)/|G|) \cdot (\alpha_x^N, \chi_i)_{G_x} \cdot \tilde{\Pi}_N(p,p).\end{aligned}$$

Given the asymptotic expansion for $\tilde{\Pi}_N(p,p)$ in [BU2] and [Z], $\tilde{\Pi}_{i,N}(p,p) \neq 0$ if $N \gg 0$, $x \notin B_{i,N}$. This clearly implies the statement.

Proof of Corollary 1.1. Let $V \subseteq M$ be the locus of points with non-trivial stabilizer. By Theorem 8.1 on page 213 of [S] and because the action is effective, V is a union of proper submanifolds of M. If $x \in M \setminus V$, then $G_x = \{e\}$ and therefore $\gamma_{i,k}(x) = \dim(V_i) \neq 0$ for every i and N. By the theorem, there exists $s \in H(M, A^{\otimes k})_i$ with $s(x) \neq 0$ if $k \gg 0$.

Before coming to the proof of Proposition 1.1, let us dwell on the previous description of the equivariant Szegö kernels $\tilde{\Pi}_{i,k}$ restricted to the diagonal. As is well known, the wave front of the Szegö kernel Π is

$$\Sigma = \big\{\big((p,p),(r\alpha_p, -r\alpha_p)\big) : p \in \mathbb{S}, r > 0)\big\} \subseteq T^*(\mathbb{S} \times \mathbb{S}) \setminus \{0\}.$$

In the notation of [BdMG], [BU2] we have in fact $\Pi \in J^{1/2}(\mathbb{S} \times \mathbb{S}, \Sigma)$. Now we have seen that

$$\tilde{\Pi}_{i,N}(p,p) = (\dim(V_i)/|G|) \cdot \sum_{g \in G} \overline{\chi}_i(g)\tilde{\Pi}_N(g^{-1}p,p).$$

For any $g \in G$ let $\alpha_g : \mathbb{S} \times \mathbb{S} \to \mathbb{S} \times \mathbb{S}$ be the diffeomorphism $(p,q) \mapsto (g\,p,q)$, and let $\tilde{\Pi}_g = \tilde{\Pi} \circ \alpha_g^* \in \mathcal{D}'(\mathbb{S} \times \mathbb{S})$, where α_g^* denotes pull-back of functions under α_g. Then $\tilde{\Pi}_g \in J^{1/2}\big(\mathbb{S} \times \mathbb{S}, \alpha_g^*(\Sigma)\big)$ and $\tilde{\Pi}_k(g\,p,q)$ is the k-th Fourier component of $\tilde{\Pi}_g$, for every integer k. One can then see, arguing as in the proofs of Lemmas 3.5 and 3.6 of [BU2], that $k^{-n}\tilde{\Pi}_k(g\,p, p)$ is bounded in $\mathcal{C}^1$ norm, say, for every $g \in G$. The same then holds for $k^{-n}\tilde{\Pi}_{i,k}(x, x)$.

Proof of Proposition 1.1. By the above, in the hypothesis of the proposition for every $x \in M$ there exists $k_x \in \mathbb{N}$ such that $x \notin \mathrm{Bs}\left(|H(M, A^{\otimes k})_i|\right)$ for every $k \geq k_x$. We now make the stronger claim that for every $x \in M$ there exist an open neighbourhood U_x of x and $k_x \in \mathbb{N}$ such that $U_x \cap \mathrm{Bs}\left(|H(M, A^{\otimes k})_i|\right) = \emptyset$ for every $k \geq k_x$. The statement will follow given the compactness of M.

If the claim was false, there would exist $x \in M$ and sequences $k_j \in \mathbb{N}$ and $x_j \in M$ $(j = 1, 2, \ldots)$ with $k_j \equiv r \pmod{|G|}$, $k_j \to +\infty$ and $x_j \to x$, such that $x_j \in \mathrm{Bs}\left(|H(M, A^{\otimes k_j})_i|\right)$ for every j. Thus,

$$\tilde{\Pi}_{i,k_j}(x_j, x_j) = 0 \quad (j = 1, 2, \ldots)$$

while

$$\tilde{\Pi}_{i,k_j}(x, x) = \frac{\dim(V_i)}{|G|} \cdot \gamma_{i,r}(x) \cdot \tilde{\Pi}_{k_j}(x, x) + \text{L.O.T.},$$

where L.O.T. denotes lower order terms in k_j. Thus, $k_j^{-n}\,\tilde{\Pi}_{i,k_j}(x, x)$ is bounded away from zero and therefore the derivatives in x of the sequence of functions $k_j^{-n}\,\tilde{\Pi}_{i,k_j}(x', x')$ are unbounded, a contradiction.

Proof of Corollary 1.2. Let us agree that the irreducible representation corresponding to $i = 1$ is just the trivial representation, so that

$$H(M, A^{\otimes N})_1 = H(M, A^{\otimes N})^G$$

for every integer N. Then $\overline{\chi}_1(g) = 1$ for every $g \in G$. Furthermore, for every $x \in M$, $g \in G_x$ and $k \in \mathbb{N}$ we have $\alpha_x^{k|G|}(g) = 1$. Thus

$$\gamma_{1,k|G|}(x) = |G_x| \neq 0 \text{ for every } x \in M,$$

and the statement follows from Proposition 1.1.

Proof of Corollary 1.3. If M is a complex projective manifold and A is ample, we have section multiplication maps

$$H^0(M, A^{\otimes \ell})^G \otimes H^0(M, A^{\otimes m})_i \longrightarrow H^0(M, A^{\otimes(\ell+m)})_i$$

for every $i = 1, \ldots, c$ and integers ℓ, m. Thus, for any residue class $0 \leq r \leq |G| - 1$ and any sequence of positive integers $k_i \gg 0$, by Corollary 1.2 we have a descending chain of base loci:

$$\begin{aligned}\mathrm{Bs}\left(|H^0(M, A^{\otimes r})_i|\right) &\supseteq \mathrm{Bs}\left(|H^0(M, A^{\otimes(r+k_1|G|)})_i|\right)\\ &\supseteq \mathrm{Bs}\left(|H^0(M, A^{\otimes(r+(k_1+k_2)|G|)})_i|\right) \supseteq \cdots .\end{aligned}$$

This implies the first statement. The rest is obvious.

Proof of Corollary 1.4. If $G_x = G$ and $k \equiv r \pmod{|G|}$, then

$$\tilde{\Pi}_{i,k}(x, x) = \frac{\dim(V_i)}{|G|} \cdot \gamma_{i,r}(x) \cdot \tilde{\Pi}_k(p, p).$$

Thus, if $\gamma_{i,r}(x) = 0$, then $s(x) = 0$ for every $s \in H(M, A^{\otimes k})_i$.

Proof of Corollary 1.5. The first statement follows from Theorem 1.1, while the second is a consequence of the proof of Proposition 1.1.

Proof of Theorem 1.2. Suppose $B_{i,N} = \emptyset$ so that, perhaps after replacing N by $N+k|G|$ for $k \gg 0$, $|H(\mathbb{S})_{i,N}|$ is base-point-free. The claim is that if $U' \subset U$ is open with compact closure in U and $N \gg 0$, then $\Phi_{i,N}$ is an immersion on U'. We shall be done by proving that $N^{-1}\Phi^*_{i,N}(\omega^{(N)}_{\mathrm{FS}}) - \omega = O(1/N)$ on connected compact subsets of U, where $\omega^{(N)}_{\mathrm{FS}}$ is the Fubini-Study symplectic form on $\mathbb{P}\big(H(M, A^{\otimes k})^*\big)$. In turn, this will follow if we prove that $N^{-1}\tilde{\Phi}^*_{i,N}(\tilde{\omega}_N) - \pi^*(\omega) = O(1/N)$ on horizontal vectors, over compact subsets of $\mathbb{S}$; here $\tilde{\omega}_N = \frac{i}{2}\overline{\partial}\partial \log|\xi|^2$ on $H(M, A^{\otimes k})^* \setminus \{0\}$ (with its natural hermitian structure), and $\pi : \mathbb{S} \to M$ is the projection.

Now, if d^1 and d^2 denote exterior differentiation on the first and second component of $\mathbb{S}\times\mathbb{S}$, respectively, then $N^{-1}\tilde{\Phi}^*_{i,N}\tilde{\omega}_N = \mathrm{diag}^*(d^1 d^2 \log \tilde{\Pi}_{i,N})$, where $\mathrm{diag} : \mathbb{S} \to \mathbb{S}\times\mathbb{S}$ is the diagonal map ([SZ2], proof of Theorem 3.1 (b)). If $x, y \in M$ lie in the same connected component V of U, $G_y = G_x$. Thus $b_x := (\alpha^N_x, \chi_i)_{G_x}$ is constant on V, say equal to b_V. Hence, if $p, q \in \pi^{-1}(V)$ and $x = \pi(p)$,

$$\tilde{\Pi}_{i,N}(p,q) = \frac{\dim(V_i)}{|G|}\cdot\Big\{b_V\cdot\tilde{\Pi}_N(p,q) + \sum_{g\notin G_x}\overline{\chi}_i(g)\tilde{\Pi}_N(gp,q)\Big\}. \tag{1}$$

By the proof of Theorem 3.1 (b) of [SZ2], $(i/2N)\,\mathrm{diag}^*\big(d^1d^2\log\Pi_N\big) \to \pi^*\omega$ in $\mathcal{C}^k$-norm for any k on M. Therefore, we shall be done by proving that

$$N^{-1}d_1d_2\big(\tilde{\Pi}_N(gp,q)/\tilde{\Pi}_N(p,q)\big) \to 0 \tag{2}$$

and

$$N^{-1}d_1\big(\tilde{\Pi}_N(gp,q)/\tilde{\Pi}_N(p,q)\big)\wedge d_2\big(\tilde{\Pi}_N(g'p,q)/\tilde{\Pi}_N(p,q)\big) \to 0 \tag{3}$$

for $g, g' \notin G_x$ near compact subsets of $\mathrm{diag}(V)$.

Then let $K \subset V$ be a compact subset, and suppose $x \in K$, $g \notin G_x$, and $u, v \in T_xM$ have unit length. Let U, V be horizontal vector fields of unit length on $\mathbb{S}$, of unit length near $\mathbb{S}_x$ and extending the horizontal lifts of u and v. We want to estimate $N^{-1}U_1\circ V_2(\tilde{\Pi}_N(gp,q)/\tilde{\Pi}_N(p,q))$ over K, where $U_1 = (U, 0)$ and $V_2 = (0, V)$ are horizontal vector fields on $\mathbb{S}\times\mathbb{S}$.

Let us consider again the distribution $\tilde{\Pi}_g = \alpha^*_g\tilde{\Pi} \in J^{1/2}(\mathbb{S}\times\mathbb{S}, g^*\Sigma)$, discussed before the proof of Proposition 1.1. If P is a horizontal differential operator of degree ℓ on $\mathbb{S}\times\mathbb{S}$, its principal symbol vanishes on $g^*\Sigma$ and therefore $P(\tilde{\Pi}_g) \in J^{(\ell+1)/2}(\mathbb{S}\times\mathbb{S}, \alpha^*_g\Sigma)$. As in [BU2], Lemma 4.5, for $k \in \mathbb{N}$ we can find $\nu_{g,P,k} \in \mathcal{C}^\infty(\mathbb{S})$, having an asymptotic expansion $\nu_{g,P,k}(p) = \sum_{j=0}^\infty k^{n+(\ell-j)/2} f^{(j)}_{g,P,k}(p)$, and real phase functions $\alpha_{g,P,k} \in \mathcal{C}^\infty(\mathbb{S}\times\mathbb{S})$ such that

$$G(p,q) = \sum_k \nu_{g,P,k}(p)e^{i\alpha_{g,P,k}(p,q)}e^{-kd(gp,q)^2/2} \in J^{(\ell+1)/2}(\mathbb{S}\times\mathbb{S}, \alpha^*_g\Sigma)$$

and $P(\tilde{\Pi}_g) - G \in J^{\ell/2}(\mathbb{S}\times\mathbb{S}, \alpha^*_g\Sigma)$. Since $P(\tilde{\Pi}_g)$ has definite (even) parity, we may assume without loss of generality that so does G. Therefore, the above asymptotic expansions may be assumed to go down by integer steps: $\nu_{g,P,k}(p) = \sum_{j=0}^\infty k^{n+\ell/2-j} f^{(j)}_{g,P,k}(p)$, and

$$\big|P\big(\tilde{\Pi}_N(gp,q)\big)\big| = \nu_{g,P,0}(p)\cdot e^{-Nd(gp,q)^2/2} + O(N^{n+\ell/2-1}). \tag{4}$$

Because $K \subset U$ is compact and $g \notin G_x$ for $x \in K$, there is $\epsilon > 0$ such that $d(gp,p) > \epsilon$ for all $p \in \pi^{-1}(K)$. Thus, $P(\tilde{\Pi}^{(g)}_N)(p,p) = O(N^{m+(\ell-1)/2})$

on $\pi^{-1}(K)$. Developing $N^{-1}U_1 \circ V_2(\tilde{\Pi}_N(gp,q)/\tilde{\Pi}_N(p,q))$, we see that $N^{-1}U_1 \circ V_2(\tilde{\Pi}_N(gp,q)/\tilde{\Pi}_N(p,q)) = O(1/N)$ over K, uniformly in U and V. This proves (2); the proof of the other estimate is similar.

Proof of Proposition 1.2. Notation being as above, we may assume that K is G-invariant. Suppose then, by contradiction, that for a sequence $k_j \to +\infty$ we can find $x_{k_j}, y_{k_j} \in K$ with $d_G(x_{k_j}, y_{k_j}) > 0$ and $\Phi_{i,N+k_j|G|}(x_{k_j}) = \Phi_{i,N+k_j|G|}(y_{k_j})$. Set $N_j = N + k_j|G|$.

I claim that $d_G(x_{k_j}, y_{k_j}) \leq C/\sqrt{N_j}$. Following [BU2], proof of Corollary 4.6, pick $p_{k_j} \in \pi^{-1}(x_{k_j})$, $q_{k_j} \in \pi^{-1}(y_{k_j})$. Then $\tilde{\Phi}_{i,N_j}(x_{k_j}) = \lambda_j \tilde{\Phi}_{i,N_j}(y_{k_j})$ for some $\lambda_j \in \mathbb{C}$; it follows that $||\tilde{\Phi}_{i,N_j}(p_{k_j})||^2 = |\lambda_j|^2 \cdot ||\tilde{\Phi}_{i,N_j}(q_{k_j})||^2$. However, $||\tilde{\Phi}_{i,N_j}(p)||^2 = \tilde{\Pi}_{i,N_j}(p,p)$ $(p \in \mathbb{S})$, and therefore by (1) above $|\lambda_j| = 1 + O(N_j^{-1/2})$. We also have $|\lambda_j|\tilde{\Pi}_{i,N_j}(p_{k_j}, p_{k_j}) = |\tilde{\Pi}_{i,N_j}(p_{k_j}, q_{k_j})|$, and on the other hand, again by (1),

$$|\tilde{\Pi}_{i,N_j}(p_{k_j}, q_{k_j})| \leq C|\tilde{\Pi}_{i,N_j}(p_{k_j}, p_{k_j})|e^{-N_j d_G(p,q)^2/2} + O(k_j^{n-1/2}).$$

We conclude that $d_G(p_{k_j}, q_{k_j}) \leq C/\sqrt{k_j}$, as claimed. Hence, after replacing x_{k_j} by $g_j \cdot x_{k_j}$ for a suitable $g_j \in G$, we may assume $d(x_{k_j}, y_{k_j}) \leq C/\sqrt{N_j}$ and $d(x_{k_j}, y_{k_j}) = d_G(x_{k_j}, y_{k_j})$ for every j.

Since $d(gx, x) > \epsilon$ for some fixed $\epsilon > 0$ and all $x \in K$ and $g \notin G_x$, x_{k_j} is the only point in $G \cdot x_{k_j}$ minimizing the distance from y_{k_j}, for every j.

We may now apply the argument of the proof of Theorem 3.2 (b) of [SZ2], with minor changes.

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