DEFINABILITY IN THE LATTICE OF EQUATIONAL THEORIES OF COMMUTATIVE SEMIGROUPS

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To Professor Ralph McKenzie

ABSTRACT. In this paper we study first-order definability in the lattice of equational theories of commutative semigroups. In a series of papers, J. Ježek, solving problems posed by A. Tarski and R. McKenzie, has proved, in particular, that each equational theory is first-order definable in the lattice of equational theories of a given type, up to automorphism, and that such lattices have no automorphisms besides the obvious syntactically defined ones (with exceptions for special unary types). He has proved also that the most important classes of theories of a given type are so definable. In a later paper, Ježek and McKenzie have "almost proved" the same facts for the lattice of equational theories of semigroups. There were good reasons to believe that the same can be proved for the lattice of equational theories of commutative semigroups. In this paper, however, we show that the case of commutative semigroups is different.

1. INTRODUCTION

In [17, 14], A. Tarski and R. McKenzie raised problems of first-order definability in a lattice L of equational theories. A subset S of L is first-order definable in L if there exists a formula $\Phi(T)$ in the first-order language of L such that $\Phi(T)$ holds if and only if $T \in S$. An equational theory $T \in L$ is definable if the set $\{T\}$ is definable. Accordingly, we speak about definable relations and properties of the theories in L.

The language in question can be thought as one whose only nonlogical symbol is that of inclusion (or equivalently, the symbols of meet and join). It seems relatively poor, but the lattices of equational theories considered are usually large, and many interesting sets, from the point of view of equational logic, can be defined in this language. For general results see [14, 6, 7, 8, 9], and for some results on equational theories of rings and groups see [4, 5, 18].

In [10], Ježek and McKenzie studied first-order definability in the lattice of equational theories of semigroups. They proved that such sets as the set of finitely axiomatizable theories, the set of locally finite theories, and the set of theories of finite semigroups are all definable. Also they proved that each individual finitely axiomatizable locally finite theory is definable up to duality (i.e., up to inverting

the order of occurrences of letters in words). They have not been able to finish
their program that would lead to the conclusion that "locally finite" in the last
statement may be dropped, and that, in consequence, the lattice of equational the-
ories of semigroups has no nontrivial automorphism except for the duality. These
questions still remain open.

During my visit at Vanderbilt University, Ralph McKenzie asked me if my de-
scription in [11] can be used to answer these kinds of questions for the lattice
$\mathcal{L}(Com)$ of equational theories of *commutative* semigroups.

In [11], I described equational theories of commutative semigroups in terms of
certain order filters, integer parameters, and the so-called *remainders*. While the fil-
ters and integer parameters lead to a very regular, distributive sublattice of $\mathcal{L}(Com)$
that is easy to handle, the remainders introduce interferences that make the whole
lattice quite complex (for example, all finite lattices are contained in $\mathcal{L}(Com)$ as
sublattices). The remainders are equivalence relations on sequences of positive in-
tegers satisfying a few technical conditions. What is however the most important
is that they have a finite character, and are relatively small parts of theories. Al-
gorithms and formulas in [11] provide a general method of computing in equational
theories and varieties of commutative semigroups. This method has already proved
to be useful in solving a number of problems [3, 1, 2, 12, 13]. In this paper we use
it to solve the problem suggested by McKenzie.

From the results of [10] it follows that each equational theory of commutative
semigroups is definable in the lattice of equational theories of all semigroups, since
each such (nontrivial) theory is finitely axiomatizable and locally finite. This does
not imply that each such theory is definable in $\mathcal{L}(Com)$, because the universe of
$\mathcal{L}(Com)$ is much smaller. However, this lattice seems rich and complicated enough
to make such a statement plausible. In particular, the facts that, on the one hand,
$\mathcal{L}(Com)$ contains all finite lattices and, on the other hand, each theory in $\mathcal{L}(Com)$
is finitely axiomatizable, made us conjecture that all the theories in $\mathcal{L}(Com)$ are
definable and that $\mathcal{L}(Com)$ itself has no nontrivial automorphism.

Although no result of [10] can be applied directly to our problem, we were trying
to follow the approach used in [10] (and earlier in [6, 7, 8]) as long as possible. The
main idea of this approach is to encode equations (and sequences of words) in the
ordered set of patterns. (A pattern is an orbit under the action of automorphisms
on the free semigroup; and the order relation on patterns is inherited from the
relation "substituble into" between words.) Now, the ordered set of patterns is
order isomorphic to the set of the so-called ideal theories, which is definable. As
a result, the ideal theories can be used like Gödel numbers to encode syntactical
properties.

In the case of $\mathcal{L}(Com)$, this must be taken modulo commutativity. The ideal
theories correspond to certain Schwabauer theories, and patterns correspond to
sets of positive integers (unordered n-tuples). Unfortunately, equations do not
correspond to pairs of patterns, since possible permutations of variables in words
can no longer be encoded within the patterns. At this point we expected that the
tools of [11] would prove to be useful.

There were two surprises while performing this program. First, unlike in [10], the
set of ideal theories turned out to be too poor to define much structure in $\mathcal{L}(Com)$.
Yet, using [11], we were able to prove that the larger set of so-called Schwabauer
theories has all the nice properties of ideal theories. Schwabauer theories, known

to form a maximal modular sublattice of $\mathcal{L}(Com)$, are definable both as a set and individually. This means, in particular, that the sublattice of Schwabauer theories remains fixed under any automorphism (which is one more reason to call this sublattice the "skeleton" of $\mathcal{L}(Com)$). Using it, we can go further, to prove that many individual theories and interesting sets of theories are definable in $\mathcal{L}(Com)$. However, we have not been able to establish the definability of one-based theories generated by certain regular and quite simple equations—because this turned out not to be true. Instead, our final section is devoted to demonstrating that $\mathcal{L}(Com)$ admits a nontrivial automorphism.

2. Preliminaries

We assume that the reader has basic knowledge in the areas of first-order logic, equational logic and lattice theory. Some necessary details can be found in the introductions to [10] and [11]. In this section we recall definitions and results from [11] that we use throughout this paper.

The crucial structure behind the equational theories of commutative semigroups is the set Γ of all finite sequences $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers such that at least one $\alpha_i \neq 0$, along with some operations and an order relation described below.

For a sequence $\mathbf{a} = (\alpha_1, \ldots, \alpha_n) \in \Gamma$ and a permutation of the indices $f \in S_n$, we put $f(\mathbf{a}) = (\alpha_{f(1)}, \ldots, \alpha_{f(n)})$. In addition, for every $\gamma \geq 0$ we denote $h_\gamma(\mathbf{a}) = (\alpha_1, \ldots, \alpha_n, \gamma)$, and for all $1 \leq i, j \leq n$, $i \neq j$, we denote $g_{ij}(\mathbf{a}) = (\alpha_1, \ldots, \alpha_i + \alpha_j, \ldots, \alpha_n)$, where the latter is the sequence obtained from $\mathbf{a}$ by replacing α_i by $\alpha_i + \alpha_j$, and deleting α_j.

For sequences $\mathbf{a}, \mathbf{b} \in \Gamma$, we define $\mathbf{a} \leq \mathbf{b}$ if and only if there exists an operation f generated by the operations g_{ij}, h_1, h_0 and permutations such that $f(\mathbf{a}) = \mathbf{b}$.

Clearly, $\leq$ is reflexive (f may be taken to be an identity permutation) and transitive, which means that $(\Gamma, \leq)$ is a quasi-ordered set. Further, note that two sequences are equivalent (the relation $\leq$ holds in both directions) if and only if they differ at most in the arrangement of elements and the number of zeros. Thus, in every equivalence class there is a unique nonincreasing sequence of *positive* integers determining the class, and $\leq$ can be viewed conveniently as the induced partial order on such sequences, treated as representatives of equivalence classes (cf. Figure 1).

It is proved in [11] (by means of another alternate definition of this order) that $(\Gamma, \leq)$ is actually a well-quasi-ordered set, which means, in particular, that there are no infinite descending chains in Γ and every (order) filter J in Γ is *finitely generated* ([11, Proposition 1.5]). A filter J generated by sequences $\mathbf{a}_1, \ldots, \mathbf{a}_n$ (i.e., the set of all sequences in Γ greater or equal to one of $\mathbf{a}_1, \ldots, \mathbf{a}_n$) is denoted by $J = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$. Obviously, each filter is a union of equivalence classes. The filters in Γ correspond to certain theories of commutative semigroups as follows.

A commutative semigroup equation

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\beta_1} \ldots x_n^{\beta_n}$$

will be identified with the pair $(\mathbf{a}, \mathbf{b})$ of the sequences $\mathbf{a} = (\alpha_1, \ldots, \alpha_n)$ and $\mathbf{b} = (\beta_1, \ldots, \beta_n)$, and denoted briefly by $(\mathbf{a}, \mathbf{b})$. Usually we will assume that $\alpha_i + \beta_i > 0$. An equation is *nontrivial* if $\alpha_i \neq \beta_i$ for some i.

An *equational theory* of commutative semigroups (in short, a *theory*) is a nonempty set of equations closed on taking logical consequences, which corresponds to applying the operations g_{ij}, h_1, h_0 and permutations to pairs $(\mathbf{a}, \mathbf{b})$, and using

$(4,1)$ $(3,2)$ $(3,1,1,1)$ $(2,2,1,1)$ $(2,1,1,1,1,1)$ $(1,1,1,1,1,1,1,1)$

(4) $(3,1,1)$ $(2,2,1)$ $(2,1,1,1,1)$ $(1,1,1,1,1,1,1)$

$(3,1)$ $(2,2)$ $(2,1,1,1)$ $(1,1,1,1,1,1)$

(3) $(2,1,1)$ $(1,1,1,1,1)$

$(2,1)$ $(1,1,1,1)$

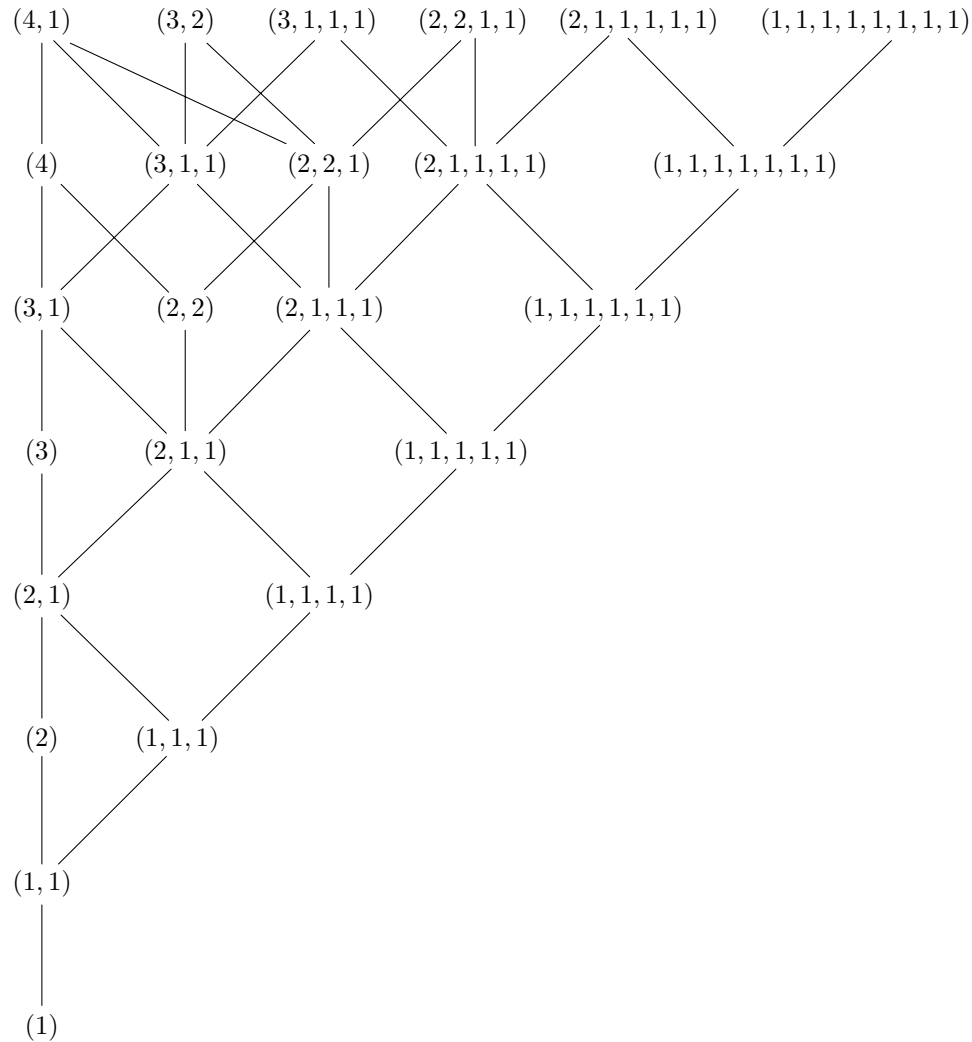(2) $(1,1,1)$

$(1,1)$

(1)

FIGURE 1. Ordered set of sequences up to equivalence.

transitivity (see [11] for more detail). The least theory consisting of the equations of the form $(\mathbf{a}, \mathbf{a})$ is called *trivial*.

Given a nonempty filter J in Γ, define $E(J)$ to be the set of all equations $(\mathbf{a}, \mathbf{b})$ such that either both $\mathbf{a}, \mathbf{b} \in J$ or $\mathbf{a} = \mathbf{b}$. Then $E(J)$ is easily seen to be an equational theory (in [10], theories of this form are called *ideal theories*).

Further, for integers $m \geq 0$ and $r > 0$ we define $E(J, m, r)$ to be the set of those pairs $(\mathbf{a}, \mathbf{b}) \in E(J)$ that satisfy the following two conditions:

   (C^r) for every i, $\alpha_i \equiv \beta_i \pmod{r}$,
   (C_m) for every i, if $\alpha_i \neq \beta_i$ then $\alpha_i, \beta_i \geq m$.

These conditions are invariant with regard to taking logical consequences. Therefore, each $E(J, m, r)$ is also an equational theory, and $E(J) = E(J, 0, 1)$. The theories $E(J, m, r)$ are called *Schwabauer theories* or *S-theories*, in brief.

Finally, we consider the following set of conditions for an equation $(\mathbf{a}, \mathbf{b})$, a filter J, and a set of equations π.

($\pi 1$) The conditions (C^r, C_m) hold.
($\pi 2$) For every permutation $f \in S_n$, $(f(\mathbf{a}), f(\mathbf{b})) \in \pi$.
($\pi 3$) For every i with $\alpha_i \neq \beta_i$, and $\gamma = \min(\alpha_i, \beta_i)$, both $h_\gamma(\mathbf{a}), h_\gamma(\mathbf{b}) \in J$.
($\pi 4$) If f is one of the operations g_{ij} or h_1, then either $(f(\mathbf{a}), f(\mathbf{b})) \in \pi$ or both $f(\mathbf{a}), f(\mathbf{b}) \in J$.

Now, let $m \geq 0, r > 0$ be integers and J a nonempty filter contained in $[(m)]$ (the latter denotes the filter generated by one-element sequence (m); for $m = 0$ we define $[(0)] = \Gamma$). Let π be an equivalence relation on the set $\Gamma_+ \setminus J$ of those finite sequences of *positive* integers that are not in J. If every block of π contains sequences of one fixed length and for all pairs $(\mathbf{a}, \mathbf{b}) \in \pi$ the conditions $(\pi 1 - \pi 4)$ above are satisfied, then π is called a *remainder of type* (J, m, r). For such a remainder we define $E(J, m, r, \pi) = E(J, m, r) \cup \pi$. Then, the main result in [11] (Theorem 4.8) can be stated as follows.

Theorem 2.1. *Every set $E(J, m, r, \pi)$ defined above is an equational theory, and every nontrivial equational theory of commutative semigroups is of this form.*

This result requires a few comments.

First, we note that in the presentation $E(J, m, r, \pi) = E(J, m, r) \cup \pi$ the union is disjoint: for $(\mathbf{a}, \mathbf{b}) \in E(J, m, r, \pi)$ either $\{\mathbf{a}, \mathbf{b}\} \subseteq J$ and $(\mathbf{a}, \mathbf{b}) \in E(J, m, r)$, or else $\{\mathbf{a}, \mathbf{b}\} \subseteq \Gamma^+ \setminus J$ and $(\mathbf{a}, \mathbf{b}) \in \pi$. In particular, $E(J, m, r)$ can be characterized as the greatest S-theory contained in $E(J, m, r, \pi)$. A theory $E = E(J, m, r, \pi)$ is an S-theory itself, if the remainder π has only trivial equations, i.e., is the least equivalence relation on $\Gamma_+ \setminus J$.

It is the main idea of [11] that the difference $\pi = E \setminus E(J, m, r)$ is a relatively small part of E with very special properties. While, in general, remainders may be very different (causing $\mathcal{L}(Com)$ itself to be quite complex), they have a "finite character", and can be described algorithmically.

We note that, in view of condition $(\pi 4)$, if $(\mathbf{a}, \mathbf{b}) \in \pi$, then the sequences $\mathbf{a}, \mathbf{b}$ are either equivalent or noncomparable in $(\Gamma_+, \leq)$. In particular, if $\mathbf{a}$ and $\mathbf{b}$ are not equal, then they have length $n \geq 2$. Moreover, by definition, the conditions $(C^r), (C_m)$ hold for each $(\mathbf{a}, \mathbf{b}) \in E(J, m, r, \pi)$. Fulfilling (C_m) is related to the fact that $J \subseteq [(m)]$ is assumed in the definition of $E(J, m, r, \pi)$.

Finally, it should also be remarked that in [11], there are actually two sets of conditions denoted (N1–N4) and $(\pi 1 - \pi 4)$. These sets play independent roles in the proofs. However, in the final presentation, (N1, N2) can be omitted, since, as it was observed by M. Grech in [1], (N1, N2) follow from $(\pi 3, \pi 4)$. That is why we have decided to replace here (N1–N4) by (C^r) and (C_m).

We will need the following description of algebraic operations in $\mathcal{L}(Com)$. Given a set J and an equivalence relation π, by $C(J, \pi)$ we denote the union of all those blocks in π which have a point in common with J, and by $\overline{C}(J, \pi)$ we denote the closure of $C(J, \pi)$ in Γ under $\leq$-equivalence. (In fact, since by $(\pi 2)$, $C(J, \pi)$ is closed under permutations, the only sequences we need to add are those obtained from sequences in $C(J, \pi)$ by adjoining zeros.) With E^S denoting the greatest S-theory contained in E, and with superscripts denoting Cartesian powers, we have the following.

Theorem 2.2. *If $E_1 = E(J_1, m_1, r_1, \pi_1)$ and $E_2 = E(J_2, m_2, r_2, \pi_2) \in \mathcal{L}(Com)$, then the following hold.*

(i) $E_1 \supseteq E_2$ *if and only if* $J_1 \supseteq J_2$, $m_1 \leq m_2$, r_1 *divides* r_2, *and* $\pi_1 \cup J_1^2 \supseteq \pi_2$.

(ii) $E_1 \vee E_2 = E(J_1 \cup J_2 \cup \overline{C}, \min(m_1, m_2), \gcd(r_1, r_2), (\pi_1 \vee \pi_2) \setminus C^2)$,
where $C = C(J_1 \cup J_2, \pi_1 \vee \pi_2)$.

(iii) $E_1 \cap E_2 = E(J_1 \cap J_2, \max(m_1, m_2), \mathrm{lcm}(r_1, r_2), \pi)$,
where $\pi = (\pi_1 \cap \pi_2) \cup (\pi_1 \cap E_2^S) \cup (\pi_2 \cap E_1^S)$.

We note that (ii) states, in particular, that $J_1 \cup J_2 \cup \overline{C}$ is a filter. The meet above is often denoted by $\wedge$ and the inclusion by $\leq$. Since this is the case in [6] and [10], in this paper we use these two notations interchangeably.

Other facts from [11], which we need to use only once in this paper, are recalled in an appropriate place (usually they follow easily from the general results stated above). Since using the above results usually leads to quite technical considerations, for the sake of clarity, more basic things, like reasoning concerning definability and order relations, will be treated in this paper rather informally.

3. Definable sets

The fact that Schwabauer theories $E(J, m, r)$ form a maximal modular sublattice of $\mathcal{L}(Com)$ has been proved by E. Nelson [15]. We call it the *skeleton* of $\mathcal{L}(Com)$, because in view of Theorem 2.2, it has a relatively simple structure reflecting the structure of the whole lattice. (Nelson used the term "the skeleton" for the sublattice of the theories $E([(k)], m, r)$.) We are going to show that both the sublattice of Schwabauer theories as a whole and each of these theories individually are definable in $\mathcal{L}(Com)$. This means, in particular, that the skeleton is fixed by all the automorphisms of $\mathcal{L}(Com)$.

First, we establish the simplest facts concerning definability of some sets and individual theories, recalling by the way some basic facts about the structure of $\mathcal{L}(Com)$. We follow the notation introduced by Nelson [15], and applied in [11], where the theories $E([(k)], m, r)$ were denoted simply $E(k, m, r)$.

The greatest element in $\mathcal{L}(Com)$ is the theory $E(1, 0, 1) = E(\Gamma, 0, 1)$ consisting of all the equations. There are the following co-atoms: $S = E(1, 1, 1)$, the theory of semilattices; $A_p = E(1, 0, p)$, the theory of Abelian groups of prime exponent p, for all $p \geq 2$; and the theory $N = E([(1, 1)], 0, 1)$ of nil-semigroups, defined by $xy = uz$. (This is generally well-known, and may be easily deduced from what is recalled in Section 2.) Obviously, the set of co-atoms is definable.

We still may distinguish (in the first-order language) the set of those theories below the co-atoms, that are contained precisely in one co-atom.

First, note that if a theory $E = E(J, m, r, \pi)$ has $J \neq [(1)]$, then $J \subseteq [(1, 1)]$, and therefore, by Theorem 2.2 (i), E is contained in N, and also in one of A_p or S, unless $m = 0$ and $r = 1$. In the latter case, E is of the form $E = E(J, 0, 1, \pi)$, and N is the unique co-atom containing E.

If $J = [(1)]$, then $m \leq 1$, since by the definition of $E(J, m, r, \pi)$ we have $J \subseteq [(m)]$. Also, by definition of the remainder, π is trivial in this case, i.e., E is of the form $E(J, m, r)$. Now, if $m = 1$, then $r > 1$, and E is contained in S and one of the A_p. If $m = 0$, then E is contained in two different co-atoms A_p and A_q, unless $r = p^k$ is a power of a prime p, in which case A_p is the unique co-atom containing E.
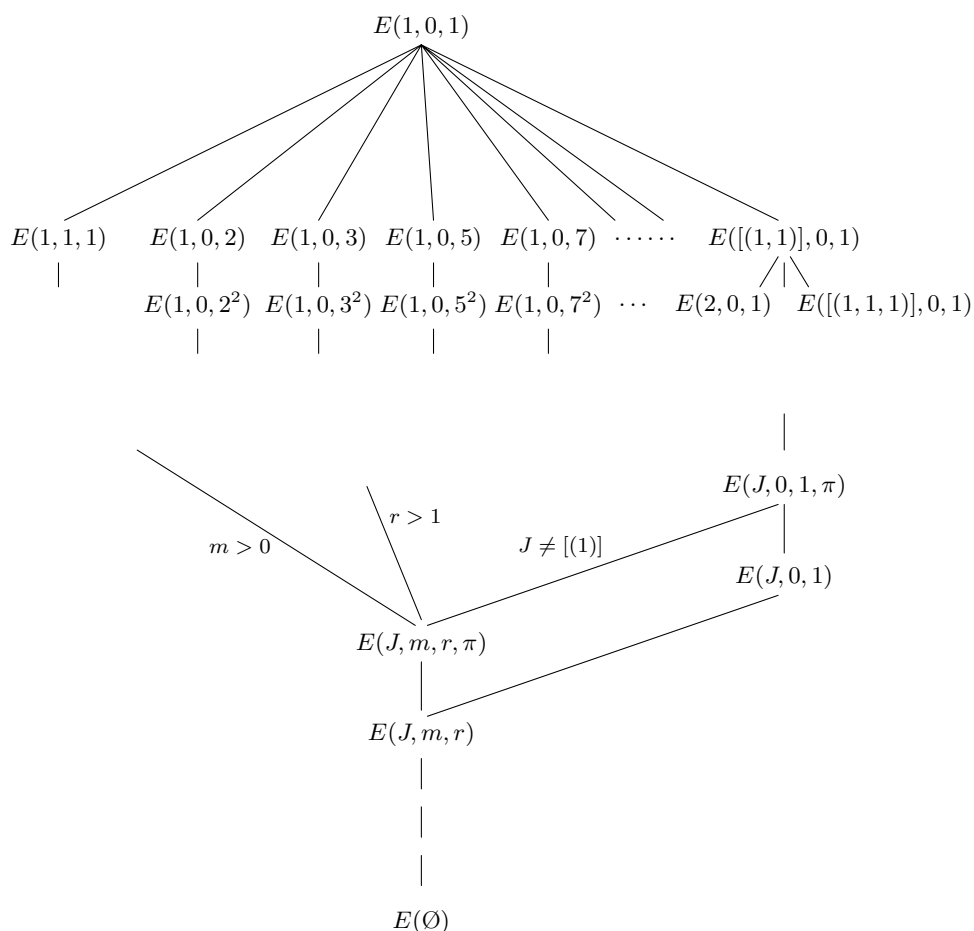
FIGURE 2. Co-atoms in $\mathcal{L}(Com)$ and the method of comparing filters.

Hence, the set $\mathcal{U}$ of the theories that are contained precisely in one co-atom consists of the theories of the form $E(J, 0, 1, \pi)$ contained in N, and for each prime p, the theories $A_{p^k} = E(1, 0, p^k)$ contained in A_p. At the present stage we are not able to distinguish (in the first-order language) between different A_p, but we can distinguish the theory N from all A_p. Indeed, while there are many noncomparable theories $E(J, 0, 1, \pi)$ in $\mathcal{U}$ contained in N, any two theories $E(1, 0, p^k)$ contained in A_p, for a fixed p, are comparable. In other words, $\mathcal{U}$ consists of chains of theories $E(1, 0, p^k)$ and a set of theories $E(J, 0, 1)$, $J \subseteq [(1, 1)]$, order-isomorphic to the lattice of filters J in Γ (without the top and bottom elements), together with the companion theories $E(J, 0, 1, \pi)$ with nontrivial remainder. The latter contains many noncomparable elements, for example, $E([(1, 1, 1)], 0, 1)$ and $E(2, 0, 1)$ (see Figure 2; cf. Figure 1).

It follows that the theory N is element-definable, and the set of theories of the form $E(J, 0, 1, \pi)$ is definable. Also S is element-definable (no element of $\mathcal{U}$ is contained in S), and therefore the set of all the theories contained in S is definable.

The latter coincides with the theories $E(J, m, r, \pi)$ with $m > 0$, which is simply the set of regular theories (generated by regular identities). This means that the set of theories with $m = 0$ is definable, as well. In turn, the theories with $r = 1$ are those not contained in any co-atom A_p (i.e. in any co-atom different from N and S). Hence, the set of these theories is definable. Consequently, the set of theories of the form $E(J, 0, r, \pi)$ with $r > 1$ is definable. This set may be partitioned into the theories contained in N (those with $J \neq [(1)]$) and the remaining, which are those of the form $E(1, 0, r)$ (π is always trivial when $J = [(1)]$). The latter are precisely the theories $A_r = E(1, 0, r)$ of Abelian groups.

We summarize these observations in the following proposition.

Proposition 3.1. *The following individual theories and sets of theories are definable in $\mathcal{L}(Com)$:*

   (a) *the theory $N = E([(1,1)], 0, 1)$ of nil-semigroups, and the set of theories of the form $E(J, 0, 1, \pi)$;*
   (b) *the theory $S = E(1, 1, 1)$ of semilattices, and the set of regular theories;*
   (c) *the set of theories $A_p = E(1, 0, p)$ of Abelian groups of prime exponent, and the set of all theories $A_r = E(1, 0, r)$ of Abelian groups;*
   (d) *the set of theories with $m = 0$, and the set of theories with $r = 1$.*

Now we proceed to demonstrate that the set of all Schwabauer theories is definable. We make use of the set of all theories of the form $E(J, 0, 1, \pi)$, which is definable by the above proposition. Let us denote this set by $\mathcal{N}$.

First we note that directly from the definition it follows that a remainder π of type (J, m, r), for any $m \geq 0$ and $r > 0$, is also a remainder of type $(J, 0, 1)$. Therefore, to every theory $E = E(J, m, r, \pi)$, we may assign uniquely a theory in $\mathcal{N}$, putting $M(E) = E(J, 0, 1, \pi)$. Observe that, in view of Theorem 2.2 (i), $M(E)$ is actually the smallest theory in $\mathcal{N}$ containing E. Hence, this assignment can be expressed in the first-order language of $\mathcal{L}(Com)$.

Given a theory $E_0 = E(J_0, 0, 1, \pi_0)$ in $\mathcal{N}$, consider now the set $M^{-1}(E_0)$ of those theories $E \in \mathcal{L}(Com)$ for which $M(E) = E_0$. It consists of all theories of the form $E_0 = E(J_0, m, r, \pi_0)$ such that $J_0 \subseteq [(m)]$ and π_0 is a remainder of type (J_0, m, r). If π_0 is trivial (contains no nontrivial equation), then the latter is no restriction, and in particular, $M^{-1}(E_0)$ has no minimal element. Namely, if $E(J_0, m, r) \in M^{-1}(E_0)$, then $E(J_0, m, 2r) \in M^{-1}(E_0)$, as well. Otherwise, there are maximal r and m such that each $(\mathbf{a}, \mathbf{b}) \in \pi$ satisfies conditions $(C^r), (C_m)$, and therefore $M^{-1}(E_0)$ has the least element $E(J_0, m, r, \pi_0)$ with maximal m and r.

It follows that for a theory $E = E(J, m, r, \pi)$, the remainder π is trivial if and only if for every theory T such that $M(T) = M(E)$ there exists a theory S properly contained in T with $M(S) = M(E)$. Thus we have

Theorem 3.2. *The set of Schwabauer theories is definable in $\mathcal{L}(Com)$.*

Finally, we apply this theorem to show that comparing filters of the theories can be expressed in the language of $\mathcal{L}(Com)$.

Given a theory $E = E(J, m, r, \pi)$, define $J(E) = E(J, 0, 1)$ (this is well-defined, since for every filter J we have $J \subseteq [(0)] = \Gamma$). Observing that $J(E)$ is the greatest Schwabauer theory contained in $M(E)$ (cf. Figure 2), we see that this assignment is definable, as well. Obviously, for any two theories E_1, E_2 with filters J_1, J_2, respectively, $J(E_1) \subseteq J(E_2)$ if and only if $J_1 \subseteq J_2$. Hence, we obtain the following.

Corollary 3.3. *There exists a formula $\Psi(E_1, E_2)$ in the first-order language of $\mathcal{L}(Com)$ such that $\Psi(E_1, E_2)$ holds for two theories $E_1 = E(J_1, m_1, r_1, \pi_1)$ and $E_2 = E(J_2, m_2, r_2, \pi_2)$ if and only if $J_1 \subseteq J_2$.*

Thus, the property that the filter of a theory E_1 is contained in the filter of a theory E_2 is definable. Proving the analogous statement for parameters m and r requires more involved arguments, and we do it later.

4. ABELIAN GROUPS AND SCHWABAUER THEORIES

The aim of this section is to prove that each individual Schwabauer theory is definable. We start from the Abelian groups. Recall that in Proposition 3.1 we have established that the set of these theories is definable.

Proposition 4.1. *Each theory $A_r = E(1, 0, r)$ of Abelian groups is definable in $\mathcal{L}(Com)$.*

Proof. Let us denote by $I(J, m, r)$ the set of all theories of the form $E(J, m, r, \pi)$, for J, m, and r fixed. Recall that this can be characterized as the set of all those theories that contain $E(J, m, r)$ as the greatest S-theory. (It is known that this is in fact an interval in $\mathcal{L}(Com)$, but we do not need this fact.)

We are going to show that there exists a definable correspondence between theories $A_r = E(1, 0, r)$ and certain S-theories, such that the linear ordering of integers r corresponds to the inclusion relation for theories. To this end, we will make use of the natural correspondence between S-theories and the intervals $I(J, m, r)$ defined above. Using it, we assign a unique filter J_r to every integer $r \geq 2$.

Given $r \geq 2$, let J be a maximal filter such that the set $I(J, 0, r)$ is nontrivial, i.e., has more than one element. This is equivalent to the fact that there exists a nontrivial remainder π of type $(J, 0, r)$. For such a remainder, and a nontrivial $(\mathbf{a}, \mathbf{b}) \in \pi$, there exists an index i such that $\alpha_i \neq \beta_i$, say, $\alpha_i < \beta_i$. Then, by (C^r), the difference $\beta_i - \alpha_i \geq r$. Since $\mathbf{a}$ and $\mathbf{b}$ are noncomparable, their length is at least 2, and therefore $\mathbf{b} \geq (1, r+1)$. It follows that $(1, r+1) \notin J$.

So, let J_r be the greatest filter not containing $(1, r+1)$, i.e., the union of all filters with this property. It is not difficult to see that $J_r \subseteq J_s$ if and only if $r \geq s$. Moreover, the equivalence relation π_r on $\Gamma_+ \setminus J_r$, whose only nontrivial block is $\{(1, r+1), (r+1, 1)\}$, is a remainder of type $(J_r, 0, r)$. It follows that J_r is actually the greatest filter J such that $I(J, 0, r)$ is nontrivial, or more generally, the greatest filter J such that $I(J, m, r)$ is nontrivial for some $m \geq 0$.

Consequently, for a fixed r, $E_r = E(J_r, 0, r)$ is the greatest S-theory $E(J, m, r)$ such that the set $I(J, m, r)$ corresponding to it is nontrivial. Since each S-theory $E(J, m, r)$ is contained in $A_r = E(1, 0, r)$, we obtain that $E_r = E(J_r, 0, r)$ may be defined as the greatest S-theory contained in $A_r = E(1, 0, r)$ such that the set $I(J, m, r)$ corresponding to it is nontrivial (i.e., there exists a theory E, other than E_r itself, having E_r as the greatest S-theory contained in it). This can be expressed in the first-order language of $\mathcal{L}(Com)$.

Therefore, also $E_r^* = E(J_r, 0, 1)$, the greatest S-theory having the same filter as E_r, may be assigned to A_r in the first-order language, in view of Corollary 3.3.

From the properties of the filters, $s \geq r$ if and only if $E_s^* \subseteq E_r^*$, for $r \geq 2$. Hence, the property that a theory of Abelian groups A_r has smaller exponent than a theory of Abelian groups A_s is definable in $\mathcal{L}(Com)$. Now, $A_2 = E(1, 0, 2)$ is the theory of Abelian groups with the smallest exponent, and therefore, in view of

Proposition 3.1 and the remark above, is definable. An easy induction yields the result. □

We are now ready to deal with integer parameters m and r. For a theory $E = E(J, m, r, \pi)$ we put $m(E) = m$ and $r(E) = r$.

Proposition 4.2. *Let $m_0 \geq 0$ and $r_0 > 0$ be fixed integers. Then the following relations are definable in $\mathcal{L}(Com)$:*

(a) $r(E) = r_0$;
(b) $r(E_1)$ *divides* $r(E_2)$;
(c) $m(E) = m_0$;
(d) $m(E_1) \leq m(E_2)$;
(e) $r(E_1) \leq r(E_2)$.

Proof. First note that, in view of Theorem 3.2, we may restrict ourselves to S-theories, since the parameters m and r are the same for a theory and the greatest S-theory contained in it.

Obviously, for a fixed $r \geq 2$, $r(E) = r$ for a theory $E = E(J', m', r')$ if and only if E is included in $A_r = E(1, 0, r)$ and it is not included in any A_s for $s > r$. Moreover, $r(E) = 1$ if and only if E is not included in any A_r. This, in view of Proposition 4.1 and the final part of its proof, establishes (a).

Obviously $r(E_1)$ divides $r(E_2)$ if and only if, for all theories A_r, the fact that E_1 is contained in A_r implies that E_2 is contained in A_r. This establishes (b).

For c), we apply induction. First, $m(E) = 0$ is definable by Proposition 3.1d. Suppose that $m(E) = n$ is definable for a fixed n. Then, for an S-theory E, $m(E) = n + 1$ if and only if there exists an S-theory E_n with the same parameters J and r as in E, and with $m(E_n) = n$, such that E is the greatest S-theory properly contained in E_n, and having the same J and r parameters as E_n. Using Theorem 2.2, it is not difficult to check that this definition is as required. In view of Corollary 3.3 and (b) above, this can be expressed in the first-order language, proving (c).

Finally, since we do not use either (d) or (e) in this paper, the proofs of these facts are left to the reader as an exercise. □

The main result of this section is

Theorem 4.3. *Each S-theory $E(J, m, r)$ is definable in $\mathcal{L}(Com)$.*

Proof. In view of Proposition 4.2, we need only find a way to define the filter J. By Corollary 3.3, it is definable that two theories have the same filter; therefore it is enough to consider a sublattice of $\mathcal{L}(Com)$ consisting of the theories $E(J, 0, 1)$. By Theorem 2.2, this is isomorphic to the lattice $\mathcal{F}$ of all filters on $(\Gamma, \leq)$ (up to the least element).

In this lattice, the set $\mathcal{F}_1$ of join-irreducible elements is just the set of one-generated filters (i.e., those generated by a single sequence). Since every filter is finitely generated, i.e., is a join of a finite number of one-generated filters, all we need to prove is that every one-generated filter is definable in the ordered set $\mathcal{F}_1$.

The latter is dually isomorphic to the ordered set Γ^* of sequences in $(\Gamma, \leq)$, taken up to equivalence. To fix notation, we assume that Γ^* consists of nonincreasing sequences of positive integers (see Figure 1), and our aim is to prove that every sequence $\mathbf{a} \in \Gamma^*$ is definable in Γ^*. Since this set is well-founded (has no infinite descending sequences), we may use induction.

The sequence (1) is the least element in Γ^*.

We claim that each sequence $\mathbf{a} = (\alpha_1, \ldots, \alpha_n) \in \Gamma^*$, other than (1), is determined (up to permutation) by its dual covers, i.e., any two different nonincreasing sequences have different collections of dual covers (note that Γ^* is not a lattice, so we cannot speak about joins and meets).

First, if $\alpha_n = 1$ then $\mathbf{a}$ has two types of dual covers: one is $\mathbf{b} = (\alpha_1, \ldots, \alpha_{n-1})$ with the property $h_1(\mathbf{b}) = \mathbf{a}$, and others $\mathbf{c}$ have the property $g_{ij}(\mathbf{c}) = \mathbf{a}$, up to permutation. While the sum of elements in each $\mathbf{c}$ is equal to $\sum \alpha_i$, the sum of elements in $\mathbf{b}$ is strictly less than that. Hence, in such a case, $\mathbf{a}$ is obtained from the dual cover with the lowest sum of elements, just by adjoining 1 to it.

Otherwise, all $\alpha_i > 1$, and $\mathbf{a}$ can be obtained from the dual cover $(\gamma_1, \ldots, \gamma_{n+1})$ for which the last entry $\gamma_{n+1} = 1$ and γ_n is the least possible. Indeed, in any case, this must be equal to the dual cover $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1, 1)$. (Originally, this part of the proof was not so straight; the simplification is due to Ralph McKenzie.)

Thus, our claim is proved. It easily follows from it that each element of Γ^* can be individually defined in the first-order language of Γ^*. $\qquad\square$

5. Definability up to permutation

A natural way to prove that each finitely axiomatizable theory is definable, is to prove that each one-based theory (i.e., axiomatized, or as we rather say, generated by a single equation) is definable. This is, obviously, equivalent to proving that for each equation $(\mathbf{a}, \mathbf{b})$ the fact that $(\mathbf{a}, \mathbf{b}) \in E$ can be expressed in the first-order language of $\mathcal{L}(Com)$.

We have not been able to prove this, because, as we show in Section 7, this is not true in general. Instead, we prove some closely related facts, which help us to prove the definability of many individual theories with nontrivial remainders. Namely, we show that, given an equation $(\mathbf{a}, \mathbf{b})$, the properties that $(\mathbf{a}, p(\mathbf{b})) \in E$ for some permutation p, and that E is generated by a set of equations of the form $(\mathbf{a}, p(\mathbf{b}))$, where p is a permutation, are definable in $\mathcal{L}(Com)$. We have already mentioned in the introduction that, in the case of commutative semigroups, there is no natural way to distinguish permutations of sequences leading to different equations. The results of this section show that this is, in fact, the crucial problem.

Let us recall first from [11] that if $E(J_0, m_0, r_0, \pi_0)$ is a theory generated by a single equation $(\mathbf{a}, \mathbf{b})$, then m_0 is the largest m such that $(\mathbf{a}, \mathbf{b})$ satisfies condition (C_m), and r_0 is the largest r such that condition (C^r) is satisfied. Moreover, if sequences $\mathbf{a}, \mathbf{b}$ are comparable and nonequivalent, say $\mathbf{a} < \mathbf{b}$, then the equation $(\mathbf{a}, \mathbf{b})$ generates an S-theory $E(J, m, r)$ with $J = [\mathbf{a}]$ (in fact, these observations may be inferred from the general results recalled in Section 2). It follows that the theories generated by a single equation $(\mathbf{a}, \mathbf{b})$ with comparable and nonequivalent sequences are individually definable in $\mathcal{L}(Com)$.

Further, recall that an equation $(\mathbf{a}, \mathbf{b})$ is *regular* if $\alpha_i, \beta_i > 0$ for all i. For nonregular equations $(\mathbf{a}, \mathbf{b})$, it is easy to observe (cf. [11, § 3.16]) that each such equation generates an S-theory $E(J, m, r)$ with $J = [\mathbf{a}, \mathbf{b}]$ and $m = 0$. Hence, each theory generated by a nonregular equation is definable, as well.

So, we may assume that equations we deal in the remainder of this section are regular, and we distinguish two cases according to whether sequences in the equation are equivalent or not. Nontrivial regular equations with equivalent sequences (i.e.,
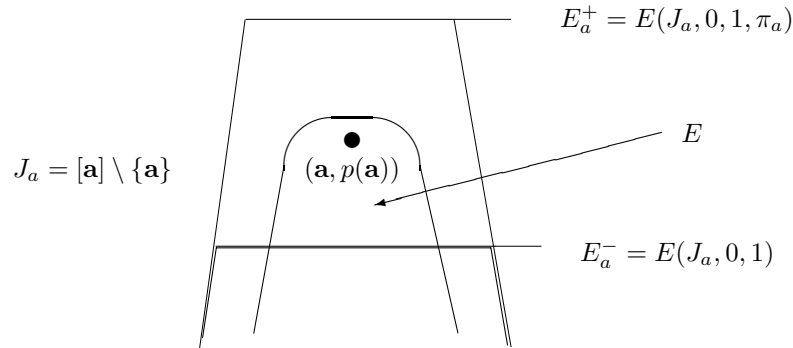
FIGURE 3. Proof of Theorem 5.1.

equations of the form $(\mathbf{a}, p(\mathbf{a}))$, where $p \in S_n$ is a nontrivial permutation and $\mathbf{a}$ is a sequence of positive integers) are called *permutational*.

For a fixed sequence $\mathbf{a}$ of positive integers, we define some filters, remainders, and theories. Let $J_a = [\mathbf{a}] \setminus \{\mathbf{a}\}$; this is the largest filter contained in $[\mathbf{a}]$ and not containing $\mathbf{a}$. Let $E_a^- = E(J_a, 0, 1)$, and let π_a be the remainder of type $(J_a, 0, 1)$ whose only nontrivial block is one consisting of all the sequences $p(\mathbf{a})$, where p is a permutation (the conditions $(\pi 1 – \pi 4)$ are obviously satisfied). Let $E_a^+ = E(J_a, 0, 1, \pi_a)$. It can be characterized as the largest theory contained in $E([\mathbf{a}], 0, 1)$ that has the same filter as E_a^-. Since all S-theories are definable, by Corollary 3.3, E_a^+ is definable. We make use of these facts in the proof below.

Theorem 5.1. *Given a sequence $\mathbf{a}$ of positive integers, the following properties are definable in $\mathcal{L}(Com)$:*

  (i) *a nontrivial equation $(\mathbf{a}, p(\mathbf{a})) \in E$ for some permutation p; and*
  (ii) *E is generated by a set of permutational equations of the form $(\mathbf{a}, p(\mathbf{a}))$.*

Proof. For (i), note that $(\mathbf{a}, p(\mathbf{a})) \in E$ for some p if and only if $E \cap E_a^+ \not\subseteq E_a^-$ (see Figure 3). Since E_a^+ and E_a^- are definable, this proves the claim.

For (ii), let $\Phi(E)$ abbreviate the following formula in the language of $\mathcal{L}(Com)$:

$$E \subseteq E_a^+, \text{ and if } E = E_1 \vee E_2 \text{ and } E_2 \subseteq E_a^-, \text{ then } E_1 = E.$$

Using Theorem 2.2 repeatedly we show that $\Phi(E)$ holds if and only if E satisfies (ii).

First, assume that $E = E(J, m, r, \pi)$ is a theory generated by a set of equations of the form $(\mathbf{a}, p(\mathbf{a}))$. Then, obviously, $E \subseteq E_a^+$, in particular $J \subseteq J_a$, and moreover, E is generated by equations in $\pi \cap \pi_a$. Suppose that $E = E_1 \vee E_2$ with $E_2 \subseteq E_a^-$. Then $E_1 \subseteq E$, and consequently, $J_1 \subseteq J \subseteq J_a$. Next, $E \subseteq E_1 \vee E_a^- = E(J_a, 0, 1, \pi_1 \setminus J_a^2)$, where π_1 is the remainder of E_1. It follows that $\pi \subseteq (\pi_1 \setminus J_a^2) \cup J_a^2$, and consequently, $\pi \cap \pi_a \subseteq \pi_1$ (since there is no nontrivial identity in $\pi_a \cap J_a^2$). The latter implies that the generating set for E is contained in E_1, which, in view of $E \supseteq E_1$, proves that $E = E_1$, and in consequence, that $\Phi(E)$ holds.

Conversely, suppose that $\Phi(E)$ is satisfied for $E = E(J, m, r, \pi)$. Then, taking $E_2 = E \cap E_a^-$, and E_1 the theory generated by $E \setminus E_a^-$, we get $E = E_1 \vee E_2$. By the second condition in $\Phi(E)$, $E = E_1$. Since by the first condition $E \setminus E_a^-$ contains only equations of the form $(q(\mathbf{a}), p(\mathbf{a}))$, condition (ii) follows. $\square$

Example 5.2. Let $\mathbf{a} = (1,2)$. Then $J_a = [(1,1,2),(3)]$, and π_a has one nontrivial block $\{(1,2),(2,1)\}$. Consequently, the only theory generated by a set of permutational equations of the form $(\mathbf{a},p(\mathbf{a}))$ is one generated by $((1,2),(2,1))$. Hence, this theory is definable. Note that it is different from E_a^+, since the equation $((3),(4))$ in one variable is not a consequence of $((1,2),(2,1))$.

Example 5.3. Let $\mathbf{a} = (1,2,3)$. Then $J_a = [(1,1,2,3),(3,3),(2,4),(1,5)]$, and the nontrivial block of π_a consists of 6 permutations of the sequence $(1,2,3)$. There are six theories generated by a set of equations of the form $(\mathbf{a},p(\mathbf{a}))$ in this case. These include: E_1, E_2, E_3, and E_4 generated by single equations $((1,2,3),(2,3,1))$, $((1,2,3),(1,3,2))$, $((1,2,3),(3,2,1))$, and $((1,2,3),(2,1,3))$, respectively, E_5 generated by all equations $(\mathbf{a},p(\mathbf{a}))$, and the trivial theory E_0. They correspond to the subgroups of the symmetric group S_3.

It is not difficult to see that all these theories are individually definable. Indeed, the trivial theory is definable, and E_5 is definable, in view of Theorem 5.1(ii), as the greatest theory generated by a set of equations of the form $(\mathbf{a},p(\mathbf{a}))$. Consequently, the set of the remaining theories E_1, E_2, E_3, E_4 is definable.

Now, among these theories, $m(E_2) = 2$, while for the other theories $m(E_i) = 1$. It follows, in view of Proposition 4.2, that E_2 is definable. Similarly, $r(E_3) = 2$, while for the other theories $r(E_i) = 1$. Hence, also E_3 is definable. Finally, applying g_{13}, we get that $((4,2),(3,3)) \in E_1$, while it may be checked directly (or using the method given in [11]) that $((4,2),(3,3)) \notin E_4$. Consequently, the theory E^* generated by $((4,2),(3,3))$ is contained in E_1, and not contained in E_4. In Example 5.5 below, we will see that E^* is definable. This implies that also E_1 and E_4 are definable.

As the second case, we consider regular equations $(\mathbf{a},\mathbf{b})$ with noncomparable sequences $\mathbf{a}$ and $\mathbf{b}$. So, for the remainder of this section, we assume that $\mathbf{a},\mathbf{b}$ are fixed, noncomparable sequences of positive integers, of the same length.

As before, we start from some analogous definitions. Let $J_{ab} = [\mathbf{a},\mathbf{b}] \setminus \{\mathbf{a},\mathbf{b}\}$, which is the largest filter contained in $[\mathbf{a},\mathbf{b}]$, not containing $\mathbf{a},\mathbf{b}$; $E_{ab}^- = E(J_{ab},0,1)$, and let π_{ab} be the remainder of type $(J_{ab},0,1)$ whose only nontrivial block is one consisting of all the sequences $p(\mathbf{a})$ and $q(\mathbf{b})$ for permutations p,q. Again, it is easily seen that π_{ab} is a well-defined remainder, and $E_{ab}^+ = E(J_{ab},0,1,\pi_{ab})$ is definable as the largest theory contained in $E([\mathbf{a},\mathbf{b}],0,1)$ that has the same filter as E_{ab}^-.

In this case we define one more theory $E_{ab} = E_a^+ \vee E_b^+ \vee E_{ab}^-$. Then, by the previous results on permutational equations, E_{ab} is definable in $\mathcal{L}(Com)$, and since $\mathbf{a},\mathbf{b}$ are noncomparable, $E_{ab} = E(J_{ab},0,1,\pi_{ab}^*)$, where π_{ab}^* has exactly two nontrivial blocks consisting of all the sequences $p(\mathbf{a})$ in one block, and all the sequences $q(\mathbf{b})$, in the other.

Theorem 5.4. *Given noncomparable sequences $\mathbf{a},\mathbf{b}$ of positive integers, the following properties are definable in $\mathcal{L}(Com)$:*

    (i) *an equation $(\mathbf{a},p(\mathbf{b})) \in E$ for some permutation p; and*
    (ii) *E is generated by a set of equations of the form $(\mathbf{a},p(\mathbf{b}))$.*

Proof. For (i), as before, it is enough to note that $(\mathbf{a},p(\mathbf{b})) \in E$ for some p if and only if $E \cap E_{ab}^+ \nsubseteq E_{ab}$ (see Figure 4).

For (ii), now let $\Phi(E)$ abbreviate the following formula:

$$E \subseteq E_{ab}^+, \text{ and if } E = E_1 \vee E_2 \text{ and } E_2 \subseteq E_{ab}^-, \text{ then } E_1 = E.$$
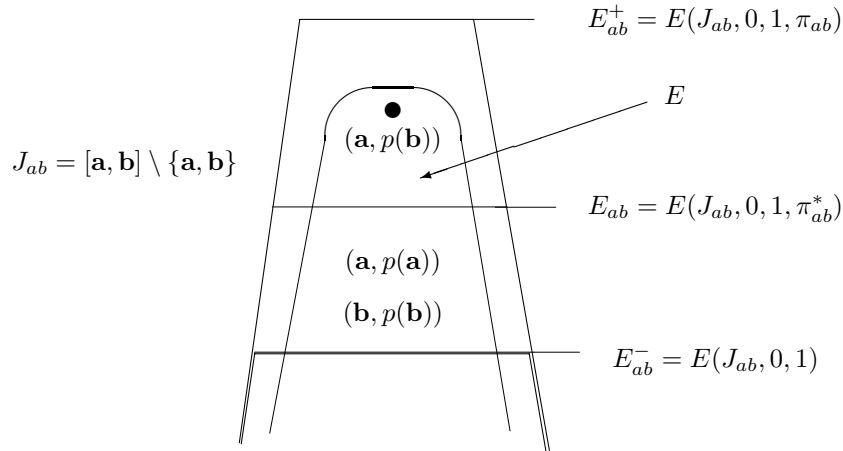
FIGURE 4. Proof of Theorem 5.4.

Let $P = P(\mathbf{a}, \mathbf{b})$ denote the set of all equations of the form $(\mathbf{a}, p(\mathbf{a}))$, $(\mathbf{a}, p(\mathbf{b}))$, or $(\mathbf{b}, q(\mathbf{b}))$, where p is a permutation. We first show that $\Phi(E)$ holds if and only if E is generated by a set of permutational equations contained in P. The proof is essentially the same as that of Theorem 5.1 (ii).

First, assume that $E = E(J, m, r, \pi)$ is a theory generated by a set of equations contained in P. Then $E \subseteq E_{ab}^+$, in particular $J \subseteq J_{ab}$, and moreover, E is generated by equations in $\pi \cap \pi_{ab}$. Suppose that $E = E_1 \vee E_2$ with $E_2 \subseteq E_{ab}^-$. Then $E \subseteq E_1 \vee E_{ab}^- = E(J_{ab}, 0, 1, \pi_1 \setminus J_{ab}^2)$, where π_1 is the remainder of E_1. It follows that $\pi \subseteq (\pi_1 \setminus J_{ab}^2) \cup J_{ab}^2$, and consequently, $\pi \cap \pi_{ab} \subseteq \pi_1$. The latter implies that $E \subseteq E_1$, and consequently $E = E_1$, as required.

Conversely, suppose that $\Phi(E)$ is satisfied for $E = E(J, m, r, \pi)$. Then, taking $E_2 = E \cap E_{ab}^-$, and E_1 the theory generated by $E \setminus E_{ab}^-$, we get $E = E_1 \vee E_2$, which implies $E = E_1$. Since by the first condition in $\Phi(E)$ we have $E \setminus E_{ab}^- \subseteq P$, the claim follows.

Now, to complete the proof, note that a pair of equations $(\mathbf{a}, p(\mathbf{b}))$, $(\mathbf{b}, q(\mathbf{b}))$ is equivalent to $(\mathbf{a}, p(\mathbf{b}))$, $(\mathbf{a}, pq(\mathbf{b}))$, while the pair $(\mathbf{a}, p(\mathbf{b}))$, $(\mathbf{a}, q(\mathbf{a}))$ is equivalent to $(\mathbf{a}, p(\mathbf{b}))$, $(\mathbf{a}, qp(\mathbf{b}))$. It follows that the theories satisfying (ii) are precisely the theories satisfying (i) and $\Phi(E)$. $\square$

Example 5.5. Let $\mathbf{a} = (4, 2)$ and $\mathbf{b} = (3, 3)$. Then $J_{ab} = [(1, 2, 4), (1, 3, 3), (6)]$; the nontrivial block of π_{ab} is $\{(4, 2), (2, 4), (3, 3)\}$; and π_{ab}^* also has only one nontrivial block $\{(4, 2), (2, 4)\}$ (since the block with $(3, 3)$ is trivial). In this case the theory E^* generated by $(\mathbf{a}, \mathbf{b})$ is the only theory contained in E_{ab}^+ generated by equations of the form $(\mathbf{a}, p(\mathbf{b}))$. Hence, this theory is definable. In Section 7, we show however that this is not generally the case for theories generated by a single equation in two variables.

Many one-based theories generated by a regular equation $(\mathbf{a}, \mathbf{b})$ with noncomparable sequences may be proved to be definable, following Example 5.3. Generally, this depends on particular properties of the set of the theories generated by a set

of equations of the form $(\mathbf{a}, p(\mathbf{b}))$ or $(\mathbf{a}, p(\mathbf{a}))$. In the next section, we look for a general result in this direction.

6. Definable theories with nontrivial remainders

From remarks at the beginning of the previous section it follows that each one-based theory generated by an irregular equation, or by an equation with comparable but nonequivalent sequences, is an S-theory (i.e., its remainder is trivial), and therefore, in view of Theorem 4.3, is definable in $\mathcal{L}(Com)$. In this section we consider one-based theories with nontrivial remainders.

According to Theorem 5.4, we are able to express the fact that a given equation $(\mathbf{c}, \mathbf{d})$ is in a theory E up to permutation, i.e., $(\mathbf{c}, p(\mathbf{d})) \in E$ for some permutation p.

The idea applied below is to make use of consequences $(\mathbf{a}, \mathbf{b})$ of a fixed equation $(\mathbf{c}, \mathbf{d})$, as in Example 5.3, in hope that the set of all pairs $(\mathbf{a}, p(\mathbf{b}))$ associated with those consequences determines the theory generated by $(\mathbf{c}, \mathbf{d})$ uniquely. This turns out to be true for a large class of theories with a nontrivial remainder.

A regular equation $(\mathbf{a}, \mathbf{b})$ is called *balanced*, if the covers of $\mathbf{a}$ are noncomparable with the covers of $\mathbf{b}$. In particular, $\mathbf{a}, \mathbf{b}$ have to be noncomparable. (In fact, we may restrict to g_{ij}-covers in this definition. This can be proved using the alternate definition of the order $(\Gamma, \leq)$ given in [11]. Since we make no use of this fact in our proofs below, we leave the proof to the reader.)

It is often the case that an equation $(\mathbf{a}, \mathbf{b})$ with noncomparable sequences is balanced. For example, if $\sum \mathbf{a} = \sum \mathbf{b}$ (the sums of elements in the sequences are equal), then it is enough that up to permutation the sequences differ in more than 3 elements. If $\sum \mathbf{a} < \sum \mathbf{b}$, then it is enough that $\mathbf{a}$ has an element α_i greater than the sum $\beta_r + \beta_s$ of any two elements in $\mathbf{b}$. On the other hand, no equation in two variables is balanced (we consider equations in two variables in the next section).

In this section, let us assume that $(\mathbf{c}, \mathbf{d})$ is a fixed balanced equation in $n > 2$ variables, and $E(\mathbf{c}, \mathbf{d})$ denotes the theory generated by $(\mathbf{c}, \mathbf{d})$. We shall consider only certain special consequences of $(\mathbf{c}, \mathbf{d})$: the set C_0 of consequences in n variables obtained from $(\mathbf{c}, \mathbf{d})$ by permutations and transitivity only, and the set C_1 of consequences in $n - 1$ variables obtained from $(\mathbf{c}, \mathbf{d})$ by identifying exactly two variables in any $(\mathbf{a}, \mathbf{b}) \in C_0$, and applying permutations and transitivity. In particular, we do not apply multiplication (i.e., operation h_1). To prove properties of C_0 and C_1 we will need later, let us start from more formal definitions.

Let $C_0 = C_0(\mathbf{c}, \mathbf{d})$ be the least set of pairs containing $(\mathbf{c}, \mathbf{d})$ that is an equivalence relation and closed under permutations (in the sense of condition $(\pi 2)$). Let $C_1 = C_1(\mathbf{c}, \mathbf{d})$ be the least set of pairs, containing all sets $g_{ij}(C_0)$, that is an equivalence relation and closed under permutations (similarly as in the case of permutations, operations g_{ij} are applied to every pair in C_0, to both the sequences in the pair simultaneously, and i, j are allowed to run over all possible indices). Since, the equation $(\mathbf{c}, \mathbf{d})$ is assumed to be fixed throughout this section, we will usually use the short notation C_0 and C_1.

Our first observation is

Lemma 6.1. *The set C_0 is the set of all equations $(\mathbf{a}, \mathbf{b}) \in E(\mathbf{c}, \mathbf{d})$ in n variables such that both $\mathbf{a}$ and $\mathbf{b}$ are equivalent to $\mathbf{c}$ or $\mathbf{d}$. The set C_1 is the set of all equations $(\mathbf{a}, \mathbf{b}) \in E(\mathbf{c}, \mathbf{d})$ in $n - 1$ variables such that both $\mathbf{a}$ and $\mathbf{b}$ are g_{ij}-covers of $\mathbf{c}$ or $\mathbf{d}$.*

Proof. Obviously C_0 and C_1 are contained, respectively, in the sets of equations described in the lemma. To show the converse inclusions it is enough to construct a theory E' containing $(\mathbf{c}, \mathbf{d})$ such that if $(\mathbf{a}, \mathbf{b}) \in E$ and both $\mathbf{a}$ and $\mathbf{b}$ either are equivalent to $\mathbf{c}$ or $\mathbf{d}$, or are g_{ij}-covers of $\mathbf{c}$ or $\mathbf{d}$, then $(\mathbf{a}, \mathbf{b}) \in C_0 \cup C_1$.

To this end, let J' be the filter generated by all the covers $g_{ij}(\mathbf{c})$, $h_1(\mathbf{c})$, $g_{ij}(\mathbf{d})$, $h_1(\mathbf{d})$, and let J be the set obtained from J' by removing all the elements $g_{ij}(\mathbf{c})$, $g_{ij}(\mathbf{d})$. Since they are pairwise noncomparable (by assumption that $(\mathbf{c}, \mathbf{d})$ is balanced), J is a filter, and moreover, both $h_1(\mathbf{c}), h_1(\mathbf{d}) \in J$.

Now, let π be the union of C_0 and C_1, endowed with trivial pairs $(\mathbf{a}, \mathbf{a})$ for all $\mathbf{a} \in \Gamma_+ \setminus J$. Then, it is easily seen that π is a remainder of type $(J, 0, 1)$. This follows from the facts that J is the largest filter not containing $g_{ij}(\mathbf{c}), g_{ij}(\mathbf{d})$, while $h_1(\mathbf{c}), h_1(\mathbf{d}) \in J$, and that C_0, C_1 are equivalence relations closed under permutations. Clearly, the theory $E' = E(J, 0, 1, \pi)$ has the required properties. $\square$

We wish to describe C_0 in a direct way. To this end, for $\mathbf{c} = (\gamma_1, \dots, \gamma_n)$ and $\mathbf{d} = (\delta_1, \dots, \delta_n)$, let Π_c be the equivalence relation on the set of indices $I = \{1, 2, \dots, n\}$ such that i, j are Π_c-equivalent if and only if $\gamma_i = \gamma_j$, and let Π_d be the equivalence relation on the set I such that i, j are Π_d-equivalent if and only if $\delta_i = \delta_j$. Let $\Pi = \Pi(\mathbf{c}, \mathbf{d})$ be the least equivalence relation containing both Π_c and Π_d.

We define $G(\mathbf{c}, \mathbf{d})$ to be the permutation group on the set of indices $I = \{1, 2, \dots, n\}$ generated by all transpositions (i, j) with i, j lying in the same block of Π. In other words, this is the group consisting of all permutations leaving all Π-blocks invariant, and it has the form of the direct product of the symmetric groups on blocks of Π.

Lemma 6.2. *The set $C_0 = C_0(\mathbf{c}, \mathbf{d})$ is the least symmetric and permutation closed relation containing all the pairs $(\mathbf{c}, p(\mathbf{d}))$, $(\mathbf{c}, p(\mathbf{c}))$, and $(\mathbf{d}, p(\mathbf{d}))$ with $p \in G(\mathbf{c}, \mathbf{d})$.*

Proof. Denote by $G(\mathbf{c})$ the group leaving all blocks of Π_c invariant, and suppose first that a permutation $p \in G(\mathbf{c})$. Then $(\gamma_{p(1)} \dots \gamma_{p(n)}) = (\gamma_1, \dots, \gamma_n)$, and therefore the following are consequences of $(\mathbf{c}, \mathbf{d})$: $(\mathbf{c}, p(\mathbf{c}))$, $(\mathbf{d}, p(\mathbf{c}))$, $(p^{-1}(\mathbf{d}), \mathbf{c})$, $(p^{-1}(\mathbf{d}), \mathbf{d})$, $(\mathbf{d}, p(\mathbf{d}))$. Analogous argument shows that the same holds for all $p \in G(\mathbf{d})$. It follows easily that $(\mathbf{c}, p(\mathbf{c}))$, $(\mathbf{d}, p(\mathbf{d}))$, and $(\mathbf{c}, p(\mathbf{d}))$ are among the consequences of $(\mathbf{c}, \mathbf{d})$ for all $p \in G(\mathbf{c}, \mathbf{d})$.

The least symmetric and permutation closed relation R containing all these pairs is the set of all pairs $(qp_1(x), qp_2(y))$, where $p_1, p_2 \in G(\mathbf{c}, \mathbf{d})$, q is any permutation, and $x, y \in \{\mathbf{c}, \mathbf{d}\}$. Obviously, R is reflexive. So, to see it equals C_0, it remains to observe that R is also transitive.

This is easy. Suppose that $(qp_1(x), qp_2(y))$ and $(fp_3(x), fp_4(y))$ are in R, and $qp_2(x) = fp_3(y)$. Then, since $\mathbf{c}, \mathbf{d}$ are noncomparable, $x = y$, and $p_2^{-1}q^{-1}fp_3 = g \in G(\mathbf{c}, \mathbf{d})$ (since $g \in G(x)$). It follows that $q^{-1}f = p_2 g p_3^{-1} \in G(\mathbf{c}, \mathbf{d})$. Hence, $(p_1(x), q^{-1}fp_4(y)) \in R$, and consequently, $(qp_1(x), fp_4(y)) \in R$, as required. $\square$

A direct description of C_1 is more complicated, but we shall need only a certain property of C_1. The following observation is crucial.

Lemma 6.3. *Let $\mathbf{a} = (\alpha_1, \dots, \alpha_n)$ be a sequence of positive integers. If $g_{ij}(\mathbf{a})$ and $g_{rs}(\mathbf{a})$ are equivalent (i.e., are the same up to permutation of elements), then $\{\alpha_i, \alpha_j\} = \{\alpha_r, \alpha_s\}$.*

Proof. Consider the sequences $\mathbf{a}$ and $g_{ij}(\mathbf{a})$. If we treat these sequences as multisets (with possible repetitions of elements), then removing simultaneously equal

elements from both the sets leaves us with $\{\alpha_i, \alpha_j\}$ in the first set, and $\{\alpha_i + \alpha_j\}$ in the second. This does not depend on the choice of a pair to be removed in a given step, since even if there is some α_m with $\alpha_m = \alpha_i + \alpha_j$, then the number of such α's in the first set is always one less than in the second set.   □

In view of this lemma, from each g_{ij}-cover $\mathbf{a}$ of sequences $\mathbf{c}$ or $\mathbf{d}$, even if the cover is given up to permutation, we may infer the indices $\{i, j\}$, at least up to Π_c- or Π_d-equivalence, respectively. Note that, in view of the assumption that $\mathbf{c}, \mathbf{d}$ are noncomparable, we are able to infer whether $\mathbf{a}$ is actually a g_{ij}-cover of $\mathbf{c}$ or a g_{ij}-cover of $\mathbf{d}$. It follows that, if $\mathbf{a} = pg_{ij}(\mathbf{c})$ or $\mathbf{a} = pg_{ij}(\mathbf{d})$, for a permutation p on $n - 1$ letters, we may infer which of these two cases occurs, and what the indices i, j are, up to Π-equivalence. Hence, for any sequence $\mathbf{a}$, as above, we may define $P(\mathbf{a})$ uniquely to be the union $[i] \cup [j]$ of Π-equivalence classes representing i and j, respectively. Note that this definition depends on the sequences $\mathbf{c}, \mathbf{d}$ (since Π does). The property of C_1 we need is the following.

Lemma 6.4. *If* $(\mathbf{a}, \mathbf{b}) \in C_1$*, then* $P(\mathbf{a}) = P(\mathbf{b})$*.*

Proof. By definition, C_1 is the least set of pairs containing all sets $g_{ij}(C_0)$ that is an equivalence relation and closed under permutations.

First suppose that $(\mathbf{a}, \mathbf{b}) \in g_{ij}(C_0)$ for some i, j, that is, $(\mathbf{a}, \mathbf{b}) = (g_{ij}(\mathbf{a}_1), g_{ij}(\mathbf{a}_2))$ for some $(\mathbf{a}_1, \mathbf{a}_2) \in C_0$. By Lemma 6.2, we may assume that $(\mathbf{a}_1, \mathbf{a}_2) = (f(\mathbf{c}), fp(\mathbf{d}))$ for some permutation $p \in G(\mathbf{c}, \mathbf{d})$, and a permutation f on n letters (other cases are analogous). Then, assuming $\mathbf{c} = (\gamma_1, \ldots, \gamma_n)$ and $\mathbf{d} = (\delta_1, \ldots, \delta_n)$, we compute:

$$\mathbf{a} = g_{ij}f(\gamma_1, \ldots, \gamma_n) = g_{ij}(\gamma_{f(1)}, \ldots, \gamma_{f(n)}) = p_1 g_{rs}(\gamma_1, \ldots, \gamma_n),$$

where $r = f(i)$, and $s = f(j)$, and p_1 is a suitable permutation on $n - 1$ letters obtained from f. Similarly,

$$\mathbf{b} = g_{ij}fp(\delta_1, \ldots, \delta_n) = g_{ij}f(\delta_{p(1)}, \ldots, \delta_{p(n)}) = g_{ij}(\delta_{pf(1)}, \ldots, \delta_{pf(n)})$$
$$= p_2 g_{uv}(\delta_1, \ldots, \delta_n),$$

where $u = pf(i)$ and $v = pf(j)$, and p_2 is a suitable permutation on $n - 1$ letters obtained from pf. (Note that, by definition, applying the product of permutations to a sequence corresponds to applying the permutations to the indices in the reverse order!)

Using the fact that $p \in G(\mathbf{c}, \mathbf{d})$, we obtain that $[f(i)] = [pf(i)]$ and $[f(j)] = [pf(j)]$, and consequently, $P(\mathbf{a}) = P(\mathbf{b})$. An analogous argument for other cases yields that if $(\mathbf{a}, \mathbf{b}) \in g_{ij}(C_0)$ for some i, j, then $P(\mathbf{a}) = P(\mathbf{b})$, as required.

Now, to see that this is true for all $(\mathbf{a}, \mathbf{b}) \in C_1$, note that by definition, $P(p(\mathbf{a})) = P(\mathbf{a})$ for any permutation p on $n - 1$ letters, and therefore, if $P(\mathbf{a}) = P(\mathbf{b})$ for $(\mathbf{a}, \mathbf{b}) \in C_1$, then the same holds for $(p(\mathbf{a}), p(\mathbf{b}))$. Finally, if $(\mathbf{a}_1, \mathbf{a}_2), (\mathbf{a}_2, \mathbf{a}_3) \in C_1$ have this property, then obviously $(\mathbf{a}_1, \mathbf{a}_3)$ also does. This completes the proof.   □

Theorem 6.5. *If E is a one-based theory generated by a balanced equation* $(\mathbf{c}, \mathbf{d})$*, then E is definable in* $\mathcal{L}(Com)$*.*

Proof. Let $\mathcal{M} = \mathcal{M}(\mathbf{c}, \mathbf{d})$ be the set of the theories generated by sets of equations of the form $(\mathbf{c}, p(\mathbf{d}))$. This set of theories is finite, and definable, by Theorem 5.4. Let $\mathcal{M}_d$ be the set of the theories $E \in \mathcal{M}$ such that, for each $T \in \mathcal{M}$, if $T \subseteq E$, then T is individually definable in $\mathcal{L}(Com)$ (the theories in $\mathcal{M}_d$ may be said to be

hereditarily definable with respect to $\mathcal{M}$). Since $\mathcal{M}_d$ is finite, it is definable. Our goal is to show that the set $\mathcal{M} \setminus \mathcal{M}_d$ of the remaining theories is empty.

Assume to the contrary, that it is not the case, and let $\mathcal{M}_0$ denote the set of minimal theories in the set $\mathcal{M} \setminus \mathcal{M}_d$. Then, $\mathcal{M}_0$ is definable and nonempty.

Let $E_0 \in \mathcal{M}_0$. Then, by definition, each proper subtheory of E_0 in $\mathcal{M}$ is definable, and therefore E_0 has an equation $(\mathbf{c}, f(\mathbf{d}))$ such that $E(\mathbf{c}, f(\mathbf{d}))$ is not definable. It follows also that $E_0 = E(\mathbf{c}, f(\mathbf{d}))$. In order to apply our lemmas directly, without loss of generality, we may assume that $E_0 = E(\mathbf{c}, \mathbf{d})$.

Consider now the following formula:

$\Psi(E)$: $E \in \mathcal{M}_0$, and for each pair $\mathbf{a}, \mathbf{b}$, such that $\mathbf{a} = p_1 g_{ij}(\mathbf{c})$ and
$\mathbf{b} = p_2 g_{rs}(\mathbf{d})$ for some permutations p_1, p_2, and $P(\mathbf{a}) \neq P(\mathbf{b})$, no
equation of the form $(\mathbf{a}, q(\mathbf{b}))$ is in E.

(Of course, it is assumed above that $1 \leq i, j, r, s \leq n$ with $i \neq j$, $r \neq s$, and we recall that the operation P is defined with respect to $\Pi = \Pi(\mathbf{c}, \mathbf{d})$ equivalence.)

Since there are finitely many sequences of the form $\mathbf{a} = g_{ij}(\mathbf{c})$ and $\mathbf{b} = g_{rs}(\mathbf{d})$, $\Psi(E)$ can be expressed as a conjunction of "$E \in \mathcal{M}_0$"and a finite conjunction of formulas of the form: "no equation of the form $(\mathbf{a}, q(\mathbf{b}))$ is in E". By assumption, and in view of Theorem 5.4, this can be expressed in the first-order language of $\mathcal{L}(Com)$. Obviously, by Lemma 6.4, $E(\mathbf{c}, \mathbf{d})$ satisfies this formula, and we show that this is actually the only theory satisfying it.

Assume that $\Psi(E)$ holds. Then $E \in \mathcal{M}_0$, and as we have already observed above, $E = E(\mathbf{c}, f(\mathbf{d}))$ for some permutation f. We show that $f \in G(\mathbf{c}, \mathbf{d})$.

Suppose that f is not a member of $G(\mathbf{c}, \mathbf{d})$. Then there is $i \leq n$ such that $f(i)$ is not Π-equivalent to i. Since $n > 2$, we may choose $r \leq n$ such that $r \neq i$ and $[i] \cup [r] \neq [f(i)] \cup [f(r)]$ (where the braces denote Π-equivalence classes). Indeed, if there is $r \in [i]$ with $r \neq i$, then such an r is as required. Otherwise, since $n > 2$, there is $r \neq i, f^{-1}(i)$. It follows that $f(r) \notin [i]$, and as $f(i) \notin [i]$, the claim follows. Consequently, if we put $(\mathbf{a}, \mathbf{b}) = (g_{ir}(\mathbf{c}), g_{ir} f(\mathbf{d}))$, then $P(\mathbf{a}) \neq P(\mathbf{b})$. Finally, since $g_{ir} f(\mathbf{d})$ may be presented in the form $f_1 g_{js}(\mathbf{d})$, for some permutation f_1 on $n - 1$ elements, by virtue of $\Psi(E)$, $(\mathbf{a}, \mathbf{b}) \notin E$, which contradicts the fact that $(\mathbf{a}, \mathbf{b})$ is a direct consequence of $(\mathbf{c}, f(\mathbf{d}))$.

Thus, $f \in G(\mathbf{c}, \mathbf{d})$, and it follows, by Lemma 6.2, that $(\mathbf{c}, f(\mathbf{d})) \in E(\mathbf{c}, \mathbf{d})$. By $\Psi(E)$ and minimality of $E(\mathbf{c}, \mathbf{d})$, $E(\mathbf{c}, f(\mathbf{d})) = E(\mathbf{c}, \mathbf{d})$, as claimed.

It follows that $E(\mathbf{c}, \mathbf{d})$ is definable. Since, by assumption, each proper subtheory of $E(\mathbf{c}, \mathbf{d})$ in $\mathcal{M}$ is definable, we infer that $E(\mathbf{c}, \mathbf{d}) \in \mathcal{M}_d$, which is a contradiction. Hence, $\mathcal{M}_d = \mathcal{M}$, and the proof is completed. $\square$

Example 6.6. Consider the theory E_0 generated by the equation

$$x^2 y^2 z^4 t^5 = x y^3 z^3 t^6.$$

We have $\mathbf{c} = (2, 2, 4, 5)$, $\mathbf{d} = (1, 3, 3, 6)$, $(\mathbf{c}, \mathbf{d})$ is balanced, and $E_0 = E(\mathbf{c}, \mathbf{d})$. The relation $\Pi = \Pi(\mathbf{c}, \mathbf{d})$ has two blocks, $\{1, 2, 3\}$ and $\{4\}$. Let $\mathbf{a} = p_1 g_{ij}(\mathbf{c})$ and $\mathbf{b} = p_2 g_{rs}(\mathbf{d})$ for some permutations p_1, p_2. Then, $P(\mathbf{a}) = P(\mathbf{b})$ holds either if both the sides equal $\{1, 2, 3\}$ or if both the sides equal $\{1, 2, 3, 4\}$. The former means that $\{i, j\}, \{r, s\} \subseteq \{1, 2, 3\}$, and the latter means that $4 \in \{i, j\} \cap \{r, s\}$.

Let $\Psi'(E)$ denote the formula obtained from $\Psi(E)$ in the proof above by deleting the conjunct $E \in \mathcal{M}_0$. It follows that if $(\mathbf{c}, p(\mathbf{d})) \in E$ and $\Psi'(E)$ holds, then p fixes 4. Yet, all equations $(\mathbf{c}, p(\mathbf{d}))$ with this property are equivalent to each other. Consequently, the formula $\Psi'(E)$ defines E_0 in this case.

Combining the facts that each equational theory in $\mathcal{L}(Com)$ is finitely axioma-tizable ([16]), and that each theory generated by a finite number of equations is a join of one-based theories, we obtain the following.

Corollary 6.7. *If a theory E is generated by a set of equations Σ such that each equation in Σ is either irregular, or with comparable nonequivalent sequences, or balanced, then E is definable in $\mathcal{L}(Com)$.*

7. Automorphism group

We look now at equations in two variables. In this case, it is not easy to distin-guish between the consequences of such an equation and its companion obtained by transposition of variables on one of the sides. In fact, our attempts to do so have led to a counterexample.

In this section we construct a nontrivial automorphism on $\mathcal{L}(Com)$, exchanging certain one-based theories generated by equations in two variables.

The idea is as follows. Let e_1 and e_2 be two equations. For a theory $E \in \mathcal{L}(Com)$ we define $\phi(E)$ to be the set of equations such that for every equation e, the formula $e \in E \leftrightarrow e \in \phi(E)$ fails if and only if $e \in \{e_1, e_2\}$ and $E \cap \{e_1, e_2\}$ is a one-element set. Thus, we can say that $\phi(E)$ is obtained from E by exchanging e_1 and e_2. Generally, $\phi(E)$ need not be a theory. Yet, if it happens to be a theory for every $E \in \mathcal{L}(Com)$, then ϕ is an automorphism of $\mathcal{L}(Com)$.

Indeed, it is a routine exercise to check that, in general,

$$\phi(E_1 \cap E_2) = \phi(E_1) \cap \phi(E_2).$$

A slightly more involved argument yields also the following fact.

Lemma 7.1. *If $\phi(E)$ is a theory for every $E \in \mathcal{L}(Com)$, then*

$$\phi(E_1 \vee E_2) = \phi(E_1) \vee \phi(E_2),$$

for all $E_1, E_2 \in \mathcal{L}(Com)$.

Proof. By assumption, $\phi(E_1 \vee E_2)$ is a theory. It is easy to see that it contains both $\phi(E_1)$ and $\phi(E_2)$. Hence, $\phi(E_1 \vee E_2) \supseteq \phi(E_1) \vee \phi(E_2)$.

For the converse inclusion, observe that, in turn, $\phi(\phi(E_1) \vee \phi(E_2))$ contains both E_1 and E_2. This, by assumption, is also a theory, and therefore $\phi(\phi(E_1) \vee \phi(E_2)) \supseteq E_1 \vee E_2$. Using the fact that $\phi(\phi(E)) = E$, we obtain $\phi(E_1) \vee \phi(E_2) \supseteq \phi(E_1 \vee E_2)$, completing the proof. $\square$

It is not easy to find a nontrivial pair of equations satisfying the assumption of the lemma above. The first example found by the author was the pair of equations

$$x^3 y^4 = x y^5 \quad \text{and} \quad x^4 y^3 = x y^5.$$

If E contains both the equations, or none of them, then obviously $\phi(E)$ is a theory. We need to prove that if a theory E contains only one of these equations, then after exchanging it with another one, we still have a theory. What makes this statement true in this particular case is that the filter of such a theory must be relatively large, while the remainder is relatively small, and there is no room for consequences that can distinguish one equation from another.

Ralph McKenzie has observed that also each pair of the equations

$$(*) \qquad\qquad x^k y^{k+1} = x y^{k+2} \quad \text{and} \quad x^{k+1} y^k = x y^{k+2}$$

for any odd $k \geq 3$ has all the properties of the first example we need. This led to the surprising conclusion that the automorphism group of $\mathcal{L}(Com)$ embedded a Boolean group of continuum order. The proof below is a modification of the original proof to cover the general case.

Lemma 7.2. *If $E = E(J, m, r, \pi)$ is a theory in $\mathcal{L}(Com)$ containing exactly one of the equations of $(*)$ for a fixed odd $k \geq 3$, then the following conditions hold:*

    (i) $(k+3), (1, k, k+1), (1, 1, k+2) \in J$.
    (ii) *If $(\mathbf{a}, \mathbf{b}) \in \pi$, $\mathbf{a} \neq \mathbf{b}$, and $\mathbf{b} = (1, k+2)$, then either $\mathbf{a} = (k, k+1)$ or $\mathbf{a} = (k+1, k)$.*
    (iii) *If $(\mathbf{a}, \mathbf{b}) \in \pi$, $\mathbf{a} \neq \mathbf{b}$, and f is one of the operations g_{ij} or h_1, then $f(\mathbf{a})$ is different from $(1, k+2)$ and $(k+2, 1)$.*

Proof. Our assumption is that either $e_1 = ((k, k+1), (1, k+2))$ or $e_2 = ((k+1, k), (1, k+2))$ is in E, and exactly one of these cases holds. These cases are not quite symmetric, but we try to treat them simultaneously.

First of all, note that, in any case, $m \leq 1$ and $r = 1$ (note that $\gcd(k, 2) = 1$, since k is odd). By definition of $E(J, m, r, \pi)$, either $e_i \in \pi$ for some $i = 1, 2$, or both $(k, k+1), (1, k+2) \in J$. In the latter case, by the same definition, both $e_1, e_2 \in E$, contradicting the assumption. Hence, $e_i \in \pi$ is the case.

By $(\pi 3)$, $(k+1, k, 1), (k+2, 1, 1) \in J$, and by $(\pi 4)$, $(k+3) \in J$, proving (i).

Suppose that $(\mathbf{a}, \mathbf{b}) \in \pi$, $\mathbf{a} \neq \mathbf{b}$, and $\mathbf{b} = (1, k+2)$. To fix attention, suppose that $e_1 \in \pi$, i.e., $(k, k+1), (1, k+2)$ are in a block of π, and suppose that $\mathbf{a} = (\alpha, \beta)$ is another element in this block. Recall that sequences in a block are incomparable or equivalent. If $\mathbf{a}$ were equivalent to any of $(k, k+1), (1, k+2)$, $\mathbf{a}$ would be $(k+1, k)$ or $(k+2, 1)$. In any of these cases, we would have in consequence (using $(\pi 2)$) that both $e_1, e_2 \in \pi$, which contradicts the assumption. Hence, $\mathbf{a}$ is incomparable with $(k, k+1)$ and $(1, k+2)$. It follows that both α and β are different from any of $k, k+1, k+2, 1$. Also, since $(k+3) \in J$, both $\alpha, \beta < k+3$. Consequently, $1 < \alpha, \beta < k$, which implies that $\mathbf{a} < (k, k+1)$, again contradicting the assumption. It follows that there are no other elements in a block containing $(k, k+1), (1, k+2)$, proving (ii) in this case. Exactly the same argument works for the case $e_2 \in \pi$.

For (iii), let $(\mathbf{a}, \mathbf{b}) \in \pi$. Obviously, for $f = h_1$, since $\mathbf{a}$ has length at least 2, $f(\mathbf{a})$ is different from $(1, k+2)$ and $(k+2, 1)$. So, we may assume that f is one of the g_{ij}. Assume also, to the contrary, that $f(\mathbf{a}) = (1, k+2)$ (for $(k+2, 1)$ the argument is the same). Then, by (ii), since $(1, k+2) \notin J$, $f(\mathbf{b}) = (k, k+1)$ or $(k+1, k)$. Assume that $f(\mathbf{b}) = (k, k+1)$ (again, for $(k+1, k)$ the argument is the same, since it does not depend on permutation of sequences).

Then $\mathbf{a}, \mathbf{b}$ must be three-element sequences with $g_{ij}(\mathbf{a}) = (1, k+2)$ and $g_{ij}(\mathbf{b}) = (k, k+1)$. It follows that, up to permutation, $\mathbf{a} = (1, k+2-r, r)$, and $\mathbf{b} = (k-s, s, k+1)$ or $\mathbf{b} = (k, k+1-t, t)$ for some $r, s, t \geq 1$ with $r < k+2$, $s < k$, and $t < k+1$.

Now, observe that while $g_{ij}(\mathbf{a})$ does not belong to J for any i, j (since in any case the resulting two-element sequence is less than or equal to $(1, k+2)$ or $(k, k+1)$), this is not the case with $g_{ij}(\mathbf{b})$. Indeed, if $\mathbf{b} = (k-s, s, k+1)$, then $g_{13}(\mathbf{b})$ or $g_{23}(\mathbf{b})$ is greater then $(k+3)$, and if $\mathbf{b} = (k, k+1-t, t)$, then $g_{13}(\mathbf{b})$ or $g_{12}(\mathbf{b})$ is greater then $(k+3)$. Since $(\mathbf{a}, \mathbf{b}) \in \pi$, this leads to a contradiction with condition $(\pi 4)$. $\qquad\square$

Now, given an odd $k \geq 3$, let $\phi_k(E)$ denote $\phi(E)$ applied to the equations $(*)$, and assume that exactly one of these equations is in E. Then E is nontrivial, and $E = E(J, m, r, \pi)$ for some J, m, r, and π. We prove that $\phi_k(E)$ is a theory of the form $E(J, m, r, \pi')$ for some remainder π' of type (J, m, r).

By Lemma 7.2, either $((k, k+1), (1, k+2))$ or $((k+1, k), (1, k+2))$ is in π, and moreover, the block containing $(1, k+2)$ consists of only two elements. By $(\pi 2)$, also the block containing $(k+2, 1)$ consists of exactly two elements, and in any case, these remaining elements are $(k, k+1)$ and $(k+1, k)$, placed in one of the two possible ways. Hence, it is enough to prove that if $\phi_k(\pi)$ denotes π after exchanging $(k, k+1)$ and $(k+1, k)$ in blocks, then $\phi_k(\pi)$ is a remainder of type (J, m, r).

Obviously, $\phi_k(\pi)$ is still an equivalence relation on $\Gamma_+ \setminus J$, every block in $\phi(\pi)$ contains the sequences of the same length, and conditions $(\pi 1, \pi 2)$ trivially hold. For condition $(\pi 3)$, note that exchanging $(k, k+1)$ and $(k+1, k)$ in two-element blocks changes nothing, and $(\pi 3)$, still holds. Finally, for $(\pi 4)$, from Lemma 7.2(iii) we see that nontrivial consequences of any $(\mathbf{a}, \mathbf{b}) \in \pi$ do not fall into the blocks with $(k+2, 1)$. Hence, $(\pi 4)$ holds for $\phi_k(\pi)$, as well.

Using Lemma 7.1, we infer that $\phi_k(E)$ is an automorphism, which has the following important consequence.

Theorem 7.3. *There are theories in $\mathcal{L}(Com)$ that are not definable in the first-order language.*

Since we now have a denumerable list of automorphism, we may infer, in addition, that the group of automorphisms is infinite. Yet, this may be still strengthened. Namely, let S be a subset of the set of odd integers $k \geq 3$. We define $\phi_S(E)$ to be the result of applying ϕ_k to E for all $k \in S$. This is well-defined, since the results of applying different ϕ_k are independent from each other. This is also why the proof of Lemma 7.1 and the remarks preceding it apply to $\phi_S(E)$ as well. In this way we obtain the result established during McKenzie's seminar at Vanderbilt.

Theorem 7.4. *The automorphism group of $\mathcal{L}(Com)$ contains a Boolean group of continuum order, as a subgroup.*

An interesting, still open problem is how complex the automorphism group of $\mathcal{L}(Com)$ actually is. Are there any automorphisms of order greater than 2? Are there automorphisms of some other type than those obtained by exchanging equations?

Acknowledgment

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