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NONLINEARIZABLE ACTIONS OF DIHEDRAL GROUPS ON AFFINE SPACE

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ABSTRACT. Let G be a reductive, non-abelian, algebraic group defined over $\mathbb{C}$. We investigate algebraic G-actions on the total spaces of non-trivial algebraic G-vector bundles over G-modules with great interest in the case that G is a dihedral group. We construct a map classifying such actions of a dihedral group in some cases and describe the spaces of those non-linearizable actions in some examples.

1. INTRODUCTION

Let G be a reductive complex algebraic group. When G is non-abelian, it is well-known that there exist non-linearizable actions of G on complex affine space $\mathbb{A}^n$ for $n \geq 4$, i.e., algebraic actions of G on $\mathbb{A}^n$ which are not conjugate to linear actions under polynomial automorphisms of $\mathbb{A}^n$. It is remarkable that non-linearizable actions on $\mathbb{A}^n$ known so far are all obtained from non-trivial algebraic G-vector bundles over G-modules. An algebraic G-vector bundle over a G-variety X is defined to be an algebraic vector bundle $p : E \to X$, where E is a G-variety, the projection p is G-equivariant, and the morphism induced by $g \in G$ from $p^{-1}(x)$ to $p^{-1}(gx)$ is linear for all g and $x \in X$. An algebraic G-vector bundle is called trivial if it is isomorphic to a product bundle $X \times Q \to X$ for some G-module Q. A total space of an algebraic G-vector bundle over a G-module is an affine space by the affirmative solution to the Serre conjecture by Quillen [19] and Suslin [21]. Thus, the G-action on a total space E of a non-trivial G-vector bundle over a G-module is a candidate for a non-linearizable action on affine space. There are a couple of known conditions for such an action to be non-linearizable (Bass and Haboush [1], M. Masuda and Petrie [15]). Schwarz [20] (Kraft and Schwarz [7] for details) first showed that an algebraic G-vector bundle over a G-module P can be non-trivial when the algebraic quotient of P is of one dimension, and that there exist families of non-linearizable actions on affine space, by using the above conditions. After Schwarz, lots of examples of non-trivial algebraic G-vector bundles have been presented, and it turns out that many of the G-actions on their total spaces are non-linearizable (Knop [5], M. Masuda, Moser-Jauslin and Petrie [11], M. Masuda and Petrie [16]). For abelian groups, there are no known examples of non-linearizable actions on complex affine space. In fact, for an abelian group G, every algebraic G-vector bundle over

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a G-module becomes trivial by the result of M. Masuda, Moser-Jauslin and Petrie [12], so, we cannot obtain non-linearizable actions from G-vector bundles. There are some affirmative results for the linearizability for torus actions (e.g. Bialynicki-Birula [2], Kaliman, Koras, Makar-Limanov and Russell [4]); however, it remains open whether or not every algebraic action of an abelian group on $\mathbb{A}^n$ ($n \geq 4$) is linearizable. Especially for a finite abelian group G, e.g. for a cyclic group $\mathbb{Z}/n\mathbb{Z}$, we never know even whether any G-action on $\mathbb{A}^3$ is linearizable or not.

For finite groups, M. Masuda and Petrie [16] showed that there exists a family of non-linearizable actions of a dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ for n even and ≥ 18 on $\mathbb{A}^4$. They considered D_n-actions derived from algebraic D_n-vector bundles which become trivial by adding certain trivial bundles, and showed that those actions form a family in some cases. Later, Mederer [18] showed that non-trivial algebraic D_n-vector bundles form a huge family of infinite dimension for n odd and ≥ 3. In this article, we investigate G-actions derived from non-trivial algebraic G-vector bundles. We are most interested in the case that G is a dihedral group. We present a new condition for such D_n-actions to be non-linearizable and construct a map which classifies such non-linearizable D_n-actions without imposing triviality on D_n-vector bundles under the addition of certain trivial bundles. We also describe the spaces of those non-linearizable D_n-actions in some examples.

2. Families of non-linearizable actions

Let G be a reductive, non-abelian algebraic group and let Z be an affine G-variety. We denote by $\mathbb{C}[Z]$ the coordinate ring of Z and by $\mathbb{C}[Z]^G$ the ring of invariants. The algebraic quotient $Z/\!/G$ is the affine variety defined by $Z/\!/G = \operatorname{Spec}\mathbb{C}[Z]^G$ and the quotient morphism $\pi_Z : Z \to Z/\!/G$ is the morphism corresponding to the inclusion $\mathbb{C}[Z]^G \hookrightarrow \mathbb{C}[Z]$. Let P and Q be G-modules and let $X \subseteq P$ be a G-subvariety containing the origin of P. We denote by $\operatorname{Vec}_G(X, Q)$ the set of algebraic G-vector bundles over X whose fiber over the origin is isomorphic to Q, and by $\operatorname{Vec}_G(X, Q)$ the set of G-isomorphism classes in $\operatorname{Vec}_G(X, Q)$. An element $E \to X$ of $\operatorname{Vec}_G(X, Q)$ is represented by the total space E, and the isomorphism class of $E \in \operatorname{Vec}_G(X, Q)$ is denoted by $[E]$. The set $\operatorname{Vec}_G(X, Q)$ is called trivial if $\operatorname{Vec}_G(X, Q)$ consists of the unique class $[\Theta_Q]$, where Θ_Q denotes the product bundle with fiber Q. When $\dim P/\!/G = 1$, Schwarz [20] showed that $\operatorname{Vec}_G(P, Q)$ has an additive group structure and is isomorphic to a vector group $\mathbb{C}^q$ for a non-negative integer q. Mederer [18] (cf. [8]) extended the result of Schwarz to the case where the base space is a G-equivariant affine cone X with $\dim X/\!/G = 1$. When $\dim P/\!/G \geq 2$, $\operatorname{Vec}_G(P, Q)$ can be non-trivial and of countably or uncountably infinite dimension ([9], [10], [18]).

We assume that $\operatorname{Vec}_G(P, Q)$ is non-trivial. Let $E \in \operatorname{Vec}_G(P, Q)$. The following are the known conditions for the G-action on the total space E to be non-linearizable.

Proposition 2.1. *Let $E, E' \in \operatorname{Vec}_G(P, Q)$.*

(1) ([15]) *Suppose that there exists a subgroup H of G such that $(P \oplus Q)^H = P$. Then E and E' are isomorphic as G-varieties if and only if E and the pull-back $\varphi^* E'$ are isomorphic as G-vector bundles for some G-automorphism φ of P.*

(2) ([1]) *If the Whitney sum $E \oplus \Theta_P$ is non-trivial, then the G-action on E is non-linearizable.*

Let $\mathrm{VAR}_G(P,Q)$ be the set of G-isomorphism classes of affine G-spaces represented as the total spaces of elements of $\mathrm{Vec}_G(P,Q)$. The group $\mathrm{Aut}(P)^G$ of G-equivariant automorphisms of P acts on $\mathrm{Vec}_G(P,Q)$ by pull-backs. There exists a surjection Ψ from the orbit space of $\mathrm{Vec}_G(P,Q)$ under the action of $\mathrm{Aut}(P)^G$ to $\mathrm{VAR}_G(P,Q)$. Under the assumption in Proposition 2.1 (1), Ψ is an isomorphism.

Example 2.1. Let $G = O(2) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ and let V_m $(m \geq 1)$ be a two-dimensional $O(2)$-module such that

$$\begin{aligned} \lambda(x,y) &= (\lambda^m x, \lambda^{-m} y) \qquad \text{for } \lambda \in \mathbb{C}^*, \\ \tau(x,y) &= (y,x) \qquad \text{for the generator } \tau \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Then $V_m//O(2) = \operatorname{Spec}\mathbb{C}[t] = \mathbb{A}^1$, where $t = xy$, and $\mathrm{Aut}(V_m)^G = \mathbb{C}^*$, namely, $\mathrm{Aut}(V_m)^G$ consisits of scalar multiplications.

Let n be odd. Then $\mathrm{Vec}_G(V_2,V_n) \cong \mathbb{C}^{(n-1)/2}$ and the Whitney sum with Θ_{V_2} induces an isomorphism between $\mathrm{Vec}_G(V_2,V_n)$ and $\mathrm{Vec}_G(V_2,V_n \oplus V_2)$ ([20]). By Proposition 2.1 (1) or (2), if $E \in \mathrm{Vec}_{O(2)}(V_2,V_n)$ is non-trivial, then the $O(2)$-action on E is non-linearizable. We shall describe $\mathrm{VAR}_{O(2)}(V_2,V_n)$. Since $(V_2 \oplus V_n)^{\mathbb{Z}/2\mathbb{Z}} = V_2$, where $\mathbb{Z}/2\mathbb{Z}$ is a subgroup of $\mathbb{C}^* \subset O(2)$, it follows from Proposition 2.1 (1) that

$$\mathrm{VAR}_{O(2)}(V_2,V_n) \cong \mathrm{Vec}_{O(2)}(V_2,V_n)/\mathbb{C}^*.$$

In order to look at the action of $\mathrm{Aut}(V_2)^G = \mathbb{C}^*$ on $\mathrm{Vec}_G(V_2,V_n)$, recall the isomorphism $\mathrm{Vec}_G(V_2,V_n) \cong \mathbb{C}^{(n-1)/2}$. For the details, we refer to Kraft and Schwarz [7]. Let $F = \pi_{V_2}^{-1}(1)$, which is the G-subvariety of V_2 defined by $xy = 1$. Then $F \cong G/H$, where $H = \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, and V_n is multiplicity-free with respect to H, namely, each irreducible H-module occurs in V_n with multiplicity at most one when V_n is viewed as an H-module. We set $\mathfrak{m} = \mathrm{Mor}(F, \operatorname{End} V_n)^G$, the module of G-equivariant morphisms from F to $\operatorname{End} V_n$. Let $\mathbb{B} = \operatorname{Spec}\mathbb{C}[s]$ be the double cover of $\mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$, where $s^2 = t$. Then the group $\Gamma := \{\pm 1\}$ acts on $\mathbb{B}$ and on F by scalar multipication. We denote by $\mathbb{B} \times_\Gamma F$ the quotient of $\mathbb{B} \times F$ by Γ which acts by $(b,f) \mapsto (b\gamma, \gamma^{-1} f)$ for $\gamma \in \Gamma$, $b \in \mathbb{B}$, and $f \in F$. The group G acts on $\mathbb{B} \times_\Gamma F$ through F. We define a G-equivariant morphism φ by

$$\begin{aligned} \varphi : \mathbb{B} \times_\Gamma F &\to V_2, \\ [b,f] &\mapsto bf, \end{aligned}$$

which is a G-isomorphism from $(\mathbb{B} - \{0\}) \times_\Gamma F$ onto $V_2 - \pi_{V_2}^{-1}(0)$. Note that $\mathbb{C}[\mathbb{B} \times_\Gamma F]^G \cong \mathbb{C}[\mathbb{B}]^\Gamma = \mathbb{C}[t] = \mathbb{C}[V_2]^G$. The morphism φ induces a homomorphism

$$\begin{aligned} \varphi_\# : \mathrm{Mor}(V_2, \operatorname{End} V_n)^G &\to \mathrm{Mor}(\mathbb{B} \times_\Gamma F, \operatorname{End} V_n)^G \\ &= \mathrm{Mor}(\mathbb{B}, \mathfrak{m})^\Gamma =: \mathfrak{m}(\mathbb{B})^\Gamma. \end{aligned}$$

The modules $\mathrm{Mor}(V_2, \operatorname{End} V_n)^G$ and $\mathfrak{m}(\mathbb{B})^\Gamma$ are finite free modules over $\mathbb{C}[t]$. In fact, a basis of $\mathrm{Mor}(V_2, \operatorname{End} V_n)^G \cong (\mathbb{C}[V_2] \otimes \operatorname{End} V_n)^G$ over $\mathbb{C}[t]$ is written in a matrix form as

$$\left\{ A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & x^n \\ y^n & 0 \end{pmatrix} \right\}$$

and a basis of $\mathfrak{m}(\mathbb{B})^\Gamma \cong (\mathbb{C}[s] \otimes \mathfrak{m})^\Gamma$ over $\mathbb{C}[t]$ is

$$\left\{ C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C_1 = s(A_1|_F) \right\}.$$

The module $\mathrm{Mor}(V_2, \mathrm{End}\, V_n)^G$ (resp. $\mathfrak{m}(\mathbb{B})^\Gamma$) inherits a grading from $\mathbb{C}[V_2]$ (resp. $\mathbb{C}[s]$), and $\varphi_\#$ is a homomorphism of degree 0. Let $\mathrm{Mor}(V_2, \mathrm{End}\, V_n)^G_1$ (resp. $\mathfrak{m}(\mathbb{B})^\Gamma_1$) be the submodule of $\mathrm{Mor}(V_2, \mathrm{End}\, V_n)^G$ (resp. $\mathfrak{m}(\mathbb{B})^\Gamma$) consisting of elements with positive degrees. Then $\mathrm{Vec}_G(V_2, V_n)$ is isomorphic to the quotient module $\mathfrak{m}(\mathbb{B})^\Gamma_1/\varphi_\# \mathrm{Mor}(V_2, \mathrm{End}\, V_n)^G_1$. Since

$$\varphi_\#(A_0) = C_0 \quad \text{and} \quad \varphi_\#(A_1) = t^{\frac{n-1}{2}} C_1,$$

$\{t^{i-1}C_1; 1 \le i \le \frac{n-1}{2}\}$ forms a $\mathbb{C}$-basis of $\mathfrak{m}(\mathbb{B})^\Gamma_1/\varphi_\# \mathrm{Mor}(V_2, \mathrm{End}\, V_n)^G_1$, and hence

$$\mathrm{Vec}_G(V_2, V_n) \cong \mathfrak{m}(\mathbb{B})^\Gamma_1/\varphi_\# \mathrm{Mor}(V_2, \mathrm{End}\, V_n)^G_1 \cong \mathbb{C}^{\frac{n-1}{2}}.$$

Note that $\deg(t^{i-1}C_1) = 2i-1$. The scalar multiplication on V_2 corresponds to a scalar multiplication on $\mathbb{B}$ via φ. Hence $\mathrm{Vec}_G(V_2, V_n) \cong \bigoplus_{i=1}^{(n-1)/2} W(2i-1)$ as a module of $\mathrm{Aut}(V_2)^G = \mathbb{C}^*$, where $W(i)$ denotes the representation space of $\mathbb{C}^*$ with weight i. Thus we obtain by Proposition 2.1 (1) that

$$\begin{aligned} \mathrm{VAR}_{O(2)}(V_2, V_n) &\cong \Big(\bigoplus_{i=1}^{(n-1)/2} W(2i-1)\Big)/\mathbb{C}^* \\ &=: \mathbb{P}_*(2i-1; 1 \le i \le \frac{n-1}{2}). \end{aligned}$$

Here $\mathbb{P}_*(2i-1; 1 \le i \le (n-1)/2)$ consisits of the "vertex" $_*$ and the weighted projective space $\mathbb{P}(2i-1; 1 \le i \le (n-1)/2)$ of dimension $(n-3)/2$ with weight $2i-1$ for $1 \le i \le (n-1)/2$. The "vertex" corresponds to the linearizable action and the weighted projective space to non-linearizable actions (cf. [16]).

Example 2.2. Let $G = SL_2$ and let R_n be the SL_2-module of binary forms of degree $n \ge 1$. Then $\mathrm{Vec}_G(R_2, R_n) \cong \mathbb{C}^{[(n-1)^2/4]}$ and $\mathrm{Aut}(R_2)^G = \mathbb{C}^*$ ([20], [7]). As a module of $\mathbb{C}^* = \mathrm{Aut}(R_2)^G$, $\mathrm{Vec}_G(R_2, R_n)$ is isomorphic to $\bigoplus_{i=1}^{n-2} m_i W(i)$ with multiplicity $m_i = [\frac{n-i}{2}]$. Suppose n is odd. Then $(R_2 \oplus R_n)^{\mathbb{Z}/2\mathbb{Z}} = R_2$. Hence by Proposition 2.1 (1),

$$\begin{aligned} \mathrm{VAR}_{SL_2}(R_2, R_n) &\cong \Big(\bigoplus_{i=1}^{n-2} m_i W(i)\Big)/\mathbb{C}^* \\ &=: \mathbb{P}_*(i, m_i; 1 \le i \le n-2). \end{aligned}$$

In this case, the space of non-linearizable SL_2-actions is isomorphic to the weighted projective space of dimension $[(n-1)^2/4] - 1$ with weight i of multiplicity m_i for $1 \le i \le n-2$.

Example 2.3. Let G be semisimple and let $\mathfrak{g}$ be the adjoint representation of G. Let Σ be a system of simple roots of G and F an irreducible G-module with the highest weight χ. Knop [5] constructed a map associated with $\alpha \in \Sigma$,

$$\Phi_\alpha : \mathrm{Vec}_G(\mathfrak{g}, F) \to \mathrm{Vec}_{SL_2}(R_2, R_m),$$

where $m = \langle \chi, \alpha \rangle$. The map Φ_α is surjective if the α-string of χ is regular ([5], [14]). We recall the construction of Φ_α. Let $T \subset G$ be a maximal torus with the Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Let L be the subgroup of G generated by T and the root subgroups U_α and $U_{-\alpha}$. We denote by L' the commutator subgroup of L and by Z the center of L. Then $L = L'Z$, and L' is isomorphic to SL_2 or SO_3. Let $\mathfrak{l}$ be the Lie algebra of L. Then $\mathfrak{l}$ is isomorphic to $\mathfrak{sl}_2 \oplus \mathbb{C}^{n-1}$ as an L'-module, where $n = \text{rank}\ \mathfrak{t}$. For $E \in \text{Vec}_G(\mathfrak{g}, F)$, the restricted bundle $E|_{\mathfrak{l}}$ is an L-vector bundle with fiber F' which is F viewed as an L-module. Take a $\xi_0 \in \mathfrak{t}$ so that the centralizer of ξ_0 is exactly L, and fix it. Then $\mathfrak{a} := \xi_0 + \text{Lie}L' \subseteq \mathfrak{g}$ is L-stable and isomorphic to $\mathfrak{sl}_2 \cong R_2$ as an L'-variety. Since Z acts trivially on $\mathfrak{l}$, hence on $\mathfrak{a}$, $E|_{\mathfrak{a}}$ decomposes to a Whitney sum of eigenbundles of Z. Let $(E|_{\mathfrak{a}})_\chi$ be the eigenbundle corresponding to the restricted weight of χ onto Z. Then the L'-vector bundle $(E|_{\mathfrak{a}})_\chi$ is considered as an element of $\text{Vec}_{SL_2}(R_2, R_m)$. The map Φ_α is defined by $\Phi_\alpha(E) = (E|_{\mathfrak{a}})_\chi$. By the construction of Φ_α, Φ_α decomposes to the maps

$$\phi_\alpha : \text{Vec}_G(\mathfrak{g}, F) \to \text{Vec}_L(\mathfrak{l}, F') \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)$$

and

$$\phi_{\xi_0} : \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m) \to \text{Vec}_{SL_2}(R_2, R_m).$$

From the choice of ξ_0, ϕ_{ξ_0} is surjective. In fact, $\phi_{\xi_0} \circ pr^* = id$, where

$$pr^* : \text{Vec}_{SL_2}(R_2, R_m) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)$$

is the induced map from the projection $R_2 \oplus \mathbb{C}^{n-1} \to R_2$. When Φ_α is surjective, ϕ_α is also surjective since ϕ_{ξ_0} is surjective. By [9],

$$\text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m) \cong \text{Vec}_{SL_2}(R_2, R_m) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{C}^{n-1}].$$

Hence we obtain the following.

Theorem 2.2. *Under the nontation above, if the α-string of χ is regular, then*

$$\begin{aligned} \phi_\alpha : \text{Vec}_G(\mathfrak{g}, F) &\to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m) \\ &\cong \mathbb{C}^{[(m-1)^2/4]} \otimes_{\mathbb{C}} \mathbb{C}[y_1, \cdots, y_{n-1}] \end{aligned}$$

is surjective. Furthermore, if there is a subgroup H such that $(\mathfrak{g} \oplus F)^H = \mathfrak{g}$, then ϕ_α induces a surjection

$$\text{VAR}_G(\mathfrak{g}, F) \to (\mathbb{C}^{[(m-1)^2/4]} \otimes_{\mathbb{C}} \mathbb{C}[y_1, \cdots, y_{n-1}])/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^{[(m-1)^2/4]}$ with weight i of multiplicity $m_i = [(m-i)/2]$ and on y_i with weight 1.

Proof. The first assertion follows from the above observation. For the second assertion, note that $\text{Aut}(\mathfrak{g})^G = \mathbb{C}^*$ ([7]). From Proposition 2.1 (1), there is an isomorphism $\text{VAR}_G(\mathfrak{g}, F) \cong \text{Vec}_G(\mathfrak{g}, F)/\mathbb{C}^*$. Hence ϕ_α induces a surjection

$$\text{VAR}_G(\mathfrak{g}, F) \to \text{Vec}_{SL_2}(R_2 \oplus \mathbb{C}^{n-1}, R_m)/\mathbb{C}^*.$$

The assertion follows from the statement in Example 2.2. □

Remark. When the α-string of χ is singular, the image of Φ_α contains a subspace of dimension $[m/2]([m/2]-1)/2$ ([14]).

By Theorem 2.2 and its remark, $\text{Vec}_G(\mathfrak{g}, F)$ is of infinite dimension if $m \geq 4$ and $n \geq 2$. Furthermore, if $(\mathfrak{g} \oplus F)^H = \mathfrak{g}$ for a subgroup H, then $\text{VAR}_G(\mathfrak{g}, F)$ is of infinite dimension.

Now, we give a new condition for the G-action on $E \in \text{Vec}_G(P, Q)$ to be nonlinearizable, which is used as a basic fact in the next section.

Proposition 2.3. *Let $E, E' \in \text{Vec}_G(P, Q)$. Suppose that there exist reductive subgroups H and K such that $H \subset K$ and satisfying the following conditions;*

(1) $Q^K = Q^H$,
(2) $\dim P^H = 1$ *and* $\dim P^K = 0$.

If $E \cong E'$ as G-varieties, then the restricted bundles $E|_X$ and $(c^*E')|_X$ are isomorphic as G-vector bundles, where $X = \overline{G \cdot P^H}$ and c is a scalar multiplication on P. In particular, if $E|_X$ is a non-trivial G-vector bundle, then the G-action on E is non-linearizable.

Proof. Let $\phi : E \cong E'$ be an isomorphism of G-varieties. Then ϕ restricts to an isomorphism $\phi_H : E^H \cong {E'}^H$. Since E^H and ${E'}^H$ are trivial (H-)vector bundles over P^H with fiber Q^H (cf. [6]), it follows that $E^H \cong {E'}^H \cong P^H \times Q^H$. Similarly, $E^K \cong {E'}^K \cong Q^K = Q^H$ since $\dim P^K = 0$. Since E^K (resp. ${E'}^K$) is a subbundle of E^H (resp. ${E'}^H$), we get $E^H = E^K \times P^H$ and ${E'}^H = {E'}^K \times P^H$. Let x be a coordinate variable of $P^H \times Q^H$ such that $P^H = \operatorname{Spec} \mathbb{C}[x]$. Then the ideal corresponding to E^K is (x), and the ideal for ${E'}^K$ is the same. Since ϕ, hence ϕ_H, maps E^K to ${E'}^K$ isomorphically, the ideal (x) must be fixed by the algebra isomorphism corresponding to ϕ_H. This implies that ϕ_H, hence ϕ, induces an isomorphism $\bar{c}$ on P^H such that $p'_H \circ \phi_H = \bar{c} \circ p_H$, where $p_H : E^H \to P^H$ and $p'_H : {E'}^H \to P^H$ are projections. Note that $\bar{c}$ is a scalar multiplication on P^H. Hence $\bar{c}$ extends to a scalar multiplication c on P. Since the following diagram commutes, ϕ restricts to a variety isomorphism $E|_{P^H} \cong E'|_{P^H}$:

$$\begin{array}{ccccccc}
E & & & \xrightarrow{\quad\phi\quad} & & & E' \\
 & \nwarrow & & & & \nearrow & \\
\Big\downarrow & & E^H & \overset{\phi_H}{\to} & {E'}^H & & \Big\downarrow \\
 & & \downarrow & & \downarrow & & \\
 & & P^H & \underset{\bar{c}}{\to} & P^H & & \\
 & \swarrow & & & & \searrow & \\
P & & & \xrightarrow[\quad c\quad]{} & & & P
\end{array}$$

where the diagonal arrows are inclusions. Furthermore, since ϕ is a G-isomorphism, ϕ in fact restricts to a G-isomorphism $\phi_X : E|_X \to E'|_X$ such that $p'_X \circ \phi_X = (c|_X) \circ p_X$, where $p_X : E|_X \to X$ and $p'_X : E'|_X \to X$ are projections. Thus $E|_X \cong (c^*E')|_X$ as G-vector bundles, and the assertion follows. □

Proposition 2.3 enables us to classify elements of $\mathrm{VAR}_G(P, Q)$.

Corollary 2.4. *Under the assumption and notation in Proposition 2.3, there exists a map*

$$\Phi : \mathrm{VAR}_G(P, Q) \to \mathrm{Vec}_G(X, Q)/\mathbb{C}^*,$$

where the target space is the orbit space of $\mathrm{Vec}_G(X, Q)$ *under the action of* $\mathbb{C}^*$*, which is a subgroup of* $\mathrm{Aut}(X)^G$ *consisting of scalar multiplications.*

When H is an isotropy group of a point $x \in P$ whose orbit is closed, then $X = \overline{G \cdot P^H}$ is a G-equivariant affine cone in P with $\dim X/\!/G = 1$. In this case, $\mathrm{Vec}_G(X, Q)$ is isomorphic to a finite-dimensional module of $\mathbb{C}^* \subset \mathrm{Aut}(X)^G$ ([18]). Hence $\mathrm{Vec}_G(X, Q)/\mathbb{C}^*$ is isomorphic to a weighted projective space with a "vertex".

Example 2.4. Let $G = O(2)$ and consider $\mathrm{Vec}_{O(2)}(V_1, V_m)$. Then applying Proposition 2.3 for $H = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and $K = \mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, we obtain $X = V_1$, and hence, a map $\mathrm{VAR}_{O(2)}(V_1, V_m) \to \mathrm{Vec}_{O(2)}(V_1, V_m)/\mathbb{C}^*$, which is an isomorphism. Since $\mathrm{Vec}_{O(2)}(V_1, V_m) \cong \bigoplus_{i=1}^{m-1} W(2i)$ ([7]), we have (cf. [16], [17])

$$\begin{aligned}\mathrm{VAR}_{O(2)}(V_1, V_m) &\cong (\bigoplus_{i=1}^{m-1} W(2i))/\mathbb{C}^* \\ &= \mathbb{P}_*(2i; 1 \leq i \leq m-1).\end{aligned}$$

We apply Proposition 2.3 and its corollary for dihedral groups and classify non-linearizable actions of dihedral groups in the next section.

3. Non-linearizable actions of dihedral groups

In this section, we investigate non-linearizable actions of dihedral groups. Let G be a dihedral group $D_n = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. By considering D_n as a finite subgroup of $O(2) = \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$, an $O(2)$-module V_m is naturally considered as a D_n-module. Since $V_m \cong V_{|m-n|}$ as a D_n-module, we may assume $m \leq n/2$; otherwise $m = n$. Let k be a positive integer such that $(k, n) = 1$ and $k \leq n/2$. Let $\{x, y\}$ be a coordinate system of V_k as in Example 2.1. Then $V_k//D_n = \operatorname{Spec}\mathbb{C}[t, u]$, where $t = xy$ and $u = x^n + y^n$. Let $X_k = D_n \cdot V_k^{\mathbb{Z}/2\mathbb{Z}}$, where $\mathbb{Z}/2\mathbb{Z}$ is the reflection subgroup. Then X_k is the D_n-subvariety of V_k defined by $x^n - y^n = 0$ for n odd, and $x^{n/2} - y^{n/2} = 0$ for n even. The algebraic quotient of X_k is

$$X_k//D_n = \begin{cases} \operatorname{Spec}\mathbb{C}[t, u]/(u^2 - 4t^n) & \text{for } n \text{ odd,} \\ \operatorname{Spec}\mathbb{C}[t] & \text{for } n \text{ even.} \end{cases}$$

The variety X_k is the D_n-equivariant affine cone in V_k with one-dimensional quotient. Hence $\mathrm{Vec}_{D_n}(X_k, V_m) \cong \mathbb{C}^q$ for some q ([18], [8]).

We shall classify elements of $\mathrm{VAR}_{D_n}(V_k, V_m)$ under a certain condition.

Proposition 3.1. *Let $E, E' \in \mathrm{Vec}_{D_n}(V_k, V_m)$ and let X_k be as above. Suppose that $(m, n) > 1$. Then, if $E \cong E'$ as D_n-varieties, then the restricted bundles $E|_{X_k}$ and $(c^*E')|_{X_k}$ are isomorphic as D_n-vector bundles, where c is a scalar multiplication on V_k.*

Proof. By taking $H = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and $K = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where $p = (m, n)$ in Proposition 2.3, the assertion follows. □

Under the assumption in Proposition 3.1, there exists a map

$$\Phi_{k,m} : \mathrm{VAR}_{D_n}(V_k, V_m) \to \mathrm{Vec}_{D_n}(X_k, V_m)/\mathbb{C}^*.$$

Let $i_k^* : \mathrm{Vec}_{D_n}(V_k, V_m) \to \mathrm{Vec}_{D_n}(X_k, V_m)$ be the restriction induced by the inclusion $i_k : X_k \hookrightarrow V_k$. There exists a sequence

$$\mathrm{Vec}_{O(2)}(V_k, V_m) \xrightarrow{d_n} \mathrm{Vec}_{D_n}(V_k, V_m) \xrightarrow{i_k^*} \mathrm{Vec}_{D_n}(X_k, V_m),$$

where d_n is the group restriction.

Theorem 3.2 (cf. [16])**.** *Let n be odd, and let $k = 2$ and $m = n$ in the notation above.*

(1) *The composite map $i_2^* \circ d_n : \mathrm{Vec}_{O(2)}(V_2, V_n) \to \mathrm{Vec}_{D_n}(X_2, V_n)$ is injective and*

$$\mathrm{Im}(i_2^* \circ d_n) \cong \mathbb{C}^{\frac{n-1}{2}}.$$

(2) *The image of $\Phi_{2,n}$ is isomorphic to $\mathbb{P}_*(2i - 1; 1 \leq i \leq (n-1)/2)$.*

(3) *The map* $\mathrm{VAR}_{O(2)}(V_2, V_n) \to \mathrm{VAR}_{D_n}(V_2, V_n)$ *is injective. Hence, if* $E \in \mathrm{Vec}_{O(2)}(V_2, V_n)$ *is a non-trivial* $O(2)$*-vector bundle, then the* D_n*-action on* E *is non-linearizable.*

Proof. (1) By applying the method of Mederer, we can show that $\mathrm{Vec}_{D_n}(X_2, V_n)$ is isomorphic to a vector group $\mathbb{C}^{n-1}$. For the detailed argument, we refer to Mederer [18]. We shall give a basis of $\mathrm{Vec}_{D_n}(X_2, V_n) \cong \mathbb{C}^{n-1}$. We use the notation in Example 2.1 and denote X_2 simply by X. Let $\nu : \mathbb{B} = \operatorname{Spec}\mathbb{C}[s] \to X//D_n = \operatorname{Spec}\mathbb{C}[t,u]/(u^2 - 4t^n)$ be the normalization, where $t = s^2$ and $u = 2s^n$, and let $F_X = \pi_X^{-1}(\nu(1))$. Then $F_X \cong D_n/H'$, where $H' = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and V_n is multiplicity free with respect to H'. There is a D_n-equivariant morphism

$$\begin{array}{rcl} \varphi^X : \mathbb{B} \times F_X & \to & X, \\ (b, f) & \mapsto & bf, \end{array}$$

which is an isomorphism from $(\mathbb{B} - \{0\}) \times F_X$ onto $X - \pi_X^{-1}(\nu(0))$. Note that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{B} \times F_X & \xrightarrow{\varphi^X} & X \\ \big\downarrow & & \big\downarrow \\ \mathbb{B} \times_\Gamma F & \xrightarrow[\varphi]{} & V_2 \end{array}$$

where the vertical maps are inclusions. Let $\mathfrak{m}_X = \mathrm{Mor}(F_X, \operatorname{End} V_n)^{D_n}$. Then $\mathfrak{m}_X(\mathbb{B}) := \mathrm{Mor}(\mathbb{B}, \mathfrak{m}_X)$ is a free $\mathbb{C}[s]$-module with a grading induced from $\mathbb{C}[s]$. The $\mathbb{C}[X]^{D_n}$-module $\mathrm{Mor}(X, \operatorname{End} V_n)^{D_n} \cong (\mathbb{C}[X] \otimes \operatorname{End} V_n)^{D_n}$ inherits a grading from $\mathbb{C}[X] \subset \mathbb{C}[V_2]$. Note that $\mathfrak{m}_X(\mathbb{B})$ is considered as a $\mathbb{C}[X]^{D_n}$-module via ν. The morphism φ^X induces

$$\varphi^X_\# : \mathrm{Mor}(X, \operatorname{End} V_n)^{D_n} \to \mathrm{Mor}(\mathbb{B} \times F_X, \operatorname{End} V_n)^{D_n} = \mathfrak{m}_X(\mathbb{B}),$$

which is a $\mathbb{C}[X]^{D_n}$-homomorphism of degree 0. Note that $\mathrm{Mor}(X, \operatorname{End} V_n)^{D_n}$ is a finite free module over $\mathbb{C}[X]^{D_n}$. In fact, $\mathrm{Mor}(V_2, \operatorname{End} V_n)^{D_n}$ is free over $\mathbb{C}[t,u]$ with a basis

$$\left\{ \bar{A}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \bar{A}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} x^n - y^n & 0 \\ 0 & -(x^n - y^n) \end{pmatrix}, \qquad \begin{pmatrix} 0 & x^n - y^n \\ -(x^n - y^n) & 0 \end{pmatrix} \right\}.$$

Hence $\mathrm{Mor}(X, \operatorname{End} V_n)^{D_n}$ is a free module over $\mathbb{C}[t,u]/(u^2 - 4t^n)$ with a basis $\{\bar{A}_0, \bar{A}_1\}$. Let $\mathrm{Mor}(X, \operatorname{End} V_n)_1^{D_n}$ (respectively $\mathfrak{m}_X(\mathbb{B})_1$) be the submodule of $\mathrm{Mor}(X, \operatorname{End} V_n)^{D_n}$ (respectively $\mathfrak{m}_X(\mathbb{B})$) of elements with positive degrees. Then $\mathrm{Vec}_{D_n}(X, V_n)$ is isomorphic to the quotient module of $\mathfrak{m}_X(\mathbb{B})_1$ by

$$\varphi^X_\# \mathrm{Mor}(X, \operatorname{End} V_n)_1^{D_n}.$$

The module $\mathfrak{m}_X(\mathbb{B})_1$ is free over $\mathbb{C}[s]$ with a basis

$$\left\{ \bar{C}_0 = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \bar{C}_1 = s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Since $\varphi^X_\#(t\bar{A}_i) = s\bar{C}_i$ and $\varphi^X_\#(u\bar{A}_i) = 2s^{n-1}\bar{C}_i$ for $i = 0, 1$,

$$\mathrm{Vec}_{D_n}(X, V_n) \cong \mathfrak{m}_X(\mathbb{B})_1 / \varphi^X_\# \mathrm{Mor}(X, \operatorname{End} V_n)_1^{D_n} \cong \mathbb{C}^{n-1}$$

with a basis $\{s^{2(j-1)}\bar{C}_i; i=0,1, 1 \le j \le (n-1)/2\}$. The inclusions $\mathbb{B}\times F_X \hookrightarrow \mathbb{B}\times_\Gamma F$ and $X \hookrightarrow V_2$ give rise to a homomorphism

$$\iota : \mathfrak{m}(\mathbb{B})_1^\Gamma/\varphi_\# \operatorname{Mor}(V_2, \operatorname{End} V_n)_1^{O(2)} \to \mathfrak{m}_X(\mathbb{B})_1/\varphi_\#^X \operatorname{Mor}(X, \operatorname{End} V_n)_1^{D_n},$$

which corresponds to $i_2^* \circ d_n$. Since $\iota(t^{i-1}C_1) = s^{2(i-1)}\bar{C}_1$, it follows that $i_2^* \circ d_n$ is injective and $\operatorname{Im}(i_2^* \circ d_n) \cong \mathbb{C}^{(n-1)/2}$ with a basis $\{s^{2(j-1)}\bar{C}_1; 1 \le j \le (n-1)/2\}$.

(2) From Proposition 3.1, there is a map

$$\Phi_{2,n} : \mathrm{VAR}_{D_n}(V_2, V_n) \to \mathrm{Vec}_{D_n}(X_2, V_n)/\mathbb{C}^*.$$

From (1), $\operatorname{Im} i_2^*$ contains a subspace

$$\bigoplus_{i=1}^{(n-1)/2} W(2i-1).$$

In fact, $\operatorname{Im} i_2^* \cong \bigoplus_{i=1}^{(n-1)/2} W(2i-1)$ (cf. [18, III 3,4]). Hence the assertion follows.

(3) follows from (1) and Proposition 3.1. □

Remark. From Theorem 3.2 (1), $d_n : \mathrm{Vec}_{O(2)}(V_2, V_n) \to \mathrm{Vec}_{D_n}(V_2, V_n)$ is an injection.

Let ε be the 1-dimensional sign representation and let ε^m be the direct sum of m copies of ε. One can show by direct calculation that the composite map $\widetilde{i_2^*} \circ \widetilde{d_n}$ given by

$$\begin{aligned} \mathrm{Vec}_{O(2)}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \ &\overset{\widetilde{d_n}}{\to}\ \mathrm{Vec}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \\ &\overset{\widetilde{i_2^*}}{\to}\ \mathrm{Vec}_{D_n}(X_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \end{aligned}$$

is an injection. In fact, since the dimensions of $V_2//O(2)$ and $X_2//D_n$ are both equal to 1, the map $\widetilde{i_2^*} \circ \widetilde{d_n}$ is a homomorphism of $\mathbb{C}$-vector groups. Since the generators of the $\mathbb{C}$-vector group $\mathrm{Vec}_{O(2)}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})$, which is isomorphic to $\mathrm{Vec}_{O(2)}(V_2, V_n)$, do not vanish by the homomorphism $\widetilde{i_2^*} \circ \widetilde{d_n}$ (cf. [7, VII 4], [18, III 5]), so $\widetilde{i_2^*} \circ \widetilde{d_n}$ is injective. The map

$$\theta_2 : \mathrm{Vec}_{D_n}(V_2, V_n) \to \mathrm{Vec}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})$$

sending $[E]$ to $[E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}]$ induces a map

$$\mathrm{VAR}_{D_n}(V_2, V_n) \to \mathrm{VAR}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})$$

which is the product map with $\mathbb{C}^{m_1} \times \varepsilon^{m_2}$.

Theorem 3.3. *Let n be odd and let m_1 and m_2 be non-negative integers. Then the map*

$$\begin{aligned} \mathrm{VAR}_{O(2)}(V_2, V_n) \ &\to\ \mathrm{VAR}_{D_n}(V_2, V_n) \\ &\to\ \mathrm{VAR}_{D_n}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \end{aligned}$$

induced by $\theta_2 \circ d_n$ is an injection.

Proof. Let $E, E' \in \mathrm{Vec}_{O(2)}(V_2, V_n)$ be such that $E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2} \cong E' \times \mathbb{C}^{m_1} \times \varepsilon^{m_2}$ as D_n-varieties. Then applying Proposition 2.3 to $E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}$ and $E' \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}$ with $H = \mathbb{Z}/2\mathbb{Z}$ (the reflection subgroup) and $K = D_n$, we have

$$(E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}})|_{X_2} \cong (c^*E' \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}})|_{X_2}$$

as D_n-vector bundles, where c is a scalar multiplication of V_2. Since $\widetilde{i_2^*} \circ \widetilde{d_n}$ is injective, $E \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}} \cong c^*E' \oplus \Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}$ as $O(2)$-vector bundles. Since the Whitney sum with $\Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}$ induces an isomorphism

$$\mathrm{Vec}_{O(2)}(V_2, V_n) \cong \mathrm{Vec}_{O(2)}(V_2, V_n \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}),$$

it follows that $E \cong c^*E'$ as $O(2)$-vector bundles, and the assertion follows. □

Remark. One of the first examples of non-linearizable actions by Schwarz is the $O(2)$-action on the total space of the non-trivial $E \in \mathrm{Vec}_{O(2)}(V_2, V_3)$. By Theorem 3.2 (3), the action of D_3 on E is non-linearizable. Furthermore, by Theorem 3.3, the D_3-action on $E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2}$ remains non-linearizable (cf. [3]). Since the map $\mathrm{Vec}_{O(2)}(V_2, V_n) \to \mathrm{Vec}_{O(2)}(V_2, V_n \oplus V_1)$ sending $[E]$ to $[E \oplus \Theta_{V_1}]$ is trivial [20], the D_3-action on $E \times V_1$ is linearizable.

By a method similar to the proof of Theorem 3.2, we can show the following.

Theorem 3.4 (cf. [16])**.** *Let m and n be even and $m \leq n/4$.*

(1) *The composite map $i_1^* \circ d_n : \mathrm{Vec}_{O(2)}(V_1, V_m) \to \mathrm{Vec}_{D_n}(X_1, V_m)$ is an isomorphism. Hence, $d_n : \mathrm{Vec}_{O(2)}(V_1, V_m) \to \mathrm{Vec}_{D_n}(V_1, V_m)$ is injective and $i_1^* : \mathrm{Vec}_{D_n}(V_1, V_m) \to \mathrm{Vec}_{D_n}(X_1, V_m)$ is surjective.*

(2) *The map*

$$\Phi_{1,m} : \mathrm{VAR}_{D_n}(V_1, V_m) \to \mathbb{P}_*(2i; 1 \leq i \leq m-1)$$

is surjective.

(3) *The map $\mathrm{VAR}_{O(2)}(V_1, V_m) \to \mathrm{VAR}_{D_n}(V_1, V_m)$ is injective. Hence, if $E \in \mathrm{Vec}_{O(2)}(V_1, V_m)$ is a non-trivial $O(2)$-vector bundle, then the D_n-action on E is non-linearizable.*

Proof. (1) By [20], $\mathrm{Vec}_{O(2)}(V_1, V_m) \cong \mathbb{C}^{m-1}$ and by [8], $\mathrm{Vec}_{D_n}(X_1, V_m) \cong \mathbb{C}^{m-1}$. We can show that $i_1^* \circ d_n$ is an isomorphism directly as in the proof of Theorem 3.2 (1).

(2) By [8], $\mathrm{Vec}_{D_n}(X_1, V_m) \cong \bigoplus_{i=1}^{m-1} W(2i)$. From this together with (1), the assertion follows.

(3) follows from (1) and Proposition 3.1. □

Remarks. (1) When m and n are even and $n/4 < m < n/2$, one can show that $\mathrm{Vec}_{D_n}(X_1, V_m) \cong \bigoplus_{i=1}^{n/2-m-1} W(2i)$ ([8]), and $i_1^* \circ d_n$ is a surjection. Hence $\Phi_{1,m}$ is a surjection from $\mathrm{VAR}_{D_n}(V_1, V_m)$ onto $\mathbb{P}_*(2i; 1 \leq i \leq n/2 - m - 1)$.

(2) When n is even, the Whitney sum maps

$$\mathrm{Vec}_{O(2)}(V_1, V_m) \to \mathrm{Vec}_{O(2)}(V_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})$$

and

$$\mathrm{Vec}_{D_n}(X_1, V_m) \to \mathrm{Vec}_{D_n}(X_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2})$$

are trivial (cf. [20], [8]).

(3) Suppose n is odd. Then the map

$$i_1^* \circ d_n : \mathrm{Vec}_{O(2)}(V_1, V_m) \to \mathrm{Vec}_{D_n}(X_1, V_m)$$

is injective and

$$\mathrm{Im}(i_1^* \circ d_n) \cong \bigoplus_{i=1}^{m-1} W(2i)$$

(cf. [18]). Hence, when $(m,n) > 1$, $\mathrm{VAR}_{O(2)}(V_1, V_m) \to \mathrm{VAR}_{D_n}(V_1, V_m)$ is injective.

Consider the commutative diagram for n odd:

$$\begin{array}{ccc} \mathrm{Vec}_{D_n}(V_1, V_m) & \xrightarrow{i_1^*} & \mathrm{Vec}_{D_n}(X_1, V_m) \\ \downarrow{\scriptstyle \theta_1} & & \downarrow \\ \mathrm{Vec}_{D_n}(V_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) & \xrightarrow[\widetilde{i_1^*}]{} & \mathrm{Vec}_{D_n}(X_1, V_m \oplus \mathbb{C}^{m_1} \oplus \varepsilon^{m_2}) \end{array}$$

where the vertical maps are the Whitney sum maps with $\Theta_{\mathbb{C}^{m_1} \oplus \varepsilon^{m_2}}$. By [18],

$$\mathrm{Im}\, i_1^* \cong (\bigoplus_{i=1}^{2m-1} W(i)) \oplus (\bigoplus_{i=1}^{(n-1)/2-2m} W(2m-1+2i))$$

for $m < n/4$,

$$\mathrm{Im}\, i_1^* \cong (\bigoplus_{i=1}^{n-2m-1} W(i)) \oplus (\bigoplus_{i=1}^{2m-(n+1)/2} W(n-2m-1+2i))$$

for $n/4 < m < n/2$, and

$$\mathrm{Im}(\widetilde{i_1^*} \circ \theta_1) \cong \bigoplus_{i=1}^{(n-2m-1)/2} W(2i-1).$$

Hence we obtain the following by applying Proposition 2.3.

Theorem 3.5. *Suppose that n is odd and $(m,n) > 1$. Then the image of $\Phi_{1,m}$ is isomorphic to the weighted projective space $\mathbb{P}_*((n-5)/2)$ with a vertex. The space $\mathbb{P}_*((n-5)/2)$ is of dimension $(n-5)/2$ and contains the weighted projective space $\mathbb{P}(2i-1; 1 \le i \le (n-2m-1)/2)$ whose inverse image under $\Phi_{1,m}$ consists of elements E such that the D_n-action on $E \times \mathbb{C}^{m_1} \times \varepsilon^{m_2}$ is non-linearizable.*

Remark. Mederer [18] showed that $\mathrm{Vec}_{D_3}(V_1, V_1) \cong \Omega_{\mathbb{C}}$, the module of Kähler differentials of $\mathbb{C}$ over $\mathbb{Q}$, and furthermore, there is a surjection from $\mathrm{Ker}\, i_1^*$ in the above diagram for $n \ge 5$ to $\mathrm{Vec}_{D_3}(V_1, V_1)$. Hence $\mathrm{Vec}_{D_n}(V_1, V_m)$ (n odd; $n \ge 5$) contains a space of uncountably-infinite dimension. Proposition 2.3 is, to our regret, not useful for classifying the D_n-actions derived from $\mathrm{Ker}\, i_k^*$ or $\mathrm{Vec}_{D_3}(V_1, V_1)$.

Suppose n is odd, and classify the D_n-actions derived from $\mathrm{Vec}_{D_n}(V_2 \oplus \varepsilon^m, V_n)$. By applying Proposition 2.3 for $H = \mathbb{Z}/2\mathbb{Z}$ and $K = D_n$, we obtain a surjection from $\mathrm{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n)$ to the orbit space of $\mathrm{Im}\, i_{2,m}^*$ under the action of $\mathbb{C}^*$, where $i_{2,m}^* : \mathrm{Vec}_{D_n}(V_2 \oplus \varepsilon^m, V_n) \to \mathrm{Vec}_{D_n}(X_2, V_n)$ is the restriction induced by $i_{2,m} : X_2 \hookrightarrow V_2 \oplus \varepsilon^m$. Let $i_m : V_2 \to V_2 \oplus \varepsilon^m$ be the inclusion. Then $i_{2,m}^* = i_2^* \circ i_m^*$. Since i_m^* is a surjection, $\mathrm{Im}\, i_{2,m}^* = \mathrm{Im}\, i_2^*$. Since $\mathrm{Im}\, i_2^* \cong \bigoplus_{i=1}^{(n-1)/2} W(2i-1)$ (cf. the proof of Theorem 3.2 (2)), we have a surjection

$$\mathrm{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n) \to \mathbb{P}_*(2i-1; 1 \le i \le (n-1)/2).$$

Theorem 3.6. *Let m be a non-negative integer and let n be odd. Then there is a surjection from $\mathrm{VAR}_{D_n}(V_2 \oplus \varepsilon^m, V_n)$ onto $\mathbb{P}_*(2i-1; 1 \le i \le (n-1)/2)$.*

Remark. Let l be a non-negative integer and let $(m,n) > 1$. Then one obtains a similar result for $\mathrm{VAR}_{D_n}(V_1 \oplus \varepsilon^l, V_m)$.

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