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DUAL RADON TRANSFORMS ON AFFINE GRASSMANN MANIFOLDS

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ABSTRACT. Fix $0 \leq p < q \leq n-1$, and let $G(p,n)$ and $G(q,n)$ denote the affine Grassmann manifolds of p- and q-planes in $\mathbb{R}^n$. We investigate the Radon transform $\mathcal{R}^{(q,p)} : C^\infty(G(q,n)) \to C^\infty(G(p,n))$ associated with the inclusion incidence relation. For the generic case $\dim G(q,n) < \dim G(p,n)$ and $p+q > n$, we will show that the range of this transform is given by smooth functions on $G(p,n)$ annihilated by a system of Pfaffian type differential operators. We also study aspects of the exceptional case $p+q=n$.

1. Introduction

In this paper, we continue our investigation of Radon transforms on affine Grassmann manifolds begun in our previous paper [GK]. One of the ways in which the classical k-plane transform and its dual on $\mathbb{R}^n$ can be generalized is by considering integral transforms on the Grassmannians $G(k,n)$ of unoriented affine k-planes on $\mathbb{R}^n$ with respect to the inclusion incidence relation. As well, the integral transforms which are studied here and in [GK] can be considered as an affine analogue of the Radon transforms on compact Grassmannians considered in Grinberg [Gr], Grinberg and Rubin [GrRu], and Kakehi [K].

Specifically, let us assume that $0 \leq p < q \leq n-1$. Fix a p-plane ℓ_0 and a q-plane ξ_0 on $\mathbb{R}^n$ such that $\ell_0 \subset \xi_0$, and let H_p and H_q be the subgroups of the Euclidean motion group $M(n)$ fixing ℓ_0 and ξ_0, respectively. Then $G(p,n) = M(n)/H_p$ and $G(q,n) = M(n)/H_q$ are homogeneous spaces in duality in the sense defined by Helgason [H4]. Under this duality, elements $\ell \in G(p,n)$ and $\xi \in G(q,n)$ are incident if $\ell \subset \xi$. Let $\mathcal{R}^{(p,q)} : C_c^\infty(G(p,n)) \longrightarrow C^\infty(G(q,n))$ and $\mathcal{R}^{(q,p)} : C^\infty(G(q,n)) \longrightarrow C^\infty(G(p,n))$ be the corresponding integral transforms.

In our previous paper, we examined the transform $\mathcal{R}^{(p,q)}$. When $p=0$, this transform reduces to the classical q-plane transform on $\mathbb{R}^n$. We obtained an explicit inversion formula ([GK], Theorem 6.4) and a range characterization theorem ([GK], Theorem 7.7) for the Radon transform $\mathcal{R}^{(p,q)}$ on a naturally defined Schwartz class $\mathcal{S}(G(p,n))$. Both theorems rely on an extension of the "projection-slice" theorem to affine Grassmannians, with respect to taking Fourier transforms on their fibers. Our results thus generalize the main range and inversion theorems in Richter [Ri], Gonzalez [G3] and Helgason [H1].

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Along the way we derived some useful facts about the algebras of invariant differential operators on both affine and compact Grassmannians. (These are restated in Theorems 3.1 and 4.1 below.) In both cases, these algebras are generated by Pfaffian-type operators; these operators are used in the inversion and range characterization of $\mathcal{R}^{(p,q)}$.

In the present paper, we examine the dual transform $\mathcal{R}^{(q,p)}$. When $\dim G(q,n) < \dim G(p,n)$ and $p+q>n$, we obtain a range characterization in terms of Pfaffian operators similar to the ones mentioned above with some significant differences. (See Remark 3 after Theorem 5.1.) The case when $p+q=n$ presents some interesting technical issues, and we treat this case in Section 5.

2. Preliminaries

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Denote the universal enveloping algebra of $\mathfrak{g}$ by $\mathfrak{u}(\mathfrak{g})$, and let $\mathfrak{z}(\mathfrak{g})$ denote the subalgebra of $Ad(G)$-invariant elements in $\mathfrak{u}(\mathfrak{g})$. We call $\mathfrak{z}(\mathfrak{g})$ the Casimir algebra of $\mathfrak{g}$. If λ is any representation of G on a topological vector space V, we let $d\lambda$ denote the associated representation of $\mathfrak{u}(\mathfrak{g})$ on the subspace V^∞ of smooth vectors in V. As an example, suppose that $(g,x)\to g\cdot x$ is a smooth G-action on a manifold M. Let λ be the associated left regular representation of G on $V=C^\infty(M)$. If $X\in\mathfrak{g}$, $d\lambda(X)$ is the vector field tangent to the curve $t\to\exp(-tX)\cdot x$. For $X_1,\cdots,X_m\in\mathfrak{g}$, we then have

$$\begin{aligned} &d\lambda(X_1\cdots X_m)f(x) \\ &\quad= \left\{\frac{\partial^m}{\partial t_1\cdots\partial t_m} f(\,\exp(-t_mX_m)\cdots\exp(-t_1X_1)\cdot x\,)\right\}_{t_1=\cdots=t_m=0}. \end{aligned} \tag{2.1}$$

Since $\lambda(g)\circ d\lambda(U)\circ\lambda(g^{-1}) = d\lambda(Ad(g)U)$ for all $g\in G$ and $U\in\mathfrak{u}(\mathfrak{g})$, we see that $d\lambda(\mathfrak{z}(\mathfrak{g}))$ consists of G-invariant differential operators on M.

Suppose that λ and ν are representations of G on topological vector spaces V and W, respectively. Let $R: V\longrightarrow W$ be a continuous linear map intertwining λ and ν. If V^∞ and W^∞ denote the subspaces of C^∞ vectors in V and W, respectively, then $R(V^\infty)\subset W^\infty$ and $R(d\lambda(U)v) = d\nu(U)Rv$ for all $U\in\mathfrak{u}(\mathfrak{g})$, $v\in V^\infty$. In other words, the diagram below commutes:

$$\begin{array}{ccc} V^\infty & \xrightarrow{R} & W^\infty \\ {\scriptstyle d\lambda(U)}\big\downarrow & & \big\downarrow{\scriptstyle d\nu(U)} \\ V^\infty & \xrightarrow[R]{} & W^\infty. \end{array} \tag{2.2}$$

In particular, if R is injective, then, $\ker(d\nu)\subset\ker(d\lambda)$. In case $\ker(d\nu)\subsetneqq\ker(d\lambda)$, we have necessary conditions satisfied by the range $R(V^\infty)\ :\ d\nu(U)Rv=0$ for all $v\in V^\infty, U\in\ker(d\lambda)\setminus\ker(d\nu)$.

In this paper, λ and ν will denote the left regular representations of a given Lie group G on a various smooth function spaces on homogeneous manifolds of G. More specifically, we will consider homogeneous spaces in duality, as follows. Let H and K be closed subgroups for G satisfying the conditions. (See [H2], page 140.)

(i) G, H, K, and $H\cap K$ are all unimodular.
(ii) HK is closed in G.
(iii) $H\cap K = \{\,h\in H\,|\,hK\cup h^{-1}K\subset KH\,\} = \{\,k\in K\,|\,kH\cup k^{-1}H\subset HK\,\}$.

The coset spaces $X = G/K$ and $\Xi = G/H$ are then said to be *homogeneous in duality*, and we have the following double fibration

$$\begin{tikzcd} & G/(H \cap K) \arrow[dl] \arrow[dr] & \\ X = G/K & & G/H = \Xi. \end{tikzcd} \tag{2.3}$$

The corresponding Radon transform $R : C_c^\infty(X) \longrightarrow C^\infty(\Xi)$ and its dual Radon transform ${}^tR : C_c^\infty(\Xi) \longrightarrow C^\infty(X)$ given respectively by

$$Rf(\gamma H) = \int_{H/H\cap K} f(\gamma hK)\, dh_{H\cap K}, \tag{2.4}$$

$${}^tR\varphi(gK) = \int_{K/H\cap K} f(gkH)\, dk_{H\cap K}. \tag{2.5}$$

Elements $x = gK \in X$ and $\xi = \gamma H \in \Xi$ are said to be *incident* if, as cosets of G, they have a nonempty intersection. The set of all $x \in X$ incident to ξ is easily seen to be $\widehat{\xi} = \{\, \gamma hK \,|\, h \in H\,\}$; similarly, the set of all $\xi \in \Xi$ incident to x is $\check{x} = \{\, gkH \,|\, k \in K\,\}$. The integral for $Rf(\xi)$ is thus taken over $\widehat{\xi}$ and that for ${}^tR\varphi(x)$ is taken over $\check{x}$.

For any $g \in G$, let $\tau(g)$ denote the left translation by g on X and Ξ, so that $\tau(g)x = g\cdot x, \tau(g)\xi = g\cdot\xi$. The *translate* $f^{\tau(g)}$ of a function f on X is then given by $f^{\tau(g)}(x) = f(g^{-1}\cdot x)$. For a function φ on Ξ, we define $\varphi^{\tau(g)}$ similarly. If D is a differential operator on X, its *translate* $D^{\tau(g)}$ is given by $D^{\tau(g)}f = (\,Df^{\tau(g^{-1})}\,)^{\tau(g)}$.

Let λ and ν denote the left regular representations of G on $C^\infty(X)$ and $C^\infty(\Xi)$, respectively, so that $\lambda(g)f = f^{\tau(g)}, \nu(g)\varphi = \varphi^{\tau(g)}$. The integral for R and ${}^tR$ above show that these transforms intertwine λ and ν. Thus $R(d\lambda(U)f) = d\nu(U)Rf$ for all $f \in C_c^\infty(X)$; as stated before, if R is injective, then $\ker(d\nu) \subset \ker(d\lambda)$ and we obtain differential equations satisfied by the range of R in case $\ker(d\nu) \subsetneqq \ker(d\lambda)$. It turns out that these equations are often sufficient to characterize the range as well.

If X is a homogeneous space of a compact Lie group, then any integral on X will be taken with respect to the unique normalized G-invariant measure on X.

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3. Pfaffian algebra and Radon transforms on compact Grassmann manifolds

Fix an integer n, $n \geq 3$. For any integer r in $\{1, 2, \cdots, n\}$, let P_r denote the set of all permutations of $\{1, 2, \cdots, n\}$ taken r at a time, and let T_r denote the collection of all increasing sequences $J = \{\, j_1, j_2, \cdots, j_r\,\}$ of length r in $\{1, 2, \cdots, n\}$. Then, of course, $T_r \subset P_r$. For $J \in T_r$, let $\mathfrak{S}(J)$ denote the set of all permutations of J (written as ordered r-tuples), so that $P_r = \bigcup_{J\in T_r} \mathfrak{S}(J)$. If $J' \in \mathfrak{S}(J)$, we denote its sign by $\epsilon(J')$. Let $A = (\,a_{ij}\,)$ be any $n \times n$ matrix with entries in some ring. If $I = \{\, i_1, i_2, \cdots, i_r\,\}$ and $J = \{\, j_1, j_2, \cdots, j_r\,\}$ belong to T_r, we let A_{IJ} denote the $r \times r$ submatrix $(\,a_{i_k\, j_l}\,)_{1\leq k,l\leq r}$.

The Lie algebra $so(n)$ of the special orthogonal group $SO(n)$ has basis given by $X_{ij} = E_{ij} - E_{ji}$ $(1 \leq i < j \leq n)$. Let X be the $n \times n$ matrix with vector entries

$$X = \begin{pmatrix} 0 & X_{12} & \cdots & X_{1n} \\ -X_{12} & 0 & \cdots & X_{2n} \\ & \cdots & \ddots & \\ -X_{1n} & -X_{2n} & \cdots & 0 \end{pmatrix}. \tag{3.1}$$

Fix an integer k, $0 \leq k \leq [\frac{n}{2}]$, and let $I \in T_{2k}, I = \{ i_1, i_2, \cdots, i_{2k} \}$. Let X_I denote the $2k \times 2k$ submatrix of X given by

$$X_I = \begin{pmatrix} 0 & X_{i_1 i_2} & \cdots & X_{i_1 i_{2k}} \\ -X_{i_1 i_2} & 0 & \cdots & X_{i_2 i_{2k}} \\ & \cdots & \ddots & \\ -X_{i_1 i_{2k}} & -X_{i_2 i_{2k}} & \cdots & 0 \end{pmatrix}, \tag{3.2}$$

and set

$$W_I = \sum_\sigma \epsilon(\sigma)\, X_{\sigma(i_1)\sigma(i_2)} \cdots X_{\sigma(i_{2k-1})\sigma(i_{2k})} \in \mathfrak{u}(so(n)), \tag{3.3}$$

where the sum is taken over all permutations σ of I such that $\sigma(i_1) < \sigma(i_2), \cdots$, $\sigma(i_{2k-1}) < \sigma(i_{2k})$ and $\sigma(i_1) < \sigma(i_3) < \cdots < \sigma(i_{2k-1})$. Thus W_I equals the Pfaffian $\mathrm{Pf}(X_I)$. Note that the factors in each of the summands in (3.3) commute.

Note that $\mathrm{rank}(SO(n)) = [\frac{n}{2}]$. For each integer k, $1 \leq k < [\frac{n}{2}]$, let us set

$$U_{2k} = \sum_{I \in T_{2k}} {W_I}^2; \tag{3.4}$$

in addition, let us set

$$U_{n-1} = \sum_{I \in T_{n-1}} {W_I}^2, \qquad \text{if } n \text{ is odd}, \tag{3.5}$$

$$U_n = W_{\{1,2,\cdots,n\}} \qquad \text{if } n \text{ is even}. \tag{3.6}$$

Theorem 3.1. *The elements $U_2, U_4, \cdots, U_{2[\frac{n}{2}]}$ are algebraically independent generators of the Casimir algebra $\mathfrak{z}(so(n))$.*

(See, for example, [KN], Volume II, Chapter XII, Theorem 2.7, or [GK], Theorem 2.3.)

The compact Grassmann manifolds provide a useful model for homogeneous spaces in duality. Let $1 \leq p \neq q \leq n-1$. Let

$$G = SO(n), \quad K = S(O(p) \times O(n-p)), \quad H = S(O(q) \times O(n-q)),$$

so that

$$X = G/K = G_{p,n} \quad \text{and} \quad \Xi = G/H = G_{q,n} \tag{3.7}$$

are the compact Grassmann manifolds of p- and q-dimensional subspaces of $\mathbb{R}^n$. The associated incidence relation is the usual one of inclusion; the corresponding Radon transform $R = R_{p,q}$ and its dual ${}^tR = R_{q,p}$ are defined on $C^\infty(X)$ and on $C^\infty(\Xi)$, respectively.

If $\mathrm{rank}(G_{p,n}) \leq \mathrm{rank}(G_{q,n})$, the Radon transform $R_{p,q}$ can be inverted. An explicit inversion formula is given by [K] in the case $q - p$ is even, and later by [GrRu] for any $q > p$ (the case $q < p$ reduces to the case $q > p$ by passing to orthogonal complements). See also Petrov [P]. We also have the following range characterization:

Theorem 3.2 ([GK], Theorem 4.2)**.** *Suppose that $l := \operatorname{rank}(G_{p,n}) < \operatorname{rank}(G_{q,n})$. Then*

$$\begin{aligned} R_{p,q}C^\infty(G_{p,n}) &= \{\, \varphi \in C^\infty(G_{q,n}) \,|\, d\nu(W_I)\varphi = 0, \;\; \textit{for all } I \in T_{2l+2} \,\} \\ &= \{\, \varphi \in C^\infty(G_{q,n}) \,|\, d\nu(U_{l+1})\varphi = 0 \,\}. \end{aligned} \tag{3.8}$$

4. Invariant differential operators and Radon transforms on affine Grassmannians

This paper and its predecessor [GK] address the question of extending the above compact results to analogous transforms on affine p- and q-planes in $\mathbb{R}^n$. Under the rubric of dual homogeneous spaces, our ambient group G is now the Euclidean motion group $M(n) = O(n) \rtimes \mathbb{R}^n$, and X and Ξ are now the affine Grassmann manifolds $G(p,n)$ and $G(q,n)$ of affine p- and q-planes, respectively, in $\mathbb{R}^n$

We assume throughout this paper that $p < q$.

Set $\ell_0 = \mathbb{R}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{e}_p$ and $\xi_0 = \mathbb{R}\mathbf{e}_1 \oplus \cdots \oplus \mathbb{R}\mathbf{e}_q$; their stabilizers in $M(n)$ are $H_p = M(p) \times O(n-p)$ and $H_q = M(q) \times O(n-q)$, respectively. Thus $G(p,n) = M(n)/H_p$ and $G(q,n) = M(n)/H_q$. Moreover, $\dim G(p,n) = (p+1)(n-p)$ and $\dim G(q,n) = (q+1)(n-q)$. It is easy to show that $M(n)$, H_p and H_q satisfy conditions (i), (ii), and (iii) in Section 2. The incidence relation is then the usual one of inclusion. We denote the corresponding integral transforms by $\mathcal{R}^{(p,q)}$ and $\mathcal{R}^{(q,p)}$ $(= {}^t\mathcal{R}^{(p,q)})$:

$$\mathcal{R}^{(p,q)} f(\xi) = \int_{\ell \subset \xi} f(\ell)\, d\ell, \qquad \xi \in G(q,n), \;\; f \in C_c^\infty(G(p,n)), \tag{4.1}$$

$$\mathcal{R}^{(p,q)} \varphi(\ell) = \int_{\xi \supset \ell} \varphi(\xi)\, d\xi, \qquad \ell \in G(p,n), \;\; \varphi \in C^\infty(G(q,n)). \tag{4.2}$$

Here $d\ell$ and $d\xi$ are the canonically defined measures from (2.4) and (2.5) on $\widehat{\xi} = \{\, \ell \in G(p,n) \,|\, \ell \subset \xi \,\}$ and $\check{\ell} = \{\, \xi \in G(q,n) \,|\, \xi \supset \ell \,\}$. Note that the integral in (4.2) takes place over a compact set. In [GK], we undertook a study of the transform $\mathcal{R}^{(p,q)}$, producing both an explicit inversion formula ([GK], Theorem 6.4) and a range characterization ([GK], Theorem 7.7).

We now undertake a similar study of the dual transform $\mathcal{R}^{(q,p)}$, focusing on characterizing its range. To this end, we introduce some additional notation and recall some results pertaining to the Casimir algebra of $M(n)$.

We define the rank of the group $M(n)$ to be $r = [\frac{n+1}{2}]$; likewise we define the rank of $G(p,n)$ (respectively $G(q,n)$) to be $s = \min(p+1, n-p)$ (respectively $m = \min(q+1, n-q)$). It turns out that the Casimir algebra of $M(n)$ is generated by r algebraically independent elements; the first s of these give rise to a generating set for $\mathbb{D}(G(p,n))$, the algebra of $M(n)$-invariant differential operators on $G(p,n)$. (See Theorem 4.1 below.)

Let λ and ν denote the left regular representations of $M(n)$ on smooth functions on $G(p,n)$ and on $G(q,n)$, respectively.

The Lie algebra $\mathfrak{m}(n) = so(n) \rtimes \mathbb{R}^n$ of the Euclidean motion group $M(n)$ has basis consisting of the elementary skew symmetric matrices X_{ij} $(1 \leq i < j \leq n)$, as well as the standard basis vectors E_i $(1 \leq i \leq n)$ of $\mathbb{R}^n$. Fix an integer d in

$\{1, 2, \cdots, [\frac{n+1}{2}]\}$ and let $J \in T_{2d-1}$, $J = \{j_1, j_2, \cdots, j_{2d-1}\}$. Define the element $V_J \in \mathfrak{u}(\mathfrak{m}(n))$ by

$$
\begin{aligned}
V_J \;&=\; \mathrm{Pf}\begin{pmatrix} 0 & E_{j_1} & E_{j_2} & \cdots & E_{j_{2d-1}} \\ -E_{j_1} & 0 & X_{j_1 j_2} & \cdots & X_{j_1 j_{2d-1}} \\ -E_{j_2} & -X_{j_1 j_2} & 0 & \cdots & X_{j_2 j_{2d-1}} \\ & \cdots & & \ddots & \\ -E_{j_{2d-1}} & -X_{j_1 j_{2d-1}} & -X_{j_2 j_{2d-1}} & \cdots & 0 \end{pmatrix} \\
&=\; \sum_{\sigma} \epsilon(\sigma)\, E_{\sigma(j_1)} X_{\sigma(j_2)\,\sigma(j_3)} \cdots X_{\sigma(j_{2d-2})\,\sigma(j_{2d-1})},
\end{aligned}
\tag{4.3}
$$

where the sum extends over all $\sigma \in \mathfrak{S}(J)$ such that $\sigma(j_2) < \sigma(j_3), \cdots, \sigma(j_{2d-2}) < \sigma(j_{2d-1})$ and $\sigma(j_2) < \sigma(j_4) < \cdots < \sigma(j_{2d-2})$. Note that the factors in each of the summands above commute, and we can also write

$$
V_J \;=\; \sum_{k=1}^{2d-1} (-1)^{k-1}\, E_{j_k} W_{J\setminus\{j_k\}} \;=\; \sum_{j \in J} \epsilon(j, J\setminus\{j\})\, E_j W_{J\setminus\{j\}}, \tag{4.4}
$$

where $W_{J\setminus\{j_k\}} \in \mathfrak{u}(so(n))$ is an order $2d-2$ Pfaffian of the type introduced in Section 2. Next let

$$
Q_{2d} \;=\; \sum_{J \in T_{2d-1}} {V_J}^2 \in \mathfrak{u}(\mathfrak{m}(n)). \tag{4.5}
$$

Theorem 4.1. (i) *$\mathfrak{z}(\mathfrak{m}(n))$ is generated by $Q_2, Q_4, \cdots, Q_{2[\frac{n+1}{2}]}$, and these are algebraically independent.*

(ii) *Let $s = \mathrm{rank}\,(G(p,n))$. Then the algebra $\mathbb{D}(G(p,n))$ of $M(n)$-invariant differential operators on $G(p,n)$ is generated by the elements $d\lambda(Q_2), d\lambda(Q_4), \cdots, d\lambda(Q_{2s})$, and these are algebraically independent.*

The proof of (i) can be found in [G2]; (ii) is in [GH].

We note that when X is a Riemannian symmetric space with connected isometry group G, then the algebra $\mathbb{D}(X)$ of left G-invariant differential operators on X is generated by $\mathrm{rank}(X)$ algebraically independent elements. This justifies our present definition of rank for affine Grassmann manifolds.

Let $\pi_p : G(p,n) \to G_{p,n}$ be the projection mapping of any p-plane ℓ onto the parallel p-plane σ through the origin. Then the intersection $\ell \cap \sigma^\perp$ consists of a single point x. Write $\ell = (\sigma, x)$, so that $G(p,n) = \{(\sigma, x) \in G_{p,n} \times \mathbb{R}^n \,|\, \sigma \perp x\}$. Similarly, $G(q,n) = \{(\eta, y) \in G_{q,n} \times \mathbb{R}^n \,|\, \eta \perp y\}$. Keeping in mind that $q > p$, the transform $\mathcal{R}^{(q,p)}$ can now be written in terms of these parametrizations as follows.

If $\ell = (\sigma, x)$, then any q-plane incident to ℓ is of the form $(\eta, Pr_{\eta^\perp}\, x)$, where $\eta \in G_{q,n}$, $\eta \supset \sigma$, and $Pr_{\eta^\perp}$ denotes the orthogonal projection onto the subspace $\eta^\perp$. Now the set consisting of such η is an orbit of the subgroup K_σ of $O(n)$ fixing σ. We let $d_\sigma\eta$ denote the normalized K_σ-invariant measure on this orbit. Now the transform Ψ given by

$$
\begin{aligned}
\Psi \,:\, C^\infty(G(q,n)) \;&\longrightarrow\; C^\infty(G(p,n)) \\
\varphi \;&\longmapsto\; \Psi\varphi(\sigma, x) \;:=\; \int_{\eta \in G_{q,n}\;\eta\supset\sigma} \varphi(\eta, Pr_{\eta^\perp}\, x)\, d_\sigma\eta
\end{aligned}
\tag{4.6}
$$

coincides with $\mathcal{R}^{(q,p)}$. Certainly the integral representing Ψ above is invariant under all $g \in M(n)$. This equivariance shows that Ψ coincides with $\mathcal{R}^{(q,p)}$ up to a constant multiple, which we declare to be 1.

5. Range characterization of $\mathcal{R}^{(q,p)}$

Assume that $p < q$ and $\operatorname{rank}(G(p,n)) > \operatorname{rank}(G(q,n))$. We will show that the range of $\mathcal{R}^{(q,p)}$ is characterized by a system of Pfaffian type equations. This system cannot be reduced to a single invariant equation, as in the case of the transform $\mathcal{R}^{(p,q)}$, unless possibly $p+q=n$.

Since $\operatorname{rank}(G(p,n)) > \operatorname{rank}(G(q,n))$, we must have $q > [\frac{n}{2}]$. Otherwise, $2q \leq n$, so $\operatorname{rank}(G(q,n)) = \min(q+1, n-q) \geq q \geq p+1 \geq \operatorname{rank}(G(p,n))$, a contradiction. Consequently, $\operatorname{rank}(G(q,n)) = n-q$. Let m denote this number. Also the inequality $n-q < p+1$ implies that $p+q \geq n$. This means that we also have $\operatorname{rank}(G_{q,n}) = n-q \leq \min(p, n-p) = \operatorname{rank}(G_{p,n})$, so that the corresponding compact transform $R_{q,p} : C^\infty(G_{q,n}) \to C^\infty(G_{p,n})$ is injective.

Now from (4.6), $\mathcal{R}^{(q,p)}$ is given by the following integral:

$$\mathcal{R}^{(q,p)}\varphi(\sigma,x) = \int_{\eta\supset\sigma} \varphi(\eta, Pr_{\eta^\perp} x)\, d\eta, \quad \varphi \in C^\infty(G(q,n)). \tag{5.1}$$

From general considerations, (See [H4] Chapter II, Proposition 2.4.) it follows that $\mathcal{R}^{(q,p)}\varphi \in C^\infty(G(p,n))$. Moreover, we have the following lemma.

Lemma 5.1. *If $p < q$ and $p+q \geq n$, then $\mathcal{R}^{(q,p)}$ is an injective operator from $C^\infty(G(q,n))$ to $C^\infty(G(p,n))$.*

Proof. For $\ell \in G(p,n)$, let $\ell = (\sigma, x)$, $\sigma \in G_{p,n}$, $x \in \sigma^\perp$. Then any $\xi \in G(q,n)$ including ℓ is written as $\xi = \eta + x$ for $\eta \in G_{q,n}$ with $\eta \supset \sigma$. Now let us denote $\varphi_x(\eta) = \varphi(\eta + x)$ and regard it as a function on $G_{q,n}$. Here we note that $\mathcal{R}^{(q,p)}$ is $M(n)$-invariant and, in particular, invariant under the translation $\tau(x)$. Thus we have

$$(\mathcal{R}^{(q,p)}\varphi)(\sigma,x) \;=\; (R_{q,p}\varphi_x)(\sigma) = \int_{\eta\supset\sigma} \varphi_x(\eta) d_\sigma \eta. \tag{5.2}$$

If $p < q$ and $p+q \geq n$, then $\operatorname{rank}(G_{q,n}) \leq \operatorname{rank}(G_{p,n})$ and therefore $R_{q,p}$ is injective. Hence, if $(\mathcal{R}^{(q,p)}\varphi)(\sigma,x) \equiv 0$, then $\varphi_x(\eta) \equiv 0$ on $G_{q,n}$ for all $x \in \sigma^\perp$. Since $\sigma^\perp \supset \eta^\perp$, $\varphi(\eta, y) = 0$ for all $\eta \in G_{q,n}$ and for all $y \in \eta^\perp$, that is, $\varphi \equiv 0$ on $G(q,n)$. $\square$

Note that the hypothesis above is equivalent to the condition that $\operatorname{rank}(G(q,n)) < \operatorname{rank}(G(p,n))$ and $p < q$. Rubin [Ru2] has obtained an important extension to this lemma as follows. In the case when $\operatorname{rank}(G(q,n)) = \operatorname{rank}(G(p,n))$, then $\mathcal{R}^{(q,p)}$ is injective on functions on $G(q,n)$ satisfying mild decay conditions at infinity. This result generalizes the injectivity of the classical dual transform on rapidly decreasing hyperplane functions (See [G1] or [So].) This extension is important because $\mathcal{R}^{(q,p)}$ is no longer injective in the absence of these decay conditions when $\operatorname{rank}(G(q,n)) = \operatorname{rank}(G(p,n))$, since, in this case, we would have $\operatorname{rank}(G_{q,n}) > \operatorname{rank}(G_{p,n})$.

The theorem below provides the range characterization for $\mathcal{R}^{(q,p)}C^\infty(G(q,n))$ in the case when $p+q > n$.

Theorem 5.1. *Assume that $p < q$, $p+q > n$, and $s = \operatorname{rank}(G(p,n)) > m = \operatorname{rank}(G(q,n))$. Then the range $\mathcal{R}^{(q,p)}C^\infty(G(q,n))$ consists precisely of those functions $f \in C^\infty(G(p,n))$ which satisfy the differential equations $d\lambda(V_J)f = 0$ and $d\lambda(W_I)f = 0$ for all $J \in T_{2m+1}$, $I \in T_{2m+2}$.*

Note that since $m < \operatorname{rank}(G(p,n))$, the differential operators $d\lambda(V_J)$ and $d\lambda(W_I)$ are not all zero on $G(p,n)$.

We first prove that the above equations are necessarily satisfied by the range. Now $\mathcal{R}^{(q,p)}$ is $M(n)$-equivariant. So for any $U \in \mathfrak{u}(\mathfrak{m}(n))$ we have the following commutative diagram:

$$\begin{array}{ccc} C^\infty(G(q,n)) & \xrightarrow{\mathcal{R}^{(q,p)}} & C^\infty(G(p,n)) \\ {\scriptstyle d\nu(U)}\Big\downarrow & & \Big\downarrow{\scriptstyle d\lambda(U)} \\ C^\infty(G(q,n)) & \xrightarrow[\mathcal{R}^{(q,p)}]{} & C^\infty(G(p,n)) \end{array} \tag{5.3}$$

From the considerations given in the preliminaries, it is sufficient to show that the differential operators $d\nu(V_J)$ and $d\nu(W_I)$ vanish on $G(q,n)$.

Let $\varphi \in C^\infty(G(q,n))$. Set $\eta_0 = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_q$. We now show that

$$(d\nu(V_J)\varphi)(\eta_0, 0) = 0, \tag{5.4}$$

for any $J \in T_{2m+1}$. Now

$$V_J = \sum \pm X_{j_1,j_2} \cdots X_{j_{2m-1},j_{2m}} E_{j_{2m+1}}, \tag{5.5}$$

where the sum is taken over indices $\{j_1, j_2, \cdots, j_{2m+1}\} \in \mathfrak{S}(J)$. Equation (5.4) will follow if we can show that

$$(\, d\nu(X_{j_1,j_2} \cdots X_{j_{2m-1},j_{2m}} E_{j_{2m+1}})\varphi\,)(\eta_0, 0) = 0. \tag{5.6}$$

To show (5.6), we essentially count indices. If $j_{2m+1} \in \{1, \cdots, q\}$, then we write the left-hand side of (5.6) as

$$d\nu(E_{j_{2m+1}})(d\nu(X_{j_1,j_2} \cdots X_{j_{2m-1},j_{2m}})\varphi\,)(\eta_0, 0), \tag{5.7}$$

which clearly vanishes. If $j_{2m+1} > q$, we claim that at least one pair $\{j_{2k-1}, j_{2k}\}$ belongs to $\{1, \cdots, q\}$. If not, then (since $m = n - q$), this forces $j_{2m+1} \leq q$, which was already excluded. We write the left-hand side of (5.6) as

$$d\nu(X_{j_{2k-1},j_{2k}})(\, d\nu(X_{j_1,j_2} \cdots \widehat{X}_{j_{2k-1},j_{2k}} \cdots X_{j_{2m-1},j_{2m}} E_{j_{2m+1}})\, \varphi\,)(\eta_0, 0), \tag{5.8}$$

which clearly also vanishes. In either case, we have proven (5.6), and thus (5.4).

In the following lemma, we now recall the conjugation rules in [GK], §2 and §5 governing the elements V_J and W_I in $\mathfrak{u}(\mathfrak{m}(n))$.

Lemma 5.2. *For $u \in SO(n)$ and $x \in \mathbb{R}^n$,*

$$\begin{aligned} Ad(u)W_I &= \sum_{L \in T_{2m+2}} \det(u_{LI})\; W_L, \qquad Ad(u)V_J = \sum_{R \in T_{2m+1}} \det(u_{RJ})\; V_R, \\ Ad(x)W_I &= W_I \;-\sum_{P \in T_{2m+1}} e_{PI}(x)\, V_P, \qquad Ad(x)V_J = V_J, \end{aligned} \tag{5.9}$$

where $u_{LI} = (u_{ij})_{i \in I, l \in L}$, and $e_{PI}(x)$ is a scalar which can be explicitly calculated.

We also recall that Lemma 3.1 in [GK] states that

$$d\nu(W_I)\varphi(\eta,0)=0, \tag{5.10}$$

for all $\eta\in G_{q,n}$ and for all $I\in T_{2m+2}$. We now show that for each such η and $J\in T_{2m+1}$,

$$d\nu(V_J)\varphi(\eta,0)=0. \tag{5.11}$$

For this, simply choose $u\in SO(n)$ for which $u\cdot\eta_0=\eta$. Then the left-hand side of (5.11) equals

$$\begin{aligned}(d\nu(V_J)\varphi)^{\tau(u^{-1})}(\eta_0,0)&=(\,d\nu(\mathrm{Ad}(u^{-1})V_J)\varphi^{\tau(u^{-1})}\,)(\eta_0,0)\\&=\sum_R d_{RJ}(u^{-1})\,(\,d\nu(V_R)\varphi^{\tau(u^{-1})}\,)(\eta_0,0)\\&=0,\end{aligned}$$

using (5.9) and (5.4) (with φ replaced by $\varphi^{\tau(u^{-1})}$). Finally, let $(\eta,x)\in G(q,n)$ be arbitrary. From (5.9), (5.10), (5.11), we obtain

$$\begin{aligned}(d\nu(W_I)\varphi)(\eta,x)&=(\,d\nu(\mathrm{Ad}(-x)W_I)\varphi^{\tau(-x)}\,)(\eta,0)\\&=(\,d\nu(W_I)\varphi^{\tau(-x)}\,)(\eta,0)-\sum_P e_{PI}(-x)\,(\,d\nu(V_P)\varphi^{\tau(-x)}\,)(\eta,0)\\&=0.\end{aligned}\tag{5.12}$$

Likewise, from (5.9) and (5.11), we have

$$(d\nu(V_J)\varphi)(\eta,x)=(d\nu(V_J)\varphi^{\tau(-x)})(\eta,0)=0. \tag{5.13}$$

(In applying (5.11), we replace φ by $\varphi^{\tau(-x)}$.)

For the converse, we now show that if $f\in C^\infty(G(p,n))$ satisfies the Pfaffian equations $d\lambda(W_I)f=0$, and $d\lambda(V_J)f=0$ for all $I\in T_{2m+2},\ J\in T_{2m+1}$, then $f=\mathcal{R}^{(q,p)}\varphi$ for some $\varphi\in C^\infty(G(q,n))$.

It will be more convenient to move our calculations to the product manifolds $G_{p,n}\times\mathbb{R}^n$ and $G_{q,n}\times\mathbb{R}^n$. $M(n)$ acts transitively on $G_{p,n}\times\mathbb{R}^n$ via $(u,x)\cdot(\sigma,y)=(u\cdot\sigma,u\cdot y+x)$. The subgroup fixing the point $(\sigma_0,0)$ (where $\sigma_0=\mathbb{R}e_1\oplus\cdots\oplus\mathbb{R}e_p$) is $O(p)\times O(n-p)\subset O(n)$, so that $G_{p,n}\times\mathbb{R}^n=M(n)/O(p)\times O(n-p)$. Let π_p denote the projection of $G_{p,n}\times\mathbb{R}^n$ onto $G(p,n)$ given by $\pi_p(\sigma,x)=(\sigma,Pr_{\sigma^\perp}x)$, the p-plane through x parallel to σ. If $h\in C^\infty(G(p,n))$, then the function $H=h\circ\pi_p$ is a smooth function on $G_{p,n}\times\mathbb{R}^n$ satisfying $H(\sigma,x)=H(\sigma,x+v)$ for all $v\in\sigma$. Conversely, by using local cross sections on $M(n)$, for example, it is easy to see that each smooth function $H(\sigma,x)$ on $G_{p,n}\times\mathbb{R}^n$ which satisfies $H(\sigma,x)=H(\sigma,x+v)$ for all $v\in\sigma$ must be of the form $H=h\circ\pi_p$ for some $h\in C^\infty(G(p,n))$. It is clear that π_p commutes with the left action of $M(n)$ on $G_{p,n}\times\mathbb{R}^n$ and $G(p,n)$, respectively. Similar considerations apply to the projection $\pi_q: G_{q,n}\times\mathbb{R}^n\to G(q,n)$.

Now $\mathfrak{m}(n)$ acts on $C^\infty(G_{p,n}\times\mathbb{R}^n)$ (and on $C^\infty(G_{q,n}\times\mathbb{R}^n)$) via vector fields: if $X\in M(n)$, and $H\in C^\infty(G_{p,n}\times\mathbb{R}^n)$, we set

$$X\cdot H(\sigma,y)=\frac{d}{dt}H(\exp(-tX)\cdot(\sigma,y))|_{t=0}. \tag{5.14}$$

In particular, if $X \in so(n)$, it follows that

$$X \cdot H(\sigma, y) = X_\sigma H(\sigma, y) + X_y H(\sigma, y). \tag{5.15}$$

In the above, X_σ and X_y (which act on the first and second arguments, respectively) are the tangent vectors to the curves $\exp(-tX) \cdot \sigma$ in $G_{p,n}$ and $\exp(-tX) \cdot y$ in $\mathbb{R}^n$ at $t = 0$. Similarly, if $v \in \mathbb{R}^n$, then

$$\begin{aligned} v \cdot H(\sigma, y) &= \frac{d}{dt} H(\sigma, y - tv)|_{t=0} \\ &= v_y H(\sigma, y). \end{aligned} \tag{5.16}$$

We extend the $\mathfrak{m}(n)$ action to an action of $\mathfrak{u}(\mathfrak{m}(n))$ on $C^\infty(G_{p,n} \times \mathbb{R}^n)$ and on $C^\infty(G_{q,n} \times \mathbb{R}^n)$.

Now assume that f satisfies the Pfaffian equations $d\lambda(W_I)f = 0$, and $d\lambda(V_J)f = 0$ for all $I \in T_{2m+2},\ J \in T_{2m+1}$. Let $F = f \circ \pi_p$. Since the $\mathfrak{u}(\mathfrak{m}(n))$-action commutes with π_p, we have $W_I \cdot F = 0,\ V_J \cdot F = 0$. Fix $I \in T_{2m+2}$ and suppose $(\sigma, x) \in G_{p,n} \times \mathbb{R}^n$. Then

$$\begin{aligned} (\,(W_I)_\sigma F\,)(\sigma, x) &= (\,(W_I)_\sigma F\,)^{\tau(-x)}(\sigma, 0) \\ &= (\,(W_I)_\sigma F^{\tau(-x)}\,)(\sigma, 0), \end{aligned} \tag{5.17}$$

because $\tau(-x)$ acts on the second argument whereas $(W_I)_\sigma$ acts on the first. Now by (5.9) and (5.15),

$$\begin{aligned} (\,(W_I)_\sigma F^{\tau(-x)}\,)(\sigma, 0) &= (\,W_I \cdot F^{\tau(-x)}\,)(\sigma, 0) \\ &= (\,(\mathrm{Ad}(x) W_I) \cdot F\,)^{\tau(-x)}(\sigma, 0) \\ &= (\,(W_I - \sum_P e_{PI}(x) V_P) \cdot F\,)^{\tau(-x)}(\sigma, 0) \\ &= 0. \end{aligned} \tag{5.18}$$

Since we are presently assuming that $p + q > n$, we must have $\mathrm{rank}(G_{q,n}) = n - q < \min(p, n - p) = \mathrm{rank}(G_{p,n})$. Let $R_{q,p}$ denote the $O(n)$-equivariant (compact) Radon transform from $C^\infty(G_{q,n})$ to $C^\infty(G_{p,n})$ corresponding to the inclusion incidence relation. Then $R_{q,p}$ is injective and by the range conditions for $R_{q,p}$ ([GK], Theorem 4.2 and subsequent remarks), there exists, for each $x \in \mathbb{R}^n$, a unique function $\Phi_x(\eta) \in C^\infty(G_{q,n})$ for which

$$F(\sigma, x) = \int_{\eta \supset \sigma} \Phi_x(\eta)\, d\eta. \tag{5.19}$$

We write $\Phi_x(\eta)$ as $\Phi(\eta, x)$ and the above as

$$F(\sigma, x) = \int_{\eta \supset \sigma} \Phi(\eta, x)\, d\eta. \tag{5.20}$$

Next we prove that $\Phi \in C^\infty(G_{q,n} \times \mathbb{R}^n)$. For $q - p$ even, this follows from the inversion formula for $R_{q,p}$ given in [K]. Since there is no simple explicit inversion formula for $R_{q,p}$ when p and q do not have the same parity. In fact, in this case there exists an inversion formula given by Grinberg and Rubin [GrRu]. However, their inversion formula contains differentiation in the sense of distributions, and therefore cannot be applied to prove the smoothness of the function Φ. So we will show that Φ is smooth in general by diagonalizing $R_{q,p}$. (See [K] or [Gr] for details.)

The spaces $L^2(G_{q,n})$ and $L^2(G_{p,n})$ decompose orthogonally into $O(n)$-invariant irreducible subspaces

$$L^2(G_{q,n}) = \sum_{\delta\in\widehat{G}_{q,n}} \mathcal{H}_\delta, \tag{5.21}$$

$$L^2(G_{p,n}) = \sum_{\delta'\in\widehat{G}_{p,n}} \mathcal{H}'_{\delta'}. \tag{5.22}$$

Here $\widehat{G}_{q,n}$ and $\widehat{G}_{p,n}$ are index sets corresponding to the equivalence classes of certain (spherical) representations of $O(n)$; we have $\widehat{G}_{q,n} \subset \widehat{G}_{p,n}$ ([Gr], [St]). Since $R_{q,p}$ is injective ([Gr]), one has $R_{q,p}\mathcal{H}_\delta = \mathcal{H}'_\delta$ when $\delta \in \widehat{G}_{q,n}$; in fact, by Schur's lemma $R_{q,p}$ acts essentially as scalar multiplication on $\mathcal{H}_\delta$.

For each $\delta \in \widehat{G}_{q,n}$, let $d(\delta) = \dim(\mathcal{H}_\delta)$ and let $\{\, Y_{\delta\ell} \,|\, \ell = 1, \ldots, d(\delta)\,\}$ be an orthonormal basis of $\mathcal{H}_\delta$. We can choose $Y_{\delta\ell}$ so that $|Y_{\delta\ell}(\eta)| \le d(\delta)^{1/2}$ for all η, by the Peter-Weyl theorem. By the injectivity of $R_{q,p}$ there exists a constant $c_\delta \neq 0$ such that $Z_{\delta\ell} := c_\delta R_{q,p} Y_{\delta\ell}$, $\ell = 1, \ldots, d(\delta)$, is an orthonormal basis of $\mathcal{H}'_\delta$. Let μ_δ denote the highest weight of δ, with respect to a choice of Weyl chamber in $G_{q,n}$. Then c_δ grows polynomially in μ_δ in the sense that, for some constants C_0 and M, we have $|c_\delta| \le C_0 \|\mu_\delta\|^M$ for all δ, $\|\ \|$ denoting the Killing form norm ([G4]). We also note that $d(\delta)$ grows polynomially in μ_δ, by the Weyl dimension formula.

From (5.22) (and (5.20)) we see that

$$F(\sigma, x) \;=\; \sum_{\delta\in\widehat{G}_{q,n}} \sum_{\ell=1}^{d(\delta)} a_{\delta\ell}(x)\, Z_{\delta\ell}(\sigma) \tag{5.23}$$

for certain coefficients $a_{\delta\ell}(x) \in C^\infty(\mathbb{R}^n)$. This series converges absolutely and uniformly on compact subsets of $G_{p,n} \times \mathbb{R}^n$ and can be differentiated term by term to any order in x and σ. In addition, the $a_{\delta\ell}(x)$ are rapidly decreasing in the weight μ_δ in the sense that for any compact set $K \subset \mathbb{R}^n$ and any positive integer N, there is a constant $C_{K,N}$ such that

$$|a_{\delta\ell}(x)| \;\le\; C_{K,N}\, \|\mu_\delta\|^{-N} \qquad x \in K. \tag{5.24}$$

The partial derivatives of $a_{\delta\ell}(x)$ (of any order) also satisfy a similar estimate.

By (5.20) and the estimates above we see that we can write $\Phi(\eta, x)$ as the absolutely convergent series

$$\Phi(\eta, x) \;=\; \sum_{\delta\in\widehat{G}_{q,n}} \sum_{\ell=1}^{d(\delta)} c_\delta\, a_{\delta\ell}(x)\, Y_{\delta\ell}(\eta). \tag{5.25}$$

This series also converges uniformly on compact subsets of $G_{q,n} \times \mathbb{R}^n$. More precisely, since c_δ grows at most polynomially in δ (or more precisely in μ_δ), the coefficients $b_{\delta\ell}(x) = c_\delta\, a_{\delta\ell}(x)$ of $Y_{\delta\ell}(\eta)$ and their partial derivatives satisfy a decay estimate similar to (5.24). It also follows that $\Phi(\eta, x)$ is smooth in (η, x), the smoothness in η being a consequence of the rapid decrease of the coefficients $b_{\delta\ell}(x)$ in δ.

We will next need to prove that the smooth function Φ satisfies the condition

$$\Phi(\eta, x) = \Phi(\eta, x + v), \qquad \text{for any } v \in \eta. \tag{5.26}$$

This will imply the existence of a function $\varphi \in C^\infty(G(q,n))$ for which $\Phi = \varphi \circ \pi_q$. Assuming that such a function φ exists, we have

$$\Phi(\eta, x) = \Phi(\eta, Pr_{\eta^\perp} x) = \varphi(\eta, Pr_{\eta^\perp} x), \tag{5.27}$$

for any $(\eta, x) \in G_{q,n} \times \mathbb{R}^n$, whence (5.20) can be written

$$\begin{aligned} f(\sigma, x) &= F(\sigma, x) \\ &= \int_{\eta \supset \sigma} \Phi(\eta, x)\, d\eta \\ &= \int_{\eta \supset \sigma} \Phi(\eta, \mathrm{Pr}_{\eta^\perp} x)\, d\eta \\ &= \int_{\eta \supset \sigma} \varphi(\eta, \mathrm{Pr}_{\eta^\perp} x)\, d\eta \\ &= (\mathcal{R}^{(q,p)}\varphi)(\sigma, x), \end{aligned}$$

for any $(\sigma, x) \in G(p,n)$, which of course shows that f belongs to the range $\mathcal{R}^{(p,q)}\, C^\infty(G(q,n))$.

(5.26) will follow if we can show that for any two points $x \neq y$ in $\mathbb{R}^n$, $\Phi(\eta, x) = \Phi(\eta, y)$ if η is any element of $G_{q,n}$ containing $w = y - x$. Define the compact Grassmannians $G_p(w) = \{\sigma \in G_{p,n} \mid w \in \sigma\}$ and $G_q(w) = \{\eta \in G_{q,n} \mid w \in \eta\}$. Since $w \neq 0$, we have $G_p(w) \approx G_{p-1,n-1}$ and $G_q(w) \approx G_{q-1,n-1}$. We now claim that $\mathrm{rank}(G_p(w)) \geq \mathrm{rank}(G_q(w))$. First, we have $2q \geq n+1$ so $n - q \leq q - 1$ and hence $\mathrm{rank}(G_q(w)) = \min(q-1, n-q) = n - q$. Moreover, we are presently assuming that $p + q \geq n+1$, so $n - q \leq p - 1$, and of course, $n - q < n - p$. These imply that $\mathrm{rank}(G_q(w)) = n - q \leq \min(p-1, n-p) = \mathrm{rank}(G_p(w))$, as claimed. Let $R_{q,p}(w) : C^\infty(G_q(w)) \to C^\infty(G_p(w))$ be the naturally defined integral transform arising from the inclusion relation. From the rank data, we see that this transform is injective.

Define smooth functions ψ_x and ψ_y on $G_q(w)$ by setting $\psi_x(\eta) = \Phi(\eta, x)$ and $\psi_y(\eta) = \Phi(\eta, y)$, for any $\eta \in G_q(w)$. Let $\sigma \in G_p(w)$. We have

$$\begin{aligned} F(\sigma, x) &= \int_{\eta \supset \sigma} \Phi(\eta, x) d\eta \\ &= \int_{\eta \supset \sigma} \psi_x(\eta) d\eta \\ &= (R_{q,p}(w)\psi_x)(\sigma), \end{aligned} \tag{5.28}$$

and also

$$F(\sigma, y) = (R_{q,p}(w)\psi_y)(\sigma).$$

Since $F(\sigma, x) = F(\sigma, y)$ for all $\sigma \in G_p(w)$, we have $R_{q,p}(w)\psi_x = R_{q,p}(w)\psi_y$ and hence (by the injectivity of $R_{q,p}$) $\psi_x = \psi_y$. This shows that $\Phi(\eta, x) = \Phi(\eta, y)$ for all $\eta \in G_q(w)$ and completes the proof of (5.26).

Remarks. 1. The proof of Theorem 5.1 (unfortunately) breaks down in the case $p + q = n$ (still assuming, of course, that $\mathrm{rank}(G(q,n)) < \mathrm{rank}(G(p,n))$), because in this case $\mathrm{rank}(G_q(w)) = \min(q-1, n-q) = n - q = p > p - 1 \geq \mathrm{rank}(G_p(w))$. Hence $R_{q,p}(w)$ is no longer injective.

Theorem 6.1 below suggests, however, that when $p + q = n$ and $\mathrm{rank}(G(q,n)) < \mathrm{rank}(G(p,n))$, a stronger set of range conditions apply. Specifically, it is likely that $\mathcal{R}^{(q,p)}C^\infty(G(q,n))$ consists of those functions $f \in C^\infty(G(p,n))$ annihilated by the operators $d\lambda(V_J)$, $J \in T_{2m+1}$, or, equivalently, by the single $M(n)$-invariant differential operator $d\lambda(\sum_{J\in T_{2m+1}} {V_J}^2)$, of order $2m+2$. We will provide a proof of this assertion in the special case $p=1$, $q=n-1$.

2. In Theorem 5.1, the order $m+1$ Pfaffian equations $d\lambda(W_I)f = 0$ and $d\lambda(V_J)f = 0$ are both essential in characterizing the range of $\mathcal{R}^{(p,q)}$. (See Remark 3 below.) This is in contrast with the likely range characterization for $\mathcal{R}^{(q,p)}$ in the case $p+q=n$, given in the preceding remark, as well as the range characterization in the case of the transform $\mathcal{R}^{(p,q)} = {}^t\mathcal{R}^{(q,p)} : \mathcal{S}(G(p,n)) \to \mathcal{S}(G(q,n))$ when $\operatorname{rank}(G(p,n)) < \operatorname{rank}(G(q,n))$, which was given in [GK], Theorem 7.7, and summarized below.

Let $\mathcal{S}(G(p,n))$ and $\mathcal{S}(G(q,n))$ denote the Schwartz spaces on $G(p,n)$ and $G(q,n)$, respectively ([Ri]). Let r denote the rank of $G(p,n)$. According to the above-quoted theorem, if $r = \operatorname{rank}(G(p,n)) < \operatorname{rank}(G(q,n))$, then the range $\mathcal{R}^{(p,q)}\mathcal{S}(G(p,n))$ consists precisely of those functions $\varphi \in \mathcal{S}(G(q,n))$ for which $d\nu(V_J)\varphi = 0$ for all $J \in T_{2r+1}$; alternatively, $\mathcal{R}^{(p,q)}\mathcal{S}(G(p,n))$ is given by the set of all $\varphi \in \mathcal{S}(G(q,n))$ annihilated by the $M(n)$-invariant differential operator $d\nu(\sum_{J\in T_{2r+1}} {V_J}^2)$. While the functions in the range of $\mathcal{R}^{(p,q)}$ also satisfy the equations $d\nu(W_I)\varphi = 0$ for $I \in T_{2r+2}$, these equations are not needed in the range characterization.

3. The following example shows why the equations $d\lambda(W_I)f = 0$ are needed in the range characterization of $\mathcal{R}^{(q,p)}$ in Theorem 5.1. Let us assume the hypotheses of Theorem 5.1, namely that $p < q$, $p+q > n$, and $\operatorname{rank}(G(p,n) > \operatorname{rank}(G(q,n)$. The relation $q > [\frac{n}{2}]$ implies that $2q > n$, so $q > n-q$, and thus $\operatorname{rank}(G_{q,n}) = n-q < \min(p, n-p) = \operatorname{rank}(G_{p,n})$.

Consider the orthogonal decompositions (5.21) and (5.22) of $L^2(G_{q,n})$ and $L^2(G_{p,n})$, respectively. As mentioned earlier, we have $R_{q,p}\mathcal{H}_\delta = \mathcal{H}'_\delta$ when $\delta \in \widehat{G}_{q,n}$. Now the rank data imply that $\widehat{G}_{q,n} \subsetneqq \widehat{G}_{p,n}$. Fix any $\delta_0 \in \widehat{G}_{p,n} \setminus \widehat{G}_{q,n}$ and any $\psi \neq 0, \in \mathcal{H}'_{\delta_0}$. Then $\psi \notin R_{q,p}(C^\infty(G_{q,n}))$. Define the function h on $G(p,n)$ by setting $h(\sigma, x) = \psi(\sigma)$ for any $\sigma \in G_{p,n}$ and $x \in \sigma^\perp$. Then h is smooth on $G(p,n)$ and invariant under the translations $\tau(v)$, $v \in \mathbb{R}^n$. As a result, we clearly have $d\lambda(V_J)h = 0$ for all $J \in T_{2m+1}$. On the other hand, there is no function $\varphi \in C^\infty(G(q,n))$ for which $h(\sigma, 0) = \int_{\eta\supset\sigma,\ \eta\in G_{q,n}} \varphi(\eta, 0)\, d\eta$; in particular, $h \notin \mathcal{R}^{(q,p)}C^\infty(G(q,n))$.

6. The special case $p+q=n$

Let us now focus our attention on the case $1 \le p < q < n$, with $p+q=n$. Then $\operatorname{rank}(G(p,n)) = p+1 > n-q = \operatorname{rank}(G(q,n))$. Suppose now that $f \in C^\infty(G(p,n))$ satisfies the Pfaffian system $d\lambda(V_J)f = 0$ for all $J \in T_{2m+1}$. As usual, we have set $m = \operatorname{rank}(G(q,n))$ $(= p$ in the present case). Consider the pull-back $F = f \circ \pi_p$. This function is of course smooth on $G_{p,n} \times \mathbb{R}^n$. Now since $\operatorname{rank}(G_{p,n}) = \operatorname{rank}(G_{q,n})$ here, the transform $R_{q,p} : C^\infty(G_{q,n}) \to C^\infty(G_{p,n})$ is a bijection ([Gr]). Thus for each $x \in \mathbb{R}^n$, there exists a function $\Phi_x \in C^\infty(G_{q,n})$ such that

$$F(\sigma, x) = \int_{\eta\supset\sigma} \Phi_x(\eta)\, d\eta. \tag{6.1}$$

Writing $\Phi_x(\eta) = \Phi(\eta, x)$, we may rewrite the above as

$$F(\sigma, x) = \int_{\eta \supset \sigma} \Phi(\eta, x)\, d\eta. \tag{6.2}$$

As in the proof of Theorem 5.1 it is easy to see (using either the inversion formula for $R_{p,q}$, or differentiating inside the integral sign) that $\Phi \in C^\infty(G_{q,n} \times \mathbb{R}^n)$.

Our objective, as with equation (5.26), is to show that

$$\Phi(\eta, x) = \Phi(\eta, x + v) \tag{6.3}$$

for any $v \in \eta$. Of course, we can no longer apply the argument in equation (5.28) because $\operatorname{rank}(G_q(w)) > \operatorname{rank}(G_p(w))$ by Remark 1 following Theorem 5.1. Nonetheless, condition (6.3) for Φ should allow us to conclude (as in equations (5.26) and (5.27) for the case $p + q > n$) that $\Phi = \varphi \circ \pi_q$ for some $\varphi \in C^\infty(G_{q,n})$, so that $\mathcal{R}^{(q,p)}\varphi = f$. Now let $\nabla_x \Phi(\eta, x)$ denote the gradient of Φ with respect to its second argument: $\nabla_x \Phi(\eta, x) = ((E_1 \cdot \Phi)(\eta, x), \cdots, (E_n \cdot \Phi)(\eta, x)) \in \mathbb{R}^n$. Then (6.3) is equivalent to the condition $\nabla_x \Phi(\eta, x) \perp \eta$.

We proceed to obtain some properties of Φ from the Pfaffian equations $d\lambda(V_J)f = 0$. These conditions will allow us to derive the equation (6.3) in the case $p = 1, q = n - 1$, thereby proving the range theorem, at least in this special case. Now the projection π_p commutes with the left action of $M(n)$ on the space $G(p, n)$ and $G_{p,n} \times \mathbb{R}^n$, so we obtain

$$V_J \cdot F = 0, \tag{6.4}$$

for each $J \in T_{2m+1}$. We provide a more explicit expression for (6.4) as follows. Let $(\sigma, x) \in G_{p,n} \times \mathbb{R}^n$ and $J : j_1 < j_2 < \cdots < j_{2m+1}$ in $T_{2m+1} = T_{2p+1}$,

$$\begin{aligned}
0 &= (V_J \cdot F)(\sigma, x) \\
&= (\,(\mathrm{Ad}(-x)V_J)F^{\tau(-x)}\,)(\sigma, 0) \\
&= (\,V_J F^{\tau(-x)}\,)(\sigma, 0) \\
&= \sum_{r=1}^{2p+1} (-1)^{r-1}\,(\,(W_{J\setminus\{j_r\}} E_{j_r})F^{\tau(-x)}\,)(\sigma, 0) \\
&= \sum_{r=1}^{2p+1} (-1)^{r-1}\,(\,((W_{J\setminus\{j_r\}})_\sigma \circ E_{j_r})F^{\tau(-x)}\,)(\sigma, 0) \\
&= \sum_{r=1}^{2p+1} (-1)^{r-1}\,(\,E_{j_r} \cdot ((W_{J\setminus\{j_r\}})_\sigma F)^{\tau(-x)}\,)(\sigma, 0) \\
&= \sum_{r=1}^{2p+1} (-1)^{r-1}\,(\,(E_{j_r})_x \circ (W_{J\setminus\{j_r\}})_\sigma\, F\,)(\sigma, x).
\end{aligned} \tag{6.5}$$

Apply (6.5) to (6.2) and differentiate inside the integral sign, noting that $R_{q,p}$ is $O(n)$-equivariant. This gives

$$0 = \int_{\eta \supset \sigma} \sum_{r=1}^{2p+1} (-1)^{r-1}\,(\,(E_{j_r})_x \circ (W_{J\setminus\{j_r\}})_\eta\, \Phi\,)\,(\eta, x)\, d\eta, \tag{6.6}$$

for each $(\sigma, x) \in G_{p,n} \times \mathbb{R}^n$. $R_{q,p}$ being injective, we conclude that the expression inside the integral sign vanishes; i.e.,

$$\sum_{r=1}^{2p+1} (-1)^{r-1} \left((E_{j_r})_x \circ (W_{J\setminus\{j_r\}})_\eta \, \Phi \right) (\eta, x) \; = 0 \tag{6.7}$$

for each $(\eta, x) \in G_{q,n} \times \mathbb{R}^n$ and each $J \in T_{2p+1}$.

Now the condition $\nabla_x F(\sigma, x) \perp \sigma$ (which is equivalent to $F(\sigma, x) = F(\sigma, x + w)$ for all $w \in \sigma$) and equation (6.2) leads us to the following integral condition on Φ:

$$0 = \int_{\eta \supset \sigma} \langle \nabla_x \Phi(\eta, x), w \rangle \, d\eta, \tag{6.8}$$

for each $w \in \sigma$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$.

Let us now fix $x \in \mathbb{R}^n$ and consider the section $\widetilde{\Phi}$ on the trivial bundle $\pi : G_{q,n} \times \mathbb{R}^n \to G_{q,n}$ given by $\widetilde{\Phi}(\eta) = \nabla_x \Phi(\eta, x)$. Write $\widetilde{\Phi}(\eta) = (\widetilde{\Phi}_1(\eta), \cdots, \widetilde{\Phi}_n(\eta))$. Equations (6.7) and (6.8) imply that $\widetilde{\Phi}$ satisfies the two conditions

$$\begin{aligned} &\sum_{r=1}^{2p+1} (-1)^{r-1} \, (W_{J\setminus\{j_r\}} \, \widetilde{\Phi}_{j_r})(\eta) = 0, \\ &\int_{\eta \supset \sigma} \langle \, \widetilde{\Phi}(\eta), w \, \rangle \, d\eta = 0, \end{aligned} \tag{6.9}$$

for each $w \in \sigma$. Our objective is to prove that $\widetilde{\Phi}(\eta) \perp \eta$, given these two conditions.

It is more convenient to orthogonally decompose $\widetilde{\Phi}(\eta)$ into components parallel and perpendicular to η: write

$$\widetilde{\Phi}(\eta) = \overline{\Phi}(\eta) + \Psi(\eta), \tag{6.10}$$

where $\overline{\Phi}(\eta) \perp \eta$ and $\Psi(\eta) \in \eta$. We of course wish to prove that $\Psi(\eta) = 0$ for each $\eta \in G_{q,n}$.

The following technical lemma allows us to reduce the problem to one involving the section Ψ:

Lemma 6.1. *Let $J \in T_{2m+1}$; $J : j_1 < j_2 < \cdots < j_{2m+1}$. For any $\eta \in G_{q,n}$, we have*

$$\sum_{r=1}^{2m+1} (-1)^{r-1} \, (W_{J\setminus\{j_r\}} \, \overline{\Phi}_{j_r})(\eta) = 0. \tag{6.11}$$

Proof. By hypothesis, $\overline{\Phi}(\eta) \perp \eta$. Let $\tau : G_{p,n} \to G_{q,n}$, $\tau(\sigma) = \sigma^\perp$ be the natural bijection of $G_{p,n}$ onto $G_{q,n}$. Then the mapping $T = \overline{\Phi} \circ \tau$ is a smooth section of the universal bundle $U_{p,n} = \{(\sigma, x) \in G_{p,n} \times \mathbb{R}^n \mid x \in \sigma\}$ of $G_{p,n}$. We identify $\mathbb{R}^n$ with the space of $n \times 1$ column vectors and we write $T(\sigma) = (T_1(\sigma), \cdots, T_n(\sigma))$. Since τ is $SO(n)$-equivariant, (6.11) is equivalent to the condition

$$\sum_{r=1}^{2p+1} (-1)^{r-1} \, (W_{J\setminus\{j_r\}} \, T_{j_r})(\sigma) = 0, \tag{6.12}$$

for any $\sigma \in G_{p,n}$, and any $J \in T_{2p+1}$.

In order to prove (6.12), we first note that $SO(n)$ acts on smooth sections of the universal bundle $U_{p,n} \to G_{p,n}$ via $(u \cdot F)(\sigma) = (uF)(u^{-1} \cdot \sigma)$, for any $u \in SO(n)$, $\sigma \in G_{p,n}$.

Now for any $J \in T_{2p+1}$, we denote the left-hand side of (6.12) by T_J. (Note that this is a slight abuse of notation.)

Claim. $(u\cdot T)_J(\sigma) = \sum_{M\in T_{2p+1}} \det(u_{JM})\, T_M(u^{-1}\cdot\sigma)$, for any $u \in SO(n)$, $\sigma \in G_{p,n}$.

We prove this claim by a straightforward, but somewhat lengthy, calculation, as follows:

(6.13)

$$\begin{aligned}
(u\cdot T)_J(\sigma) &= \sum_{r=1}^{2p+1} (-1)^{r-1}\, W_{J\setminus\{j_r\}}\, (u\cdot T)_{j_r}(\sigma)\\
&= \sum_{r=1}^{2p+1} (-1)^{r-1}\, (W_{J\setminus\{j_r\}})_\sigma \,(\sum_{l=1}^{n} u_{j_r,l}\, T_l(u^{-1}\cdot\sigma))\\
&= \sum_{r=1}^{2p+1}\sum_{l=1}^{n} (-1)^{r-1}\, u_{j_r,l}\, (W_{J\setminus\{j_r\}})_\sigma T_l(u^{-1}\cdot\sigma)\\
&= \sum_{r=1}^{2p+1}\sum_{l=1}^{n} (-1)^{r-1}\, u_{j_r,l}\, (\,(\mathrm{Ad}(u^{-1})W_{J\setminus\{j_r\}})T_l\,)(u^{-1}\cdot\sigma)\\
&= \sum_{r=1}^{2p+1}\sum_{l=1}^{n}\sum_{L\in T_{2p}} (-1)^{r-1}\, u_{j_r,l}\, \det(u^{-1})_{L,J\setminus\{j_r\}}\, (W_L T_l)(u^{-1}\cdot\sigma)\\
&= \sum_{l=1}^{n}\sum_{L\in T_{2p}}\sum_{r=1}^{2p+1} (-1)^{r-1}\, u_{j_r,l}\, \det(u)_{J\setminus\{j_r\},L}\, (W_L T_l)(u^{-1}\cdot\sigma).
\end{aligned}$$

Let $L \in T_{2p}$ be given by $L : l_1 < l_2 < \cdots < l_{2p}$. Then

$$\sum_{r=1}^{2p+1} (-1)^{r-1}\, u_{j_r,l}\, \det(u)_{J\setminus\{j_r\},L} \;=\; \begin{vmatrix} u_{j_1,l} & u_{j_1,l_1} & \cdots & u_{j_1,l_{2p}}\\ u_{j_2,l} & u_{j_2,l_1} & \cdots & u_{j_2,l_{2p}}\\ & \cdots & \cdots & \\ u_{j_{2p+1},l} & u_{j_{2p+1},l_1} & \cdots & u_{j_{2p+1},l_{2p}} \end{vmatrix}. \tag{6.14}$$

The latter determinant equals zero if $l \in L$, and equals $\epsilon(l,L)\,\det(u_{J,L\cup\{l\}})$ if $l \notin L$. Thus the last expression in (6.13) equals

$$\sum_{\substack{1\le l\le n\\ L\in T_{2p},\ l\notin L}} \epsilon(l,L)\,\det(u_{J,L\cup\{l\}})\,(W_L T_l)(u^{-1}\cdot\sigma). \tag{6.15}$$

We can rewrite the above sum by setting $M = L\cup\{l\}$ (for $l \notin L$, so $M \in T_{2p+1}$), to obtain

$$\begin{aligned}
&\sum_{M\in T_{2p+1}} \det(u_{JM}) \sum_{k=1}^{2p+1} (-1)^{k-1}\,(W_{M\setminus\{m_k\}}T_{m_k})(u^{-1}\cdot\sigma)\\
&= \sum_{M\in T_{2p+1}} \det(u_{JM})\,(T_M)(u^{-1}\cdot\sigma).
\end{aligned} \tag{6.16}$$

In the above, we have written $M : m_1 < m_2 < \cdots < m_{2p+1}$. This completes the proof of the claim.

The condition (6.12), which is all that we need in order to prove Lemma 6.1, is equivalent to the condition $T_J(\sigma) = 0$ for each $J \in T_{2p+1}$, $\sigma \in G_{p,n}$. So fix J and

σ. Let $\sigma_0 = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_p \in G_{p,n}$, and choose an element $u \in SO(n)$ such that $u^{-1} \cdot \sigma_0 = \sigma$. Then by the claim above

$$\begin{aligned} T_J(\sigma) &= (u \cdot (u^{-1} \cdot T))_J(\sigma) \\ &= \sum_{M \in T_{2p+1}} \det(u_{JM})\, (u^{-1} \cdot T)_M(\sigma_0). \end{aligned} \tag{6.17}$$

Let $S = u^{-1} \cdot T$; S is a section of the universal bundle $U_{p,n} \to G_{p,n}$. It suffices for us to prove that $S_M(\sigma_0) = 0$ for each $M \in T_{2p+1}$.

Again writing $M : m_1 < m_2 < \cdots < m_{2p+1}$, we have

$$S_M(\sigma_0) = \sum_{k=1}^{2p+1} (-1)^{k-1} (W_{M \setminus \{m_k\}} S_{m_k})(\sigma_0). \tag{6.18}$$

We show that each summand in the right-hand side above equals zero. This simply involves an index count. Now $(W_{M\setminus\{m_k\}} S_{m_k})(\sigma_0)$ is a sum of terms of the form

$$(X_{r_1,r_2} \cdots X_{r_{2p-1},r_{2p}} S_{m_k})(\sigma_0), \tag{6.19}$$

where $(r_1, \cdots, r_{2p}) \in \mathfrak{S}(M \setminus \{m_k\})$. If $m_k > p$, then we write

(6.20)

$$\begin{aligned} &(X_{r_1,r_2} \cdots X_{r_{2p-1},r_{2p}} S_{m_k})(\sigma_0) \\ &= \left\{ \frac{\partial^p}{\partial t_1 \cdots \partial t_p} S_{m_k}(\exp(-t_1 X_{r_1,r_2}) \cdots \exp(-t_p X_{r_{2p-1},r_{2p}}) \cdot \sigma_0) \right\}_{t_1 = \cdots = t_p = 0}. \end{aligned}$$

In this case, $\exp(-t_1 X_{r_1,r_2}) \cdots \exp(-t_p X_{r_{2p-1},r_{2p}}) \cdot \sigma_0$ is a p-plane in $e_{m_k}^\perp$, so

$$S_{m_k}(\exp(-t_1 X_{r_1,r_2}) \cdots \exp(-t_p X_{r_{2p-1},r_{2p}}) \cdot \sigma_0) = 0. \tag{6.21}$$

If $m_k \le p$, it is easy to see that there exists a pair (r_{2k-1}, r_{2k}) of indices belonging to $\{p+1, \cdots, n\}$. Then in this case we write

$$\begin{aligned} &(X_{r_1,r_2} \cdots X_{r_{2p-1},r_{2p}} S_{m_k})(\sigma_0) \\ &= \frac{d}{dt} (X_{r_1,r_2} \cdots \widehat{X}_{r_{2k-1},r_{2k}} \cdots X_{r_{2p-1},r_{2p}} S_{m_k})(\exp(-t X_{r_{2k-1},r_{2k}}) \cdot \sigma_0)|_{t=0}. \end{aligned} \tag{6.22}$$

But $\exp(-t X_{r_{2k-1},r_{2k}}) \cdot \sigma_0 = \sigma_0$, so the above term also equals zero. This completes the proof of Lemma 6.1. $\square$

Let us write $\Psi(\eta) = (\Psi_1(\eta), \cdots, \Psi_n(\eta))$. Ψ is of course a smooth section of the universal bundle $U_{q,n} \to G_{q,n}$; by (6.9) and (6.11), Ψ also satisfies the Pfaffian system

$$\sum_{r=1}^{2p+1} (-1)^{r-1} (W_{J\setminus\{j_r\}}\, \Psi_{j_r})(\eta) = 0, \tag{6.23}$$

for each $\eta \in G_{q,n}$ and $J \in T_{2p+1}$. In the notation introduced before the claim during the proof of Lemma 6.1, Ψ thus satisfies the system $\Psi_J = 0$ for each $J \in T_{2p+1}$. In addition, the section $\overline{\Phi}$ given in (6.10) trivially satisfies the integral condition

$$\int_{\eta \supset \sigma} \langle\, \overline{\Phi}(\eta), w \,\rangle\, d\eta = 0 \qquad (\sigma \in G_{p,n}), \tag{6.24}$$

for each $w \in \sigma$, since $\overline{\Phi}(\eta) \perp \eta$, so by (6.9), we must have

$$\int_{\eta \supset \sigma} \langle \Psi(\eta), w \rangle \, d\eta = 0 \qquad (\sigma \in G_{p,n}), \tag{6.25}$$

for all $w \in \sigma$.

The proof of the range theorem for $\mathcal{R}^{(p,q)}$, where $p + q = n$, thus reduces to showing that any smooth section Ψ of the universal bundle $U_{q,n} \to G_{q,n}$ satisfying

$$\Psi_J = 0 \qquad \text{for all } \; J \in T_{2p+1} \tag{6.26}$$

and

$$\int_{\eta \supset \sigma} \langle \Psi(\eta), w \rangle \, d\eta = 0 \qquad (\sigma \in G_{p,n}, \; w \in \sigma) \tag{6.27}$$

must vanish.

We prove this last statement in the case $q = n - 1$. Assume then that Ψ is a section of the universal bundle $U_{n-1,n} \to G_{n-1,n}$ satisfying $\Psi_J = 0$ for all $J \in T_3$ and

$$\int_{\eta \supset \sigma} \langle \Psi(\eta), v \rangle \, d\eta = 0 \qquad (\sigma \in G_{1,n}, \; v \in \sigma). \tag{6.28}$$

We wish to show that Ψ vanishes.

We identify $G_{n-1,n}$ with $\mathbb{RP}^{n-1}$. Let $\widetilde{\pi}$ denote the (twofold) projection of $\mathbf{S}^{n-1}$ onto $G_{n-1,n}$ given by $\widetilde{\pi}(\omega) = \omega^{\perp}$, and let $T = \Psi \circ \widetilde{\pi}$. Since $T(\omega) \perp \omega$ and $T(-\omega) = T(\omega)$ for all $\omega \in \mathbf{S}^{n-1}$, we may consider T to be a smooth, even vector field on $\mathbf{S}^{n-1}$. Since $\widetilde{\pi}$ commutes with the $SO(n)$-actions on $\mathbf{S}^{n-1}$ and $G_{n-1,n}$, respectively, the conditions (6.26) and (6.27) on Ψ imply that

$$T_J = 0, \qquad\qquad J \in T_3, \tag{6.29}$$

$$\int_{\omega \in \mathbf{S}^{n-1}, \omega \perp v} \langle T(\omega) \, , \, v \rangle d_v\omega \;=\; 0, \tag{6.30}$$

for each $v \in \mathbf{S}^{n-1}$. Here $d_v\omega$ denotes the area element on the totally geodesic submanifold $A(v) = v^{\perp} \cap \mathbf{S}^{n-1}$. Writing $T(\omega) = (T_1(\omega), \cdots, T_n(\omega))$, the condition (6.29) becomes

$$(X_{ij}T_k - X_{ik}T_j + X_{jk}T_i)(\omega) = 0, \tag{6.31}$$

for all $1 \leq i < j < k \leq n$.

First we show that these equations imply that $T = \text{grad} f$ for a smooth function f on $\mathbf{S}^{n-1}$. For this, we extend T to a smooth vector field $\widetilde{T}$ on a small spherical shell $1 - \epsilon < ||x|| < 1 + \epsilon$ in $\mathbb{R}^n$, with $\widetilde{T}(r\omega) = T(\omega)$ for all $r \in (1 - \epsilon, 1 + \epsilon)$. Since $\widetilde{T}$ is radially constant, equation (6.31) still holds on the spherical shell, with $\widetilde{T}$ replacing T and x replacing ω. From the natural identification of vector fields and 1-forms on $\mathbb{R}^n$, $\widetilde{T}$ corresponds to the 1-form $\widetilde{\tau} = \sum_{i=1}^n T_i dx_i$, and T corresponds to the pull-back τ of $\widetilde{\tau}$ to $\mathbf{S}^{n-1}$. Now $d\widetilde{\tau} = \sum_{i<j} (\frac{\partial T_j}{dx_i} - \frac{\partial T_i}{dx_j}) dx_i dx_j$; $d\tau$ has the same expression. We show that $d\tau = 0$ by showing that $d\tau$ vanishes on each set of a finite open cover of $\mathbf{S}^{n-1}$. On the open subset $x_n \neq 0$ of $\mathbf{S}^{n-1}$, we can replace

dx_n by $-\frac{1}{x_n}\sum_{k=1}^{n-1} x_k dx_k$; the coefficient of $dx_i dx_j$ then becomes

$$\begin{aligned}
&\left(\frac{\partial T_j}{dx_i}-\frac{\partial T_i}{dx_j}\right)-\frac{x_j}{x_n}\left(\frac{\partial T_n}{dx_i}-\frac{\partial T_i}{dx_n}\right)+\frac{x_i}{x_n}\left(\frac{\partial T_n}{dx_j}-\frac{\partial T_j}{dx_n}\right)\\
&=\frac{1}{x_n}\left[\left(x_i\frac{\partial T_n}{dx_j}-x_j\frac{\partial T_n}{dx_i}\right)-\left(x_i\frac{\partial T_j}{dx_n}-x_n\frac{\partial T_j}{dx_i}\right)+\left(x_j\frac{\partial T_i}{dx_n}-x_n\frac{\partial T_i}{dx_j}\right)\right]\\
&=\frac{1}{x_n}\left[X_{ij}\cdot T_n-X_{in}\cdot T_j+X_{jn}\cdot T_i\right]\\
&=0, \qquad\qquad \text{(by (6.31))}.
\end{aligned} \tag{6.32}$$

We can likewise show that $d\tau$ vanishes on each open subset $x_i \neq 0$ of $\mathbf{S}^{n-1}$; hence $d\tau = 0$ on $\mathbf{S}^{n-1}$. Since the first cohomology group of $\mathbf{S}^{n-1}$ vanishes, this implies that $\tau = df$ for some smooth function f on $\mathbf{S}^{n-1}$; under the identification of T with τ, we get $T = \mathrm{grad} f$. Resolving f into odd and even components, we note that the even component must be constant, so we can assume that f is an odd function on $\mathbf{S}^{n-1}$.

For each $v \in \mathbf{S}^{n-1}$, let $H(v)$ denote the hemisphere opposite to v: $H(v) = \{\omega \in \mathbf{S}^{n-1} \mid \langle \omega, v\rangle < 0\}$. Since v is the outward-pointing unit normal on its boundary in $\mathbf{S}^{n-1}$, we can apply the divergence theorem (on $\mathbf{S}^{n-1}$) to the integral in (6.30) to obtain

$$0 = \int_{A(v)} \langle\, v\,, \mathrm{grad} f(\omega)\,\rangle \; d_v\omega = \int_{H(v)} L_{\mathbf{S}^{n-1}}\, f(\omega) d\omega. \tag{6.33}$$

Here $L_{\mathbf{S}^{n-1}}$ denotes the Laplace-Beltrami operator and $d\omega$ denotes the volume element on $\mathbf{S}^{n-1}$. (6.33) shows that $L_{\mathbf{S}^{n-1}} f$ is a smooth odd function on $\mathbf{S}^{n-1}$ whose integral vanishes on all hemispheres. The following lemma shows that the only such function is zero. Since $\mathbf{S}^{n-1}$ is compact, this will imply that f is constant, hence zero, so that $T = \mathrm{grad}_{\mathbf{S}^{n-1}} f = 0$.

Lemma 6.2. *Let γ denote the hemisphere transform on $\mathbf{S}^{n-1}$,*

$$(\gamma(h))(v) = \int_{H(v)} h(\omega)\, d\omega, \qquad \textit{for } h \in C^\infty(\mathbf{S}^{n-1}).$$

Then $\gamma(h) = 0$ only if h is even.

Proof. Since γ commutes with the action of $SO(n)$ on $\mathbf{S}^{n-1}$, it can be diagonalized. Let $\mathcal{H}_j$ denote the space of degree j spherical harmonics on $\mathbf{S}^{n-1}$, for $j = 0, 1, 2, \cdots$. $SO(n)$ acts irreducibly on each $\mathcal{H}_j$ so γ must equal a scalar operator c_j there. c_j can be calculated by integrating a zonal harmonic in $\mathcal{H}_j$ over an appropriate hemisphere; this gives

$$c_j = \int_0^1 C_j^{\frac{n-2}{2}}(x)\,(1-x^2)^{\frac{n-3}{2}}\, dx,$$

where C_j^ν is a Gegenbauer polynomial of degree j. From Formula 7.311, page 826 of the book by Gradshteyn and Ryzhik [GR], we obtain

$$c_j = \begin{cases} \Gamma(\frac{n-1}{2}) & j = 0,\\ 0 & j \ \text{even},\\ \frac{\Gamma(n-2+j)\Gamma(\frac{n-1}{2})}{j!\Gamma(n-2)\Gamma(-\frac{j}{2})\Gamma(\frac{n+j}{2})} & j \ \text{odd}. \end{cases} \tag{6.34}$$

This establishes the lemma. ∎

Here we note that more information about the hemispherical transform can be found in [Ru1].

From the argument above, we obtain the following range characterization theorem for $\mathcal{R}^{(n-1,1)}$.

Theorem 6.1. *The image* $\mathcal{R}^{(n-1,1)}\left(C^\infty(G(n-1,n))\right)$ *equals the space* $\{\, f \in C^\infty(G(1,n)) \,|\, d\lambda(V_J)f = 0, \text{ for } J \in T_3 \,\}$.

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