FINITE TIME BLOW-UP FOR A DYADIC MODEL
OF THE EULER EQUATIONS

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ABSTRACT. We introduce a dyadic model for the Euler equations and the
Navier-Stokes equations with hyper-dissipation in three dimensions. For the
dyadic Euler equations we prove finite time blow-up. In the context of the
dyadic Navier-Stokes equations with hyper-dissipation we prove finite time
blow-up in the case when the dissipation degree is sufficiently small.

1. INTRODUCTION

The purpose of the paper is to discuss a dyadic model for the unforced in-
compressible constant density Euler equations in three dimensions and to show
conditions under which this model and certain dissipative modifications blow up.

The Euler equations are a system of equations for an unknown time-varying
vector field u on $\mathbf{R}^3$, the velocity field of a fluid and an unknown function p, its
pressure. The equations are

$$(1.1) \qquad \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p,$$

together with the divergence free condition

$$\nabla \cdot u = 0.$$

We let T be the L^2 orthogonal projection into divergence free vector fields, and
applying it to the equation (1.1), we get

$$\frac{\partial u}{\partial t} + T((u \cdot \nabla)u) = 0.$$

Thus we think of the Euler equations as saying that the change in the velocity u is
given as a certain bilinear operator of u. Of course, the Euler equations conserve
energy which is defined as $||u||_{L^2}^2$. More precisely, for every $t > 0$ one has

$$||u(.,t)||_{L^2} = ||u_0||_{L^2},$$

with $u_0 = u(x,0)$. This follows from the identity

$$\langle T((u \cdot \nabla)u), u \rangle = 0,$$

where $\langle, \rangle$ is the L^2 pairing.

Some basic questions concerning Euler equations are still unsolved. For example,
it is an open problem to find out if solutions of the Euler equations in 3 dimensions
satisfying bounded energy condition (i.e., $||u_0||_{L^2} = C$) form singularities in finite

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time. However, the local in time existence of solutions for 3D Euler equations is known in the Sobolev space H^s, $s > \frac{5}{2}$; see, for example, [7] or [11].

The Euler equations describe the motion of an inviscid fluid. To describe a viscous fluid, one adds a viscosity term. The resulting equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p - (-\Delta)u,$$

are called the Navier-Stokes equations.

Sometimes to interpolate these equations one can study a family of intermediate equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p - (-\Delta)^\alpha u,$$

where $\alpha > 0$. These are called the Navier-Stokes equations with dissipation α.

It is easy to see that classical solutions to the Navier-Stokes equations with dissipation α satisfy decay of energy, i.e., for every $t > 0$ one has

$$||u(.,t)||_{L^2}^2 = ||u_0||_{L^2}^2 - 2\int_0^t \nu\langle(-\Delta)^\alpha u, u\rangle.$$

As in the case of the Euler equations the theory of the Navier-Stokes equations in three dimensions is far from being complete. The major open problem is the question of global existence of smooth solutions of the Navier-Stokes equations in 3 dimensions satisfying bounded energy condition, where by a smooth solution in $\mathbb{R}^3$ one means a solution to the Navier-Stokes equations such that $u(x,t) \in C^\infty[\mathbb{R}^3 \times \mathbb{R}^+]$. For the precise formulation of this open problem and a review of known results (in particular, local in time existence results, global existence results with small initial data and partial regularity results) see [4].

Many results which can be obtained about Euler and Navier-Stokes equations use principally the conservation (decay) of energy and the scaling properties of the nonlinear term $\langle T((u \cdot \nabla)u), u\rangle$, most notably partial regularity results for the Navier-Stokes equations of Caffarelli, Kohn and Nirenberg [2], and a criterion for loss of regularity for solutions of the Euler equations of Beale, Kato, and Majda [1]. We discuss certain dyadic models for the equations which share these properties.

The dyadic model presented here is an infinite system of nonlinearly coupled ODEs. Each ODE illustrates time evolution of a wavelet coefficient which describes behavior of the velocity that is localized to a certain frequency range. Therefore, the dyadic model could be understood in a general context of Littlewood-Paley theory, which allows localization of a function into frequency ranges. Recently Littlewood-Paley theory was extensively used in studying the Navier-Stokes equations. One of the reasons why Littlewood-Paley techniques are useful in applications to fluid equations is because they allow paraproduct decomposition of the nonlinear term $\langle T((u \cdot \nabla)u), u\rangle$, which in turn helps isolate certain frequency interactions. For an overview of existence results for the Navier-Stokes equations that use such an approach see, for example, [3]. Also each ODE in our model reflects behavior of the velocity localized in space on a dyadic cube. Here we used a standard harmonic analysis approach of decomposing $\mathbb{R}^3$ into dyadic cubes. Using such a decomposition we study flows on the dyadic tree.

We first used dyadic models in our work on partial regularity for the Navier-Stokes equations with hyper-dissipation [8] where they were a helpful test case for our ideas. On the other hand, by following harmonic analysis techniques one can obtain existence results for dyadic models that correspond to the known existence

results for the actual Euler and Navier-Stokes equations. This is possible thanks to the fact that our model equations share the energy conservation and scaling properties of the actual equations. An example of such a local existence result for the dyadic Euler equations is given in [5]. Also we note that the dyadic models belong to a general class of shell models introduced by Gledzer [6] and Ohkitani and Yamada [10] and are used for investigating turbulence.

In this paper, we obtain blow-up for models corresponding to the Euler equations and to the Navier-Stokes equations with dissipation α with $\alpha < \frac{1}{4}$. Our results work because in our models energy travels always from low frequency to high frequency. However, in the Euler equations, one of the principal ways that energy travels through frequency space is by the way in which the flow deforms space, because it is incompressible. We believe that it is likely the study of geometric behavior could lead to a real understanding of the Euler and Navier-Stokes equations.

The paper is organized as follows. In section 2 we introduce dyadic models. In section 3 we give a blow-up result for the dyadic Euler equations. Then in section 4 we present a proof of finite time blow-up for the dyadic Navier-Stokes equations with small dissipation.

2. Dyadic model

Here we shall introduce a dyadic model for the equations of fluid motion in three dimensions.

We define a dyadic cube in a standard way. A cube Q in $\mathbb{R}^3$ is a dyadic cube if its side length is an integer power of 2, 2^l, and the corners of the cube are on the lattice $2^l \mathbb{Z}^3$.

We let $\mathcal{D}$ denote the set of dyadic cubes in $\mathbb{R}^3$. We let $\mathcal{D}_j$ denote the subset of dyadic cubes having side length 2^{-j}. Abusing notation slightly we define the function

$$j : \mathcal{D} \longrightarrow \mathbb{Z},$$

by letting $j(Q) = j$ if $Q \in \mathcal{D}_j$. We define $\tilde{Q}$, the parent of Q, to be the unique dyadic cube in $\mathcal{D}_{j(Q)-1}$ which contains Q. For $m \geq 1$ we define $\mathcal{C}^m(Q)$, the mth order grandchildren of Q to be the set of those cubes in $\mathcal{D}_{j(Q)+m}$ which are contained in Q. We sometimes refer to the first order grandchildren of Q as the children of Q.

In our dyadic model we consider a scalar-valued function $u(x,t)$. It is represented by a wavelet expansion

$$u(x,t) = \sum_Q u_Q(t) w_Q(x),$$

where $\{w_Q\}$ is an orthonormal family in $L^2(\mathbb{R}^3)$ of wavelets such that the wavelet w_Q is localized at the spatial dyadic cube $Q \in \mathcal{D}_j$. The wavelet coefficient corresponding to the cube Q is denoted by $u_Q(t)$. We will refer to the values $u_Q(t)$ as the coefficients of the function u.

We define the dyadic Laplacian Δ by

$$\Delta(w_Q) = 2^{2j(Q)} w_Q.$$

We define $||u||_{L^2}$ to denote the L^2 norm of u and for any $\alpha > 0$, we define

$$||u||_{H^{2\alpha}} = ||u||_{L^2} + ||(\Delta)^\alpha u||_{L^2}.$$

We define $\langle u, v \rangle$ to denote the L^2 pairing of u and v. Since our wavelets w_Q are orthonormal in $L^2(\mathbb{R}^3)$, we notice that

$$\langle u, v \rangle = \sum_Q u_Q(t) v_Q(t).$$

We would like to have an operator which will mimic the behavior of the nonlinear term $T(u \cdot \nabla u)$. We define a bilinear operator, the cascade operator, by defining two pieces from which it is built. The cascade down operator is defined by

$$(C_d(u, v))_Q = 2^{\frac{5j(Q)}{2}} u_{\tilde{Q}} v_{\tilde{Q}}.$$

We define the cascade up operator by

$$(C_u(u, v))_Q = 2^{\frac{5(j(Q)+1)}{2}} u_Q \sum_{Q' \in C^1(Q)} v_{Q'}.$$

We define the cascade operator

$$C(u, v) = C_u(u, v) - C_d(u, v).$$

Obviously,

(2.1) $$\langle C_u(u, u), u \rangle = \langle C_d(u, u), u \rangle,$$

which implies

(2.2) $$\langle C(u, u), u \rangle = 0.$$

The choice of the exponent $\frac{5}{2}$ comes from the scaling properties of $u \cdot \nabla u$. A heuristic motivation is as follows. If w_Q were an L^2 normalized (vector-valued) wavelet supported on $Q \in \mathcal{D}_j$, it would have

$$\|w_Q\|_{L^\infty} \sim 2^{\frac{3j}{2}}.$$

Moreover, it would satisfy

$$\|\nabla w_Q\|_{L^\infty} \sim 2^{\frac{5j}{2}},$$

since differentiating adds a factor of 2^j. Thus one would have

$$|w_Q \cdot \nabla w_Q| \lesssim 2^{4j} \lesssim 2^{\frac{5j}{2}} \|w_Q\|_{L^\infty}.$$

Having defined operator $C(u, v)$ we can speak about the dyadic version of the Euler as well as the Navier-Stokes equations. More precisely, we shall say that a time varying function u satisfies the dyadic Euler equations provided that

(2.3) $$\frac{du}{dt} + C(u, u) = 0.$$

Also we shall say that u satisfies the dyadic Navier-Stokes equations with hyperdissipation if

(2.4) $$\frac{du}{dt} + C(u, u) + (\Delta)^\alpha u = 0.$$

We will restrict our attention to functions u all of whose coefficients u_Q are initially nonnegative. This class of functions is preserved by both flows (2.3) and (2.4). Let us verify that for the flow (2.3). Fix a dyadic cube Q. We rewrite the equation (2.3) in terms of wavelet coefficients as follows:

(2.5) $$\frac{du_Q(t)}{dt} + 2^{\frac{5(j(Q)+1)}{2}} u_Q \sum_{Q' \in C^1(Q)} u_{Q'}(t) = 2^{\frac{5j(Q)}{2}} u_{\tilde{Q}}^2(t).$$

Now we remark that the equation (2.5) is a first order linear ordinary differential equation in $u_Q(t)$ and its solution is

$$(2.6) \qquad u_Q(t) = \frac{1}{\mu(t)} \left(u_Q(0) + \int_0^t 2^{\frac{5j(Q)}{2}} u_{\tilde{Q}}^2(\tau) \mu(\tau) d\tau \right),$$

where

$$\mu(t) = e^{\int_0^t 2^{\frac{5(j(Q)+1)}{2}} \sum_{Q' \in C^1(Q)} u_{Q'}(\tau) d\tau}.$$

Since $\mu(t) > 0$ for all t, (2.6) implies that $u_Q(t) \geq 0$ for all $t > 0$, provided that $u_Q(0)$ is nonnegative.

3. The dyadic Euler equations

3.1. Energy flow. One of the most important features of the flow (2.3) is that it conserves energy. To be more precise,

$$(3.1) \qquad \frac{d}{dt}(\langle u, u \rangle) = 0.$$

This can be obtained by pairing (2.3) with u. Energy can be thought of as divided up amongst the nodes Q. To be more precise, if we write

$$E = \langle u, u \rangle,$$

then

$$E = \sum_{Q \in \mathcal{D}} E_Q,$$

where

$$E_Q = u_Q^2.$$

The flow (2.3) gives rise to an extremely local description of energy flow along the tree $\mathcal{D}$.

To be precise

$$(3.2) \qquad \frac{d}{dt} E_Q = E_{Q,in} - E_{Q,out},$$

where

$$E_{Q,in} = 2^{\frac{5j(Q)}{2}} (2E_{\tilde{Q}} \sqrt{E_Q}),$$

and

$$E_{Q,out} = \sum_{Q' \in \mathcal{C}^1(Q)} E_{Q',in}.$$

Thus energy is flowing always from larger squares to smaller ones and indeed it flows along the edges of the tree $\mathcal{D}$.

We define a Carleson box by

$$\mathcal{C}(Q) = \bigcup_{k=1}^{\infty} \mathcal{C}^k(Q),$$

and the energy of a Carleson box by

$$E_{\mathcal{C}(Q)} = \sum_{Q' \in \mathcal{C}(Q)} E_{Q'}.$$

Also we shall introduce an extended Carleson box by

$$\mathcal{C}_0(Q) \equiv Q \cup \mathcal{C}(Q).$$

Then we define the energy of $\mathcal{C}_0(Q)$ by

$$E_{\mathcal{C}_0(Q)} = \sum_{Q' \in \mathcal{C}_0(Q)} E_{Q'}.$$

By using (2.1) we sum (3.2) over all cubes $Q \in \mathcal{C}(Q)$ and $Q \in \mathcal{C}_0(Q)$ respectively, to obtain the following proposition.

Proposition 3.1.1. *Let u be a time-varying "function" with positive coefficients evolving according to the flow (2.3). Then for any Q, the functions in time given by $E_{\mathcal{C}(Q)}$ and $E_{\mathcal{C}_0(Q)}$ are monotone increasing.*

3.2. **The heart.** We begin with an easy lemma about Carleson boxes.

Lemma 3.2.1. *For any $\epsilon > 0$, there is $\delta(\epsilon) > 0$, so that if we know that*

$$E_{\mathcal{C}(Q)} > (1 - \delta)2^{-(3+\epsilon)j(Q)},$$

then there exists $Q' \in \mathcal{C}^1(Q)$, so that

$$E_{\mathcal{C}_0(Q')} \geq 2^{-(3+\epsilon)j(Q')}.$$

Proof. Let us suppose the conclusion of the lemma is false, i.e.,

$$(3.3) \qquad\qquad E_{\mathcal{C}_0(Q')} < 2^{-(3+\epsilon)j(Q')},$$

for all $Q' \in \mathcal{C}^1(Q)$.

On the other hand, we know

$$(3.4) \qquad\qquad E_{\mathcal{C}(Q)} = \sum_{Q' \in \mathcal{C}^1(Q)} E_{\mathcal{C}_0(Q')}.$$

Now because we are in dimension 3, there are exactly 2^3 elements $Q' \in \mathcal{C}^1(Q)$ with $j(Q') = j(Q) + 1$. Thus (3.4) combined with (3.3) implies

$$E_{\mathcal{C}(Q)} \leq 2^3 \cdot 2^{-(3+\epsilon)(j(Q)+1)},$$

and therefore,

$$E_{\mathcal{C}(Q)} \leq 2^{-\epsilon} \cdot 2^{-(3+\epsilon)j(Q)}$$

which is a contradiction provided we have chosen δ sufficiently small such that $2^{-\epsilon} < 1 - \delta$. $\qquad\square$

Now we prove the main lemma.

Lemma 3.2.2. *Fix j_0 sufficiently large. Then there is a sufficiently small $0 < \epsilon < 1$ so that if at time t_0 we have*

$$(3.5) \qquad\qquad E_{\mathcal{C}_0(Q)} \geq 2^{-(3+\epsilon)j(Q)},$$

with $j(Q) > j_0$, then there is some t with $t < t_0 + 2^{-\epsilon j(Q)}$ and a cube $Q' \in \mathcal{C}^1(Q)$ so that at time t we have $E_{\mathcal{C}_0(Q')} \geq 2^{-(3+\epsilon)j(Q')}$.

Proof. We assume that the conclusion of the lemma is false, i.e.,

$$E_{\mathcal{C}_0(Q')} < 2^{-(3+\epsilon)j(Q')},$$

for all $Q' \in \mathcal{C}^1(Q)$ and for all $t \in [t_0, t_0 + 2^{-\epsilon j(Q)}]$.

In light of Lemma 3.2.1 and Proposition 3.1.1, it must be the case that for all $t \in [t_0, t_0 + 2^{-\epsilon j(Q)}]$ we have $E_Q \geq \delta 2^{-(3+\epsilon)j(Q)}$, since otherwise because of the

hypothesis (3.5) we would have $E_{\mathcal{C}(Q)} > (1-\delta)2^{-(3+\epsilon)j(Q)}$, which would by Lemma 3.2.1 lead to a contradiction.

Moreover, since energy flows only in the direction of smaller cubes and is conserved, it must be that for any child $Q' \in \mathcal{C}^1(Q)$, it is the case that

$$E_{\mathcal{C}(Q)} \geq \int_{t_0}^{t_0+2^{-\epsilon j(Q)}} E_{Q',in},$$

at time $t_0 + 2^{-\epsilon j(Q)}$. Thus since

$$E_{Q',in} = 2^{\frac{5(j(Q)+1)}{2}} 2E_Q u_{Q'},$$

we must have

$$(3.6) \qquad \int_{t_0}^{t_0+2^{-\epsilon j(Q)}} u_{Q'} \lesssim \frac{1-\delta}{\delta} 2^{\frac{-5j(Q)}{2}}.$$

However, we know that

$$(3.7) \qquad \frac{du_{Q'}}{dt} = 2^{\frac{5(j(Q)+1)}{2}} E_Q - 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big).$$

Thus integrating, applying the fact that for all $t \in [t_0, t_0 + 2^{-\epsilon j(Q)}]$, we have $E_Q > \delta 2^{-(3+\epsilon)j(Q)}$, and $u_{Q''} \lesssim 2^{-\frac{3j(Q)}{2}}$ (we can afford to give back the ϵ), and using (3.6), we see that the first term of (3.7) dominates (for ϵ sufficiently large and $j(Q)$ sufficiently small) and

$$\int_{t_0}^{t_0+2^{-\epsilon j(Q)}} \frac{du_{Q'}}{dt} \gtrsim \delta 2^{-(\frac{1}{2}+2\epsilon)j(Q)},$$

which is a contradiction of our upper bound on $u_{Q'}$ by the fundamental theorem of calculus. $\qquad\square$

Corollary 3.2.3. *Let u be a solution to (2.3) which has initially all positive coefficients, and at time 0 has $E_Q > 2^{-(3+\epsilon)j(Q)}$, for some Q with $j(Q) > j_0$ with j_0 as in the previous lemma. Then the $H^{\frac{3}{2}+\epsilon}$ norm of u becomes unbounded in finite time.*

Proof. We apply the lemma. We find a cube Q_1 properly contained in Q and a time $t_1 < 2^{-\epsilon j(Q)}$ so that at t_1 we have $E_{\mathcal{C}_0(Q_1)} > 2^{-(3+\epsilon)j(Q_1)}$.

We iterate this procedure finding a cube Q_k properly contained in Q_{k-1} and a time t_k so that $t_{k-1} \leq t_k < t_{k-1} + 2^{-\epsilon j(Q_{k-1})}$ and at time t_k we have $E_{\mathcal{C}_0(Q_k)} > 2^{-(3+\epsilon)j(Q_k)}$.

Estimating just using the coefficients of $\mathcal{C}_0(Q_k)$, we see that at time t_k, we have that

$$\begin{aligned}
||u||^2_{H^{\frac{3}{2}+\epsilon}} &\geq \sum_{j \geq j(Q_k)} 2^{(3+2\epsilon)j} \sum_{Q \in \mathcal{D}_j} u_Q^2(t_k) \\
&\geq 2^{(3+2\epsilon)j(Q_k)} \sum_{j \geq j(Q_k)} \sum_{Q \in \mathcal{D}_j} u_Q^2(t_k) \\
&\geq 2^{(3+2\epsilon)j(Q_k)} E_{\mathcal{C}_0(Q_k)}(t_k) \\
&\geq 2^{\epsilon j(Q_k)}.
\end{aligned}$$

Since $j(Q_k)$ is an increasing sequence of integers, this is going to ∞. However, since

$$t_k = (t_k - t_{k-1}) + (t_{k-1} - t_{k-2}) + \cdots + t_1 \leq 2^{-\epsilon j(Q)} + \sum_{l=1}^{k-1} 2^{-\epsilon j(Q_l)}$$

and the $j(Q_l)$'s are an increasing sequence of integers, we see that the sequence $\{t_k\}$ converges to a finite limit. $\qquad\square$

We point out here that under certain assumptions on the initial $E_Q(0)$ Corollary 3.2.3 guarantees finite time blow-up of $||u||_{H^{\frac{3}{2}+\epsilon}}$. This was obtained using an extended Carleson box $\mathcal{C}_0(Q)$. In the next section we shall prove for the Navier-Stokes equations with small dissipation, finite time blow-up of slightly weaker $||u||_{H^{2+\epsilon}}$ norm. The reason for this is the fact that for the Navier-Stokes equations with hyper-dissipation the energy itself is not conserved. Instead part of it dissipates at each level of our dyadic tree $\mathcal{D}$.

However, from Fedor Nazarov [9] we learned from his proof that the solution to the dyadic Navier-Stokes equations with enough dissipation stays bounded in a certain C^k space provided that it started in the same C^k space. The main tool in his proof is the following observation. Let us truncate the system of ODEs which describe the dyadic Navier-Stokes equations with hyper-dissipation. If provided with enough dissipation, the system will scatter all energy. Therefore, energy cannot become concentrated over the first few levels. For example, this is true for the dyadic Navier-Stokes itself. A natural question is: what does "enough dissipation" mean? In the following section we prove that if the dissipation exponent α is less than $\frac{1}{4}$, one obtains finite time blow-up.

4. The dyadic Navier-Stokes equations with hyper-dissipation

4.1. **Energy flow.** We consider the dyadic Navier-Stokes equations with hyper-dissipation (2.4). Since

$$\langle C(u, u), u \rangle = 0,$$

we have

$$\frac{1}{2} \frac{d}{dt}(\langle u, u \rangle) + \langle (\Delta)^\alpha u, u \rangle = 0,$$

and therefore, we have energy decay

$$(4.1) \qquad \langle u, u \rangle + 2 \int_0^t \langle (\Delta)^\alpha u, u \rangle = \langle u_0, u_0 \rangle,$$

where $u_0 = u(x, 0)$.

Let us imagine that each node Q along our tree $\mathcal{D}$ has a wastebasket which is at time t filled with $2 \int_{t_0}^t 2^{2\alpha j(Q)} u_Q^2$.

We define the energy of a cube Q at time t as in the case of the dyadic Euler equation

$$E_Q(t) = u_Q^2(t).$$

For t greater than or equal to some fixed time t_0 we introduce the energy of a wastebasket of a cube Q at time t as

$$W_{Q,t_0}(t) = 2 \int_{t_0}^t 2^{2\alpha j(Q)} E_Q.$$

Then (4.1) is saying that the sum of energy at nodes plus energy at wastebaskets is controlled.

Also we have a local description of energy flow along the tree $\mathcal{D}$:

$$(4.2) \qquad \frac{d}{dt}E_Q = E_{Q,in} - E_{Q,out} - 2^{2\alpha j(Q)}2E_Q,$$

where

$$E_{Q,in} = 2^{\frac{5j(Q)}{2}}2E_{\tilde{Q}}\sqrt{E_Q},$$

and

$$E_{Q,out} = \sum_{Q'\in\mathcal{C}^1(Q)} E_{Q',in}.$$

Thus energy is flowing always from larger squares to smaller ones and, except for portions which go to wastebaskets, energy flows along the edges of the tree $\mathcal{D}$.

We define the energy of a Carleson box by

$$E_{\mathcal{C}(Q)} = \sum_{Q'\in\mathcal{C}(Q)} E_{Q'},$$

and the waste of a Carleson box by

$$W_{\mathcal{C}(Q),t_0} = \sum_{Q'\in\mathcal{C}(Q)} W_{Q',t_0}.$$

We immediately get the following proposition.

Proposition 4.1.1. *Let u be a time-varying "function" with positive coefficients evolving according to the flow (2.4). Then for any Q, the function in time given by $E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0}$ is monotone increasing.*

4.2. **Energy concentration.** As in the Euler case we begin with a lemma about Carleson boxes.

Lemma 4.2.1. *For any $\epsilon > 0$, there is $\delta(\epsilon) > 0$, so that if we know that*

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} > (1-\delta)2^{-(4+\epsilon)j(Q)},$$

then there exists $Q' \in \mathcal{C}(Q)$, so that

$$E_{Q'} + W_{Q',t_0} \geq 2^{-(4+\epsilon)j(Q')}.$$

Proof. Let us assume the lemma is false, i.e.,

$$(4.3) \qquad E_{Q'} + W_{Q',t_0} < 2^{-(4+\epsilon)j(Q')},$$

for all $Q' \in \mathcal{C}(Q)$.

On the other hand, we have

$$
\begin{aligned}
E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} &= \sum_{Q'\in\mathcal{C}(Q)} [E_{Q'} + W_{Q',t_0}] \\
(4.4) \qquad &= \sum_{k=1}^{\infty} \sum_{Q'\in\mathcal{C}^k(Q)} [E_{Q'} + W_{Q',t_0}].
\end{aligned}
$$

Now because we are in dimension 3, there are exactly 2^{3k} elements $Q' \in \mathcal{C}^k(Q)$ with $j(Q') = j(Q) + k$. Thus by using (4.3) we can bound (4.4) from above by

$$\sum_{k=1}^{\infty} 2^{3k} \cdot 2^{-(4+\epsilon)(j(Q)+k)},$$

which is in turn the same as

$$(4.5) \qquad 2^{-(4+\epsilon)j(Q)} \sum_{k=1}^{\infty} 2^{-(\epsilon+1)k}.$$

Since

$$\sum_{k=1}^{\infty} 2^{-(\epsilon+1)k} < 1$$

we can choose $0 < \delta < 1$ such that

$$\sum_{k=1}^{\infty} 2^{-(\epsilon+1)k} < 1 - \delta.$$

Therefore, (4.5) is bounded above by $(1 - \delta)2^{-(4+\epsilon)j(Q)}$, and we obtain

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} < (1 - \delta)2^{-(4+\epsilon)j(Q)},$$

which contradicts the assumption of our lemma. □

Now we prove the main lemma.

Lemma 4.2.2. *Fix j_0 sufficiently large. Then there exists an ϵ, $0 < \epsilon < 1 - 4\alpha$, so that if at time t_0, we have*

$$(4.6) \qquad E_Q + W_{Q,t_0} \geq 2^{-(4+\epsilon)j(Q)}$$

with $j(Q) > j_0$, then there is some t with $t < t_0 + T$, where

$$2^{\frac{(\epsilon-1)j(Q)}{2}} < T < 2^{-2\alpha j},$$

and a cube $Q' \in \mathcal{C}(Q)$ so that at time t we have

$$E_{Q'} + W_{Q',t_0} \geq 2^{-(4+\epsilon)j(Q')}.$$

Proof. We assume that the conclusion of the lemma is false, i.e.,

$$(4.7) \qquad E_{Q'} + W_{Q',t_0} < 2^{-(4+\epsilon)j(Q')},$$

for all $Q' \in \mathcal{C}(Q)$, and for all $t \in [t_0, t_0 + T]$.

In light of Lemma 4.2.1 and Proposition 4.1.1, it must be the case that for all $t \in [t_0, t_0 + T]$, we have

$$(4.8) \qquad E_Q(t) + W_{Q,t_0}(t) \geq \delta 2^{-(4+\epsilon)j(Q)},$$

since otherwise because of the hypothesis (4.6) we would have

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q)} > (1 - \delta)2^{-(4+\epsilon)j(Q)}, \text{ at some } t \in [t_0, t_0 + T],$$

which would lead to a contradiction by Lemma 4.2.1.

Since $W_{Q,t_0}(t)$ is a monotone increasing function of t, (4.8) implies that

$$E_Q(t) + W_{Q,t_0}(t_0 + T) \geq \delta 2^{-(4+\epsilon)j(Q)}, \text{ for all } t \in [t_0, t_0 + T],$$

and therefore, we have either

$$(4.9) \qquad E_Q(t) \geq \frac{1}{2}\delta 2^{-(4+\epsilon)j(Q)}, \text{ for all } t \in [t_0, t_0 + T],$$

or

$$(4.10) \qquad W_{Q,t_0}(t_0 + T) \geq \frac{1}{2}\delta 2^{-(4+\epsilon)j(Q)}.$$

We shall analyze those cases separately.

First let us assume (4.10). Let Q' be an element of $\mathcal{C}^1(Q)$. Then $u_{Q'}$ satisfies

$$(4.11) \qquad \frac{du_{Q'}}{dt} = 2^{\frac{5(j(Q)+1)}{2}} u_Q^2 - 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big) - 2^{2\alpha(j(Q)+1)} u_{Q'}.$$

We shall integrate (4.11) on the time interval $[t_0, t_0 + T]$.

In order to simplify our notation we introduce the following integrals:

$$I_1 \; := \; \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+1)}{2}} u_Q^2,$$

$$I_2 \; := \; \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big),$$

$$I_3 \; := \; \int_{t_0}^{t_0+T} 2^{2\alpha(j(Q)+1)} u_{Q'}.$$

By using (4.10) we estimate I_1 and obtain

$$(4.12) \qquad I_1 := \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+1)}{2}} u_Q^2 \gtrsim \delta 2^{-(\frac{3}{2}+2\alpha+\epsilon)j}.$$

We bound I_2 as follows:

$$\begin{aligned}
I_2 \; &:= \; \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big) \\
&\leq \; \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big| \sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big| \\
&\lesssim \; 2^{\frac{5j}{2}} \cdot T \cdot 2^{-(4+\epsilon)j} \cdot 2^3,
\end{aligned}$$

where the last inequality follows from (4.7). Thus

$$(4.13) \qquad I_2 \lesssim T \cdot 2^{-(\frac{3}{2}+\epsilon)j}.$$

Similarly, we use (4.7) in order to get

$$(4.14) \qquad I_3 := \int_{t_0}^{t_0+T} 2^{2\alpha(j(Q)+1)} u_{Q'} \lesssim T \cdot 2^{(2\alpha-2-\frac{\epsilon}{2})j}.$$

One easily checks from (4.12), (4.13) and (4.14) that if $0 < \epsilon < 1 - 4\alpha$ and $T < 2^{-2\alpha j}$, then

$$I_1 \geq I_2,$$

as well as

$$I_1 \geq I_3.$$

Therefore, after integrating (4.11) on the time interval $[t_0, t_0 + T]$ we conclude that

$$\int_{t_0}^{t_0+T} \frac{du_{Q'}}{dt} \gtrsim \delta 2^{-(\frac{3}{2}+2\alpha+\epsilon)j},$$

which is a contradiction by the fundamental theorem of calculus, since $\epsilon < 1 - 4\alpha$.

Now we are left to verify the contradiction in the case that one has (4.9). Let us assume (4.9). Moreover, since energy flows only in the direction of smaller cubes

and is conserved up to waste baskets, it must be that for any child $Q' \in \mathcal{C}^1(Q)$, it is the case that

$$E_{\mathcal{C}(Q)} + W_{\mathcal{C}(Q),t_0} \geq \int_{t_0}^{t_0+T} E_{Q',in},$$

at time $t_0 + T$. Thus since

$$E_{Q',in} = 2^{\frac{5(j(Q)+1)}{2}} 2 E_Q u_{Q'},$$

we must have

(4.15)
$$\int_{t_0}^{t_0+T} u_{Q'} \lesssim \frac{1-\delta}{\delta} 2^{\frac{-5j(Q)}{2}}.$$

We integrate (4.11) on the time interval $[t_0, t_0 + T]$. By using (4.9) we bound I_1 as follows:

(4.16)
$$I_1 := \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+1)}{2}} u_Q^2 dt \gtrsim T \cdot \delta \cdot 2^{-(\frac{3}{2}+\epsilon)j}.$$

We bound I_2 as

$$
\begin{aligned}
I_2 \quad &:= \quad \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big(\sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big) \\
&\leq \quad \int_{t_0}^{t_0+T} 2^{\frac{5(j(Q)+2)}{2}} u_{Q'} \Big| \sum_{Q'' \in \mathcal{C}^1(Q')} u_{Q''} \Big| \\
&\lesssim \quad \frac{1-\delta}{\delta} 2^{-(2+\frac{\epsilon}{2})j},
\end{aligned}
$$

where the last inequality follows from (4.15) and (4.7). Thus

(4.17)
$$I_2 \lesssim \frac{1-\delta}{\delta} 2^{-(2+\frac{\epsilon}{2})j}.$$

By using (4.15) we obtain the following bound on I_3

(4.18)
$$I_3 := \int_{t_0}^{t_0+T} 2^{2\alpha(j(Q)+1)} u_{Q'} \lesssim 2^{(2\alpha-\frac{5}{2})j}.$$

Again one easily checks from (4.16), (4.17) and (4.18) that I_1 dominates provided that $0 < \epsilon < 1 - 4\alpha$ and $T > 2^{\frac{(\epsilon-1)j}{2}}$.

Therefore, after integrating (4.11) on the time interval $[t_0, t_0 + T]$ we conclude that

$$\int_{t_0}^{t_0+T} \frac{du_{Q'}}{dt} \gtrsim T \cdot \delta \cdot 2^{-(\frac{3}{2}+\epsilon)j},$$

which is a contradiction by the fundamental theorem of calculus, since $T > 2^{\frac{(\epsilon-1)j(Q)}{2}}$.
$\square$

Lemma 4.2.3. *Fix j_0 sufficiently large. Then there exists an ϵ, $0 < \epsilon < 1 - 4\alpha$, such that if at time t_0 we have*

(4.19)
$$E_Q \gtrsim 2^{-(4+\epsilon)j(Q)}$$

with $j(Q) > j_0$, then there is some time t with $t < t_0 + T$, where

$$2^{\frac{(\epsilon-1)j(Q)}{2}} < T < 2^{-2\alpha j(Q)},$$

and a cube $Q' \in \mathcal{C}(Q)$ so that at time t we have

$$E_{Q'} \gtrsim 2^{-(4+\epsilon)j(Q')}.$$

Proof. We shall prove the lemma by contradiction. Assume the lemma is false, i.e.,

(4.20) $E_{Q'} < 2^{-(4+\epsilon)j(Q')}$, for all $Q' \in \mathcal{C}(Q)$, and for all $t \in [t_0, t_0 + T]$.

However, from the hypothesis (4.19) and Lemma 4.2.2 we can find time $t_1 < t_0 + T$, where

(4.21) $$2^{\frac{(\epsilon-1)j(Q)}{2}} < T < 2^{-2\alpha j(Q)}$$

and a cube $Q_1 \in \mathcal{C}(Q)$ so that

(4.22) $$E_{Q_1}(t_1) + W_{Q_1,t_0}(t_1) \geq 2^{-(4+\epsilon)j(Q_1)}.$$

On the other hand, by using monotonicity of the function $W_{Q_1,t_0}(t)$ and (4.20) we calculate

$$
\begin{aligned}
E_{Q_1}(t_1) + W_{Q_1,t_0}(t_1) &< E_{Q_1}(t_1) + W_{Q_1,t_0}(t_0 + T) \\
&= E_{Q_1}(t_1) + 2\int_{t_0}^{t_0+T} 2^{2\alpha j(Q_1)} E_{Q_1} \\
&< 2^{-(4+\epsilon)j(Q_1)} + 2 \cdot 2^{2\alpha j(Q_1)} \cdot 2^{-(4+\epsilon)j(Q_1)} \cdot T \\
&\lesssim 2^{-(4+\epsilon)j(Q_1)},
\end{aligned}
$$

where the last inequality follows from (4.21).

Thus

$$E_{Q_1}(t_1) + W_{Q_1,t_0}(t_1) \lesssim 2^{-(4+\epsilon)j(Q_1)},$$

which contradicts (4.22), and the lemma is proved.   □

Corollary 4.2.4. *Let u be a solution to (2.4) which has all positive coefficients and at time 0 has some Q with $E_Q > 2^{-(4+\epsilon)j(Q)}$, for $j(Q) > j_0$ with j_0 as in the previous lemma. Then the $H^{2+\epsilon}$ norm of u becomes unbounded in finite time.*

Proof. We apply Lemma 4.2.3. We find a cube Q_1 properly contained in Q and a time $t_1 < T < 2^{-2\alpha j(Q)}$ so that at t_1 we have $E_{Q_1} \gtrsim 2^{-(4+\epsilon)j(Q_1)}$.

We iterate this procedure finding a cube Q_k properly contained in Q_{k-1} and a time t_k so that $t_{k-1} \leq t_k < t_{k-1} + T < t_{k-1} + 2^{-2\alpha j(Q_{k-1})}$, and at time t_k we have $E_{Q_k} \gtrsim 2^{-(4+\epsilon)j(Q_k)}$.

Estimating just using the coefficient at Q_k, we see that at time t_k we have that

$$\|u\|_{H^{2+\epsilon}} \geq 2^{(2+\epsilon)j(Q_k)} E_{Q_k}^{1/2} \geq 2^{\frac{\epsilon j(Q_k)}{2}}.$$

Since $j(Q_k)$ is an increasing sequence of integers, this is going to ∞. However, since

$$t_k = (t_k - t_{k-1}) + (t_{k-1} - t_{k-2}) + \cdots + t_1 \leq 2^{-2\alpha j(Q)} + \sum_{l=1}^{k-1} 2^{-2\alpha j(Q_l)},$$

and the $j(Q_l)$'s are an increasing sequence of integers, we see that the sequence $\{t_k\}$ converges to a finite limit.   □

References

[1] J.T. Beale, T. Kato, and A. Majda: *Remarks on the breakdown of smooth solutions for the 3D Euler equations*, Comm. Math. Phys. 94 , No. 1 (1984), 61–66. MR **85j:**35154

[2] L. Caffarelli, R. Kohn, and L. Nirenberg: Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.* **35**, No. 6 (1982), 771–831. MR **84m:**35097

[3] M. Cannone: Harmonic analysis tools for solving the incompressible Navier-Stokes equations, To appear in *Handbook of Mathematical Fluid Dynamics* **3** (2004).

[4] C. Fefferman: Existence and smoothness of the Navier-Stokes equation, *http://www. claymath.org*, (2000).

[5] S. Friedlander, and N. Pavlović: Blow up in a three-dimensional vector model for the Euler equations, *Preprint*, (2003).

[6] E. B. Gledzer: System of hydrodynamic type admitting two quadratic integrals of motion, *Sov. Phys. Dokl.* **18**, No. 4 (1973), 216–217.

[7] Kato, T.: Quasi-linear equations of evolution with applications to partial differential equations, *Lecture Notes in Mathematics* **448**, Berlin, Heidelberg, New York, Springer, (1975), 25–70. MR **53:**11252

[8] N.H. Katz and N. Pavlović: A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes equation with hyper-dissipation, *GAFA* **12** (2002), 355–379. MR **2003e:**35243

[9] F. Nazarov: *Personal communication*, (2001).

[10] K. Ohkitani and M. Yamada: Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully developed model turbulence, *Prog. Theor. Phys.* **81**, No. 2 (1989), 329–341. MR **90j:**76065

[11] R. Temam: Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids,*Lecture Notes in Mathematics* **565**, Berlin, Heidelberg, New York,Sp ringer, (1976), 184–195. MR **57:**6902

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