

## A THETA FUNCTION IDENTITY AND ITS IMPLICATIONS

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ABSTRACT. In this paper we prove a general theta function identity with four parameters by employing the complex variable theory of elliptic functions. This identity plays a central role for the cubic theta function identities. We use this identity to re-derive some important identities of Hirschhorn, Garvan and Borwein about cubic theta functions. We also prove some other cubic theta function identities. A new representation for  $\prod_{n=1}^{\infty}(1 - q^n)^{10}$  is given. The proofs are self-contained and elementary.

### 1. INTRODUCTION

Throughout this paper we will use  $q$  to denote  $\exp(2\pi i\tau)$  with  $\text{Im}(\tau) > 0$ . We will use the familiar notation

$$(1.1) \quad (z; q)_{\infty} = \prod_{n=0}^{\infty} (1 - zq^n),$$

and sometimes write

$$(1.2) \quad (a, b, c, \dots; q)_{\infty} = (a; q)_{\infty} (b; q)_{\infty} (c; q)_{\infty} \cdots.$$

It is easy to see that for any integer  $m > 0$ ,

$$(1.3) \quad (z, zq, \dots, zq^{m-1}; q^m)_{\infty} = (z; q)_{\infty}.$$

If we define  $\omega$  to be the primitive cube root of unity given by  $\omega = \exp(\frac{2\pi i}{3})$ , then, using the identity  $(1 - x)(1 - x\omega)(1 - x\omega^2) = 1 - x^3$ , we find that

$$(1.4) \quad (z, z\omega, z\omega^2; q)_{\infty} = (z^3; q^3)_{\infty}.$$

The well-known Jacobi triple product identity is

$$(1.5) \quad (q, z, q/z; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n$$

(see [1, pp. 21-22], [2, p. 35], [7], and [10]).

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Jacobi's theta function  $\theta_1(z|\tau)$  is defined as

$$\begin{aligned}\theta_1(z|\tau) &= -iq^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{(2n+1)iz} \\ (1.6) \quad &= 2q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z\end{aligned}$$

(see, for example, [16, p. 463]).

Using the triple product identity, we can find the infinite product representation for  $\theta_1(z|\tau)$ , namely,

$$\begin{aligned}\theta_1(z|\tau) &= 2q^{1/8}(\sin z)(q, qe^{2iz}, qe^{-2iz}; q)_{\infty} \\ (1.7) \quad &= iq^{1/8}e^{-iz}(q, e^{2iz}, qe^{-2iz}; q)_{\infty}\end{aligned}$$

(see, for example, [16, p. 469]).

In [13], we use the complex variable theory of elliptic functions to establish a general theta function identity. We then derive some remarkable theta function identities related to the modular equations of degree 5; in particular, we give new proofs of the two fundamental identities satisfied by the Rogers-Ramanujan continued fraction. In this paper we set up the following general theta function identity with four parameters by employing the same method. This is a notable identity, which contains many interesting cubic theta function identities as special cases.

**Theorem 1.** *Suppose  $f(z)$  is an entire function satisfying the functional equations*

$$(1.8) \quad f(z + \pi) = -f(z) \quad \text{and} \quad f(z + \pi\tau) = -q^{-3/2}e^{-6iz}f(z).$$

*Then the following identity holds:*

$$\begin{aligned}(1.9) \quad &\{f(z) - f(-z)\} \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau) \\ &= \{f(y) - f(-y)\} \theta_1(x|\tau) \theta_1(z|\tau) \theta_1(x+z|\tau) \theta_1(x-z|\tau) \\ &- \{f(x) - f(-x)\} \theta_1(y|\tau) \theta_1(z|\tau) \theta_1(y+z|\tau) \theta_1(y-z|\tau).\end{aligned}$$

The contents are organized as follows. In Section 2, we prove Theorem 1 using the classical theory of elliptic functions. In Section 3, the following theta function identity is first proved using Theorem 1.

**Theorem 2.** *Let  $\theta_1(z|\tau)$  be the Jacobi theta function defined by (1.6). Then for any  $x, y$ , and  $z$ , we have*

$$\begin{aligned}(1.10) \quad &\theta_1(y|\frac{\tau}{3}) \theta_1(x|\tau) \theta_1(z|\tau) \theta_1(x-z|\tau) \theta_1(x+z|\tau) \\ &- \theta_1(x|\frac{\tau}{3}) \theta_1(y|\tau) \theta_1(z|\tau) \theta_1(y-z|\tau) \theta_1(y+z|\tau) \\ &= \theta_1(z|\frac{\tau}{3}) \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x-y|\tau) \theta_1(x+y|\tau).\end{aligned}$$

Then we use Theorem 2 to derive a interesting theta function identity given in Theorem 5 and from which we establish a new representation for  $(q; q)_{\infty}^{10}$ . In Section 4, we derive some identities for the Hirschhorn-Garvan-Borwein two-variable cubic theta functions; our method is different from that of Hirschhorn, Garvan, and Borwein. In Section 5, we use Theorem 1 to prove the following remarkable theta function identities.

**Theorem 3.** Let  $\theta_1(x|\tau)$  be the Jacobi theta function defined as in (1.6), and let  $a(q)$  be the Ramanujan function (see [4]) defined by

$$(1.11) \quad a(q) := 1 + 6 \sum_{n=0}^{\infty} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

Then, we have

$$(1.12) \quad \theta_1^3(x + \frac{\pi}{3}|\tau) + \theta_1^3(x - \frac{\pi}{3}|\tau) - \theta_1^3(x|\tau) = 3a(q)\theta_1(3x|3\tau)$$

and

$$(1.13) \quad \theta_1^3(x|3\tau) - q^{1/2}e^{2ix}\theta_1^3(x + \pi\tau|3\tau) - q^{1/2}e^{-2ix}\theta_1^3(x - \pi\tau|3\tau) = a(q)\theta_1(x|\tau).$$

Equality (1.13) is [3, p.142, Entry 3]. In Section 6, we set up the following identities using Theorem 1.

**Theorem 4.** We have

$$(1.14) \quad \begin{aligned} & (q; q)_{\infty}^3 \theta_1^3(x|\tau) \theta_1(3y|3\tau) - (q; q)_{\infty}^3 \theta_1^3(y|\tau) \theta_1(3x|3\tau) \\ &= 3q^{1/4}(q^3; q^3)_{\infty}^3 \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x - y|\tau) \theta_1(x + y|\tau) \end{aligned}$$

and

$$(1.15) \quad \begin{aligned} & (q; q)_{\infty}^3 \theta_1^3(x|\tau) \theta_1(y|\frac{\tau}{3}) - (q; q)_{\infty}^3 \theta_1^3(y|\tau) \theta_1(x|\frac{\tau}{3}) \\ &= q^{-1/(12)}(q^{1/3}; q^{1/3})_{\infty}^3 \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x - y|\tau) \theta_1(x + y|\tau). \end{aligned}$$

## 2. THE PROOF OF THEOREM 1

To prove the theorem, we require the following Lemma 1. Lemma 1 is a fundamental theorem of elliptic functions and can be found in [6, p. 22]. Recently, in [9, 11, 12, 13, 14, 15], we have used Lemma 1 to set up many important theta function identities.

**Lemma 1.** The sum of all the residues of an elliptic function vanishes in the period parallelogram.

*Proof.* Suppose that  $f(u)$  is the given function satisfying the functional equations (1.8). Then we consider the function

$$(2.1) \quad g(u) = \frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u-x|\tau)\theta_1(u+x|\tau)\theta_1(u-y|\tau)\theta_1(u+y|\tau)\theta_1(u-z|\tau)\theta_1(u+z|\tau)}.$$

Here we temporarily assume that  $0 < x, y, z < \pi$  are three distinct parameters different from the zero points of  $\theta_1(2u|\tau)f(u)$ . Using the functional equations

$$(2.2) \quad \theta_1(z + \pi|\tau) = -\theta_1(z|\tau) \quad \text{and} \quad \theta_1(z + \pi\tau|\tau) = -q^{-1/2}e^{-2iz}\theta_1(z|\tau),$$

we can verify that  $g(u + \pi) = g(u)$  and  $g(u + \pi\tau) = g(u)$ . Hence  $g(u)$  is an elliptic function with periods  $\pi$  and  $\pi\tau$ . We readily find that  $x, \pi - x, y, \pi - y, z, \pi - z$  are its only poles and all its poles are simple poles. In this paper we will use  $\text{res}(g; \alpha)$  to denote the residue of  $g$  at  $\alpha$ . Then Lemma 1 gives

$$(2.3) \quad \text{res}(g; x) + \text{res}(g; \pi - x) + \text{res}(g; y) + \text{res}(g; \pi - y) + \text{res}(g; z) + \text{res}(g; \pi - z) = 0.$$

In this paper, we will also use the prime to denote the partial derivative with respect to  $u$ . Now, we have the following elementary calculation:

$$\begin{aligned}
 \text{res}(g; x) &= \lim_{u \rightarrow x} (u - x)g(u) \\
 &= \lim_{u \rightarrow x} \frac{\theta_1(2u|\tau)f(u)}{\theta_1(u|\tau)\theta_1(u+x|\tau)\theta_1(u-y|\tau)\theta_1(u+y|\tau)\theta_1(u-z|\tau)\theta_1(u+z|\tau)} \\
 &\quad \times \lim_{u \rightarrow x} \frac{u-x}{\theta_1(u-x|\tau)} \\
 (2.4) \quad &= \frac{f(x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)}.
 \end{aligned}$$

In the same way we can show that

$$(2.5) \quad \text{res}(g; \pi - x) = \frac{-f(-x)}{\theta_1'(0|\tau)\theta_1(x|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)}.$$

Noting that  $g(u)$  is symmetric in  $x, y$ , and  $z$ , we interchange  $x$  and  $y$  in (2.4) and (2.5) to obtain

$$(2.6) \quad \text{res}(g; y) = \frac{-f(y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}$$

and

$$(2.7) \quad \text{res}(g; \pi - y) = \frac{f(-y)}{\theta_1'(0|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}.$$

Similarly we find that

$$(2.8) \quad \text{res}(g; z) = \frac{f(z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}$$

and

$$(2.9) \quad \text{res}(g; \pi - z) = \frac{-f(-z)}{\theta_1'(0|\tau)\theta_1(z|\tau)\theta_1(x-z|\tau)\theta_1(x+z|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}.$$

Substituting (2.4)-(2.9) into (2.3), after some simplification, we obtain (2.1). By analytic continuation, we know (2.1) holds for all  $x, y$ , and  $z$ , and so this completes the proof of Theorem 1.  $\square$

### 3. A NEW REPRESENTATION FOR $(q; q)_\infty^{10}$

In this section we first give a proof of Theorem 2 using Theorem 1.

*Proof.* It is easy to verify that  $\theta_1(x|\frac{\tau}{3})$  satisfies all the conditions of Theorem 1. By taking  $f(x) = \theta_1(x|\frac{\tau}{3})$  in Theorem 1, we immediately obtain Theorem 2; and thus we complete the proof of Theorem 2.  $\square$

Using the identity  $(z, z\omega, z\omega^2; q)_\infty = (z^3; q^3)_\infty$ , where  $\omega = \exp(\frac{2\pi i}{3})$ , and the infinite product representation for  $\theta_1(z|\tau)$ , we readily find that

$$(3.1) \quad \theta_1(3z|3\tau) = -\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3} \theta_1(z|\tau) \theta_1(z + \frac{\pi}{3}|\tau) \theta_1(z - \frac{\pi}{3}|\tau).$$

Appealing to the infinite product representation for  $\theta_1(z|\tau)$ , we can also find that

$$(3.2) \quad \theta_1(\frac{\pi}{3}|\tau) = \theta_1(\frac{2\pi}{3}|\tau) = \sqrt{3}q^{1/8}(q^3; q^3)_\infty.$$

Taking  $z = \frac{\pi}{3}$  in Theorem 2 and then using (3.1) and (3.2) in the resulting equation, we obtain the following identity.

**Theorem 5.** *For any  $x, y$  we have the identity*

$$(3.3) \quad q^{1/(12)}(q; q)_{\infty}^2 \theta_1(x|\frac{\tau}{3}) \theta_1(3y|3\tau) - q^{1/(12)}(q; q)_{\infty}^2 \theta_1(y|\frac{\tau}{3}) \theta_1(3x|3\tau) \\ = \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x-y|\tau) \theta_1(x+y|\tau).$$

Differentiating both sides of (1.6) with respect to  $z$  and then setting  $z = 0$ , we readily find that

$$(3.4) \quad \theta_1'(0|\tau) = q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2},$$

and by successive differentiations, we obtain

$$(3.5) \quad \theta_1'''(0|\tau) = -q^{1/8} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^3 q^{n(n+1)/2}.$$

Differentiation of (1.7) gives

$$(3.6) \quad \theta_1'(0|\tau) = 2q^{1/8}(q; q)_{\infty}^3.$$

Comparing (3.4) and (3.6), we obtain Jacobi's identity

$$(3.7) \quad (q; q)_{\infty}^3 = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

In the next theorem we will give a new representation for  $(q; q)_{\infty}^{10}$ . An entirely different proof of this new representation has been given in the paper [5]. Using this identity, the authors have also given a short proof of Ramanujan's famous congruence  $p(11n+6) \equiv 0 \pmod{11}$ , where  $p(n)$  denotes the number of unrestricted partitions of the positive integer  $n$ .

**Theorem 6.** *We have*

$$(3.8) \quad 32(q; q)_{\infty}^{10} \\ = 9 \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^3 q^{3n(n+1)/2} \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{n(n+1)/6} \right) \\ - \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n+1) q^{3n(n+1)/2} \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n (2n+1)^3 q^{n(n+1)/6} \right). \\ (3.9)$$

*Proof.* Dividing both sides of (3.3) by  $y$  and then allowing  $y \rightarrow 0$ , we obtain

$$(3.10) \quad 3q^{1/(12)}(q; q)_{\infty}^2 \theta_1'(0|3\tau) \theta_1(x|\frac{\tau}{3}) - q^{1/(12)}(q; q)_{\infty}^2 \theta_1'(0|\frac{\tau}{3}) \theta_1(3x|3\tau) = \theta_1'(0|\tau) \theta_1^3(x|\tau).$$

Using (3.6) in this equation, we obtain the following interesting identity:

$$(3.11) \quad (q; q)_{\infty} \theta_1^3(x|\tau) = 3q^{1/3}(q^3; q^3)_{\infty}^3 \theta_1(x|\frac{\tau}{3}) - (q^{1/3}; q^{1/3})_{\infty}^3 \theta_1(3x|3\tau).$$

Differentiating both sides of the identity with respect to  $x$  three times, and then setting  $x = 0$ , we obtain

$$(3.12) \quad 6(q; q)_{\infty} \theta_1'(0|\tau)^3 = 3q^{1/3}(q^3; q^3)_{\infty}^3 \theta_1'''(0|\frac{\tau}{3}) - 27(q^{1/3}; q^{1/3})_{\infty}^3 \theta_1'''(0|3\tau).$$

Using (3.6) in the left side of this equation, we find that

$$(3.13) \quad 48q^{3/8}(q; q)_{\infty}^{10} = 3q^{1/3}(q^3; q^3)_{\infty}^3 \theta_1'''(0|\frac{\tau}{3}) - 27(q^{1/3}; q^{1/3})_{\infty}^3 \theta_1'''(0|3\tau).$$

Using (3.5) and (3.7) in the right side of the equation, we arrive at (3.9). This completes the proof of Theorem 6.  $\square$

#### 4. SOME IDENTITIES FOR THE HIRSCHHORN-GARVAN-BORWEIN TWO-VARIABLE CUBIC THETA FUNCTIONS

Noting the fact that  $\theta_1(x - y|\tau) = -\theta_1(y - x|\tau)$ , we can rewrite (3.3) as

$$(4.1) \quad \begin{aligned} & q^{1/(12)}(q; q)_{\infty}^2 \theta_1(y|\frac{\tau}{3}) \theta_1(3x|3\tau) - q^{1/(12)}(q; q)_{\infty}^2 \theta_1(x|\frac{\tau}{3}) \theta_1(3y|3\tau) \\ &= \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(y - x|\tau) \theta_1(x + y|\tau). \end{aligned}$$

Using the second equation of (1.7) in the above equation and then replacing  $e^{2ix}$  by  $x$  and  $e^{2iy}$  by  $y$  in the resulting equation, we obtain the following infinite product identity:

**Theorem 7.** *We have*

$$(4.2) \quad \begin{aligned} & (q; q)_{\infty}^2 (q^{1/3}, x, q^{1/3}/x; q^{1/3})_{\infty} (q^3, y^3, q^3/y^3; q^3)_{\infty} \\ & - yx^{-1}(q; q)_{\infty}^2 (q^{1/3}, y, q^{1/3}/y; q^{1/3})_{\infty} (q^3, x^3, q^3/x^3; q^3)_{\infty} \\ & = (q, x, q/x; q)_{\infty} (q, y, q/y; q)_{\infty} (q, y/x, qx/y; q)_{\infty} (q, xy, q/xy; q)_{\infty}. \end{aligned}$$

Appealing to the Jacobi triple product identity, we have

$$(4.3) \quad \begin{aligned} & (q; q)_{\infty}^2 (q^{1/3}, x, q^{1/3}/x; q^{1/3})_{\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(n-1)/2} y^{3n} \\ & - yx^{-1}(q; q)_{\infty}^2 (q^3, x^3, q^3/x^3; q^3)_{\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/6} y^n \\ & = (q, x, q/x; q)_{\infty} \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} y^n \right) \\ & \times \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} y^n x^{-n} \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} y^n x^n \right). \end{aligned}$$

It is easy to see that the coefficient of  $y$  on the right side of the above equation is

$$(4.4) \quad -(q, x, q/x; q)_{\infty} \sum_{m, n=-\infty}^{\infty} q^{m^2+mn+n^2-m-n} x^{n-m}$$

and the coefficient of  $y$  on the left side is

$$(4.5) \quad -x^{-1}(q; q)_{\infty}^2 (q^3, x^3, q^3/x^3; q^3)_{\infty}.$$

By equating the above two quantities, we arrive at

$$(4.6) \quad \begin{aligned} & \sum_{m, n=-\infty}^{\infty} q^{m^2+mn+n^2-m-n} x^{n-m} \\ & = (q; q)_{\infty} (q^3; q^3)_{\infty} (1 + x + x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_{\infty}}{(qx, q/x; q)_{\infty}}. \end{aligned}$$

Making the changes of indices  $m \rightarrow -m$  and  $n \rightarrow -n$ , we obtain

$$(4.7) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n} = (q; q)_{\infty} (q^3; q^3)_{\infty} (1+x+x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_{\infty}}{(qx, q/x; q)_{\infty}}.$$

This is the same as in [8, p.675, Equation (1.23)].

Equating the terms independent of  $y$  in (4.3), we find that

$$(4.8) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n} = q^{1/3} (q; q)_{\infty} (q^3; q^3)_{\infty} (1+x+x^{-1}) \frac{(q^3 x^3, q^3/x^3; q^3)_{\infty}}{(qx, q/x; q)_{\infty}} + (q; q)_{\infty} (q^{1/3}; q^{1/3})_{\infty} \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_{\infty}}{(qx, q/x; q)_{\infty}}.$$

Combining the above two equations, we find that

$$(4.9) \quad (q; q)_{\infty} (q^{1/3}; q^{1/3})_{\infty} \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_{\infty}}{(qx, q/x; q)_{\infty}} = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n} - q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n}.$$

Let  $\omega$  be the primitive cube root of unity given by  $\omega = \exp(\frac{2\pi i}{3})$ . Then we readily find that the right side of (4.9) is equal to

$$(4.10) \quad \sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^n \omega^{m-n}$$

(see [8, p. 678] for the details). Therefore, we have

$$(4.11) \quad \sum_{m,n=-\infty}^{\infty} q^{(m^2+mn+n^2)/3} x^n \omega^{m-n} = (q; q)_{\infty} (q^{1/3}; q^{1/3})_{\infty} \frac{(q^{1/3} x, q^{1/3} x^{-1}; q^{1/3})_{\infty}}{(qx, q/x; q)_{\infty}}.$$

Writing  $q$  as  $q^3$ , we obtain

$$(4.12) \quad \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^n \omega^{m-n} = (q; q)_{\infty} (q^3; q^3)_{\infty} \frac{(qx, qx^{-1}; q)_{\infty}}{(q^3 x, q^3/x; q^3)_{\infty}}.$$

This is the same as in [8, p. 675, equation (1.22)].

If we denote

$$(4.13) \quad a(q, x) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^{m-n},$$

$$(4.14) \quad b(q, x) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} x^n \omega^{m-n},$$

$$(4.15) \quad c(q, x) = q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} x^{m-n},$$

then (4.8) can be written as

$$(4.16) \quad a(q, x) = c(q, x) + b(q^{1/3}, x),$$

from which one can easily derive the following important identity of Hirschhorn, Garvan, and Borwein (see [8, p.682] for the details).

**Theorem 8.** *We have*

$$(4.17) \quad a^3(q, x) = c^3(q, x) + b^2(q, 1)b(q, x^3).$$

## 5. THE PROOF OF THEOREM 3

Using the identity  $(z, zq, zq^2; q^3)_{\infty} = (z; q)_{\infty}$  and the infinite product representation for  $\theta_1(z|\tau)$ , we readily find that

$$(5.1) \quad \theta_1(z|\frac{\tau}{3}) = \frac{(q^{1/3}; q^{1/3})_{\infty}}{(q; q)_{\infty}^3} \theta_1(z|\tau) \theta_1(z + \frac{\pi\tau}{3}|\tau) \theta_1(z - \frac{\pi\tau}{3}|\tau),$$

and we can also find that

$$(5.2) \quad \theta_1(\frac{\pi\tau}{3}|\tau) = iq^{-1/(24)}(q^{1/3}; q^{1/3})_{\infty} \quad \text{and} \quad \theta_1(\frac{2\pi\tau}{3}|\tau) = iq^{-5/(24)}(q^{1/3}; q^{1/3})_{\infty}.$$

We recall the definition of the Ramanujan function

$$(5.3) \quad a(q) := 1 + 6 \sum_{n=0}^{\infty} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right).$$

Using logarithmic differentiation on (1.7), we obtain

$$(5.4) \quad \frac{\theta'_1}{\theta_1}(z|\tau) = -i - 2i \sum_{n=0}^{\infty} \frac{q^n e^{2iz}}{1 - q^n e^{2iz}} + 2i \sum_{n=1}^{\infty} \frac{q^n e^{-2iz}}{1 - q^n e^{-2iz}}.$$

Comparing the above two equations, we infer that

$$(5.5) \quad a(q) = -2 + 3i \frac{\theta'_1}{\theta_1}(\pi\tau|3\tau).$$

It is well-known that the trigonometric series expansion for the logarithmic derivative of  $\theta_1(z|\tau)$  is

$$(5.6) \quad \frac{\theta'_1}{\theta_1}(z|\tau) = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^n} \sin 2nz$$

(see [16, p.489]).



Thus, we have

$$(5.7) \quad \frac{\theta'_1}{\theta_1}\left(\frac{\pi}{3}|\tau\right) = \frac{1}{\sqrt{3}}a(q).$$

Now we are ready to prove Theorem 3.

*Proof.* Differentiating both sides of (1.9) with respect to  $z$  and then setting  $z = 0$ , we obtain

$$(5.8) \quad \begin{aligned} & \theta'_1(0|\tau) \{f(y) - f(-y)\} \theta_1^3(x|\tau) - \theta'_1(0|\tau) \{f(x) - f(-x)\} \theta_1^3(y|\tau) \\ &= 2f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

Inserting  $\theta'_1(0|\tau) = 2q^{1/8}(q; q)_\infty^3$  in the above equation, we have

$$(5.9) \quad \begin{aligned} & q^{1/8}(q; q)_\infty^3 \{f(y) - f(-y)\} \theta_1^3(x|\tau) - q^{1/8}(q; q)_\infty^3 \{f(x) - f(-x)\} \theta_1^3(y|\tau) \\ &= f'(0)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

It is easy to verify that  $\theta_1^3(x + \frac{\pi}{3}|\tau)$  satisfies (1.8), so we can take  $f(x) = \theta_1^3(x + \frac{\pi}{3}|\tau)$  in (5.9). By (3.2) and (5.7) and a direct computation, we have

$$(5.10) \quad f'(0) = 3\theta_1^3\left(\frac{\pi}{3}|\tau\right) \left(\frac{\theta'_1}{\theta_1}\right)\left(\frac{\pi}{3}|\tau\right) = 9q^{3/8}(q^3; q^3)_\infty^3 a(q);$$

and thus we have

$$(5.11) \quad \begin{aligned} & q^{1/8}(q; q)_\infty^3 \left\{ \theta_1^3\left(y + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(y - \frac{\pi}{3}|\tau\right) \right\} \theta_1^3(x|\tau) \\ & - q^{1/8}(q; q)_\infty^3 \left\{ \theta_1^3\left(x + \frac{\pi}{3}|\tau\right) + \theta_1^3\left(x - \frac{\pi}{3}|\tau\right) \right\} \theta_1^3(y|\tau) \\ &= 9q^{3/8}(q^3; q^3)_\infty^3 a(q)\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau). \end{aligned}$$

Taking  $y = \frac{\pi}{3}$  in the above equation and noting that  $\theta_1(\frac{\pi}{3}|\tau) = \theta_1(\frac{2\pi}{3}|\tau) = \sqrt{3}q^{1/8}(q^3; q^3)_\infty$ , we find that

$$(5.12) \quad \begin{aligned} & \theta_1^3(x|\tau) - \theta_1^3\left(x + \frac{\pi}{3}|\tau\right) - \theta_1^3\left(x - \frac{\pi}{3}|\tau\right) \\ &= 3a(q)\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3}\theta_1(x|\tau)\theta_1\left(x + \frac{\pi}{3}|\tau\right)\theta_1\left(x - \frac{\pi}{3}|\tau\right). \end{aligned}$$

From (3.1), we have

$$(5.13) \quad \theta_1(3x|3\tau) = -\frac{(q^3; q^3)_\infty}{(q; q)_\infty^3}\theta_1(x|\tau)\theta_1\left(x + \frac{\pi}{3}|\tau\right)\theta_1\left(x - \frac{\pi}{3}|\tau\right).$$

We substitute the above equation into the right side of (5.12) to obtain (1.12).

Proceeding through the same steps as before, by taking  $f(x) = e^{2ix}\theta_1^3(x + \frac{\pi\tau}{3}|\tau)$  in (5.9) and then setting  $y = \frac{\pi\tau}{3}$  and appealing to (5.1), (5.2), and (5.5), we can find that

$$(5.14) \quad \theta_1^3(x|\tau) - q^{1/6}e^{2ix}\theta_1^3\left(x + \frac{\pi\tau}{3}|\tau\right) - q^{1/6}e^{-2ix}\theta_1^3\left(x - \frac{\pi\tau}{3}|\tau\right) = a(q^{1/3})\theta_1\left(x|\frac{\tau}{3}\right).$$

Replacing  $q$  by  $q^3$ , we obtain (1.13). Thus, we complete the proof of Theorem 3.  $\square$

## 6. THE PROOF OF THEOREM 4

*Proof.* Taking  $z = \frac{\pi}{3}$  in Theorem 1 and appealing to (3.1) and (3.2), we obtain

$$\begin{aligned}
 & \sqrt{3}q^{1/8}(q; q)_{\infty}^3 \{f(x) - f(-x)\} \theta_1(3y|3\tau) \\
 &= \sqrt{3}q^{1/8}(q; q)_{\infty}^3 \{f(y) - f(-y)\} \theta_1(3x|3\tau) \\
 (6.1) \quad &+ \left\{f\left(\frac{\pi}{3}\right) - f\left(-\frac{\pi}{3}\right)\right\} \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau).
 \end{aligned}$$

It is easy to see that  $\theta_1^3(x|\tau)$  satisfies all the conditions of Theorem 1. We choose  $f(x) = \theta_1^3(x|\tau)$  and then use (3.2) in the resulting equation to obtain (1.14).

Taking  $z = \frac{\pi\tau}{3}$  in Theorem 1 and appealing to (5.1) and (5.2), we obtain

$$\begin{aligned}
 & iq^{-1/(24)}(q; q)_{\infty}^3 \{f(x) - f(-x)\} \theta_1(y|\frac{\tau}{3}) \\
 &= iq^{-1/(24)}(q; q)_{\infty}^3 \{f(y) - f(-y)\} \theta_1(x|\frac{\tau}{3}) \\
 (6.2) \quad &- \left\{f\left(\frac{\pi\tau}{3}\right) - f\left(-\frac{\pi\tau}{3}\right)\right\} \theta_1(x|\tau) \theta_1(y|\tau) \theta_1(x+y|\tau) \theta_1(x-y|\tau).
 \end{aligned}$$

Choosing  $f(x) = \theta_1^3(x|\tau)$  and then using (5.2) in the resulting equation, we obtain (1.15). This completes the proof of Theorem 4.  $\square$

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