TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 4, Pages 1469–1496
S 0002-9947(04)03535-4
Article electronically published on September 2, 2004

HARDY SPACE OF EXACT FORMS ON $\mathbb{R}^N$

ZENGJIAN LOU AND ALAN McINTOSH

ABSTRACT. We show that the Hardy space of divergence-free vector fields on $\mathbb{R}^3$ has a divergence-free atomic decomposition, and thus we characterize its dual as a variant of BMO. Using the duality result we prove a "div-curl" type theorem: for b in $L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$, $\sup \int b \cdot (\nabla u \times \nabla v)\, dx$ is equivalent to a BMO-type norm of b, where the supremum is taken over all $u, v \in W^{1,2}(\mathbb{R}^3)$ with $\|\nabla u\|_{L^2},\ \|\nabla v\|_{L^2} \leq 1$. This theorem is used to obtain some coercivity results for quadratic forms which arise in the linearization of polyconvex variational integrals studied in nonlinear elasticity. In addition, we introduce Hardy spaces of exact forms on $\mathbb{R}^N$, study their atomic decompositions and dual spaces, and establish "div-curl" type theorems on $\mathbb{R}^N$.

1. Statement of the main theorem

In this paper we show that divergence-free vector fields in the Hardy space $\mathcal{H}^1(\mathbb{R}^3, \mathbb{R}^3)$ can be decomposed as a sum of divergence-free atoms. This enables us to characterize the dual space of the divergence-free Hardy space $\mathcal{H}^1_{div}(\mathbb{R}^3, \mathbb{R}^3)$ as a new variant of BMO. In turn, we obtain a "div-curl" type theorem which generalizes a result proved by Coifman, Lions, Meyer and Semmes in [CLMS].

Our paper answers questions in two separate but closely related areas of analysis: harmonic analysis, and elliptic systems of partial differential equations. In particular we obtain applications to coercivity problems for the class of linear elliptic systems under the polyconvex condition used in the study of nonlinear elasticity by Ball in [B].

The theorem of "div-curl" type which we prove, concerns an estimate for quadratic forms on $\mathbb{R}^3$ in terms of a variant of BMO in the following way.

Theorem 4.1. *Let $b \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}^3)$. Then*

$$\sup_{u,v \in W} \int_{\mathbb{R}^3} b \cdot (\nabla u \times \nabla v)\, dx \sim \|b\|_{BMO_d(\mathbb{R}^3,\mathbb{R}^3)}, \tag{1.1}$$

where $W = \{w \in W^{1,2}(\mathbb{R}^3) : \|\nabla w\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \leq 1\}$ and

$$\|b\|_{BMO_d(\mathbb{R}^3,\mathbb{R}^3)} = \sup_{B \subset \mathbb{R}^3} \inf_g \left(\frac{1}{|B|} \int_B |b-g|^2\, dx \right)^{1/2}, \tag{1.2}$$

Received by the editors May 16, 2003 and, in revised form, October 19, 2003.

2000 *Mathematics Subject Classification.* Primary 42B30; Secondary 35J45, 58A10.

Key words and phrases. Divergence-free Hardy space, Hardy space of exact forms, atomic decomposition, BMO, div-curl, coercivity.

The authors are supported by the Australian Government through the Australian Research Council. This paper was written when both authors were at the Center for Mathematics and its Applications of the Mathematical Sciences Institute at the Australian National University.

the supremum in (1.2) *being taken over all balls B in $\mathbb{R}^3$, the infimum being taken over all $g \in L^2(B, \mathbb{R}^3)$ with* curl $g = 0$ *in B. The implicit constants in* (1.1) *are absolute constants.*

Let us compare this result with Theorem III.2 of [CLMS]. It implies that for $b \in L^2_{loc}(\mathbb{R}^N)$,

$$\sup_u \int_{\mathbb{R}^N} b \det Du \, dx \sim \|b\|_{BMO(\mathbb{R}^N)} \tag{1.3}$$

where the supremum is taken over all $u \in W^{1,N}(\mathbb{R}^N, \mathbb{R}^N)$ with $\|\nabla u_i\|_{L^N(\mathbb{R}^N, \mathbb{R}^N)} \leq 1$, $i = 1, \cdots, N$. When $N = 2$, $\det Du$ is a quadratic form in (u_1, u_2), and we see that this result is precisely a 2-dimensional version of Theorem 4.1, where b is a scalar-valued BMO function.

Thus, Theorem 4.1 is a 3-dimensional generalization of the 2-dimensional case of (1.3), in which we stay with quadratic forms and L^2 norms on ∇u, but are required to introduce a new BMO-type norm on vector-valued functions.

Indeed we also prove an N-dimensional version of Theorem 4.1, where b is an l-form ($0 \leq l \leq N - 2$). See Theorem 6.6.

We emphasize the three-dimensional case however, because of its application to questions of Zhang concerning coercivity. He previously obtained analogous 2-dimensional coercivity results in [Z1].

Extensions of these results to the case of forms on $\mathbb{R}^N_+$ and on strongly Lipschitz domains in $\mathbb{R}^N$ is contained in the sequels [LM1] and [LM2] to this paper.

The paper is organized as follows. Section 2 provides the definition of divergence-free Hardy space on $\mathbb{R}^3$ and the proof of its divergence-free atomic decomposition. Using the atomic decomposition, we characterize its dual in Section 3. The proof of the "div-curl" theorem is in Section 4. Section 5 is devoted to applications of the main theorem to coercivity properties and Gårding's inequality of certain polyconvex quadratic forms. In Section 6, we introduce Hardy spaces of exact forms on $\mathbb{R}^N$, give their atomic decompositions, characterize their dual spaces and establish a "div-curl" theorem on $\mathbb{R}^N$. In addition, we give a decomposition theorem of these Hardy spaces into "$du \wedge dv$" quantities which is a generalization of a similar decomposition theorem by Coifman, Lions, Meyer and Semmes in [CLMS].

In this paper, unless otherwise specified, C denotes a constant independent of functions and domains related to the inequalities. Such C may differ at different occurrences. The notation "$A(f) \sim B(f)$" means that there exists a constant $C > 0$ such that $C^{-1}A(f) \leq B(f) \leq CA(f)$ for all f in the space referred to.

2. Divergence-free atomic decomposition

In this section we introduce the divergence-free Hardy space and prove its divergence-free atomic decomposition. A similar decomposition was obtained by Gilbert, Hogan and Lakey in [GHL] by using a result of divergence-free wavelet decomposition of $L^2(\mathbb{R}^3, \mathbb{R}^3)$ due to Lemarié-Rieusset [Le]. Our proof is different from that in [GHL] and is valid for forms of all degrees as is shown in Section 6.

We first recall briefly some definitions and results of Hardy spaces and tent spaces which are used in this paper.

The *Hardy space* $\mathcal{H}^1(\mathbb{R}^3)$ is the space of locally integrable functions f for which

$$M(f)(x) = \sup_{t>0} |\varphi_t * f(x)|$$

belongs to $L^1(\mathbb{R}^3)$, where $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi_t(x) = \frac{1}{t^3}\varphi(\frac{x}{t})$, $t > 0$, $\int_{\mathbb{R}^3} \varphi(x)\ dx = 1$, supp $\varphi \subset B(0,1)$, a ball centered at the origin with radius 1. The norm of $\mathcal{H}^1(\mathbb{R}^3)$ is defined by

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^3)} = \|M(f)\|_{L^1(\mathbb{R}^3)},$$

where M is the maximal function. Among many characterizations of Hardy spaces, the atomic decomposition is an important one. An $L^2(\mathbb{R}^3)$ function a is an $\mathcal{H}^1(\mathbb{R}^3)$-*atom* if there exists a ball (or a cube) $B = B_a$ in $\mathbb{R}^N$ satisfying:

1) supp $a \subset B$;
2) $\|a\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \le |B|^{-1/2}$;
3) $\int_B a(x)\ dx = 0$.

It is obvious that any $\mathcal{H}^1(\mathbb{R}^3)$-atom a is in $\mathcal{H}^1(\mathbb{R}^3)$. The basic result about atoms is the following atomic decomposition theorem (see [CW] and [La]): A function f on $\mathbb{R}^3$ belongs to $\mathcal{H}^1(\mathbb{R}^3)$ if and only if f has a decomposition

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k's are $\mathcal{H}^1(\mathbb{R}^3)$-atoms and $\sum_{k=0}^\infty |\lambda_k| < \infty$. Furthermore,

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^3)} \sim \inf \left(\sum_{k=0}^{\infty} |\lambda_k| \right),$$

where the infimum is taken over all such decompositions, the constants of the proportionality are absolute constants.

Define the *tent space* $\mathcal{N}^p(\mathbb{R}^4_+)$ $(1 \le p < \infty)$ to consist of all measurable functions F on $\mathbb{R}^4_+$ for which $S(F) \in L^p(\mathbb{R}^3)$, where $S(F)$ is the square function defined by

$$S(F)(x) = \left(\int_{\Gamma(x)} |F(y,t)|^2 \frac{dydt}{t^4} \right)^{1/2},$$

$\Gamma(x) = \{(y,t) \in \mathbb{R}^4_+ : |y - x| < t\}$, $\|F\|_{\mathcal{N}^p(\mathbb{R}^4_+)} = \|S(F)\|_{L^p(\mathbb{R}^3)}$.

An $\mathcal{N}^p(\mathbb{R}^4_+)$-*atom* is a function α supported in a tent $T(B) = \{(x,t) : |x - x_0| \le r - t\}$ of a ball $B = B(x_0, r)$ in $\mathbb{R}^3$, for which

$$\int_{T(B)} |\alpha(x,t)|^2\ \frac{dxdt}{t} \le |B|^{1-2/p}.$$

When $1 \le p < \infty$, Coifman, Meyer and Stein proved the following atomic decomposition theorem in [CMS]: any $F \in \mathcal{N}^p(\mathbb{R}^4_+)$ can be written as

$$F = \sum_{k=0}^{\infty} \lambda_k \alpha_k,$$

where the α_k are $\mathcal{N}^p(\mathbb{R}^4_+)$-atoms and $\sum_{k=0}^\infty |\lambda_k| \le C\|F\|_{\mathcal{N}^p(\mathbb{R}^4_+)}$.

In the proof of Theorem 2.3, we use the following two facts: one is that the operator defined by

$$g \quad \mapsto \quad S_\psi(g) := \left(\int_{\Gamma(x)} |g * \psi_t(y)|^2\ \frac{dydt}{t^4} \right)^{1/2}$$

is bounded from $\mathcal{H}^1(\mathbb{R}^3)$ to $L^1(\mathbb{R}^3)$ for $\psi \in \mathcal{S}(\mathbb{R}^3)$ (the space of test functions) with $\int \psi \, dx = 0$, and

$$\|S_\psi(g)\|_{L^1(\mathbb{R}^3)} \le C\|g\|_{\mathcal{H}^1(\mathbb{R}^3)} \tag{2.1}$$

(see Theorems 3 and 4 in Chapter III of [St2]). The other is that the operator

$$\pi_\psi(a) = \int_0^\infty a(\cdot, t) * \psi_t \, \frac{dt}{t}$$

is bounded from $\mathcal{N}^2(\mathbb{R}^4_+)$ to $L^2(\mathbb{R}^3)$ and

$$\|\pi_\psi(a)\|_{L^2(\mathbb{R}^3)} \le C\|a\|_{\mathcal{N}^2(\mathbb{R}^4_+)} \tag{2.2}$$

([CMS, Theorem 6]).

Now we define the divergence-free Hardy space. Let $\mathcal{H}^1(\mathbb{R}^3, \mathbb{R}^3)$ denote the space of vector-valued functions with each component in $\mathcal{H}^1(\mathbb{R}^3)$.

Definition 2.1. The *divergence-free Hardy space on* $\mathbb{R}^3$ is defined as

$$\mathcal{H}^1_{div}(\mathbb{R}^3, \mathbb{R}^3) = \{f \in \mathcal{H}^1(\mathbb{R}^3, \mathbb{R}^3) : \operatorname{div} f = 0 \ \text{ in } \ \mathbb{R}^3\}$$

with norm

$$\|f\|_{\mathcal{H}^1_{div}(\mathbb{R}^3,\mathbb{R}^3)} = \|f\|_{\mathcal{H}^1(\mathbb{R}^3,\mathbb{R}^3)},$$

where div f is defined in the sense of distributions.

Definition 2.2. A function $a \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ is said to be an $\mathcal{H}^1_{div}(\mathbb{R}^3, \mathbb{R}^3)$-*atom* if there is a ball (or a cube) $B = B_a$ in $\mathbb{R}^3$ satisfying

(i) supp $a \subset B$;
(ii) $\|a\|_{L^2(\mathbb{R}^3,\mathbb{R}^3)} \le |B|^{-1/2}$;
(iii) $\int_B a(x)\, dx = 0$;
(iv) div $a = 0$ in $\mathbb{R}^3$.

The properties of tent spaces can be used to clarify various points in the theory of Hardy spaces, for example, atomic decompositions. Combining this with an idea of Auscher we prove a divergence-free atomic decomposition of the divergence-free Hardy space.

For simplicity, we prove the result by using the language of forms. Let us interpret vector fields as two-forms. Then $\mathcal{H}^1(\mathbb{R}^3, \mathbb{R}^3)$ and $\mathcal{H}^1_{div}(\mathbb{R}^3, \mathbb{R}^3)$ become $\mathcal{H}^1(\mathbb{R}^3, \wedge^2)$ and $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$ respectively. Similarly we can define tent spaces and their atoms for forms. See Appendix B for definitions of exterior operator d and its formal adjoint δ and information on forms.

Theorem 2.3. *A function f on $\mathbb{R}^3$ is in $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$ if and only if it has a decomposition*

$$f = \sum_{k=0}^\infty \lambda_k a_k,$$

where the a_k's are $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$-atoms and $\sum_{k=0}^\infty |\lambda_k| < \infty$. Furthermore,

$$\|f\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)} \sim \inf\left(\sum_{k=0}^\infty |\lambda_k|\right),$$

where the infimum is taken over all such decompositions. The constants of the proportionality are absolute constants.

Proof. The easy part is the "if" part, that is, assuming f has such a decomposition in $\mathcal{D}'(\mathbb{R}^3, \wedge^2)$ (the space of distributions). For then, if the sum is finite,

$$\|f\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)} \le \sum_k |\lambda_k| \|a_k\|_{\mathcal{H}^1(\mathbb{R}^3,\wedge^2)} \le C \sum_k |\lambda_k|, \tag{2.3}$$

where we used the fact that $\|a\|_{\mathcal{H}^1(\mathbb{R}^3,\wedge^2)} \le C$ when a is an $\mathcal{H}^1(\mathbb{R}^3,\wedge^2)$-atom.

We now prove the "only if" part. Suppose $f = \sum_{1\le i<j\le 3} f_{ij}\ e^i \wedge e^j \in \mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)$. Choose a function $\varphi \in C_0^\infty(\mathbb{R}^3)$ with support in the unit ball, which satisfies

$$\int_0^\infty t|\xi|^2 \hat{\varphi}(t\xi)^2\ dt = 1, \quad \xi \in \mathbb{R}^3. \tag{2.4}$$

Define

$$F(x,t) = t\delta\Big(f * \varphi_t(x)\Big), \quad x \in \mathbb{R}^3, \quad t > 0.$$

Then $F(x,t)$ can be written as

$$\begin{aligned} F(x,t) &= \sum_{1\le i<j\le 3} \sum_{l=1}^{3} t \frac{\partial}{\partial x_l} (f_{ij} * \varphi_t)(x)\ \mu_l^*(e^i \wedge e^j) \\ &= \sum_{1\le i<j\le 3} \sum_{l=1}^{3} f_{ij} * (\partial_l \varphi)_t(x) (\delta_{li} e^j - \delta_{lj} e^i). \end{aligned} \tag{2.5}$$

Since $f_{ij} * (\partial_l \varphi)_t \in \mathcal{N}^1(\mathbb{R}^4_+)$ by (2.1), then $F \in \mathcal{N}^1(\mathbb{R}^4_+, \wedge^1)$ with

$$\|F\|_{\mathcal{N}^1(\mathbb{R}^4_+,\wedge^1)} \le C \|f\|_{\mathcal{H}^1(\mathbb{R}^3,\wedge^2)}.$$

By the atomic decomposition theorem for tent spaces, F has a decomposition

$$F = \sum_{k=0}^{\infty} \lambda_k \alpha_k \tag{2.6}$$

with

$$\sum_{k=0}^{\infty} |\lambda_k| \le C \|F\|_{\mathcal{N}^1(\mathbb{R}^4_+,\wedge^1)},$$

where the α_k's are $\mathcal{N}^1(\mathbb{R}^4_+,\wedge^1)$-atoms, i.e. there exist balls B_k such that supp $\alpha_k \subset T(B_k)$ and

$$\int_{T(B_k)} |\alpha_k(y,t)|^2\ \frac{dydt}{t} \le \frac{1}{|B_k|}. \tag{2.7}$$

Define

$$\pi F = -\int_0^\infty t d\Big(F(\cdot,t) * \varphi_t\Big)\ \frac{dt}{t}.$$

Then (2.6) gives $\pi F = \sum_{k=0}^\infty \lambda_k a_k$, where $a_k = \pi \alpha_k$. Let $\alpha_k := \sum_{i=1}^3 \alpha_k^i\ e^i,\ a_k^{i,l} := -\pi_{\partial_l \varphi}(\alpha_k^i)$, where the α_k^i's are $\mathcal{N}^1(\mathbb{R}^4_+)$-atoms, then

$$a_k = \sum_{i,l=1}^{3} a_k^{i,l}\ \mu_l(e^i).$$

It is easy to check that the function a_k satisfies the following conditions: 1) supp $a_k \subset 2B_k$, since $\partial_l \varphi$ is supported in the unit ball and supp $\alpha_k \subset T(B_k)$, where $2B_k$ denotes a ball with the same center and twice the radius of B_k; 2) $\int a_k\ dx = 0$, since $\partial_l \varphi \in \mathcal{S}(\mathbb{R}^3)$ and $\int \partial_l \varphi\ dx = 0$; 3) $da_k = 0$ in $\mathbb{R}^3$. We now prove that a_k also satisfies the size condition: 4) $\int_{2B_k} |a_k|^2\ dx \leq C|2B_k|^{-1}$, where C is independent of a_k and B_k. Since α_k^i are $\mathcal{N}^1(\mathbb{R}^4_+)$-atoms, then $\alpha_k^i \in \mathcal{N}^2(\mathbb{R}^4_+)$. The boundedness of π_ψ in (2.2) and (2.7) imply that $a_k^{i,l} \in L^2(\mathbb{R}^3)$ and

$$\begin{aligned}
\|a_k^{i,l}\|^2_{L^2(2B_k)} &\leq C\|\alpha_k^i\|^2_{\mathcal{N}^2(\mathbb{R}^4_+)} \\
&= C\int_{\mathbb{R}^3}\int_{\mathbb{R}^4_+} |\alpha_k^i(y,t)|^2 \chi\left(\frac{x-y}{t}\right)\ \frac{dydt}{t^4}\ dx \\
&\leq C\int_{T(B_k)} |\alpha_k^i(y,t)|^2\ \frac{dydt}{t} \\
&\leq C|2B_k|^{-1},
\end{aligned}$$

where χ denotes the characteristic function in the unit ball. We proved that a_k are $\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)$-atoms. Moreover we next show that

$$f = \sum_{k=0}^{\infty} \lambda_k a_k$$

is an atomic decomposition of f, where the λ_k's are the same as those in (2.6). To see this we only need to prove that

$$f = \pi F.$$

Applying the Fourier transform to $\Delta f = (d\delta + \delta d)f$ and the following two facts

$$\widehat{df}(\xi) = i\xi \wedge \hat{f}(\xi), \quad \widehat{\delta f}(\xi) = -i\xi \vee \hat{f}(\xi), \quad \xi \in \mathbb{R}^3,$$

we obtain

$$\begin{aligned}
\widehat{\Delta f}(\xi) &= i\xi \wedge \widehat{\delta f}(\xi) \\
&= -i\xi \wedge \big(i\xi \vee \hat{f}(\xi)\big) \\
&= \xi \wedge \big(\xi \vee \hat{f}(\xi)\big),
\end{aligned}$$

where we used the fact that $df = 0$. In addition $\widehat{\Delta f}(\xi) = -|\xi|^2 \hat{f}(\xi)$, so

$$\xi \wedge \big(\xi \vee \hat{f}(\xi)\big) = -|\xi|^2\hat{f}(\xi). \tag{2.8}$$

The assumption (2.4) on φ and (2.8) give

$$\begin{aligned}
\widehat{\pi F}(\xi) &= -\int_0^\infty it\xi \wedge \Big(t\widehat{\delta(f*\varphi_t)}(\xi)\hat{\varphi}(t\xi)\Big)\, dt \\
&= \int_0^\infty it\xi \wedge \Big(it\xi \vee \hat{f}(\xi)\hat{\varphi}(t\xi)^2\Big)\, \frac{dt}{t} \\
&= -\int_0^\infty t^2\xi \wedge \Big(\xi \vee \hat{f}(\xi)\Big)\, \hat{\varphi}(t\xi)^2\ \frac{dt}{t} \\
&= \int_0^\infty t|\xi|^2\hat{\varphi}(t\xi)^2\hat{f}(\xi)\ dt = \hat{f}(\xi).
\end{aligned}$$

The desired result follows. The proof of Theorem 2.3 is completed □

3. The dual space

We now characterize the dual space of $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$. Recall that under the duality

$$(f, g) = \int_{\mathbb{R}^3} f(x)g(x)\ dx,$$

when suitably defined, the dual of $\mathcal{H}^1(\mathbb{R}^3)$ is the real-valued $BMO(\mathbb{R}^3)$ space of functions f of bounded mean oscillation, i.e.,

$$\|f\|_{BMO(\mathbb{R}^3)} = \sup_B \inf_{c\in\mathbb{R}} \left(\frac{1}{|B|}\int_B |f-c|^2\ dx\right)^{1/2} < \infty,$$

the supremum being taken over all balls B in $\mathbb{R}^3$, as in [St2]. So we have $(\mathcal{H}^1(\mathbb{R}^3, \wedge^2))' = BMO(\mathbb{R}^3, \wedge^1)$ under the pairing

$$(f, g) = \int_{\mathbb{R}^3} f \wedge g,$$

when suitably defined.

Definition 3.1. Let $BMO_d(\mathbb{R}^3, \wedge^1)$ be the space of measurable functions G for which

$$\|G\|_{BMO_d(\mathbb{R}^3,\wedge^1)} = \sup_B \inf_{g_B} \left(\frac{1}{|B|}\int_B |G-g_B|^2\ dx\right)^{1/2} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^3$, the infimum is taken over all functions $g_B \in L^2(B, \wedge^1)$ with $dg_B = 0$ in B.

Let $X_0 = \{G \in BMO_d(\mathbb{R}^3, \wedge^1) : \|G\|_{BMO_d(\mathbb{R}^3,\wedge^1)} = 0\}$. Consider the Banach space $BMO_d(\mathbb{R}^3, \wedge^1)/X_0$ with the norm

$$\|G + X_0\|_{BMO_d(\mathbb{R}^3,\wedge^1)/X_0} = \|G\|_{BMO_d(\mathbb{R}^3,\wedge^1)}.$$

We show that it is the dual of $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$. To prove this we first prove a lemma. Let $D(\mathbb{R}^3, \wedge^2)$ denote the vector space finitely generated by $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$-atoms. By Theorem 2.3, one has that $D(\mathbb{R}^3, \wedge^2)$ is dense in $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$.

Lemma 3.2. *For $g \in BMO(\mathbb{R}^3, \wedge^1)$,*

$$\int_{\mathbb{R}^3} g \wedge h = 0 \quad \textit{for all} \;\; h \in D(\mathbb{R}^3, \wedge^2)$$

if and only if

$$dg = 0 \quad \textit{in} \;\; \mathbb{R}^3.$$

Proof. Note that $d\varphi$ is an $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$-atom if $\varphi \in C_0^\infty(\mathbb{R}^3, \wedge^1)$. Then $\int_{\mathbb{R}^3} g \wedge d\varphi = 0$ if $\int_{\mathbb{R}^3} g \wedge h = 0$ for all $h \in D(\mathbb{R}^3, \wedge^2)$. Hence $dg = 0$ in $\mathbb{R}^3$.

Suppose $h \in D(\mathbb{R}^3, \wedge^2)$ and

$$h = \sum_k \lambda_k a_k,$$

where the a_k's are $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$-atoms, i.e., $a_k \in L^2(\mathbb{R}^3, \wedge^2)$ supported in balls B_k and $da_k = 0$ in B_k. By Proposition A.1 in Appendix A, there exist functions $\varphi_k \in W_0^{1,2}(B_k, \wedge^1)$ such that $a_k = d\varphi_k$, where $W_0^{1,2}(B_k, \wedge^1)$ denotes the Sobolev space $W^{1,2}(B_k, \wedge^1)$ with zero boundary values. From Green's formula, for $g \in BMO(\mathbb{R}^3, \wedge^1)$ with $dg = 0$ in $\mathbb{R}^3$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} g \wedge h &= \sum_k \lambda_k \int_{B_k} g \wedge a_k \\ &= \sum_k \lambda_k \int_{B_k} g \wedge d\varphi_k \\ &= \sum_k \lambda_k \int_{B_k} dg \wedge \varphi_k = 0\end{aligned}$$

for all $h \in D(\mathbb{R}^3, \wedge^2)$. □

Now we are ready to prove our main result of this section.

Theorem 3.3. *If $G+X_0 \in BMO_d(\mathbb{R}^3, \wedge^1)/X_0$, then the linear functional L defined by*

$$L(h) = \int_{\mathbb{R}^3} G \wedge h, \tag{3.1}$$

initially defined on $D(\mathbb{R}^3, \wedge^2)$, has a unique bounded extension to $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$. Conversely, if L is in $(\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2))'$, then there exists a unique $G+X_0 \in BMO_d(\mathbb{R}^3, \wedge^1)/X_0$ such that (3.1) *holds. The map $G + X_0 \mapsto L$ given by* (3.1) *is a Banach isomorphism between $BMO_d(\mathbb{R}^3, \wedge^1)/X_0$ and $(\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2))'$.*

Proof. Let $G \in BMO_d(\mathbb{R}^3, \wedge^1)$. Define

$$L(h) = \int_{\mathbb{R}^3} G \wedge h, \qquad h \in D(\mathbb{R}^3, \wedge^2).$$

If $\|G\|_{BMO_d(\mathbb{R}^3,\wedge^1)} = 0$, then for any ball B,

$$\inf_{g \in L^2(B,\wedge^1), dg=0} \int_B |G - g|^2 \, dx = 0. \tag{3.2}$$

Since $\{g \in L^2(B, \wedge^1) : \ dg = 0 \text{ in } B\}$ is a closed subspace of $L^2(B, \wedge^1)$ (see, for example, [ISS, Corollary 5.2]), (3.2) implies that $G \in L^2(B, \wedge^1)$ with $dG = 0$ in B for all $B \subset \mathbb{R}^3$. Hence $dG = 0$ in $\mathbb{R}^3$. By Lemma 3.2 we have

$$L(h) = \int_{\mathbb{R}^3} G \wedge h = 0 \quad \text{ for all } \quad h \in D(\mathbb{R}^3, \wedge^2).$$

Therefore we can define $\rho_1 : BMO_d(\mathbb{R}^3, \wedge^1)/X_0 \to (D(\mathbb{R}^3, \wedge^2))'$ by

$$\rho_1(G + X_0)(h) = \int_{\mathbb{R}^3} G \wedge h, \quad h \in D(\mathbb{R}^3, \wedge^2).$$

The proof that, for every $G \in BMO_d(\mathbb{R}^3, \wedge^1)$, the linear functional (3.1) is defined and bounded on $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$ depends on the inequality

$$\Big| \int_{\mathbb{R}^3} G \wedge h \Big| \le C \|G\|_{BMO_d(\mathbb{R}^3,\wedge^1)} \|h\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)} \tag{3.3}$$

for $G \in BMO_d(\mathbb{R}^3, \wedge^1)$ and h in the dense subspace $D(\mathbb{R}^3, \wedge^2) \subset \mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$.

Similar to the proof of Lemma 3.2, write $h \in D(\mathbb{R}^3, \wedge^2)$ as a finite sum of atoms a_k supported in balls B_k. For all $g_k \in L^2(B_k, \wedge^1)$ with $dg_k = 0$ in B_k, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} G \wedge h\Big| &\le \sum_k |\lambda_k| \Big|\int_{B_k} (G - g_k) \wedge a_k\Big| \\ &\le \sum_k |\lambda_k| \left(\int_{B_k} |G - g_k|^2 \, dx\right)^{1/2} \|a_k\|_{L^2(B_k, \wedge^2)} \\ &\le \sum_k |\lambda_k| \left(\frac{1}{|B_k|}\int_{B_k} |G - g_k|^2 \, dx\right)^{1/2},\end{aligned}$$

where we used the size condition of a_k. This gives

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} G \wedge h\Big| &\le \sum_k |\lambda_k| \inf_{g_k} \left(\frac{1}{|B_k|}\int_{B_k} |G - g_k|^2 \, dx\right)^{1/2} \\ &\le C\|h\|_{\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)} \|G\|_{BMO_d(\mathbb{R}^3, \wedge^1)}.\end{aligned}$$

Then (3.3) is proved. Therefore each $G \in BMO_d(\mathbb{R}^3, \wedge^1)$ gives a bounded linear functional on the dense subspace $D(\mathbb{R}^3, \wedge^2)$, and thus on $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$.

Let

$$Y_0 := \{g \in BMO(\mathbb{R}^3, \wedge^1) : dg = 0 \text{ in } \mathbb{R}^3\}.$$

Note that $\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$ is a closed subspace of $\mathcal{H}^1(\mathbb{R}^3, \wedge^2)$. Applying the Hahn-Banach Theorem and Lemma 3.2, one finds that Y_0 is a closed subspace of $BMO(\mathbb{R}^3, \wedge^1)$ and the map

$$\rho_2 : \quad L \ \mapsto \ G + Y_0$$

is a Banach isomorphism between $\left(\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)\right)'$ and $Y := BMO(\mathbb{R}^3, \wedge^1)/Y_0$, and

$$\|L\|_{op} \sim \|G + Y_0\|_Y,$$

where $L \in \left(\mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)\right)'$ is defined as in (3.1), $\|L\|_{op}$ is the operator norm of L.

Define

$$\rho_3 : \ G + Y_0 \ \mapsto \ G + X_0$$

from Y to $BMO_d(\mathbb{R}^3, \wedge^1)/X_0$. We next show that ρ_3 is well defined and bounded. For $G \in BMO(\mathbb{R}^3, \wedge^1)$, $g \in Y_0$ and a ball $B \subset \mathbb{R}^3$, we have

$$\inf_{g_B} \left(\frac{1}{|B|}\int_B |G - g_B|^2 \, dx\right)^{1/2} \le \inf_{c \in \wedge^1} \left(\frac{1}{|B|}\int_B |G - g - c|^2 \, dx\right)^{1/2},$$

where the infimum in the left-hand side is taken over all $g_B \in L^2(B, \wedge^1)$ with $dg_B = 0$ in B and that in the right-hand side is taken over all constant forms $c \in \wedge^1$. Hence

$$\|G\|_{BMO_d(\mathbb{R}^3, \wedge^1)} \le \inf_{g \in Y_0} \|G - g\|_{BMO(\mathbb{R}^3, \wedge^1)} = \|G + Y_0\|_Y.$$

It is straightforward to check that

$$\rho_3 \circ \rho_2 \circ \rho_1 = I, \quad \rho_1 \circ \rho_3 \circ \rho_2 = I,$$

where I denotes the identity map. Hence the theorem is proved. □

4. Proof of the main theorem

Using the duality result in the previous section we next prove our main theorem of this paper: the following "div-curl" type theorem on $\mathbb{R}^3$. The N-dimensional case is studied in Section 6.

Theorem 4.1. *Let $b \in L^2_{loc}(\mathbb{R}^3, \wedge^1)$. Then*

$$\sup_{u,v\in W} \int_{\mathbb{R}^3} b \wedge du \wedge dv \sim \|b\|_{BMO_d(\mathbb{R}^3,\wedge^1)}, \tag{4.1}$$

where $W = \{w \in W^{1,2}(\mathbb{R}^3) : \|dw\|_{L^2(\mathbb{R}^3,\wedge^1)} \le 1\}$, dw is the gradient of w. The implicit constants in (4.1) *are absolute constants.*

Proof. Suppose $b \in BMO_d(\mathbb{R}^3, \wedge^1)$. From Theorem II.1 in [CLMS] (see also Lemma 6.9 in Section 6), $du \wedge dv \in \mathcal{H}^1_d(\mathbb{R}^3, \wedge^2)$ for $u,\ v \in W^{1,2}(\mathbb{R}^3)$ and there exists an absolute constant C such that

$$\|du \wedge dv\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)} \le C\|du\|_{L^2(\mathbb{R}^3,\wedge^1)}\|dv\|_{L^2(\mathbb{R}^3,\wedge^1)}. \tag{4.2}$$

From (3.3), (4.2) and the assumption on u and v, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} b \wedge du \wedge dv\Big| &\le \|b\|_{BMO_d(\mathbb{R}^3,\wedge^1)}\|du \wedge dv\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)}\\ &\le C\|b\|_{BMO_d(\mathbb{R}^3,\wedge^1)}.\end{aligned}$$

On the other hand, we need to prove that there exists an absolute constant C such that for all balls $B \subset \mathbb{R}^3$,

$$\inf_{g_B}\left(\frac{1}{|B|}\int_B |b-g_B|^2\ dx\right)^{1/2} \le C \sup_{u,v\in W}\Big|\int_{\mathbb{R}^3} b \wedge du \wedge dv\Big|, \tag{4.3}$$

where the infimum is taken over all $g_B \in L^2(B, \wedge^1)$ with $dg_B = 0$ in B. Since the left-hand side of (4.3) is invariant by scaling, to prove (4.3) we need only to show that for the unit ball B_0, there exist $u_0 \in W^{1,2}_0(B_0)$, $v_0 \in W^{1,2}_0(2B_0)$ with $\|du_0\|_{L^2(\mathbb{R}^3,\wedge^1)}$, $\|dv_0\|_{L^2(\mathbb{R}^3,\wedge^1)} \le 1$ such that

$$\inf_{g_0}\left(\int_{B_0} |b-g_0|^2\ dx\right)^{1/2} \le C\Big|\int_{B_0} b \wedge du_0 \wedge dv_0\Big|, \tag{4.4}$$

where the infimum is taken over all $g_0 \in L^2(B_0, \wedge^1)$ with $dg_0 = 0$ in B_0.

We now prove (4.4). Let

$$H = \{h \in L^2(B_0, \wedge^1) : \delta h = 0 \ \text{ in } \ B_0,\ n \vee h|_{\partial B_0} = 0\}.$$

Since H is a closed subspace of $L^2(B_0, \wedge^1)$, we have the decomposition

$$L^2(B_0, \wedge^1) = H \oplus H^\perp, \tag{4.5}$$

where $H^\perp$ denotes the orthogonal complement of H in $L^2(B_0, \wedge^1)$ and

$$H^\perp = \{dq : q \in W^{1,2}(B_0)\}$$

(refer to [GR, Theorem 2.7]). Let $b \in L^2(B_0, \wedge^1)$, (4.5) gives that $b = h + dq$, where $h \in H$, $q \in W^{1,2}(B_0)$. Then we have

$$\int_{B_0} b \wedge du_0 \wedge dv_0 = \int_{B_0} h \wedge du_0 \wedge dv_0$$

and

$$\inf_{g_0}\left(\int_{B_0}|b-g_0|^2\,dx\right)^{1/2}\le\|h\|_{L^2(B_0,\wedge^1)}.$$

Therefore to prove (4.4) it is sufficient to prove that there exists an absolute constant C such that

$$\|h\|_{L^2(B_0,\wedge^1)}\le C\Big|\int_{B_0}h\wedge du_0\wedge dv_0\Big| \tag{4.6}$$

for all $h\in H$.

Applying the remark after Proposition A.1 in Appendix A, for $h\in H$, there exists $\varphi\in W_0^{1,2}(B_0,\wedge^1)$ and an absolute constant C_0 such that

$$*h=d\varphi$$

and

$$\|D\varphi\|_{L^2(B_0,\wedge^1)}\le C_0\|h\|_{L^2(B_0,\wedge^1)}, \tag{4.7}$$

where $*$ is the Hodge star operator, $D\varphi=(\partial_1\varphi,\cdots,\partial_N\varphi)$, $\partial_i\varphi=\sum_I\partial_i\varphi_I dx_I$ for $\varphi=\sum_I\varphi_I dx_I$. Thus we have

$$\begin{aligned}\|h\|^2_{L^2(B_0,\wedge^1)}&=\int_{B_0}h\wedge d\varphi\\&=\int_{B_0}h\wedge d(\varphi_1dx_1+\varphi_2dx_2+\varphi_3dx_3)\\&\le 3\max_{1\le i\le 3}\Big|\int_{B_0}h\wedge d\varphi_i\wedge dx_i\Big|\\&:=3\Big|\int_{B_0}h\wedge d\varphi_{i_0}\wedge dx_{i_0}\Big| \qquad (4.8)\end{aligned}$$

for some choice of i_0 $(1\le i_0\le 3)$. Define

$$u_0=\frac{\varphi_{i_0}}{C_0\|h\|_{L^2(B_0,\wedge^1)}}.$$

It is obvious that $u_0\in W_0^{1,2}(B_0)$ and $\|du_0\|_{L^2(\mathbb{R}^3,\wedge^1)}\le 1$ by (4.7). We now construct v_0. Let $\psi_0\in C_0^\infty(\mathbb{R}^3)$ such that

$$\psi_0=\begin{cases}1 & \text{in } B_0;\\ 0 & \text{outside } \overline{2B_0}.\end{cases}$$

Define

$$v_0=\gamma x_{i_0}\psi_0,\qquad 1\le i_0\le 3,$$

where $\gamma>0$ is a constant so that $\|dv_0\|_{L^2(\mathbb{R}^3,\wedge^1)}\le 1$. It is easy to check that $v_0\in C_0^\infty(2B_0)$ and $dv_0=\gamma dx_{i_0}$ in B_0. So (4.8) and the construction of u_0 and v_0 give

$$\begin{aligned}\|h\|_{L^2(B_0,\wedge^1)}&\le 3C_0\gamma^{-1}\Big|\int_{B_0}h\wedge\frac{d\varphi_{i_0}}{C_0\|h\|_{L^2(B_0,\wedge^1)}}\wedge\gamma dx_{i_0}\Big|\\&=3C_0\gamma^{-1}\Big|\int_{B_0}h\wedge du_0\wedge dv_0\Big|.\end{aligned}$$

This proves (4.6). The proof of Theorem 4.1 is completed. □

We are especially interested in the three-dimensional case, because, as shown in the following section, Theorem 4.1 can be used to give coercivity properties and Gårding's inequality of some polyconvex quadratic forms.

5. Applications

In the study of homogenization of linearized elasticity, Geymonat, Müller and Triantafyllidis considered the following system in [GMT]

$$\begin{cases} \operatorname{div}_\alpha A^{i,j}_{\alpha,\beta}(\frac{x}{\varepsilon})\partial_\beta u_j = f \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{5.1}$$

where $A^{i,j}_{\alpha,\beta}(x)$ is a periodic measurable function, $1 \le i,\ j,\ \alpha,\ \beta \le N$. A quantity Λ is introduced which gives a criterion of whether an elliptic system satisfying the Legendre-Hadamard condition can be homogenized, namely

$$\Lambda = \inf\left\{ \frac{\int_{\mathbb{R}^N} A^{i,j}_{\alpha,\beta}(x)\partial_\alpha u_i \partial_\beta u_j dx}{\int_{\mathbb{R}^N} |Du|^2 dx} : u \in C_0^\infty(\mathbb{R}^N, \wedge^1) \right\}.$$

It was proved in [GMT] that if $\Lambda > 0$, some homogenization results can be obtained for the system (5.1). If $\Lambda < 0$, the system cannot be homogenized. Zhang asked the following question: what conditions on the coefficient $A^{i,j}_{\alpha,\beta}$ of the system imply that $\Lambda \ge 0$? For $N = 2$, this question was answered by Zhang in [Z1].

Motivated by [Z1] and using the idea there we answer the question for $N = 3$. Suppose that $A^{i,j}_{\alpha,\beta}(x)\partial_\alpha u_i \partial_\beta u_j$ can be written in the form

$$A^{i,j}_{\alpha,\beta}(x)\partial_\alpha u_i \partial_\beta u_j = B^{i,j}_{\alpha,\beta}(x)\partial_\alpha u_i \partial_\beta u_j + b_{ij}(x)(\text{adj } Du)_{i,j}, \tag{5.2}$$

where $A^{i,j}_{\alpha,\beta}$, $B^{i,j}_{\alpha,\beta} \in L^\infty(\mathbb{R}^3)$, $B^{i,j}_{\alpha,\beta}\partial_\alpha u_i \partial_\beta u_j \ge C|Du|^2$, adj Du denotes the adjoint matrix of Du for $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$ (the summation convention is understood). We are interested in forms of this type in three dimensions, because they arrive naturally from the linearization of polyconvex variational integrals studied in nonlinear elasticity by Ball in [B].

When $i = 1$, the last term in (5.2) becomes

$$\begin{aligned} &\sum_{j=1}^3 b_{1j}(x)(\text{adj } Du)_{1,j} \\ &\qquad = b_{11}(x)(\text{adj } Du)_{1,1} + b_{12}(x)(\text{adj } Du)_{1,2} + b_{13}(x)(\text{adj } Du)_{1,3} \\ &\qquad = \det\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix} \\ &\qquad := b_1 \wedge du_2 \wedge du_3. \end{aligned}$$

For $i = 2,\ 3$, we have similarly

$$\sum_{j=1}^3 b_{2j}(x)(\text{adj } Du)_{2,j} = b_2 \wedge du_1 \wedge du_3$$

and

$$\sum_{j=1}^3 b_{3j}(x)(\text{adj } Du)_{3,j} = b_3 \wedge du_1 \wedge du_2,$$

where $b_i = (b_{i1}, b_{i2}, b_{i3})$. Let $a(u)$ denote the following polyconvex quadratic form

$$a(u) = |Du|^2 + b_1 \wedge du_2 \wedge du_3 + b_2 \wedge du_1 \wedge du_3 + b_3 \wedge du_1 \wedge du_2, \tag{5.3}$$

where b_1, b_2 and b_3 are one-forms. So the question when $\Lambda \geq 0$ becomes: find necessary conditions of b_i such that $\int_{\mathbb{R}^3} a(u)\ dx \geq 0$ for all $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$. In fact the conditions are that $\|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)}$ cannot be too large. We prove this by using Theorem 4.1. Another application of Theorem 4.1 is to prove the weak coercivity property—Gårding's inequality.

5.1. Coercivity. For coercivity we give an "almost" necessary and sufficient condition on b_i such that $\int_{\mathbb{R}^3} a(u)\ dx \geq 0$ for all $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$.

Proposition 5.1. *Let $a(u)$ be the expression shown in* (5.3)*.*

(1) *There exists an absolute constant C_1 such that* $\max_{1\leq i\leq 3} \|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)} \leq C_1$ *implies that*

$$\int_{\mathbb{R}^3} a(u)\ dx \geq \frac{1}{2}\|Du\|^2_{L^2(\mathbb{R}^3,\wedge^1)}$$

for all $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$.

(2) *If $\int_{\mathbb{R}^3} a(u)\ dx \geq 0$ for all $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$, then there exists an absolute constant C_2 such that*

$$\max_{1\leq i\leq 3} \|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)} \leq C_2. \tag{5.4}$$

Proof. (1) Let $b_i \in BMO_d(\mathbb{R}^3, \wedge^1)$ and $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$. From (3.3) and (4.2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} a(u)\ dx \\
&\quad\geq \|Du\|^2_{L^2(\mathbb{R}^3,\wedge^1)} - \Big(\|b_1\|_{BMO_d(\mathbb{R}^3,\wedge^1)}\|du_2 \wedge du_3\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)} \\
&\qquad\qquad + \|b_2\|_{BMO_d(\mathbb{R}^3,\wedge^1)}\|du_1 \wedge du_3\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)} \\
&\qquad\qquad\qquad + \|b_3\|_{BMO_d(\mathbb{R}^3,\wedge^1)}\|du_1 \wedge du_2\|_{\mathcal{H}^1_d(\mathbb{R}^3,\wedge^2)}\Big) \\
&\quad\geq \|Du\|^2_{L^2(\mathbb{R}^3,\wedge^1)} - C \max_{1\leq i\leq 3}\|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)}\Big(\|du_2\|_{L^2(\mathbb{R}^3,\wedge^1)}\|du_3\|_{L^2(\mathbb{R}^3,\wedge^1)} \\
&\qquad\qquad + \|du_1\|_{L^2(\mathbb{R}^3,\wedge^1)}\|du_3\|_{L^2(\mathbb{R}^3,\wedge^1)} \\
&\qquad\qquad\qquad + \|du_1\|_{L^2(\mathbb{R}^3,\wedge^1)}\|du_2\|_{L^2(\mathbb{R}^3,\wedge^1)}\Big) \\
&\quad\geq \Big(1 - C \max_{1\leq i\leq 3}\|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)}\Big)\|Du\|^2_{L^2(\mathbb{R}^3,\wedge^1)}.
\end{aligned}$$

So

$$\max_{1\leq i\leq 3} \|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)} \leq C_1 = \frac{1}{2C}$$

implies that

$$\int_{\mathbb{R}^3} a(u)\ dx \geq \frac{1}{2}\|Du\|^2_{L^2(\mathbb{R}^3,\wedge^1)}$$

for all $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$.

(2) From Theorem 4.1, there exist absolute constants C and C' such that

$$C\|b\|_{BMO_d(\mathbb{R}^3,\wedge^1)} \leq \sup_{u,v\in W}\int_{\mathbb{R}^3} b \wedge du \wedge dv \leq C'\|b\|_{BMO_d(\mathbb{R}^3,\wedge^1)}.$$

For any $\varepsilon > 0$, there exist $u^\varepsilon,\ v^\varepsilon \in W^{1,2}(\mathbb{R}^3)$ with $\|du^\varepsilon\|_{L^2(\mathbb{R}^3,\wedge^1)},\ \|dv^\varepsilon\|_{L^2(\mathbb{R}^3,\wedge^1)} \leq 1$ such that

$$C\|b\|_{BMO_d(\mathbb{R}^3,\wedge^1)} - \varepsilon \leq \int_{\mathbb{R}^3} b \wedge du^\varepsilon \wedge dv^\varepsilon. \tag{5.5}$$

For b_i and $\varepsilon = 1$, (5.5) gives

$$\int_{\mathbb{R}^3} b_i \wedge d(-u^1) \wedge dv^1 \leq -\ C\|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)} + 1. \tag{5.6}$$

Let $w^1 = (0, -u^1, v^1)$. Since $\int_{\mathbb{R}^3} a(u)\ dx \geq 0$ for all $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$, in particular $\int_{\mathbb{R}^3} a(w^1)\ dx \geq 0$. Combining this with (5.6) we get

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^3} |Dw^1|^2\ dx + \int_{\mathbb{R}^3} b_i \wedge d(-u^1) \wedge dv^1 \\ &\leq \|Dw^1\|^2_{L^2(\mathbb{R}^3,\wedge^1)} - C\|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)} + 1 \\ &\leq 3 - C\|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)}. \end{aligned}$$

Hence

$$\max_{1\leq i\leq 3} \|b_i\|_{BMO_d(\mathbb{R}^3,\wedge^1)} \leq C_2 = \frac{3}{C}.$$

Proposition 5.1 is proved. □

5.2. Gårding's inequality. Usually Gårding's inequality is a consequence of ellipticity (see, for example, [Gi, Chapter 1]). In [Z2] and [Z3], counterexamples were given showing that Gårding's inequality may not hold in general for systems with L^∞ coefficients which satisfy the Legendre-Hadamard ellipticity condition. In this section, we establish a necessary condition for Gårding's inequality. The condition is that a certain type of BMO seminorm of b_i (defined in (5.8)) cannot be too large.

The following lemma can be implied from Theorem 4.1.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^3$ be an open domain and $b \in L^2_{loc}(\Omega, \wedge^1)$. Then there exists an absolute constant C_3 such that*

$$\sup_B \inf_g \left(\frac{1}{|B|}\int_B |b - g|^2\ dx\right)^{1/2} \leq C_3 \sup_{u,v\in W} \int_\Omega b \wedge du \wedge dv, \tag{5.7}$$

where the supremum in the left-hand side is taken over all balls B with $2B \subset \Omega$, the infimum is taken over all $g \in L^2(B, \wedge^1)$ with $dg = 0$ in B, and $W = \{w \in W^{1,2}_0(\Omega) : \|dw\|_{L^2(\Omega,\wedge^1)} \leq 1\}$.

Remark. The two sides of (5.7) are actually equivalent when Ω is a special Lipschitz domain or a bounded strongly Lipschitz domain. See Theorem 6.1 of [LM2].

Let us denote the left-hand side of (5.7) by $\|b\|_{BMO^H_d(\Omega,\wedge^1)}$.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^3$ be an open domain, let $a(u)$ be the expression shown in (5.3) and $\int_\Omega a(u)\ dx \geq 0$ for all $u \in W^{1,2}_0(\Omega, \wedge^1)$. Then there exists an absolute constant C_4 such that*

$$\max_{1\leq i\leq 3} \|b_i\|_{BMO^H_d(\Omega,\wedge^1)} \leq C_4.$$

Proof. Using Lemma 5.2, similar to the proof of Proposition 5.1 (2), we can prove the proposition. The details are omitted. □

For $b \in L^2_{loc}(\mathbb{R}^3, \wedge^1)$, define

$$\|b\|_* = \lim_{l\to 0} \sup_B \inf_g \left(\frac{1}{|B|} \int_B |b-g|^2 \, dx \right)^{1/2}, \tag{5.8}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^3$ with radius less than $l > 0$, the infimum is taken over all $g \in L^2(B, \wedge^1)$ with $dg = 0$ in B.

Proposition 5.4. *Assuming Gårding's inequality holds for $\int_{\mathbb{R}^3} a(u)\, dx$, that is, there exist constants $\lambda_0 > 0$, $\lambda_1 \geq 0$ such that*

$$\int_{\mathbb{R}^3} a(u)\, dx \geq \lambda_0 \int_{\mathbb{R}^3} |Du|^2 \, dx - \lambda_1 \int_{\mathbb{R}^3} |u|^2 \, dx \tag{5.9}$$

for all $u \in W^{1,2}(\mathbb{R}^3, \wedge^1)$. Then

$$\max_{1\leq i\leq 3} \|b_i\|_* \leq C_4,$$

where C_4 is the same constant in Lemma 5.3.

Proof. From (5.8), there exists a sequence of balls $B_{r_k} = B(x_k, r_k) \subset \mathbb{R}^3$ with $r_k \to 0$ such that

$$\inf_g \left(\frac{1}{|B_{r_k}|} \int_{B_{r_k}} |b_i - g|^2 \, dx \right)^{1/2} \to \|b_i\|_*. \tag{5.10}$$

Suppose that $v := v_1 dx_1 + v_2 dx_2 + v_3 dx_3 \in W^{1,2}(\mathbb{R}^3, \wedge^1)$ is supported in $2B_{r_k}$. By (5.9),

$$\begin{aligned} \int_{2B_{r_k}} |Dv(x)|^2 \, dx + \int_{2B_{r_k}} \Big(& b_1(x) \wedge dv_2(x) \wedge dv_3(x) \\ & + b_2(x) \wedge dv_1(x) \wedge dv_3(x) + b_3(x) \wedge dv_1(x) \wedge dv_2(x) \Big) \\ \geq{} & \lambda_0 \int_{2B_{r_k}} |Dv(x)|^2 \, dx - \lambda_1 \int_{2B_{r_k}} |v(x)|^2 \, dx. \end{aligned} \tag{5.11}$$

Set $x = x_k + 2r_k y$ and let $v^k(y) = v(x_k + 2r_k y)$, $b_i^k(y) = b_i(x_k + 2r_k y)$ in (5.11) we have

$$\begin{aligned} \int_{B(0,1)} |Dv^k(y)|^2 \, dy + \int_{B(0,1)} \Big(& b_1^k(y) \wedge dv_2^k(y) \wedge dv_3^k(y) \\ & + b_2^k(y) \wedge dv_1^k(y) \wedge dv_3^k(y) + b_3^k(y) \wedge dv_1^k(y) \wedge dv_2^k(y) \Big) \\ \geq{} & \lambda_0 \int_{B(0,1)} |Dv^k(y)|^2 \, dy - \lambda_1 \int_{B(0,1)} (2r_k)^2 |v^k(y)|^2 \, dy \end{aligned} \tag{5.12}$$

for all $v^k \in W_0^{1,2}\big(B(0,1), \wedge^1\big)$. For r_k sufficiently small, by Poincaré's inequality the right-hand side of (5.12) is nonnegative, i.e.,

$$\int_{B(0,1)} a(v^k)\, dy \geq 0$$

for all $v^k \in W_0^{1,2}\big(B(0,1), \wedge^1\big)$. This yields

$$\max_{1\leq i\leq 3} \inf_g \left(\frac{1}{|B(0,1/2)|} \int_{B(0,1/2)} |b_i^k(y) - g(y)|^2 \, dy \right)^{1/2} \leq C_4 \tag{5.13}$$

by Lemma 5.3, where the infimum is taken over all $g \in L^2\big(B(0,1/2), \wedge^1\big)$ with $dg = 0$ in $B(0,1/2)$. Let $y = \frac{x-x_k}{2r_k}$, (5.13) gives

$$\max_{1\le i\le 3} \inf_{\tilde g} \left(\frac{1}{|B_{r_k}|} \int_{B_{r_k}} |b_i(x) - \tilde g(x)|^2 \, dx \right)^{1/2} \le C_4, \tag{5.14}$$

where $\tilde g(x) \in L^2(B_{r_k}, \wedge^1)$ with $d\tilde g = 0$. Combining (5.12) with (5.14) we have

$$\max_{1\le i\le 3} \|b_i\|_* \le C_4.$$

The proof of Proposition 5.4 is finished. □

Remark. Following Remark 2 of [Z1], we know that when b_i in (5.3) are in $L^\infty(\mathbb{R}^3)$ and $\|b_i\|_*$ is sufficiently small, then the Gårding's inequality in (5.9) holds with constants λ_0, λ_1 depending on b_i. This can be proved by following the approach in [Gi, pages 9-11].

6. Hardy spaces of exact forms on $\mathbb{R}^N$

In this section we introduce Hardy spaces of exact forms on $\mathbb{R}^N$ and study their atomic decompositions and dual spaces. Using duality results we prove that Theorem 4.1 holds on $\mathbb{R}^N$ when $b,\ u,\ v$ are respectively $k,\ m,\ l$-forms with $k + m + l + 2 = N$.

6.1. Definitions.

Definition 6.1. For $0 \le l \le N$, the *Hardy space of l-forms* is defined as

$$\mathcal{H}^1(\mathbb{R}^N, \wedge^l) = \{f : \mathbb{R}^N \to \wedge^l : \text{each component of } \ f \text{ is in } \ \mathcal{H}^1(\mathbb{R}^N)\}$$

with the norm

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^l)} = \sum_I \|f_I\|_{\mathcal{H}^1(\mathbb{R}^N)}$$

for $f = \sum_I f_I dx_I$.

Definition 6.2. Let $1 \le l \le N$. The *Hardy space of exact l-forms* is defined as

$$\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l) = \{f \in \mathcal{H}^1(\mathbb{R}^N, \wedge^l) : f = dg \ \text{ for some } \ g \in \mathcal{D}'(\mathbb{R}^N, \wedge^{l-1})\}$$

with the norm

$$\|f\|_{\mathcal{H}^1_d(\mathbb{R}^N,\wedge^l)} = \|f\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^l)}.$$

Remark. When $l = N$, $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ is isomorphic to the usual Hardy space $\mathcal{H}^1(\mathbb{R}^N)$. When $l = N-1$, $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ is isomorphic to the divergence-free Hardy space $\mathcal{H}^1_{div}(\mathbb{R}^N, \mathbb{R}^N) := \{f \in \mathcal{H}^1(\mathbb{R}^N, \mathbb{R}^N) : \operatorname{div} f = 0 \text{ in } \ \mathbb{R}^N\}$.

Definition 6.3. We say that a is an $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$-*atom* if

(i) there exists $b \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in a ball (or a cube) B in $\mathbb{R}^N$ such that $a = db$;

(ii) a and b satisfy size conditions: $\|a\|_{L^2(B,\wedge^l)} \le |B|^{-1/2}$, $\|b\|_{L^2(B,\wedge^{l-1})} \le r(B)|B|^{-1/2}$, where $r(B)$ denotes the radius of B.

6.2. Atomic decompositions and dual spaces. The main result of this section is the following atomic decomposition theorem for $\mathcal{H}_d^1(\mathbb{R}^N, \wedge^l)$. We also characterize its dual by using the decomposition.

Theorem 6.4. *Let $1 \le l \le N$. An $l-$form f on $\mathbb{R}^N$ is in $\mathcal{H}_d^1(\mathbb{R}^N, \wedge^l)$ if and only if it has a decomposition*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{6.1}$$

where the a_k's are $\mathcal{H}_d^1(\mathbb{R}^N, \wedge^l)$-atoms and $\sum_{k=0}^{\infty} |\lambda_k| < \infty$. Furthermore,

$$\|f\|_{\mathcal{H}_d^1(\mathbb{R}^N, \wedge^l)} \sim \inf \left(\sum_{k=0}^{\infty} |\lambda_k| \right),$$

where the infimum is taken over all such decompositions. The constants of the proportionality depend only on the dimension N.

Proof. We first prove the "if" part. Suppose that f can be written as (6.1) with

$$\sum_{k=0}^{\infty} |\lambda_k| < \infty, \tag{6.2}$$

where the a_k's are $\mathcal{H}_d^1(\mathbb{R}^N, \wedge^l)$-atoms, i.e., there exist $b_k \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in balls B_k such that $a_k = db_k$ and $\|b_k\|_{L^2(\mathbb{R}^N, \wedge^{l-1})} \le r(B_k)|B_k|^{-1/2}$. We need to show that

$$g := \sum_{k=0}^{\infty} \lambda_k b_k$$

exists in $\mathcal{D}'(\mathbb{R}^N, \wedge^{l-1})$, for then $f = dg \in \mathcal{H}_d^1(\mathbb{R}^N, \wedge^l)$.

To prove $g \in \mathcal{D}'(\mathbb{R}^N, \wedge^{l-1})$, it is sufficient to show that the sum $\sum_{k=0}^{\infty} \lambda_k b_k$ is convergent in the sense of distributions. From (6.2),

$$\sum_{k=m}^{n} |\lambda_k| \to 0 \qquad \text{as } m,\ n \to \infty.$$

Combining this with the size condition of b_k, for any $\varphi \in C_0^\infty(\mathbb{R}^N, \wedge^{N-l+1})$ with compact support Ω,

$$\begin{aligned}
\Big| \int_{\mathbb{R}^N} \left(\sum_{k=m}^{n} \lambda_k b_k \right) \wedge \varphi \Big| &\le \sum_{k=m}^{n} |\lambda_k| \Big| \int_{B_k \cap \Omega} b_k \wedge \varphi \Big| \\
&\le C \sum_{k=m}^{n} |\lambda_k| \|b_k\|_{L^2(B_k \cap \Omega, \wedge^{l-1})} |B_k \cap \Omega|^{1/2} \\
&\le C \sum_{k=m}^{n} |\lambda_k| r(B_k) |B_k|^{-1/2} |B_k \cap \Omega|^{1/2} \\
&\le C \sum_{k=m}^{n} |\lambda_k| \to 0 \qquad \text{as } \ m,\ n \to \infty,
\end{aligned}$$

where the constant C depends only on φ. The convergence of $\sum_{k=0}^{\infty} \lambda_k b_k$ is proved.

The proof of the "only if" part is similar to that of Theorem 2.3 in Section 2, so we only give an outline of it. From the proof of Theorem 2.3, we know that any $f \in \mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ can be written as

$$f = -\int_0^\infty td\Big(t\delta(f * \varphi_t) * \varphi_t\Big)\frac{dt}{t}, \tag{6.3}$$

where $\varphi \in C_0^\infty(\mathbb{R}^N)$ with support in the unit ball and $\int_0^\infty t|\xi|^2\hat{\varphi}(t\xi)^2\ dt = 1$. For $y \in \mathbb{R}^N,\ t > 0$, define

$$F(y,t) = t\delta(f * \varphi_t)(y).$$

Let $f = \sum_I f_I e_I \in \wedge^l$, then

$$F(y,t) = \sum_{i,I} t\partial_i(f_I * \varphi_t)(y)\mu_i^*(e_I) = \sum_{i,I} f_I * (\partial_i\varphi)_t(y)\mu_i^*(e_I).$$

Applying (2.1), we get $F \in \mathcal{N}^1(\mathbb{R}^{N+1}_+, \wedge^{l-1})$ and

$$\|F\|_{\mathcal{N}^1(\mathbb{R}^{N+1}_+, \wedge^{l-1})} \le C\|f\|_{\mathcal{H}^1(\mathbb{R}^N, \wedge^l)}.$$

From atomic decompositions for tent spaces,

$$F = \sum_{k=0}^\infty \lambda_k \alpha_k \tag{6.4}$$

with

$$\sum_{k=0}^\infty |\lambda_k| \le C\|F\|_{\mathcal{N}^1(\mathbb{R}^{N+1}_+, \wedge^{l-1})},$$

where the α_k's are $\mathcal{N}^1(\mathbb{R}^{N+1}_+, \wedge^{l-1})$-atoms. Define

$$a_k = -\int_0^\infty td\Big(\alpha_k(\cdot, t) * \varphi_t\Big)\frac{dt}{t}.$$

From (6.3) and (6.4), we have

$$f = \sum_{k=0}^\infty \lambda_k a_k,$$

where a_k is supported in a ball $2B_k$ and satisfies the size condition: $\|a_k\|_{L^2(2B_k, \wedge^l)} \le C|2B_k|^{-1/2}$ for a constant C independent of k. Applying Lemma 6.7 (1) in Section 6 to a_k, there exists $b_k \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in $2B_k$ such that $a_k = db_k$ and

$$\|b_k\|_{L^2(2B_k, \wedge^{l-1})} \le C\ r(2B_k)|2B_k|^{-1/2},$$

where C is independent of k. We have proved that a_k is an $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$-atom. The proof of Theorem 6.4 is finished. □

Now we consider dual spaces of $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$. Let $BMO_d(\mathbb{R}^N, \wedge^k)$ $(0 \le k \le N)$ be the space of all locally integrable functions $G : \mathbb{R}^N \to \wedge^k$ with

$$\|G\|_{BMO_d(\mathbb{R}^N, \wedge^k)} = \sup_B \inf_{g_B} \left(\frac{1}{|B|}\int_B |G - g_B|^2\ dx\right)^{1/2} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^N$ and the infimum is taken over all $g_B \in L^2(B, \wedge^k)$ with $dg_B = 0$ in B. Consider $BMO_d(\mathbb{R}^N, \wedge^k)/X_0$ with the norm

$$\|G + X_0\|_{BMO_d(\mathbb{R}^N, \wedge^k)/X_0} = \|G\|_{BMO_d(\mathbb{R}^N, \wedge^k)},$$

where $X_0 = \{G \in BMO_d(\mathbb{R}^N, \wedge^k) : \|G\|_{BMO_d(\mathbb{R}^N,\wedge^k)} = 0\}$. We see that when $k = 0$, $BMO_d(\mathbb{R}^N, \wedge^0)/X_0$ reduces to the usual BMO-space on $\mathbb{R}^N$.

The following theorem is an analogue of Theorem 3.3, which reveals that the dual of $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ is the space $BMO_d(\mathbb{R}^N, \wedge^{N-l})/X_0$. It is a generalization of the well-known duality result $(\mathcal{H}^1(\mathbb{R}^N))' = BMO(\mathbb{R}^N)$. Its proof is similar to that of Theorem 3.3, so we skip the details. Let $D(\mathbb{R}^N, \wedge^l)$ denote the vector space finitely generated by $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$-atoms. Theorem 6.4 implies that $D(\mathbb{R}^N, \wedge^l)$ is dense in $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$.

Theorem 6.5. *Let $1 \le l \le N$. If $G+X_0 \in BMO_d(\mathbb{R}^N, \wedge^{N-l})/X_0$, then the linear functional L defined by*

$$L(h) = \int_{\mathbb{R}^N} G \wedge h, \tag{6.5}$$

initially defined in $D(\mathbb{R}^N, \wedge^l)$, has a unique bounded extension to $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$. Conversely, if $L \in (\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l))'$, then there exists a unique $G+X_0 \in BMO_d(\mathbb{R}^N, \wedge^{N-l})/X_0$ such that (6.5) *holds. The map $G + X_0 \mapsto L$ given by* (6.5) *is a Banach isomorphism between $BMO_d(\mathbb{R}^N, \wedge^{N-l})/X_0$ and $(\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l))'$.*

6.3. The "div-curl" type theorem on $\mathbb{R}^N$. We now prove the "div-curl" type theorem on $\mathbb{R}^N$, which is a generalization of Theorem 4.1 to N-dimensions and to forms of all degrees.

Theorem 6.6. *Let $b \in L^2_{loc}(\mathbb{R}^N, \wedge^l)$, $l,\ m,\ n \ge 0$ and $l + m + n + 2 = N$. Then*

$$\sup_{u,v} \int_{\mathbb{R}^N} b \wedge du \wedge dv \sim \|b\|_{BMO_d(\mathbb{R}^N,\wedge^l)}, \tag{6.6}$$

where the supremum is taken over all u and v with

$$\begin{cases} u \in W^{1,2}(\mathbb{R}^N, \wedge^m), \quad \|du\|_{L^2(\mathbb{R}^N,\wedge^{m+1})} \le 1; \\ v \in W^{1,2}(\mathbb{R}^N, \wedge^n), \quad \|dv\|_{L^2(\mathbb{R}^N,\wedge^{n+1})} \le 1. \end{cases} \tag{6.7}$$

The implicit constants in (6.6) *depend only on N.*

To prove Theorem 6.6 we need the following lemmas.

Lemma 6.7. *Let B be a ball in $\mathbb{R}^N$.*

(1) *If u satisfies either of the following conditions:*

1) *$u \in L^2(B, \wedge^l)$, $du = 0$ in B and $n \wedge u|_{\partial B} = 0$, where $0 < l < N$;*

2) *$u \in L^2(B, \wedge^l)$ with $\int u = 0$, where $l = N$,*

then there exists $\varphi \in W^{1,2}_0(B, \wedge^{l-1})$ and a constant C independent of u and B such that

$$u = d\varphi,$$

$$\|D\varphi\|_{L^2(B,\wedge^{l-1})} \le C\|u\|_{L^2(B,\wedge^l)} \tag{6.8}$$

and

$$\|\varphi\|_{L^2(B,\wedge^{l-1})} \le C\ r(B)\|u\|_{L^2(B,\wedge^l)}. \tag{6.9}$$

(2) *If u satisfies either of the following conditions:*

1) $u \in L^2(B, \wedge^l)$, $\delta u = 0$ *in* B *and* $n \vee u|_{\partial B} = 0$, *where* $0 < l < N$*;*

2) $u \in L^2(B, \wedge^l)$ *with* $\int u = 0$, *where* $l = 0$,

then there exists $\psi \in W_0^{1,2}(B, \wedge^{l+1})$ *and a constant* C *independent of* u *and* B *such that*

$$u = \delta\psi$$

and

$$\|D\psi\|_{L^2(B,\wedge^{l+1})} \leq C\|u\|_{L^2(B,\wedge^l)}.$$

Proof. (1) When u satisfies the conditions of 2), Lemma 6.7 (1) is a special case of Nečas' result in [N, Lemma 7.1, Chapter 3]. So we only prove 1). From Theorem 3.3.3 in Chapter 3 of [Sc], there exists $\varphi \in W^{1,2}(B, \wedge^{l-1})$ such that

$$u = d\varphi, \qquad \varphi|_{\partial B} = 0$$

and

$$\|\varphi\|_{W^{1,2}(B,\wedge^{l-1})} \leq C\|u\|_{L^2(B,\wedge^l)}$$

for some constants C independent of u. So the only thing we need to check is that the constants in (6.8) and (6.9) are independent of balls B. We only prove this for (6.9). Let $B = B(x_0, r)$, x_0 be the center of B, $r = r(B)$. It is easy to check that $u(x_0 + ry)$, y is in the unit ball B_0, satisfies the conditions of 1) for B_0. So there exists $\frac{\varphi(x_0+ry)}{r} \in W_0^{1,2}(B_0, \wedge^{l-1})$ and a constant C independent of u and r such that

$$u(x_0 + ry) = d\frac{\varphi(x_0 + ry)}{r}$$

and

$$\frac{1}{r}\|\varphi(x_0 + ry)\|_{L^2(B_0,\wedge^{l-1})} \leq C \; \|u(x_0 + ry)\|_{L^2(B_0,\wedge^l)}.$$

This is equivalent to (6.9) by a simple computation.

(2) can be derived from Corollary 3.3.4 of [Sc, Chapter 3], Nečas' lemma and a similar discussion to (1). □

Remark. Lemma 6.7 holds when B is replaced by any smooth and contractible domain in $\mathbb{R}^N$, although with constants C which depend on the domain.

Lemma 6.8. *For a ball* $B \subset \mathbb{R}^N$ *and* $0 \leq l \leq N$, *let* $H = \{u \in L^2(B, \wedge^l) : \; du = 0 \text{ in } \; B, \; n \wedge u|_{\partial B} = 0\}$ *and* $\bar{H} = \{u \in L^2(B, \wedge^l) : \; \delta u = 0 \text{ in } \; B, \; n \vee u|_{\partial B} = 0\}$. *Then*

$$\begin{aligned} L^2(B, \wedge^l) &= H \oplus \{\delta w : w \in W^{1,2}(B, \wedge^{l+1})\} \\ &= \bar{H} \oplus \{dw : w \in W^{1,2}(B, \wedge^{l-1})\} \end{aligned}$$

(refer to [GR, *Theorem* 2.7]*).*

The following result is a generalized version of the "div-curl" lemma by Coifman, Lions, Meyer and Semmes in [CLMS, Theorem II.1] (see also [Gr] for related results). When $m + l = N$, it can be found in [HLMZ, Proposition 4.1].

Lemma 6.9. *If* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < m + l \leq N$, $u \in L^p(\mathbb{R}^N, \wedge^m)$, $v \in L^q(\mathbb{R}^N, \wedge^l)$, $du = 0$, $dv = 0$ *in* $\mathbb{R}^N$. *Then* $u \wedge v \in \mathcal{H}^1(\mathbb{R}^N, \wedge^{m+l})$ *and there exists a constant* C *independent of* u *and* v *such that*

$$\|u \wedge v\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^{m+l})} \leq C\|u\|_{L^p(\mathbb{R}^N,\wedge^m)}\|v\|_{L^q(\mathbb{R}^N,\wedge^l)}.$$

Proof. Suppose $m+l=N$. When $l=1$, Lemma 6.9 becomes Theorem II.1 of [CLMS]. The proof of the case $l\neq 1$ is completely similar to the case of $l=1$.

If $m+l<N$, we know that $u\wedge v\in\mathcal{H}^1(\mathbb{R}^N,\wedge^{m+l})$ if and only if $u\wedge v\wedge dx_{i_{m+l+1}}\wedge\cdots\wedge dx_{i_N}\in\mathcal{H}^1(\mathbb{R}^N,\wedge^N)$. Set

$$V=v\wedge dx_{i_{m+l+1}}\wedge\cdots\wedge dx_{i_N}.$$

Then $v\in L^q(\mathbb{R}^N,\wedge^l)$ and $dv=0$ imply that $V\in L^q(\mathbb{R}^N,\wedge^{N-m})$ and $dV=0$. For u and V, applying the result when $m+l=N$, we have $u\wedge V\in\mathcal{H}^1(\mathbb{R}^N,\wedge^N)$ and

$$\begin{aligned}\|u\wedge V\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^N)}&\le C\|u\|_{L^p(\mathbb{R}^N,\wedge^m)}\|V\|_{L^q(\mathbb{R}^N,\wedge^{N-m})}\\&=C\|u\|_{L^p(\mathbb{R}^N,\wedge^m)}\|v\|_{L^q(\mathbb{R}^N,\wedge^l)}.\end{aligned}$$

This proves the result. □

We are now ready to prove Theorem 6.6. There are some similarities between its proof and that of Theorem 4.1.

Proof of Theorem 6.6. Suppose that u and v satisfy (6.7). Lemma 6.9 yields that $du\wedge dv\in\mathcal{H}^1_d(\mathbb{R}^N,\wedge^{m+n+2})$ and

$$\begin{aligned}\|du\wedge dv\|_{\mathcal{H}^1_d(\mathbb{R}^N,\wedge^{N-l})}&=\|du\wedge dv\|_{\mathcal{H}^1(\mathbb{R}^N,\wedge^{m+n+2})}\\&\le C\|du\|_{L^2(\mathbb{R}^N,\wedge^{m+1})}\|dv\|_{L^2(\mathbb{R}^N,\wedge^{n+1})}\le C.\end{aligned}$$

Let $b\in BMO_d(\mathbb{R}^N,\wedge^l)$ and $h\in\mathcal{H}^1_d(\mathbb{R}^N,\wedge^{N-l})$. By using a similar argument as in Theorem 3.3, we have

$$\Big|\int_{\mathbb{R}^N}b\wedge h\Big|\le C\|b\|_{BMO_d(\mathbb{R}^N,\wedge^l)}\|h\|_{\mathcal{H}^1_d(\mathbb{R}^N,\wedge^{N-l})}.$$

Therefore,

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}b\wedge du\wedge dv\Big|&\le\|b\|_{BMO_d(\mathbb{R}^N,\wedge^l)}\|du\wedge dv\|_{\mathcal{H}^1_d(\mathbb{R}^N,\wedge^{N-l})}\\&\le C\|b\|_{BMO_d(\mathbb{R}^N,\wedge^l)}.\end{aligned}$$

We now prove the reversed inequality in (6.6). By scaling we need only to show that for the unit ball B_0, there exist u_0 and v_0 with

$$\left\{\begin{array}{ll}u_0\in W_0^{1,2}(B_0,\wedge^m), & \|du_0\|_{L^2(\mathbb{R}^N,\wedge^{m+1})}\le 1,\\ v_0\in W_0^{1,2}(2B_0,\wedge^n), & \|dv_0\|_{L^2(\mathbb{R}^N,\wedge^{n+1})}\le 1\end{array}\right.\tag{6.10}$$

such that

$$\inf_{g_0}\left(\int_{B_0}|b-g_0|^2\,dx\right)^{1/2}\le C\Big|\int_{B_0}b\wedge du_0\wedge dv_0\Big|,\tag{6.11}$$

where the infimum is taken over all $g_0\in L^2(B_0,\wedge^l)$ with $dg_0=0$ in B_0 and C is a constant independent of b, u_0 and v_0.

Let $b\in L^2(B_0,\wedge^l)$, Lemma 6.8 gives $b=h+dq$ for $h\in\bar{H}=\{h\in L^2(B_0,\wedge^l):\delta h=0,\ n\vee h|_{\partial B_0}=0\}$ and $q\in W^{1,2}(B_0,\wedge^{l-1})$. We have

$$\int_{B_0}b\wedge du_0\wedge dv_0=\int_{B_0}h\wedge du_0\wedge dv_0$$

and

$$\inf_{g_0}\left(\int_{B_0}|b-g_0|^2\,dx\right)^{1/2}\le\|h\|_{L^2(B_0,\wedge^l)}.$$

Thus to prove (6.11) it is sufficient to show that there exist u_0 and v_0 satisfying (6.10) such that

$$\|h\|_{L^2(B_0,\wedge^l)} \leq C\Big|\int_{B_0} h\wedge du_0 \wedge dv_0\Big| \tag{6.12}$$

for all $h \in \bar{H}$, where C does not depend on h, u_0 and v_0.

Applying Lemma 6.7 (2) to $h \in \bar{H}$, there exists $\varphi \in W_0^{1,2}(B_0, \wedge^{l+1})$ and a constant C_0 independent of h such that

$$h = \delta\varphi$$

and

$$\|D\varphi\|_{L^2(B_0,\wedge^{l+1})} \leq C_0\|h\|_{L^2(B_0,\wedge^l)}. \tag{6.13}$$

Let $\varphi = \sum_I \varphi_I dx_I$, where $I = (i_1, \cdots, i_{l+1})$, $1 \leq i_1 < \cdots < i_{l+1} \leq N$. Then

$$*h = *\delta\varphi = \pm d * \varphi = \pm \sum_I d\varphi_I \wedge (*dx_I).$$

Denote $*dx_I$ by dx_J, where $J = \{j_1, \cdots, j_{m+n+1}\}$, $1 \leq j_1 < \cdots < j_{m+n+1} \leq N$. Thus

$$\begin{aligned}\|h\|^2_{L^2(B_0,\wedge^l)} &\leq \sum_I \Big|\int_{B_0} h\wedge d\varphi_I \wedge dx_J\Big| \\ &\leq C \max_I \Big|\int_{B_0} h\wedge d\varphi_I \wedge dx_J\Big| \\ &:= C\Big|\int_{B_0} h\wedge d\varphi_{I_0} \wedge dx_{J_0}\Big| \end{aligned}\tag{6.14}$$

for some choice of I_0, where $dx_{J_0} = *dx_{I_0}$ and $C = \binom{N}{l+1}$. Let $J_0 = \{j_{0_1}, \cdots, j_{0_{m+n+1}}\}$. Define

$$u_0 = \frac{\varphi_{I_0} dx_{j_{0_1}} \wedge \cdots \wedge dx_{j_{0_m}}}{C_0\|h\|_{L^2(B_0,\wedge^l)}}.$$

It is easy to check that $u_0 \in W_0^{1,2}(B_0, \wedge^m)$ and $\|du_0\|_{L^2(\mathbb{R}^N,\wedge^{m+1})} \leq 1$ by (6.13).

Now we construct v_0. Let $\psi_0 \in C_0^\infty(\mathbb{R}^N)$, ψ_0 equals 1 in B_0 and 0 outside $\overline{2B_0}$. Setting

$$v_0 = \gamma\psi_0 x_{j_{0_{m+1}}} dx_{j_{0_{m+2}}} \wedge \cdots \wedge dx_{j_{0_{m+n+1}}},$$

where $\gamma > 0$ is a constant so that $\|dv_0\|_{L^2(\mathbb{R}^N,\wedge^{n+1})} \leq 1$. So $v_0 \in C_0^\infty(2B_0, \wedge^n)$ and

$$dv_0 = \gamma dx_{j_{0_{m+1}}} \wedge \cdots \wedge dx_{j_{0_{m+n+1}}} \text{ in } B_0.$$

Combining (6.14) with the construction of u_0 and v_0, we obtain

$$\|h\|_{L^2(B_0,\wedge^l)} \leq C\Big|\int_{B_0} h\wedge du_0 \wedge dv_0\Big|,$$

where $C = \gamma^{-1}\binom{N}{l+1}C_0$. This proves (6.12). The proof of Theorem 6.6 is completed. □

Remark. From the proof of Theorem 6.6, we see that the equivalence in (6.7) is also true if the supremum is taken over all $u \in W^{1,2}(\mathbb{R}^N, \wedge^m)$, $v \in W^{1,2}(\mathbb{R}^N, \wedge^n)$ with $\|Du\|_{L^2(\mathbb{R}^N,\wedge^{m+1})}$, $\|Dv\|_{L^2(\mathbb{R}^N,\wedge^{n+1})} \leq 1$.

Let $l = m = 0$, $n = N - 2$. Theorem 6.6 becomes

Corollary 6.10. *For $b \in L^2_{loc}(\mathbb{R}^N)$,*

$$\|b\|_{BMO(\mathbb{R}^N)} \sim \sup_{\alpha,\beta} \int_{\mathbb{R}^N} b\ \alpha \wedge \beta,$$

where the supremum is taken over all $\alpha \in L^2(\mathbb{R}^N, \wedge^1)$, $\beta \in L^2(\mathbb{R}^N, \wedge^{N-1})$ with $d\alpha = d\beta = 0$ and $\|\alpha\|_{L^2(\mathbb{R}^N,\wedge^1)},\ \|\beta\|_{L^2(\mathbb{R}^N,\wedge^{N-1})} \leq 1$.

It is easy to see that Corollary 6.10 is equivalent to the following result by Coifman, Lions, Meyer and Semmes in [CLMS, page 262]: for $b \in L^2_{loc}(\mathbb{R}^N)$,

$$\|b\|_{BMO(\mathbb{R}^N)} \sim \sup_{E,F} \int_{\mathbb{R}^N} b\ E \cdot F\ dx,$$

the supremum being taken over all E, $F \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ with div $E = 0$, curl $F = 0$ and $\|E\|_{L^2(\mathbb{R}^N,\mathbb{R}^N)}$, $\|F\|_{L^2(\mathbb{R}^N,\mathbb{R}^N)} \leq 1$.

6.4. A decomposition theorem. In [CLMS, Theorem III.2], Coifman, Lions, Meyer and Semmes proved a decomposition of $\mathcal{H}^1(\mathbb{R}^N)$ into "div-curl" quantities. The next theorem generalizes their result by showing that $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ can be decomposed into a sum of terms of the form $du \wedge dv$.

Theorem 6.11. *Let $1 \leq l \leq N$ and $0 \leq m \leq l-2$. Then any $f \in \mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ can be written as*

$$f = \sum_{k=0}^{\infty} \lambda_k\ du_k \wedge dv_k$$

with

$$\sum_{k=0}^{\infty} |\lambda_k| \leq C\|f\|_{\mathcal{H}^1_d(\mathbb{R}^N,\wedge^l)}$$

for constants C depending only on N, where $u_k \in W^{1,2}(\mathbb{R}^N, \wedge^m)$ and $v_k \in W^{1,2}(\mathbb{R}^N, \wedge^{l-m-2})$ with $\|du_k\|_{L^2(\mathbb{R}^N,\wedge^{m+1})}$, $\|dv_k\|_{L^2(\mathbb{R}^N,\wedge^{l-m-1})} \leq 1$.

Proof. By Theorem 6.4, any $f \in \mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$ has a decomposition

$$f = \sum_{i=0}^{\infty} \mu_i a_i, \tag{6.15}$$

where the a_i's are $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$-atoms and

$$\sum_{i=0}^{\infty} |\mu_i| \leq C\|f\|_{\mathcal{H}^1_d(\mathbb{R}^N,\wedge^l)}$$

for constants C depending only on N. For simplicity we drop the subscript i of a_i temporarily. Since $a := a_i$ is an $\mathcal{H}^1_d(\mathbb{R}^N, \wedge^l)$-atom, i.e., there exists $b \in L^2(\mathbb{R}^N, \wedge^{l-1})$ supported in a ball B such that $a = db$ and $\|a\|_{L^2(B,\wedge^l)} \leq |B|^{-1/2}$. Applying Lemma 6.7 (1), there exists $\varphi \in W^{1,2}_0(B, \wedge^{l-1})$ and a constant C_0 independent of a and B such that

$$a = d\varphi \tag{6.16}$$

and

$$\|D\varphi\|_{L^2(B,\wedge^{l-1})} \leq C_0\|a\|_{L^2(B,\wedge^l)}.$$

Let $\varphi = \sum_I \varphi_I dx_I$, $I = (i_1, \cdots, i_{l-1})$, $1 \le i_1 < \cdots < i_{l-1} \le N$. From (6.16) the atom a can be written as

$$\begin{aligned} a &= \sum_I d\varphi_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{l-1}} \\ &= \sum_I d(C_0^{-1}|B|^{1/2}\varphi_I) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_m} \wedge d(C_0|B|^{-1/2}x_{i_{m+1}}) \wedge \cdots \wedge dx_{i_{l-1}}. \end{aligned}$$

For any I, define

$$u_{(I)} = C_0^{-1}|B|^{1/2}\varphi_I dx_{i_1} \wedge \cdots \wedge dx_{i_m}.$$

Then $u_{(I)} \in W_0^{1,2}(B, \wedge^m)$ and $\|du_{(I)}\|_{L^2(B,\wedge^{m+1})} \le 1$. As in the proof of Theorem 6.6, define $\psi_0 \in C_0^\infty(\mathbb{R}^N)$. Let

$$v_{(I)} = \gamma C_0 |B|^{-1/2} \psi_B(x_{i_{m+1}} - x^0_{i_{m+1}}) dx_{i_{m+2}} \wedge \cdots \wedge dx_{i_{l-1}},$$

where $\psi_B(x) = \psi_0\left(\frac{x-x^0}{r}\right)$, x^0 denotes the center of the ball B, $r = r(B)$, γ is a constant independent of x^0 and r such that $\|dv_{(I)}\|_{L^2(B,\wedge^{l-m-1})} \le 1$. We see that $v_{(I)} \in C_0^\infty(2B, \wedge^{l-m-2})$ and

$$dv_{(I)} = \gamma C_0 |B|^{-1/2} dx_{i_{m+1}} \wedge \cdots \wedge dx_{i_{l-1}} \quad \text{in} \quad B.$$

Thus any atom a can be written as

$$a = \gamma^{-1} \sum_I du_{(I)} \wedge dv_{(I)}.$$

Combining this with the atomic decomposition of f in (6.15). We proved Theorem 6.11. □

Let $l = N$, $m = 0$ in Theorem 6.11 we get

Corollary 6.12 ([CLMS, Theorem III.2])**.** *Any $f \in \mathcal{H}^1(\mathbb{R}^N)$ can be written as*

$$f = \sum_{k=0}^{\infty} \lambda_k \; E_k \cdot F_k$$

with

$$\sum_{k=0}^{\infty} |\lambda_k| \le C\|f\|_{\mathcal{H}^1(\mathbb{R}^N)}$$

for a constant C depending only on N, where $E_k,\ F_k \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ with div $E_k =$ curl $F_k = 0$ *and* $\|E_k\|_{L^2(\mathbb{R}^N,\mathbb{R}^N)}$, $\|F_k\|_{L^2(\mathbb{R}^N,\mathbb{R}^N)} \le 1$.

The proof of Corollary 6.12 in [CLMS] is based on two results from functional analysis (see [CLMS, Lemmas III.1 and III.2]). The proof we have given is more natural in the context of the theory of Hardy spaces.

Appendix A. Surjectivity of the curl operator

In this appendix we present an unpublished result of Costabel [C], that in three dimensions, the operator curl is surjective from $W_0^{1,2}(\Omega, \mathbb{R}^3)$ to a closed subspace of $L^2(\Omega)$ when Ω is a bounded contractible strongly Lipschitz domain in $\mathbb{R}^3$ (see [LM2] for the definition of strongly Lipschitz domains). For $\psi \in \mathcal{D}'(\Omega)$, we adopt the notation $\nabla\psi = (\partial_1\psi, \partial_2\psi, \partial_3\psi)$, while for $v = (v_1, v_2, v_3) \in \mathcal{D}'(\Omega, \mathbb{R}^3)$, we define the divergence and curl operators by

$$\text{div } v = \sum_{i=1}^{3} \partial_i v_i$$

and

$$\text{curl } v = (\partial_2 v_3 - \partial_3 v_2,\ \partial_3 v_1 - \partial_1 v_3,\ \partial_1 v_2 - \partial_2 v_1)\,.$$

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, the divergence operator is a continuous map from $W_0^{1,2}(\Omega, \mathbb{R}^N)$ onto $L_0^2(\Omega)$, where $W_0^{1,2}(\Omega, \mathbb{R}^N)$ denotes the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^N)$ with zero boundary values, and $L_0^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f\ dx = 0\}$. This is a result by Nečas in [N, Lemma 7.1, Chapter 3]. We now consider the operator curl : $W_0^{1,2}(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3)$. The next proposition shows that the operator curl is surjective from $W_0^{1,2}(\Omega, \mathbb{R}^3)$ to a closed subspace of $L^2(\Omega, \mathbb{R}^3)$.

Proposition A.1. *Let Ω be a bounded contractible strongly Lipschitz domain in $\mathbb{R}^3$, $u \in L^2(\Omega, \mathbb{R}^3)$, $\text{div } u = 0$ in Ω and $n \cdot u|_{\partial\Omega} = 0$. Then there exists $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ such that*

$$\text{curl } \varphi = u$$

and

$$\|\varphi\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C\|u\|_{L^2(\Omega,\mathbb{R}^3)},$$

where the constant C depends only on the domain Ω.

Proof. Let U denote the extension by zero of u to $\mathbb{R}^3$. Then $\text{div } U = 0$ on all of $\mathbb{R}^3$. Therefore there exists $V \in W_{\text{loc}}^{1,2}(\mathbb{R}^3, \mathbb{R}^3)$ such that $U = \text{curl } V$. On letting B be a large ball containing $\overline{\Omega}$, then $U \in W^{1,2}(B)$. In the simply connected domain $B\backslash\overline{\Omega}$, $\text{curl } V = 0$, so there exists $\psi \in W^{2,2}(B\backslash\overline{\Omega})$ such that $\nabla\psi = V$.

Let $E : W^{2,2}(B\backslash\overline{\Omega}) \to W^{2,2}(B)$ be a bounded extension operator (see [St1]). The vector field $\Psi = V - \nabla(E\psi) \in W^{1,2}(B, \mathbb{R}^3)$ has support in $\overline{\Omega}$ and satisfies $\text{curl } \Psi = U$. Thus the restriction $\varphi = \Psi|_\Omega \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ solves the equation $\text{curl } \varphi = u$ as required.

The mapping $u \mapsto \varphi$ is a bounded linear mapping depending only on E. $\square$

Remark. (1) In Proposition A.1, when $\Omega = B$, a ball in $\mathbb{R}^3$, we have

$$\|D\varphi\|_{L^2(B,\mathbb{R}^3)} \le C\|u\|_{L^2(B,\mathbb{R}^3)}$$

for an absolute constant C, where $D\varphi = (\partial_j\varphi_i)$ for $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.

(2) In N dimensions, the proposition applies to solving $d\varphi = u$ for $\varphi \in W_0^{1,2}(\Omega, \wedge^1)$ when u is a 2–form satisfying $du = 0$ and $n \wedge u|_\Omega = 0$. The proof does not apply to general k–forms. However when the boundary of Ω is smooth, a slight adaptation of the above argument gives an alternative proof of Lemma 6.7.

Appendix B. Review of differential forms

The setting of this section is that of forms on open domain $\Omega \subset \mathbb{R}^N$. We give a brief outline of the basic formalism.

Let $\{e_1, \cdots, e_N\}$ denote the basis of Euclidean space $\mathbb{R}^N$ and $l = 1, \cdots, N$. The space of all l-linear, alternating functions $\xi : (\mathbb{R}^N)^l \to \mathbb{R}$ is denoted by $\wedge^l(\mathbb{R}^N)$, or just $\wedge^l$ where there is no possibility of confusion. In particular $\wedge^1(\mathbb{R}^N)$ is the dual of $\mathbb{R}^N$ and $\wedge^0(\mathbb{R}^N) = \mathbb{R}$. The dual base to $\{e_1, \cdots, e_N\}$ will be denoted by

$e^1, \cdots, e^N$ and referred to as the standard base for $\wedge^1(\mathbb{R}^N)$. The vector space of all forms $\wedge(\mathbb{R}^N) = \bigoplus_{l=0}^N \wedge^l(\mathbb{R}^N)$ is equipped with the inner product

$$\langle \alpha, \beta \rangle = \sum \alpha_{i_1 \cdots i_l} \beta_{i_1 \cdots i_l}$$

for $\alpha = \sum \alpha_{i_1 \cdots i_l} e^{i_1} \wedge \cdots \wedge e^{i_l}$ and $\beta = \sum \beta_{i_1 \cdots i_l} e^{i_1} \wedge \cdots \wedge e^{i_l}$. For $w \in \wedge(\mathbb{R}^N)$ the associated norm is denoted by $|w| = \langle w, w \rangle^{1/2}$. The inner product induces a dual pairing between $\wedge^l(\mathbb{R}^N)$ and $\wedge^{N-l}(\mathbb{R}^N)$ which results from the action of the Hodge star operator $*$ defined by

$$*1 = e^1 \wedge \cdots \wedge e^N;$$

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle e^1 \wedge \cdots \wedge e^N$$

for all $\alpha,\ \beta \in \wedge^l(\mathbb{R}^N)$. The exterior and interior multiplication operators on $\wedge(\mathbb{R}^N)$ are linear operators defined by

$$\mu_k : \wedge^l \to \wedge^{l+1}; \quad \mu_k(1) = e^k,\ \mu_k(e^i) = e^k \wedge e^i,\ \cdots$$

and

$$\mu_k^* : \wedge^l \to \wedge^{l-1}; \quad \mu_k^*(e^i) = \delta_{ki},\ \mu_k^*(e^i \wedge e^l) = \delta_{ki} e^l - \delta_{kl} e^i,\ \cdots$$

respectively. The exterior and interior operators can be written as (see [GHL])

$$d = \sum_{k=1}^N \mu_k \frac{\partial}{\partial x_k}, \qquad \delta = \sum_{k=1}^N \mu_k^* \frac{\partial}{\partial x_k}.$$

We define the interior product between a 1-form α and an $l-$form u by setting

$$\alpha \vee u = (-1)^{(l-1)N} * (\alpha \wedge * u).$$

Suppose Ω is a bounded Lipschitz domain in $\mathbb{R}^N$. We denote by $L^2(\Omega, \wedge^l)$ the space of square integrable l-forms on Ω.

Definition B.1. Let $0 \le l \le N$. For $u \in L^2(\Omega, \wedge^l)$, we say that $du = 0$ on Ω if

$$\int_\Omega u \wedge d\varphi = 0$$

for all $\varphi \in C_0^\infty(\Omega, \wedge^{N-l-1})$.

Definition B.2. For $u \in L^2(\Omega, \wedge^l)$ with $du = 0$ on Ω, we define $n \wedge u|_{\partial\Omega} \in W^{-1/2,2}(\partial\Omega, \wedge^{l+1})$ by

$$\langle n \wedge u|_{\partial\Omega}, \psi \rangle_{\partial\Omega} = (-1)^l \int_\Omega u \wedge d\Psi,$$

where $\Psi \in C^1(\bar{\Omega}, \wedge^{N-l-1})$, $\psi = \Psi|_{\partial\Omega}$, $W^{-1/2,2}(\partial\Omega, \wedge^{l+1})$ is the space of $(l+1)$-forms each of whose components is in $W^{-1/2,2}(\partial\Omega)$.

See [A] for the definition of Sobolev spaces $W^{s,p}(\partial\Omega)$.

Remark. It is easy to show that the definition of $\langle n \wedge u|_{\partial\Omega}, \psi \rangle_{\partial\Omega}$ is independent of the choice of the extension Ψ. Note that

$$\|n \wedge u|_{\partial\Omega}\|_{W^{-1/2,2}(\partial\Omega, \wedge^{l+1})} \le C \|u\|_{L^2(\Omega, \wedge^l)}$$

for all $u \in L^2(\Omega, \wedge^l)$ such that $du = 0$ ([HLMZ]).

The Green's formula is as follows: if $u \in L^2(\Omega, \wedge^l)$ with $du \in L^2(\Omega, \wedge^{l+1})$ and $\varphi \in W^{1,2}(\Omega, \wedge^{N-l-1})$, then

$$\int_\Omega du \wedge \varphi + (-1)^l \int_\Omega u \wedge d\varphi = \langle n \wedge u|_{\partial\Omega}, \varphi\rangle_{\partial\Omega}.$$

The formula follows from Stokes' theorem and the fact that $C_0^\infty(\bar{\Omega}, \wedge^l)$ is dense both in the space $\{u \in L^2(\Omega, \wedge^l) : du \in L^2(\Omega, \wedge^{l+1})\}$ and in $W^{1,2}(\Omega, \wedge^k)$ (see, for example, [ISS, Corollary 3.6]).

Acknowledgment

The authors are grateful to Kewei Zhang for his motivation in raising the problems concerning the atomic decompositions of $\mathcal{H}^1_{div}(\mathbb{R}^3, \mathbb{R}^3)$ and explaining their significance to questions of coercivity for elliptic systems. We would like to thank Pascal Auscher, Martin Costabel and Tom ter Elst for helpful discussions, and also the referee for suggestions and comments.

References

[A] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975. MR 56:9247

[B] J. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63**(1977), 337-403. MR 57:14788

[C] M. Costabel, Personal communication.

[CLMS] R. Coifman, P. L. Lions, Y. Meyer, S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures and Appl. **72**(1993), 247-286. MR 95d:46033

[CMS] R. Coifman, Y. Meyer, E. M. Stein, *Some new function spaces and their application to harmonic analysis*, J. Funct. Anal. **62**(1985), 304-335. MR 86i:46029

[CW] R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83**(1977), 569-645. MR 56:6264

[GHL] J. E. Gilbert, J. A. Hogan, J. D. Lakey, *Atomic decomposition of divergence-free Hardy spaces*, Mathematica Moraviza, Special Volume, 1997, Proc. IWAA, 33-52.

[Gi] M. Giaquinta, Introduction to Regularity Theory for Nonlinear Elliptic Systems, Lectures in Mathematics, ETH Zürich Birkhäuser Verlag, Basel, 1993. MR 94g:49002

[GMT] G. Geymonat, S. Müller, N. Triantafyllidis, *Homogenization of nonlinear elastic materials, microscopic bifurcation and microscopic loss of rank-one convexity*, Arch. Rational Mech. Anal. **122**(1993), 231-290. MR 94i:73009

[Gr] L. Grafakos, *Hardy space estimates for multilinear operators, II*, Rev. Mat. Iber. **8**(1992), 69-92. MR 93j:42012

[GR] V. Girault, P-A. Raviart, Finite element methods for Navier-Stokes equations, theory and algorithms, Springer-Verlag, Berlin, Heidelberg, 1986. MR 88b:65129

[HLMZ] J. A. Hogan, C. Li, A. M^{c}Intosh, K. Zhang, *Global higher integrability of Jacobians on bounded domains*, Ann. Inst. H. Poincaré Anal. Non linéaire **17**(2000), 193-217. MR 2001h:31003

[ISS] T. Iwaniec, C. Scott, B. Stroffolini, *Nonlinear Hodge theory on manifolds with boundary*, Annali di Matematica pura ed applicata **177**(1999), 37-115. MR 2001f:58052

[La] R. H. Latter, *A decomposition of $\mathcal{H}^p(\mathbb{R}^N)$ in term of atoms*, Studia Math. **62**(1978), 92-101. MR 58:2198

[Le] P. G. Lemarié-Rieusset, *Ondelettes vecteurs á divergence nulle*, *C. R. Acad. Sci. Paris* **313**(1991), 213-216. MR 92i:42018

[LM1] Z. Lou, A. M^{c}Intosh, *Divergence-free Hardy space on $\mathbb{R}^N_+$*, Science in China, Ser. A, **47** (2004), 198-208.

[LM2] Z. Lou, A. M^{c}Intosh, *Hardy spaces of exact forms on Lipschitz domains in $\mathbb{R}^N$*, Indiana Univ. Math. J. **53** (2004), 583-611.

[LMWZ] C. Li, A. M^{c}Intosh, Z. Wu, K. Zhang, *Compensated compactness, paracommutators and Hardy spaces*, J. Funt. Anal. **150**(1997), 289-306. MR 99c:42038

[N] J. Nečas, Les méthodes directes en théorie des équations elliptiques. Masson et Cie, Eds., Paris; Academia, Editeurs, Prague 1967. MR 37:3168

[Sc] G. Schwarz, Hodge Decomposition - A method for solving boundary value problems, Lecture Notes in Mathematics Vol. 1607, Springer-Verlag, Berlin Heidelberg, 1995. MR 96k:58222

[St1] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, 1970. MR 44:7280

[St2] E. M. Stein, Harmonic Analysis, real-variable methods, orthogonality, and oscillatory integrals, Princeton Univ. Press, New Jersey, 1993. MR 95c:42002

[SW] E. M. Stein, G. Weiss, *On the theory of harmonic functions of several variables I: The theory of $\mathcal{H}^p$ spaces*, Acta Math. **103**(1960), 25-62. MR 22:12315

[Z1] K. Zhang, *On the coercivity of elliptic systems in two dimensional spaces*, Bull. Austral. Math. Soc. **54**(1996), 423-430. MR 98c:35037

[Z2] K. Zhang, *A counterexample in the theory of coerciveness for elliptic systems*, J. PDEs, **2**(1989), 79-82. MR 91d:35067

[Z3] K. Zhang, *A further comment on the coerciveness theory for elliptic systems*, J. PDEs, **2**(1989), 62-66. MR 90k:35095

Institute of Mathematics, Shantou University, Shantou Guangdong 515063, People's Republic of China

E-mail address: zjlou@stu.edu.cn

Center for Mathematics and its Applications, Mathematical Sciences Institute, the Australian National University, Canberra, Australian Capital Territory 0200, Australia

E-mail address: alan@maths.anu.edu.au