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A SHARP WEAK TYPE (p,p) INEQUALITY $(p>2)$ FOR MARTINGALE TRANSFORMS AND OTHER SUBORDINATE MARTINGALES

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ABSTRACT. If $(d_n)_{n\geq 0}$ is a martingale difference sequence, $(\varepsilon_n)_{n\geq 0}$ a sequence of numbers in $\{1,-1\}$, and n a positive integer, then

$$P(\max_{0\leq m\leq n} |\sum_{k=0}^{m} \varepsilon_k d_k| \geq 1) \leq \alpha_p \| \sum_{k=0}^{n} d_k \|_p^p.$$

Here α_p denotes the best constant. If $1 \leq p \leq 2$, then $\alpha_p = 2/\Gamma(p+1)$ as was shown by Burkholder. We show here that $\alpha_p = p^{p-1}/2$ for the case $p > 2$, and that $p^{p-1}/2$ is also the best constant in the analogous inequality for two martingales M and N indexed by $[0,\infty)$, right continuous with limits from the left, adapted to the same filtration, and such that $[M,M]_t - [N,N]_t$ is nonnegative and nondecreasing in t. In Section 7, we prove a similar inequality for harmonic functions.

1. Introduction

Most of the paper is devoted to the proof of this sharp inequality for $p > 2$ in the simple setting described below. The biconcave function that we construct in Section 3 is used to obtain the upper estimate $p^{p-1}/2$ for the best constant in this setting. In Section 6 the same function is used to show that $p^{p-1}/2$ is also an upper estimate of the best constant in the general case. Section 2 contains the proof that $p^{p-1}/2$ is a lower estimate in the simple case and therefore must also be a lower estimate in the general case. Therefore, $p^{p-1}/2$ is the best constant in both cases.

Let $(f_n)_{n\geq 0}$ be a sequence, denoted by f, of real integrable functions on a probability space $(\Omega, \mathcal{F}, P)$ and $(d_n)_{n\geq 0}$ its difference sequence: $f_n = \sum_{k=0}^{n} d_k$, $n \geq 0$. If for all $n \geq 1$, the expectation of the product of d_n and $\varphi(d_0, \cdots, d_{n-1})$ is zero for all real bounded continuous functions φ on $\mathbf{R}^n$, equivalently, the conditional expectation $E(d_n|d_0, \cdots, d_{n-1}) = 0$ almost everywhere, then f is a *martingale*. Given such a martingale f and a sequence of numbers $\varepsilon_n \in \{1,-1\}$, the *transform g of f by $(\varepsilon_n)_{n\geq 0}$* is defined by $g_n = \sum_{k=0}^{n} \varepsilon_k d_k$. Notice that g is also a martingale. The maximal function of g is defined by $g^*(\omega) = \sup_n |g_n(\omega)|$, $\omega \in \Omega$, and the p-norm of f by $\|f\|_p = \sup_n \|f_n\|_p$.

Here are some typical martingale results specialized to this setting. For martingales f as above (see Doob [16] and the references there to earlier work):

(i) $\|f\|_1 < \infty \Rightarrow f$ converges a.e.,

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(ii) $\lambda^p P(f^* \geq \lambda) \leq \|f\|_p^p$ if $1 \leq p < \infty$ and $\lambda > 0$,
(iii) $\|f^*\|_p \leq q\|f\|_p$ if $1/p + 1/q = 1$ and $1 < p < \infty$.

For f and g as above, Burkholder [6], [7] proved that:

(i)′ $\|f\|_1 < \infty \Rightarrow g$ converges a.e.,
(ii)′ $\lambda^p P(g^* \geq \lambda) \leq \frac{2}{\Gamma(p+1)}\|f\|_p^p$ if $1 \leq p \leq 2$ and $\lambda > 0$,
(iii)′ $\|g\|_p \leq (p^* - 1)\|f\|_p$ if $1 < p < \infty$ and $p^* = \max\{p, q\}$,

and the constants in (ii)′ and (iii)′ are the best possible. See Burkholder [8], [9], Bañuelos and Wang [3], and Wang [22] for some of the later related work. Notice that (ii) and (ii)′ follow from the special case in which λ is replaced by 1. Also notice that the best constants in (ii) and (ii)′ are strikingly different. Inequality (iii)′ implies that the best constant in (ii)′ for $p > 2$ is less than or equal to $(p-1)^p$. But what is the best constant? In the first part of the paper, we study this natural question. Our answer is contained in the following theorem.

Theorem 1.1. *Let $p > 2$ and $\lambda > 0$. If f is a martingale and g is a transform of f by $(\varepsilon_k)_{k\geq 0}$ as above, then*

$$\lambda^p P(g^* \geq \lambda) \leq \frac{p^{p-1}}{2}\|f\|_p^p \tag{1.1}$$

and $p^{p-1}/2$ is the best constant.

Proof. Let $\beta < p^{p-1}/2$. In the next section, we shall show that there is a probability space, a martingale f defined on this space, a transform g of f as above, and a positive integer n such that

$$P(|g_n| \geq 1) > \beta\|f_n\|_p^p.$$

This and the inequality $g^* \geq |g_n|$ imply that $p^{p-1}/2$ is a lower estimate of the best constant.

To prove that $p^{p-1}/2$ is also an upper estimate, let f be a martingale and g the transform of f by $(\varepsilon_k)_{k\geq 0}$ where $\varepsilon_k \in \{1, -1\}$ as above. We can and do assume that $\|f\|_p$ is finite. Let $Z_n = (X_n, Y_n)$ for $n \geq 0$ where

$$X_n = f_n + g_n = \sum_{k=0}^{n}(1 + \varepsilon_k)d_k,$$
$$Y_n = f_n - g_n = \sum_{k=0}^{n}(1 - \varepsilon_k)d_k,$$

so $f_n = \frac{X_n+Y_n}{2}$ and $g_n = \frac{X_n-Y_n}{2}$. Define the function v on $\mathbf{R}^2$ by

$$\begin{aligned} v(x,y) &= 1 - \frac{p^{p-1}}{2}\left|\frac{x+y}{2}\right|^p \quad \text{if} \quad |\tfrac{x-y}{2}| \geq 1, \\ &= -\frac{p^{p-1}}{2}\left|\frac{x+y}{2}\right|^p \quad \text{if} \quad |\tfrac{x-y}{2}| < 1. \end{aligned}$$

Then

$$P(|g_n| \geq 1) - \frac{p^{p-1}}{2}\|f_n\|_p^p = E\,v(Z_n).$$

But $E\,v(Z_n) \leq 0$ for all $n \geq 0$, as we shall show, so

$$P(|g_n| \geq 1) \leq \frac{p^{p-1}}{2}\|f_n\|_p^p \tag{1.2}$$

and (1.1) easily follows by a standard stopping-time argument, which here goes as follows. Let $\tau(\omega) = \inf\{n \geq 0 : |g_n(\omega)| \geq 1\}$ so that $P(\max_{0\leq k\leq n} |g_k| \geq 1) = P(|g_{\tau\wedge n}| \geq 1)$, apply (1.2) to the martingale $(f_{\tau\wedge n})_{n\geq 0}$ and its transform $(g_{\tau\wedge n})_{n\geq 0}$ and then use the inequality $\|f_{\tau\wedge n}\|_p \leq \|f_n\|_p$. This yields $P(g^* > 1) \leq p^{p-1}/2\|f\|_p$ for all f and g as above and gives (1.1) with strict inequality on the left side. The limit of $\eta P(g^* > \eta)$ as $\eta \uparrow \lambda$ is $\lambda P(g^* \geq \lambda)$ so (1.1) holds as stated. In Section 3, we show that there exists a biconcave majorant u of v on $\mathbf{R}^2$ satisfying $u(0,0) = 0$. The function u has the further property that $u(0,y) = u(0,-y)$ and $u(x,0) = u(-x,0)$. Therefore, $E\,v(Z_n) \leq E\,u(Z_n)$ and $u(x,y) \leq 0$ for all $x, y \in \mathbf{R}$ satisfying $xy = 0$. The next step is to show that $E\,u(Z_n) \leq E\,u(Z_{n-1})$ for all $n \geq 1$. To do this assume that $\varepsilon_n = 1$, the case $\varepsilon_n = -1$ being similar. Then by the conditional form of the Jensen inequality for concave functions, we have that almost everywhere

$$\begin{aligned} E\left[u(Z_n)|d_0,\cdots,d_{n-1}\right] &= E\left[u(X_{n-1}+2d_n, Y_{n-1})|d_0,\cdots,d_{n-1}\right] \\ &\leq u(X_{n-1}+2E\left(d_n|d_0,\cdots,d_{n-1}\right), Y_{n-1}) \\ &= u(Z_{n-1}). \end{aligned}$$

Taking expectations of both sides gives $E\,u(Z_n) \leq E\,u(Z_{n-1})$. So

$$E\,u(Z_n) \leq \cdots \leq E\,u(Z_0).$$

But $E\,u(Z_0) \leq 0$ since $u(Z_0) = u((1+\varepsilon_0)d_0, (1-\varepsilon_0)d_0) \leq 0$, in which the product of $(1+\varepsilon_0)d_0$ and $(1-\varepsilon_0)d_0$ is zero. Therefore, $E\,v(Z_n) \leq E\,u(Z_n) \leq 0$. □

The function u that we have used in this proof is concave along horizontal and vertical lines; it is also concave along every line of positive slope as we prove in Section 4. We use this property in Section 5 to generalize Theorem 1.1 to differentially subordinate martingales. A sharp weak type inequality for martingale transforms more general than the plus-and-minus-one transforms of Theorem 1.1 follows at once, and by approximation, so does a similar inequality for stochastic integrals. The results of Section 4 and 5 lead to the proof in Section 6 of the following theorem, the main result of this paper. In this theorem the probability space $(\Omega, \mathcal{F}, P)$ is complete, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, and $\mathcal{F}_0$ contains all the sets of measure 0.

Theorem 1.2. *Let $p > 2$ and $\lambda > 0$. If M and N are right-continuous martingales with limits from the left, adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and $[M,M]_t - [N,N]_t$ is nonnegative and nondecreasing in t for $t \in [0,\infty)$, then*

$$\lambda^p P(N^* \geq \lambda) \leq \frac{p^{p-1}}{2}||M||_p^p \tag{1.3}$$

and the constant $p^{p-1}/2$ is the best possible.

For background on the quadratic-variation process $[M,M]$, see Dellacherie and Meyer [15] or Protter [20]. The condition that $[M,M]_t - [N,N]_t$ is nonnegative and nondecreasing in t for $t \geq 0$ was introduced and used by Bañuelos and Wang [3] and Wang [22]; see also [4], [5].

In the differential subordination setting of Section 5, the Bañuelos-Wang condition takes the form

$$\sum_{k=0}^{n}(d_k^2 - e_k^2)$$

is nonnegative and nondecreasing in n and is equivalent to the condition (5.1). In the stochastic integral setting of [9] where the L^p-norm of $\int_0^t U_s dM_s$ and $\int_0^t V_s dM_s$ are compared under the condition that the predictable processes U and V satisfy

$$|V_s(\omega)| \leq |U_s(\omega)| \quad \text{for all} \quad \omega \in \Omega \quad \text{and} \quad s \geq 0, \tag{1.4}$$

it takes the form

$$\int_0^t (|U_s|^2 - |V_s|^2) d[M, M]_s$$

is nonnegative and nondecreasing in t. The Bañuelos-Wang condition makes it possible to obtain sharp inequalities for a larger class of martingale pairs (M, N). In this setting the special functions needed for the stochastic integral and discrete-time cases again come into play. It is not known for most sharp martingale inequalities, however, if the less restrictive condition (i) $[N, N]_t \leq [M, M]_t$ for all $t \geq 0$, or the even less restrictive condition (ii) $[N, N]_\infty \leq [M, M]_\infty$ would suffice; see Section 6 of [9].

In Section 7, we use the inequality (1.3) to prove an analogous inequality for harmonic functions.

2. A lower estimate of the best constant

Let $p > 2$ and $0 < \beta < p^{p-1}/2$. Then choose $\theta \in (0, \frac{p-1}{p})$ so that

$$\beta < (1-\theta)^p p^{p-1}/2. \tag{2.1}$$

Finally, choose an odd positive integer M satisfying both $M > p$ and

$$(M-1)^2 > \frac{4}{\theta^p} \left(\frac{p-1}{p}\right)^{p-1} p(p-1)(p-2)(\theta^{p-3} + \theta^{3-p}). \tag{2.2}$$

In this section, we construct a martingale F such that G, the transform of F by the sequence $(-1, 1, -1, 1, \cdots)$, satisfies

$$P(|G_M| \geq 1) > \beta \|F\|_p^p,$$

showing that the best constant for the inequality (1.1) must be greater than or equal to $p^{p-1}/2$. For convenience in the proof, we define δ by $(M-1)\delta = \frac{p-1}{p} - \theta$.

Consider the following $\mathbf{R}^2$-valued martingale Z, one that is also Markov. Here $Z_n = (X_n, Y_n)$, a function with all of its values in the set

$$\{c_1, \cdots, c_M, c_*, z_0, \cdots, z_{M-1}\} \subset \mathbf{R}^2$$

where the elements of this set are defined as follows: $z_0 = (\theta, -\theta)$; if n is an odd integer such that $1 \leq n \leq M-2$, then

$$z_n = (\theta + n\delta, -\theta - (n-1)\delta);$$

if n is an even integer such that $2 \leq n \leq M-1$, then

$$z_n = (\theta + (n-1)\delta, -\theta - n\delta);$$

if n is an odd integer such that $1 \leq n \leq M$, then

$$c_n = \left(\frac{p-3}{p-1}(\theta + (n-1)\delta), -\theta - (n-1)\delta\right);$$

if n is an even integer such that $2 \leq n \leq M-1$, then

$$c_n = (\theta + (n-1)\delta, -\frac{p-3}{p-1}(\theta + (n-1)\delta));$$

and

$$c_* = \left(\frac{p+1}{p}, -\frac{p-1}{p} \right).$$

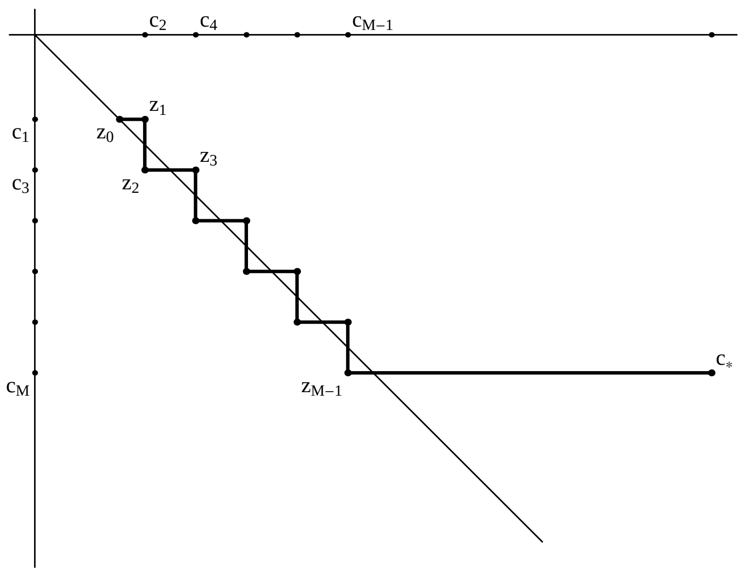

FIGURE 1. One of the paths of Z for the case $p = 3$

Before describing the probability framework, we remark that intuitively Z_n represents a Markov particle's location at time n; Z starts at z_0 and then moves to z_1 or c_1; if not c_1, then it moves successively through $z_1, z_2, \cdots, z_{n-1}$ and then to z_n or c_n if $1 \le n < M-1$; once it arrives at a c_n it stops; if it ever arrives at z_{M-1}, then it moves to c_M or to c_* and then stops. Figure 1 illustrates one of the paths of Z in the case $p = 3$.

Let Ω denote the set $\{c_1, c_2, \cdots, c_M, c_*\}$, $\mathcal{F}$ the σ-algebra of all subsets of Ω, and P the probability measure that assigns weight π_n to c_n if $1 \le n \le M$ and weight π_* to c_*. The weights depend on p, θ, and M as does Ω. These weights need not be given explicitly here because the probability structure will be described below in a more useful way.

We define Z on Ω as follows: $Z_0(\omega) = z_0$ for all $\omega \in \Omega$; if $1 \le m \le n \le M$, then $Z_n(c_m) = c_m$; if $1 \le n \le M-1$, then $Z_n(\omega) = z_n$ for all $\omega \in \{c_{n+1}, \cdots, c_M, c_*\}$; $Z_M(c_*) = c_*$; if $n > M$, then $Z_n = Z_M$.

It is clear that Z is Markov with $P(Z_0 = z_0) = 1$. To force Z to be a martingale, we assume that (here $(a_n, b_n) = c_n$, $(a_*, b_*) = c_*$, and $(x_n, y_n) = z_n$)

$$\begin{aligned} P(Z_1 = z_1) &= \frac{x_0 - a_1}{x_1 - a_1}, \\ P(Z_1 = c_1) &= \frac{\delta}{x_1 - a_1}; \end{aligned}$$

if n is an even integer such that $2 \le n \le M-1$, then

$$\begin{aligned} P(Z_n = z_n | Z_{n-1} = z_{n-1}) &= \frac{b_n - y_{n-1}}{b_n - y_n}, \\ P(Z_n = c_n | Z_{n-1} = z_{n-1}) &= \frac{2\delta}{b_n - y_n}; \end{aligned}$$

if n is an odd integer such that $3 \le n \le M-2$, then

$$\begin{aligned} P(Z_n = z_n | Z_{n-1} = z_{n-1}) &= \frac{x_{n-1} - a_n}{x_n - a_n}, \\ P(Z_n = c_n | Z_{n-1} = z_{n-1}) &= \frac{2\delta}{x_n - a_n}; \end{aligned}$$

and

$$\begin{aligned} P(Z_M = c_* | Z_{M-1} = z_{M-1}) &= \frac{x_{M-1} - a_M}{a_* - a_M}, \\ P(Z_M = c_M | Z_{M-1} = z_{M-1}) &= \frac{a_* - x_{M-1}}{a_* - a_M}. \end{aligned}$$

Observe that $E[Z_n|Z_{n-1}] = Z_{n-1}$ so by the Markov property

$$E[Z_n|Z_0, \ldots, Z_{n-1}] = Z_{n-1} \text{ if } n \ge 1.$$

Therefore, Z is a martingale. Also, $Z_0 = (X_0, Y_0) = (\theta, -\theta)$ and if $n \ge 1$, then

$$X_n = \theta + \sum_{k=1}^{n} (1+\varepsilon_k) d_k \quad \text{and} \quad Y_n = -\theta + \sum_{k=1}^{n} (1-\varepsilon_k) d_k,$$

where $\varepsilon_k = (-1)^{k+1}$ and d_n is given by $X_n - X_{n-1} = (1+\varepsilon_n)d_n$ if n is odd and by $Y_n - Y_{n-1} = (1-\varepsilon_n)d_n$ if n is even. Consequently, f and g defined by $f_0 = g_0 = 0$ and, for $n \ge 1$, by

$$f_n = \sum_{k=1}^{n} d_k \quad \text{and} \quad g_n = \sum_{k=1}^{n} \varepsilon_k d_k,$$

are also martingales with g being a transform of f by $(\varepsilon_k)_{k\ge 0}$. For $n \ge 0$,

$$f_n = \frac{X_n + Y_n}{2} \quad \text{and} \quad \theta + g_n = \frac{X_n - Y_n}{2}.$$

Using the function v defined in (1.2), we have that

$$\begin{aligned} E[v(Z_M)] &= P\left(\left| \frac{X_M - Y_M}{2} \right| \ge 1 \right) - \frac{p^{p-1}}{2} E\left[\left| \frac{X_M + Y_M}{2} \right|^p \right] \\ &= P(|\theta + g_M| \ge 1) - \frac{p^{p-1}}{2} \|f\|_p^p. \end{aligned} \tag{2.3}$$

Here $\|f_M\|_p = \|f\|_p$ because $\|f_0\|_p \le \cdots \le \|f_M\|_p = \|f_{M+1}\|_p = \cdots$. The main step in the proof that $p^{p-1}/2$ is a lower estimate for the best constant is to show that

$$E[v(Z_M)] > 0. \tag{2.4}$$

If (2.4) holds, then by (2.3),

$$P(|g_M| \ge 1 - \theta) \ge P(|\theta + g_M| \ge 1) > \frac{p^{p-1}}{2} \|f\|_p^p.$$

Using this and (2.1), we obtain

$$P(|G_M| \geq 1) > (1-\theta)^p \frac{p^{p-1}}{2} \|F\|_p^p > \beta \|F\|_p^p \tag{2.5}$$

where F is the martingale $(1-\theta)^{-1} f$, and $G = (1-\theta)^{-1} g$, the transform of F by $(-1, 1, -1, 1, \cdots)$. So provided (2.4) holds, $p^{p-1}/2$ is a lower estimate of the best constant.

To prove (2.4), define $\phi : \mathbf{R} \times [-\frac{p-1}{p}, 0) \to \mathbf{R}$ and $\psi : (0, \frac{p-1}{p}] \times \mathbf{R} \to \mathbf{R}$ by

$$\begin{aligned} \phi(x,y) &= [(p-2)y + px](-y)^{p-1}, \\ \psi(x,y) &= [(2-p)x - py]x^{p-1}, \end{aligned}$$

and notice, for example, that $\phi_{xx}(x,y) = 0$ but that

$$\psi_{xx}(x,y) = -p(p-1)(p-2)(x+y)x^{p-3}, \tag{2.6}$$

$$\psi_{xxx}(x,y) = -p(p-1)(p-2)[-y + (x+y)(p-2)]x^{p-4}. \tag{2.7}$$

These imply that $\psi_{xx}(-y, y) = 0$ and $\psi_{xxx}(x,y) < 0$ if $x \geq -y > 0$. If $-\frac{p-1}{p} < y < 0$ and $\frac{3-p}{p-1} y \leq x \leq -y$, let

$$u(x,y) = \frac{1}{4} \left(\frac{p}{p-1} \right)^{p-1} \phi(x,y)$$

and also if $y = -\frac{p-1}{p}$ and $\frac{p-3}{p} \leq x \leq \frac{p+1}{p}$. If $0 < x \leq \frac{p-1}{p}$ and $-x < y \leq \frac{3-p}{p-1} x$, let

$$u(x,y) = \frac{1}{4} \left(\frac{p}{p-1} \right)^{p-1} \psi(x,y).$$

By the definitions of ϕ and ψ, the function u is continuous, and has continuous first and second partial derivatives on the interior of its domain of definition.

Observe that $u(\omega) = v(\omega)$ for all $\omega \in \Omega$: both $u(c_*)$ and $v(c_*)$ are equal to $(2p-1)/(2p)$ and if $1 \leq n \leq M$, then both $u(c_n)$ and $v(c_n)$ are equal to

$$\frac{-p^{p-1}}{2(p-1)^p} (\theta + (n-1)\delta)^p.$$

Therefore, $E[v(Z_M)] = E[u(Z_M)]$ and the problem is reduced to showing that

$$E[u(Z_M)] > 0. \tag{2.8}$$

The calculation of $E[u(Z_0)]$ shows that

$$E[u(Z_M)] = \frac{\theta^p}{2} \left(\frac{p}{p-1} \right)^{p-1} + \sum_{n=1}^{M} E[u(Z_n) - u(Z_{n-1})].$$

So (2.8) will follow if we can prove that

$$\sum_{n=1}^{M} E[u(Z_n) - u(Z_{n-1})] > -\frac{\theta^p}{4} \left(\frac{p}{p-1} \right)^{p-1}. \tag{2.9}$$

By Taylor's formula with Lagrange's remainder,

$$\begin{aligned} E[u(Z_1)-u(Z_0)] &= [u(z_0+(\delta,0))-u(z_0)]\frac{x_0-a_1}{x_1-a_1} \\ &\quad +[u(c_1)-u(z_0)]\frac{\delta}{x_1-a_1} \\ &= [u_x(z_0)\delta+\frac{\delta^2}{2}R_1]\frac{x_0-a_1}{x_1-a_1} \\ &\quad +[u_x(z_0)(a_1-x_0)]\frac{\delta}{x_1-a_1} \\ &= \frac{\delta^2}{2}R_1\frac{x_0-a_1}{x_1-a_1} \qquad (2.10) \end{aligned}$$

where, for some $\delta^* \in (0,\delta)$, $R_1 = u_{xx}(z_0+(\delta^*,0))$. By (2.6) and (2.7),

$$0 > R_1 > u_{xx}(z_0+(\delta,0)) = -p(p-1)(p-2)\delta x_1^{p-3}.$$

So (2.10), $(x_0-a_1)/(x_1-a_1) \in (0,1)$, and $\theta < x_1 \leq \frac{p-1}{p} < 1 < \frac{1}{\theta}$ imply that

$$E[u(Z_1)-u(Z_0)] > -\frac{1}{2}p(p-1)(p-2)\delta^3 x_1^{p-3} > -C_{p,\theta}\delta^3 \qquad (2.11)$$

where $C_{p,\theta} = p(p-1)(p-2)(\theta^{3-p}+\theta^{p-3})$. If $2 \leq n \leq M$, then

$$E[u(Z_n)-u(Z_{n-1})] > -C_{p,\theta}\delta^3 \qquad (2.12)$$

also holds as we shall see. Since $Z_n(\omega) \neq Z_{n-1}(\omega)$ if and only if $Z_{n-1}(\omega) = z_{n-1}$,

$$\begin{aligned} E[u(Z_n)-u(Z_{n-1})] &= E[u(Z_n)-u(Z_{n-1})|Z_n \neq Z_{n-1}]P(Z_n \neq Z_{n-1}) \\ &\quad +E[u(Z_n)-u(Z_{n-1})|Z_n = Z_{n-1}]P(Z_n = Z_{n-1}) \\ &= E[u(Z_n)-u(z_{n-1})|Z_{n-1} = z_{n-1}]P(Z_{n-1} = z_{n-1}). \end{aligned}$$

The case $n = M$ is easy: u is affine on the line segment with endpoints c_M and c_*, so $E[u(Z_M)-u(z_{M-1})|Z_{M-1} = z_{M-1}] = 0$ and

$$E[u(Z_M)-u(Z_{M-1})] = 0. \qquad (2.13)$$

Now let $2 \leq n \leq M-1$ and $\gamma_n = \frac{z_n+z_{n-1}}{2}$. If n is odd, then

$$\begin{aligned} E[u(Z_n)|Z_{n-1} = z_{n-1}] &= u(z_n)\frac{x_{n-1}-a_n}{x_n-a_n} + u(c_n)\frac{2\delta}{x_n-a_n} \\ &= [u(\gamma_n)+u_x(\gamma_n)\delta+\frac{\delta^2}{2}R_n]\frac{x_{n-1}-a_n}{x_n-a_n} \\ &\quad +[u(\gamma_n)-u_x(\gamma_n)(x_{n-1}+\delta-a_n)]\frac{2\delta}{x_n-a_n} \\ &= u(\gamma_n)-u_x(\gamma_n)\delta+\frac{\delta^2}{2}R_n\frac{x_{n-1}-a_n}{x_n-a_n} \end{aligned}$$

where, for some $\delta_n^* \in (0,\delta)$, $R_n = u_{xx}(\gamma_n+(\delta_n^*,0))$. Here

$$0 > R_n > u_{xx}(\gamma_n+(\delta,0)) = -p(p-1)(p-2)\delta x_n^{p-3}$$

and $0 > E[u(Z_n)-u(z_{n-1})|Z_{n-1} = z_{n-1}] > -C_{p,\theta}\delta^3$ follows. But this implies that

$$\begin{aligned} E[u(Z_n)-u(z_{n-1})] &> -C_{p,\theta}\delta^3 P(Z_{n-1} = z_{n-1}) \\ &> -C_{p,\theta}\delta^3. \end{aligned}$$

The same inequality is obtained by similar means if n is even.

Consequently, by (2.11), (2.12), (2.13), (2.2), and $(M-1)\delta = \frac{p-1}{p} - \theta < 1$,

$$\begin{aligned} \sum_{n=1}^{M} E[u(Z_n) - u(Z_{n-1})] &> -(M-1)C_{p,\theta}\delta^3 \\ &> -C_{p,\theta}\left(\frac{p}{p-1} - \theta\right)^3 \frac{1}{(M-1)^2} \\ &> -\frac{\theta^p}{4}\left(\frac{p}{p-1}\right)^{p-1} \end{aligned}$$

and (2.9), hence (2.8), holds. This completes the proof that if $p > 2$, then the best constant for the inequality (1.1) is greater than or equal to $p^{p-1}/2$.

3. A biconcave majorant of v

Here $p > 2$ and v is the function on $\mathbf{R}^2$ defined in the proof of Theorem 1.1. The function identically 1 on $\mathbf{R}^2$ is a biconcave majorant of v but more is needed as can be seen from that proof. In this section we show there is a biconcave majorant u of v on $\mathbf{R}^2$ with $u(0,0) = 0$. Our first step is to prove that there is a function

$$g : [\tfrac{p-1}{p}, \infty) \to [-\tfrac{p-1}{p}, \infty)$$

with the following properties:

(3.1) g is strictly increasing and continuous on $[\frac{p-1}{p}, \infty)$,

(3.2) $g(x) > x - 2$ for all $x \geq \frac{p-1}{p}$,

(3.3) g is differentiable on $(\frac{p-1}{p}, \infty)$ and satisfies

$g'(x) = \frac{4}{p^p(p-1)}(g(x) - x + 2)^{-2}(g(x) + 1)^{2-p}$ for all $x > \frac{p-1}{p}$,

(3.4) $g(\frac{p-1}{p}) = -\frac{p-1}{p}$, $g'(\frac{p-1}{p}+) = \frac{1}{p-1}$.

An equivalent problem is to show there is a function

$$G : [-\tfrac{p-1}{p}, \infty) \to [\tfrac{p-1}{p}, \infty)$$

such that

(3.5) G is strictly increasing and continuous on $[-\frac{p-1}{p}, \infty)$,

(3.6) $G(y) < y + 2$ for all $y \geq -\frac{p-1}{p}$,

(3.7) G is differentiable on $(-\frac{p-1}{p}, \infty)$ and satisfies

$G'(y) = \frac{p^p(p-1)}{4}(y + 2 - G(y))^2(y+1)^{p-2}$ for all $y > -\frac{p-1}{p}$,

(3.8) $G(-\frac{p-1}{p}) = \frac{p-1}{p}$, $G'(-\frac{p-1}{p}+) = p - 1$.

To see the equivalence of the two problems, note that if g satisfies (3.1)–(3.4), then its inverse function satisfies (3.5)–(3.8), and if G satisfies (3.5)–(3.8), then its inverse satisfies (3.1)–(3.4).

Notice that the differential equation for G in (3.7) has the Riccati form. Using the transformation

$$h(y) = \exp\left[\int_{-\frac{p-1}{p}}^{y} \frac{p^p(p-1)}{4}(z+1)^{p-2}(z+2-G(z))\,dz\right] \tag{3.9}$$

(see, for example, Polyanin and Zaitsev [19]) we obtain the following differential equation for $h(y)$:

$$(y+1)h''(y)+(2-p)h'(y)-\frac{p^p(p-1)}{4}(y+1)^{p-1}h(y)=0. \tag{3.10}$$

Two linearly independent solutions of this differential equation on the interval $(-1,\infty)$ are given by

$$\begin{aligned} h_1(y) &= \sqrt{(1+y)^{p-1}}I_{-\frac{p-1}{p}}(z_0), \\ h_2(y) &= \sqrt{(1+y)^{p-1}}I_{\frac{p-1}{p}}(z_0), \end{aligned}$$

where $z_0=\sqrt{p^{p-2}(p-1)(1+y)^p}$ and I_α is the modified Bessel function of the first kind (see Abramowitz [1]). Here $\alpha\in\{(p-1)/p,-(p-1)/p\}$,

$$I_\alpha(z)=\sum_{k=0}^{\infty}\frac{(\frac{z}{2})^{2k+\alpha}}{k!\Gamma(\alpha+k+1)},$$

and

$$z^2I''_\alpha(z)+zI'_\alpha(z)-(z^2+\alpha^2)I_\alpha(z)=0. \tag{3.11}$$

If $\alpha>-1$, then I_α is infinitely differentiable on $(0,\infty)$, which implies that h_1 and h_2 are infinitely differentiable on $(-1,\infty)$. By (3.11), h_1 and h_2 satisfy (3.10): substitute h_i for h on the left side of (3.10) to obtain

$$\frac{1}{4}p^2(1+y)^{\frac{p-3}{2}}[z_0^2I''_\alpha(z_0)+z_0I'_\alpha(z_0)-(z_0^2+\alpha^2)I_\alpha(z_0)]=0. \tag{3.12}$$

Let $h=a_1h_1+a_2h_2$ where a_1, a_2 are real numbers such that

$$h(-\tfrac{p-1}{p})=1 \text{ and } h'(-\tfrac{p-1}{p})=\tfrac{p(p-1)}{2}. \tag{3.13}$$

Then, for $y>-1$,

$$(y+1)h''(y)=(p-2)h'(y)+\frac{p^p(p-1)}{4}(y+1)^{p-1}h(y) \tag{3.14}$$

so if we can show that $h>0$ and $h'>0$ on $[-\frac{p-1}{p},\infty)$, then $h''>0$ on $[-\frac{p-1}{p},\infty)$ and

$$h \text{ is strictly increasing and convex on } [-\tfrac{p-1}{p},\infty). \tag{3.15}$$

By (3.13) and the continuity of h', there is an interval $[-\frac{p-1}{p},c)$ on which $h'>0$. Let

$$b=\sup\{c: h'>0 \text{ on } [-\tfrac{p-1}{p},c)\}.$$

Then $h'>0$, hence $h>0$, on $[-\frac{p-1}{p},b)$, implying, by (3.14), that $h''>0$ on $[-\frac{p-1}{p},b)$. Therefore, h' is strictly increasing on $[-\frac{p-1}{p},b)$. Consequently, b is infinite; otherwise, there is a $c>b$ such that $h'\geq\frac{p(p-1)}{2}>0$ on $[-\frac{p-1}{p},c)$ contradicting the maximality of b. So h,h', and h'' are strictly positive on $[-\frac{p-1}{p},\infty)$.

These properties of h imply the existence of a function G with the properties (3.5)–(3.8). This function is defined on $[-\frac{p-1}{p},\infty)$ by

$$G(y)=y+2-\frac{4h'(y)}{p^p(p-1)h(y)(y+1)^{p-2}}. \tag{3.16}$$

The function $g : [\frac{p-1}{p}, \infty) \to \mathbf{R}$ is the inverse of G. We now define a biconcave majorant u of v on $\mathbf{R}^2$, first on the subdomains $D_0, \cdots, D_5$:

$$\begin{aligned}
D_1 &= \{(x,y)\,|\,0 < x < \tfrac{p-1}{p},\ -x < y < \tfrac{3-p}{p-1}x\},\\
D_2 &= \{(x,y)\,|\,x > \tfrac{p-1}{p},\ g(x) < y < 2g(x) - x + 2\},\\
D_3 &= \{(x,y)\,|\,x > \tfrac{p-1}{p},\ \tfrac{1-p}{p} \vee (x-2) < y < g(x)\},\\
D_4 &= \{(x,y)\,|\,x > \tfrac{p-1}{p},\ (-x) \vee (x-2) < y < \tfrac{1-p}{p}\},\\
D_5 &= \{(x,y)\,|\,-x < y < x-2\},\\
D_0 &= \{(x,y)\,|\,x > 0,\ -x < y < x\} \cap \left(\bar{D}_1 \cup \cdots \cup \bar{D}_5\right)^c.
\end{aligned}$$

On $D_0 \cup \cdots \cup D_5$, the function u is defined as follows:

$$u(x,y) = \begin{cases}
-\frac{p^{p-1}}{2}\left|\frac{x+y}{2}\right|^p & \text{if } (x,y) \in D_0,\\[2ex]
\frac{1}{4}\left(\frac{p}{p-1}\right)^{p-1}[(2-p)x - py]x^{p-1} & \text{if } (x,y) \in D_1,\\[2ex]
\frac{p^{p-1}}{4}(g(x)+1)^{p-1}\left((p-2)g(x) - px + 2p - 2\right)\frac{2(g(x)+1)-x-y}{g(x)+2-x} & \\
\qquad -\frac{p^{p-1}}{2}(g(x)+1)^p\frac{y-g(x)}{g(x)+2-x} & \text{if } (x,y) \in D_2,\\[2ex]
\frac{p^{p-1}}{4}(y+1)^{p-1}\left((p-2)y + 2p - 2 - pG(y)\right)\frac{y+2-x}{y+2-G(y)} & \\
\qquad +\left(1 - \frac{p^{p-1}}{2}(y+1)^p\right)\frac{x-G(y)}{y+2-G(y)} & \text{if } (x,y) \in D_3,\\[2ex]
\frac{1}{2p(p-1)}\left(p^p(y+1)^p - p^3(y+1) + 2p(p-1)\right)\frac{y+2-x}{2y+2} & \\
\qquad +\left(1 - \frac{p^{p-1}}{2}(y+1)^p\right)\frac{x+y}{2y+2} & \text{if } (x,y) \in D_4,\\[2ex]
1 - \frac{p^{p-1}}{2}\left|\frac{x+y}{2}\right|^p & \text{if } (x,y) \in D_5.
\end{cases}$$

This function u can be extended to a continuous function on $\{(x,y)| -x \le y \le x\}$. We use the same letter u to denote this extension and define it further on $\mathbf{R}^2$ by

$$u(x,y) = u(y,x) = u(-x,-y) = u(-y,-x). \tag{3.17}$$

The function u is continuous on $\mathbf{R}^2$ and the partial derivatives u_x and u_y exist and are continuous on the set $S = \{(x,y) \in \mathbf{R}^2 : |x-y| \neq 2\}$. Furthermore, $u(0,0) = 0$. The proof of these properties of u is routine and is omitted. Note that u is not C^1 on $\mathbf{R}^2$. For example, if $y > -\frac{p-1}{p}$, then

$$\begin{aligned}
u_x((y+2)-,y) &= u_x((y+2)+,y) + \frac{1}{y+2-G(y)}\\
&> u_x((y+2)+,y).
\end{aligned}\tag{3.18}$$

We now show that u is a biconcave majorant of v on $\mathbf{R}^2$ and begin by showing that on each of the subdomains $D_0, \cdots, D_5$ the function u is biconcave. On D_0 and D_5, the function u is biconcave since on these subdomains

$$u_{xx}(x,y) = u_{yy}(x,y) = -\frac{p^p(p-1)}{8}\left|\frac{x+y}{2}\right|^{p-2} < 0.$$

We notice that u is a linear function of y for $(x,y) \in D_1 \cup D_2$, and is a linear function of x for $(x,y) \in D_3 \cup D_4$. So $u_{yy} = 0$ on $D_1 \cup D_2$ and $u_{xx} = 0$ on $D_3 \cup D_4$.

Therefore the biconcavity of u on D_1 and D_4 follows from the calculations

$$\begin{aligned} u_{xx}(x,y) &= -\frac{1}{4}\left(\frac{p}{p-1}\right)^{p-1}[p(p-1)(p-2)x^{p-3}](x+y)<0 \quad \text{on} \quad D_1, \\ u_{yy}(x,y) &= -\frac{1}{4}p^p(p-2)(1+y)^{p-3}(x+y)<0 \quad \text{on} \quad D_4. \end{aligned}$$

Here we have used the assumption that $p>2$. It remains to show that $u_{xx}\le 0$ on D_2 and $u_{yy}\le 0$ on D_3. The common boundary of D_2 and D_3 is the graph of g and the common boundary of D_0 and D_2 is the graph of f, where f is defined by $f(x)=2g(x)-x+2$ (so $g(x)$ is the average of $f(x)$ and $x-2$).

We now show that $u_{xx}\le 0$ on D_2. If $x\in(\frac{p-1}{p},\infty)$ and $g(x)<y<f(x)$, then

$$\begin{aligned} u_{xx}(x,y) &= \frac{p^p(p-1)}{4}(g(x)+1)^{p-3}[-2g'(x)(g(x)+1) \\ &\quad -(g'(x))^2(-2p+px+2-2x-2(p-1)g(x)) \\ &\quad -(-2g(x)+x-2)g''(x)(g(x)+1) \\ &\quad -[(g'(x))^2(p-2)+(g(x)+1)g''(x)]y]. \end{aligned} \tag{3.19}$$

This is linear in y. Therefore, to show that $u_{xx}(x,y)\le 0$ for all $(x,y)\in D_2$, we need to show only that $u_{xx}(x,f(x)-)\le 0$ and $u_{xx}(x,g(x)+)\le 0$. Indeed, $u_{xx}(x,g(x)+)=0$ as can be seen from (3.19) by using the differential equation (3.3) for g. Similarly,

$$u_{xx}(x,f(x)-)=\frac{1}{2}p^p(p-1)(g(x)+1)^{p-2}(g'(x)-1)g'(x).$$

By (3.1) and (3.4), $g(x)+1\ge\frac{1}{p}>0$ so, by (3.2) and (3.3), $g'(x)>0$. Also, as we shall show, $g'(x)\le 1$. Therefore $u_{xx}(x,f(x)-)\le 0$.

The inequality $g'(x)\le 1$ for all $x\in(\frac{p-1}{p},\infty)$ is equivalent to the inequality

$$G'(y)\ge 1 \text{ for all } y\in(-\tfrac{p-1}{p},\infty). \tag{3.20}$$

Suppose that (3.20) does not hold. Then there exist numbers a and b such that

$$\inf\{G'(y):y>-\tfrac{p-1}{p}\}<b<a<1.$$

Let

$$\begin{aligned} y_1 &= \inf\{y>-\tfrac{p-1}{p}:G'(y)=b\}, \\ y_0 &= \sup\{y>-\tfrac{p-1}{p}:G'(y)=a,\, y<y_1\}. \end{aligned}$$

Recall that $G'(-\frac{p-1}{p})=p-1>1$. This and the continuity of G' imply that $-\frac{p-1}{p}<y_0<y_1$. Moreover,

$$b<G'(y)<a=G'(y_0)<1 \tag{3.21}$$

for all $y\in(y_0,y_1)$. But there is a number $\varepsilon>0$ such that

$$\frac{G(y)-G(y_0)}{y-y_0}<1 \text{ for all } y\in(y_0,y_0+\varepsilon)$$

which, by (3.7), implies that for such y,

$$\begin{aligned}\frac{4}{p^p(p-1)}G'(y) &= (y+2-G(y))^2(y+1)^{p-2}\\ &= \left(y_0+2-G(y_0)+(y-y_0)\left[1-\frac{G(y)-G(y_0)}{y-y_0}\right]\right)^2(y+1)^{p-2}\\ &> (y_0+2-G(y_0))^2(y_0+1)^{p-2}\\ &= \frac{4}{p^p(p-1)}G'(y_0),\end{aligned}$$

and gives $G'(y) > G'(y_0)$. For $y \in (y_0, y_0+\varepsilon) \cap (y_0, y_1)$, this is a contradiction to (3.21). Therefore, (3.20) holds.

Now we check that $u_{yy} \le 0$ on D_3. Fix $y \in (-\frac{p-1}{p}, \infty)$. Then $u_{yy}(G(y)+, y) = 0$ and, on D_3, $u(x,y) = A(y) + B(y)x$ where $A(y)$ and $B(y)$ are functions of y only. For $G(y) < x < y+2$, $u_{yy}(x,y)$ is a linear function of x. Since $u_{yy}(G(y)+, y) = 0$ and

$$u_{yy}((y+2)-, y) = \frac{2}{(y+2-G(y))^2} - \frac{1}{2}(p-1)p^p(y+1)^{p-2} = \frac{2(1-G'(y))}{(y+2-G(y))^2},$$

we see from (3.20) that $u_{yy} \le 0$ on D_3.

The function u is biconcave not only on $D_0, \cdots, D_5$, but also, by (3.17), on each of the reflected subdomains. As can be checked, the function u is concave on each of the horizontal and vertical line segments included in the complement of the union of these open sets. The following elementary lemma easily yields the completion of the proof that u is biconcave on all of $\mathbf{R}^2$.

Lemma 3.1. *Let $a < c < b$. If $\phi : (a,b) \to \mathbf{R}$ is concave on (a,c), concave on (c,b), continuous at c, differentiable on $(a,c) \cup (c,b)$, and $\phi'(c-) \ge \phi'(c+)$, then ϕ is concave on (a,b).*

We now prove that u is a majorant of v on $\mathbf{R}^2$. By (3.17), it is enough to prove this for $\{(x,y) : -x \le y \le x\}$. On $\bar{D}_0$ and $\bar{D}_5$, we have equality: $u = v$. So by the continuity of u and v on $\{(x,y) : |\frac{x-y}{2}| < 1\}$, it is enough to show in the following that $u \ge v$ on $D_1, \cdots, D_4$. To do this, we use that $u_{yy} = 0$ on $D_1 \cup D_2$ and $u_{xx} = 0$ on $D_3 \cup D_4$ as well as $v_{xx} < 0$ and $v_{yy} < 0$ on $D_1 \cup \cdots \cup D_4$. For example, consider D_1. On the upper part of its boundary, that is, on $\bar{D}_0 \cap \bar{D}_1$,

$$u_y(x,y) - v_y(x,y) = 0. \tag{3.22}$$

This follows from the continuity of u_y on the closure of $D_1 \cup D_2$ and the equality $u = v$ on $\bar{D}_0$. On D_1, $u_{yy}(x,y) - v_{yy}(x,y) = -v_{yy}(x,y) > 0$, so $u_y(x,y) - v_y(x,y)$ is strictly increasing in y. Therefore, by (3.22),

$$u_y(x,y) - v_y(x,y) < 0 \text{ on } D_1.$$

This implies that $u(x,y) - v(x,y)$ is strictly decreasing in y, so the equality $u = v$ on $\bar{D}_0 \cap \bar{D}_1$ gives

$$u(x,y) - v(x,y) > 0 \text{ on } D_1.$$

The proof that $u > v$ on D_2 is the same if D_1 is replaced by D_2.

Now consider D_3. Let $y > -\frac{p-1}{p}$ and let w be the linear function on the interval $[G(y), y+2]$ such that $w(G(y)) = u(G(y), y)$ and $w(y+2) = v((y+2)-, y)$. Then, as we shall show,

$$v(x,y) < w(x) < u(x,y) \tag{3.23}$$

for all $x \in (G(y), y+2)$, proving that $u > v$ on D_3. Because $u(\cdot, y)$ is also linear on $[G(y), y+2]$ and

$$w(y+2) = u(y+2,y) - 1 < u(y+2,y), \tag{3.24}$$

the right side of (3.23) holds. The left side of (3.23) also holds since $v(\cdot, y)$ is strictly concave and

$$w_x((y+2)-) = v_x((y+2)-, y). \tag{3.25}$$

This equality can be checked by using

$$v_x((y+2)-, y) = v_x((y+2)+, y) = u_x((y+2)+, y),$$

and (3.18), which is equivalent to

$$\frac{u(y+2,y) - u(G(y),y)}{y+2-G(y)} = v_x((y+2)-, y) + \frac{1}{y+2-G(y)}.$$

By the equality in (3.24), this is equivalent to

$$\frac{w(y+2) - w(G(y))}{y+2-G(y)} = v_x((y+2)-, y).$$

Therefore, (3.25) holds and $u > v$ on D_3.

Now consider the remaining subdomain D_4. If $y \in (-1, -\frac{p-1}{p})$, then $u(\cdot, y)$ is linear and $v(\cdot, y)$ is concave on $[-y, y+2]$. Therefore, $u(\cdot, y) - v(\cdot, y)$ is a convex function on $[-y, y+2]$. To finish the proof that $u > v$ on D_4, it is enough to observe that

$$\begin{aligned} u(-y,y) - v(-y,y) &= \frac{1}{2p(p-1)}(p^p(y+1)^p - p^3(y+1) + 2p(p-1)) \\ &\geq \frac{1}{2p(p-1)}[2p(p-1) - p^3(y+1)] \\ &\geq \frac{p-2}{2(p-1)} > 0 \end{aligned}$$

and

$$u_x(-y,y) - v_x(-y,y) = \frac{p^2 - p^p(y+1)^{p-1}}{4(p-1)} > 0.$$

This completes the proof that u is a biconcave majorant of v on $\mathbf{R}^2$ with $u(0,0) = 0$ and the proof of Theorem 1.1.

4. Concavity along the lines of positive slope

This property of u, stronger than biconcavity, will be used in the proof of Theorem 5.1. Here of course u is the function defined in Section 3 and $p > 2$.

Lemma 4.1. *If $x, y, h, k \in \mathbf{R}$ and $hk \geq 0$, then the function $\varphi : \mathbf{R} \to \mathbf{R}$ defined by*

$$\varphi(t) = u(x+ht, y+kt)$$

is concave on $\mathbf{R}$.

Proof. If $(x, y) \in D_0 \cup \cdots \cup D_5$, then $u_{xx}(x, y) \leq 0$, $u_{yy}(x, y) \leq 0$, and $u_{xy}(x, y) \leq 0$. The first two inequalities have been proved in Section 3, and the third also holds:

$$u_{xy}(x,y) = \begin{cases} -\frac{p^2}{4}(\frac{p}{p-1})^{p-2}x^{p-2} & \text{for } (x,y) \in D_1, \\ -\frac{1}{(2-x+g(x))^2} & \text{for } (x,y) \in D_2, \\ -\frac{1}{(2+y-G(y))^2} & \text{for } (x,y) \in D_3, \\ -\frac{p^p}{4}(1+y)^{p-2} & \text{for } (x,y) \in D_4, \\ -\frac{p^p(p-1)}{8}(\frac{x+y}{2})^{p-2} & \text{for } (x,y) \in D_0 \cup D_5. \end{cases}$$

Therefore, if $(x + ht, y + kt) \in D_0 \cup \cdots \cup D_5$, then

$$\varphi''(t) = u_{xx}(x+ht, y+kt)h^2+2u_{xy}(x+ht, y+kt)hk+u_{yy}(x+ht, y+kt)k^2 \leq 0. \tag{4.1}$$

As mentioned in Section 3, u_x and u_y exist and are continuous in the open set $S = \{(x, y) \in \mathbf{R}^2 : |x - y| \neq 2\}$. Therefore, the first derivatives of φ exist and are continuous on the set $\{t \in \mathbf{R} : (x + ht, y + kt) \in S\}$, so by (4.1) and Lemma 3.1, φ is locally concave on this set. It follows from Lemma 3.1 and calculations such as (3.18) that u is concave on the graph $\{(x + ht, y + kt) : t \in \mathbf{R}\}$ of any line with $(x, y) \in S$ and $hk \geq 0$. The continuity of u then assures the concavity of u on the graph of each of the lines with $(x, y) \notin S$ and $h = k = 1$. Therefore, φ is concave on $\mathbf{R}$ under the condition $hk \geq 0$. □

In the next section, we shall use the smooth approximation u^m of u obtained by convoluting u with the Gaussian density $m \exp[-m\pi(x^2 + y^2)]$. The function u^m is infinitely differentiable, and $u^m \to u$ pointwise as $m \to \infty$. Moreover, u^m has the concavity property of Lemma 4.1 since the integration that gives u^m preserves this property of u. Let $hk \geq 0$. Then the function φ_m associated with u^m satisfies $\varphi_m'' \leq 0$ on $\mathbf{R}$, so

$$u^m_{xx}(x, y)h^2 + 2u^m_{xy}(x, y)hk + u^m_{yy}(x, y)k^2 \leq 0. \tag{4.2}$$

Thus, φ_m' is decreasing which gives $\varphi_m(1) - \varphi_m(0) \leq \varphi_m'(0)$ by the mean value theorem. Therefore,

$$u^m(x + h, y + k) - u^m(x, y) \leq u^m_x(x, y)h + u^m_y(x, y)k. \tag{4.3}$$

Using the equality $u^m_x(0, 0) = u^m_y(0, 0) = 0$, which follows from the analogue of (3.17) for u^m, and the inequality (4.3), we obtain

$$u^m(h, k) \leq u^m(0, 0). \tag{4.4}$$

The condition $hk \geq 0$ is necessary for (4.2), (4.3), and (4.4) to hold. Also, there is a positive real number c_p, the same number for each m, such that

$$|u^m(x, y)| \leq c_p(|x|^p + |y|^p) + c_p, \tag{4.5}$$

$$|u^m_x(x, y)| \leq c_p(|x|^{p-1} + |y|^{p-1}) + c_p, \tag{4.6}$$

with a similar bound for $|u^m_y|$. To prove (4.5) and (4.6), use, for example, the biconcavity of u^m so that $u^m_x(\cdot, y)$ is nonincreasing on $\mathbf{R}$, and the definition of u on the set $\{(x, y) \in \mathbf{R}^2 : |\frac{x-y}{2}| > 1\}$.

5. Differential subordination

In this section, we give a simple application of the results of the previous section to martingales f and g where g is not necessarily a transform of f. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ a filtration of $\mathcal{F}$: a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. We assume that f_n and g_n are $\mathcal{F}_n$-measurable, $f_n = \sum_{k=0}^n d_k$ and $g_n = \sum_{k=0}^n e_k$ for all nonnegative integers n, and that

$$E\,(d_n|\mathcal{F}_{n-1}) = E\,(e_n|\mathcal{F}_{n-1}) = 0$$

almost everywhere for all $n \geq 1$, that is, f and g are martingales adapted to this filtration. If f and g satisfy

$$|e_n(\omega)| \leq |d_n(\omega)| \tag{5.1}$$

for all $\omega \in \Omega$ and $n \geq 0$, then g is *differentially subordinate* to f. One example is given by $e_n = \varepsilon_n d_n$, $n \geq 0$, as in Theorem 1.1. Another example, which leads in many contexts to sharp inequalities for stochastic integrals (see, for example, [7], [9], [13]), is given by $e_n = v_n d_n$ where v_n is $\mathcal{F}_{(n-1)\vee 0}$-measurable and satisfies $|v_n(\omega)| \leq 1$ for all $\omega \in \Omega$ and all $n \geq 0$.

Theorem 5.1. *Let $p > 2$ and $\lambda > 0$. If f and g are martingales adapted to the same filtration and g is differentially subordinate to f, then*

$$\lambda^p P(g^* \geq \lambda) \leq \frac{p^{p-1}}{2}\|f\|_p^p \tag{5.2}$$

and the constant $p^{p-1}/2$ is the best possible.

Proof. That $p^{p-1}/2$ is a lower estimate for the best constant is a consequence of Theorem 1.1. That it is also an upper estimate can be seen as follows. Assume as before that $\|f\|_p$ is finite. As in the proof of Theorem 1.1, it is enough to show that

$$P(|g_n| \geq 1) \leq \frac{p^{p-1}}{2}\|f_n\|_p^p. \tag{5.3}$$

This is equivalent to proving that $E\,v(Z_n) \leq 0$, where v is defined in Section 1, and Z_n is defined here by $Z_n = (X_n, Y_n)$ with $X_n = \sum_{k=0}^n (d_k + e_k)$ and $Y_n = \sum_{k=0}^n (d_k - e_k)$. The function u of Sections 3 and 4 is a majorant of v, therefore the theorem will be proved if we can show that

$$E\,u(Z_n) \leq 0 \tag{5.4}$$

for all $n \geq 0$. The first step of the proof of this is to show that

$$E\,u^m(Z_n) \leq \cdots \leq E\,u^m(Z_0) \tag{5.5}$$

for all positive integers m and n. Let $H_n = d_n + e_n$ and $K_n = d_n - e_n$. By (5.1), $H_n K_n \geq 0$, and by (4.3),

$$\begin{aligned} u^m(Z_n) &= u^m(X_{n-1} + H_n, Y_{n-1} + K_n) \\ &\leq u^m(Z_{n-1}) + u_x^m(Z_{n-1})H_n + u_y^m(Z_{n-1})K_n. \end{aligned}$$

By (4.5), (4.6), and the finiteness of $\|f\|_p$, each term of this inequality has finite expectation. By the martingale condition, $E\,H_n = E\,K_n = 0$. Therefore, $E\,[u^m(Z_n)|\mathcal{F}_{n-1}] \leq u^m(Z_{n-1})$, which implies that $E\,u^m(Z_n) \leq E\,u^m(Z_{n-1})$. Consequently, (5.5) holds for $m, n \geq 1$. By (5.1), $X_0 Y_0 \geq 0$ which gives, by (4.4), that $E\,u^m(Z_0) \leq u^m(0,0)$. Thus, by (5.5),

$$E\,u^m(Z_n) \leq u^m(0,0) \tag{5.6}$$

for all $m \geq 1$ and $n \geq 0$. By (4.5),

$$E\,[\sup_{m\geq 1} |u^m(Z_n)|] \leq c_p(E\,|X_n|^p + E\,|Y_n|^p) + c_p,$$

in which $E\,|X_n|^p$ and $E\,|Y_n|^p$ are finite, a consequence of the finiteness of $\|f\|_p$. Therefore, taking the limit of both sides of (5.6) gives $E\,u(Z_n) \leq u(0,0) \leq 0$ and (5.4) is proved, which completes the proof of the theorem. □

6. Proof of Theorem 1.2

The proof has the same pattern as the proof of Theorem 5.1. Let $Z = (X, Y)$ where

$$X = M + N \quad \text{and} \quad Y = M - N.$$

The inequality (1.3) will follow if we can show that for $\|M\|_p$ finite,

$$E\,u^m(Z_t) \leq u^m(0,0).$$

This holds for $t = 0$ just as in Section 5, so it remains to show that for $t > 0$,

$$E\,u^m(Z_t) \leq E\,u^m(Z_0). \tag{6.1}$$

By Itô's formula as extended by Kunita and Watanabe [18] and Meyer (see [15] and the references given there),

$$u^m(Z_t) = u^m(Z_0) + I_t + J_t + Q_t/2 + S_t, \tag{6.2}$$

where

$$\begin{aligned}
I_t &= \int_{(0,t]} u_x^m(Z_{s-})dX_s, \\
J_t &= \int_{(0,t]} u_y^m(Z_{s-})dY_s, \\
Q_t &= \int_{(0,t]} (u_{xx}^m(Z_{s-})d[X^c, X^c]_s + 2u_{xy}^m(Z_{s-})d[X^c, Y^c]_s + u_{yy}^m(Z_{s-})d[Y^c, Y^c]_s), \\
S_t &= \sum_{0<s\leq t} (u^m(Z_s) - u^m(Z_{s-}) - u_x^m(Z_{s-})\Delta X_s - u_y^m(Z_{s-})\Delta Y_s).
\end{aligned}$$

Here X^c is the continuous part of X and $\Delta X_s = X_s - X_{s-}$ for $s > 0$. The Bañuelos-Wang condition that $[M,M]_t - [N,N]_t$ is nonnegative and nondecreasing in t implies that $(\Delta M_s)^2 - (\Delta N_s)^2 \geq 0$ for all $s > 0$ (see Lemma 1 of Wang [22]). Therefore, $(\Delta X_s)(\Delta Y_s) \geq 0$ and, by (4.3), $S_t \leq 0$. Also, $[X^c, Y^c]_s$, which is equal to $[M^c, M^c]_s - [N^c, N^c]_s$, is nonnegative and nondecreasing in s. Therefore, $Q_t \leq 0$. We now show that $E\,I_t = 0$. A similar argument shows that $E\,J_t = 0$ so the expectation of the right side of (6.2) is less than or equal to $E\,u^m(Z_0)$ and (6.1) holds. Using the Bañuelos-Wang condition, we see that $[N,N]_\infty \leq [M,M]_\infty$ so

$$\begin{aligned}
\|N\|_2 &= E\,[N,N]_\infty \leq E\,[M,M]_\infty \\
&= \|M\|_2 \leq \|M\|_p < \infty,
\end{aligned}$$

and X, the sum of M and N, is a martingale satisfying $\|X\|_2 < \infty$. Thus, $(I_t)_{t>0}$ is a local martingale (see Theorem 20 on page 56 of [20]). In fact, $(I_t)_{t>0}$ is a martingale as we now show. By (4.6),

$$\begin{aligned}
|u_x^m(Z_{s-})| &\leq c_p(|X_{s-}|^{p-1} + |Y_{s-}|^{p-1}) + c_p \\
&\leq c_p((X^*)^{p-1} + (Y^*)^{p-1}) + c_p.
\end{aligned}$$

We denote the last expression by W. Let $q = p/(p-1)$. By Doob's maximal inequality,

$$\|(M^*)^{p-1}\|_q^q = E[(M^*)^p] \leq q^p \|M\|_p^p < \infty.$$

Therefore, $(M^*)^{p-1} \in L^q$. By a special case of the Burkholder-Davis-Gundy inequality (see, for example, page 287 of [15]),

$$\begin{aligned} E\,[(N^*)^p] &\leq c'_p E\,([N,N]_\infty^{\frac{p}{2}}) \\ &\leq c'_p E\,([M,M]_\infty^{\frac{p}{2}}) \\ &\leq c''_p E\,[(M^*)^p], \end{aligned}$$

so $(N^*)^{p-1}$ also belongs to L^q. This implies that $(X^*)^{p-1}$ and $(Y^*)^{p-1}$ belong to L^q. Thus, $W \in L^q$. Using the Burkholder-Davis-Gundy inequality again, we see that

$$\begin{aligned} E\,[\sup_{0<s\leq t} |I_s|] &\leq cE\,[(\int_{(0,t]} W^2 d[X,X]_s)^{\frac{1}{2}}] \\ &= cE\,[W(\int_{(0,t]} d[X,X]_s)^{\frac{1}{2}}] \\ &\leq cE\,[W([X,X]_t - [X,X]_0)^{\frac{1}{2}}]. \end{aligned}$$

By Hölder's inequality, $E[W[X,X]_\infty^{\frac{1}{2}}] \leq \|W\|_q \|[X,X]_\infty^{\frac{1}{2}}\|_p \leq c'\|W\|_q \|X^*\|_p < \infty$. Therefore, by the dominated convergence theorem, the right continuity of $[X,X]$, and

$$|E\,I_t| \leq cE[W([X,X]_t - [X,X]_0)],$$

$E\,I_t$ converges to 0 as $t \to 0$. Moreover,

$$E\,[\sup_{t>0} |I_t|] \leq E\,[W[X,X]_\infty^{\frac{1}{2}}] < \infty,$$

which implies that the local martingale $(I_t)_{t>0}$ is a martingale. Consequently, $E\,I_s = E\,I_t$ for all $s, t > 0$ so $E\,I_t = \lim_{s\to 0} E\,I_s = 0$ completing the proof of (6.1) and Theorem 1.2.

7. An inequality for harmonic functions

Let D be an open connected set of points $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$, u and v harmonic functions on D, and $|\nabla u(x)|$ the Euclidean norm of the gradient vector $(u_{x_1}(x), \cdots, u_{x_n}(x))$. Then v is *differentially subordinate* to u if, for all $x \in D$,

$$|\nabla v(x)| \leq |\nabla u(x)|.$$

Fix a point $\xi \in D$ and let D_0 be a bounded connected subdomain of D satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$. Denote by $\mu_{D_0}^{\xi}$ the harmonic measure on ∂D_0 with respect to ξ. If $1 \leq p < \infty$, let

$$||u||_p = \sup_{D_0} [\int_{\partial D_0} |u(x)|^p d\mu_{D_0}^{\xi}(x)]^{\frac{1}{p}} \tag{7.1}$$

where the supremum is taken over all such D_0. Assume that $|v(\xi)| \leq |u(\xi)|$.

For any such D_0, let $\tau(\omega) = \inf\{t \geq 0 : |Z_t| \notin D_0\}$, $M_t = u(Z_{t\wedge\tau})$, and $N_t = v(Z_{t\wedge\tau})$, where $(Z_t)_{t\geq 0}$ is a Brownian motion in $\mathbf{R}^n$ that starts at $\xi : Z_0 = \xi$.

Then M and N are martingales. They also satisfy the Bañuelos-Wang condition as can be seen from

$$\begin{aligned} [M,M]_t &= |u(\xi)|^2 + \int_0^{t\wedge\tau} |\nabla u(Z_s)|^2 ds, \\ [N,N]_t &= |v(\xi)|^2 + \int_0^{t\wedge\tau} |\nabla v(Z_s)|^2 ds. \end{aligned}$$

Theorem 7.1. *Let $p > 2$. If u and v are harmonic functions on D such that $|v(\xi)| \le |u(\xi)|$ for some $\xi \in D_0$ as above, and v is differentially subordinate to u, then*

$$\mu^{\xi}_{D_0}(\{x \in \partial D_0 : |v(x)| \ge 1\}) \le \frac{p^{p-1}}{2}\|u\|_p^p. \tag{7.2}$$

Proof. Since D_0 is bounded, we have that $P(\tau < \infty) = 1$, $P(Z_\tau \in \partial D_0) = 1$, M is a uniformly integrable martingale, and $\|M\|_p = \|u(Z_\tau)\|_p < \infty$. The distribution of Z_τ on ∂D_0 is the harmonic measure $\mu^{\xi}_{D_0}$. Therefore, by Theorem 1.2,

$$\begin{aligned} \mu^{\xi}_{D_0}(\{x \in \partial D_0 : |v(x)| \ge 1\}) &= P(|v(Z_\tau)| \ge 1) \\ &\le P(N^* \ge 1) \\ &\le \frac{p^{p-1}}{2}\|M\|_p^p \\ &= \frac{p^{p-1}}{2}\int_{\partial D_0} |u(x)|^p d\mu^{\xi}_{D_0}(x) \\ &\le \frac{p^{p-1}}{2}\|u\|_p^p. \end{aligned}$$

□

Remark. The best constant for the inequality (7.2) is unknown if $n > 2$ and $p > 2$. It is also unknown if $n > 2$ and $1 < p < 2$ (but see [21]). If $n \ge 1$ and $p = 1$, it is known to be 2 (see [10] and page 1023 of [12]). In the classical case in which $n = 2$, D is the open unit disk, u is harmonic on D, v is its conjugate with $v(0) = 0$, and $\xi = 0$, Davis proved that for $p = 1$, the best constant is

$$\frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots}.$$

Later Baernstein [2] gave another proof. Modifying Baernstein's method, Tomaszewski [21] found the best constant in a related case in which $1 < p < 2$. In the conjugate harmonic function case, the best constant for $p > 2$ is still unknown as far as we know, although Essén [17] has some deep results that may lead to the answer.

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