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THE TITS BOUNDARY OF A CAT(0) 2-COMPLEX

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Abstract. We investigate the Tits boundary of CAT(0) 2-complexes that have only a finite number of isometry types of cells. In particular, we show that away from the endpoints, a geodesic segment in the Tits boundary is the ideal boundary of an isometrically embedded Euclidean sector. As applications, we provide sufficient conditions for two points in the Tits boundary to be the endpoints of a geodesic in the 2-complex and for a group generated by two hyperbolic isometries to contain a free group. We also show that if two CAT(0) 2-complexes are quasi-isometric, then the cores of their Tits boundaries are bi-Lipschitz.

1. Introduction

In this paper we study the Tits boundary of CAT(0) 2-complexes that have only a finite number of isometry types of cells. A CAT(0) 2-complex in this paper is a CAT(0) piecewise Riemannian 2-complex. Here each closed 2-cell is equipped with a Riemannian metric so that it is convex and its boundary is a broken geodesic. The metric on the 2-complex is the induced path metric.

Hadamard manifolds are simply connected complete Riemannian manifolds of nonpositive sectional curvature. CAT(0) spaces are counterparts of Hadamard manifolds in the category of metric spaces. A CAT(0) space is a complete simply connected geodesic metric space so that all its triangles are at least as thin as the triangles in the Euclidean space. CAT(0) spaces have many of the geometric properties enjoyed by Hadamard manifolds including convexity of distance functions, uniqueness of geodesic segments and contractibility. As in the case of Hadamard manifolds, a CAT(0) space X has a well-defined ideal boundary $\partial_\infty X$. There is a topology on $\partial_\infty X$ called the cone topology and a metric d_T on $\partial_\infty X$ called the Tits metric. The topology induced by d_T is usually different from the cone topology.

Given a CAT(0) space X, the Tits boundary $\partial_T X$ means the ideal boundary equipped with the Tits metric d_T. $\partial_T X$ reflects the large scale geometry of X. In particular, $\partial_T X$ encodes information on large flat subspaces of X as well as the amount of negative curvature in X (hyperbolicity). Tits metric and Tits boundary are closely related to many interesting questions in geometry. They play an important role in many rigidity theorems (J. Heber [H], W. Ballmann [B], G. Mostow [M], B. Leeb [L]). They are also very important to the question of whether a group

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of isometries of X contains a free group of rank two (K. Ruane [R], X. Xie [X1], [X2]).

For an arbitrary CAT(0) space X, $\partial_T X$ is complete and CAT(1) (see Section 2.4 or [B]). The Tits boundary of the product of two CAT(0) spaces is the spherical join of the Tits boundaries of the two factors. When a locally compact CAT(0) space X admits a cocompact isometric action, B. Kleiner ([K]) showed the geometric dimension of $\partial_T X$ is 1 less than the maximal dimension of isometrically embedded Euclidean spaces in X. Aside from these general results not much is known about the Tits boundary of CAT(0) spaces. Tits boundary is not even well understood for Hadamard manifolds or 2-dimensional (nonmanifold) CAT(0) spaces. Tits boundary is well understood only for a few classes of CAT(0) spaces: $\partial_T X$ is discrete if X is a Gromov hyperbolic CAT(0) space; $\partial_T X$ is a spherical building if X is a higher rank symmetric space or Euclidean building; $\partial_T X$ contains interval components if X is the universal cover of a nonpositively curved graph manifold (S. Buyalo and V. Schroeder [BS], C. Croke and B. Kleiner [CK2]); the Tits boundary was also studied for real analytic Hadamard 4-manifolds admitting cocompact actions (C. Hummel and V. Schroeder [HS1], [HS2]) and the universal covers of certain torus complexes (C. Croke and B. Kleiner [CK1]).

Let X be a CAT(0) 2-complex. B. Kleiner's theorem (Theorem 7.1 of [K]) implies that $\partial_T X$ has geometric dimension at most 1. It follows that any closed metric ball with radius r ($r < \pi/2$) in $\partial_T X$ is an $\mathbb{R}$-tree. For an $\mathbb{R}$-tree we can talk about geodesic segments and branch points. Our first goal is to understand geodesic segments and branch points in the Tits boundary.

A *sector* is a closed convex subset of the Euclidean plane $\mathbb{E}^2$ whose boundary is the union of two rays emanating from the origin. We equip a sector with the induced metric. The image of an isometric embedding from a sector into a CAT(0) space X is called a *flat sector* in X. We notice if S is a flat sector in a CAT(0) space X, then the Tits boundary $\partial_T S$ of S is a closed interval and isometrically embeds into the Tits boundary $\partial_T X$ of X. The following theorem says that away from the endpoints, a segment in the Tits boundary is the Tits boundary of a flat sector. For a CAT(0) 2-complex X, we say Shape(X) is finite if X has only finitely many isometry types of cells.

Theorem 3.1. *Let X be a* CAT(0) 2*-complex with* Shape(X) *finite, and $\gamma : [0, h] \to \partial_T X$ a geodesic in $\partial_T X$ with length $h \leq \pi$. Then for any $\epsilon > 0$, there exists a flat sector S in X with $\partial_T S = \gamma([\epsilon, h - \epsilon])$.*

Theorem 3.1 does not hold without the assumption that Shape(X) is finite (see the example in Section 3). Theorem 3.1 also cannot be improved to include the case $\epsilon = 0$: let X be the universal cover of a torus complex considered in [CK1], then $\partial_T X$ contains interval components; some of these intervals are not the Tits boundaries of any flat sectors.

It follows from Theorem 3.1 that (see Proposition 4.12) a branch point in $\partial_T X$ is represented by a ray where two flat sectors branch off.

Let X be a CAT(0) 2-complex with Shape(X) finite. Then small metric balls in $\partial_T X$ are $\mathbb{R}$-trees. An $\mathbb{R}$-tree may have lots of branch points. One may ask if there is a constant $c = c(X) > 0$ such that the distance between any two branch points in $\partial_T X$ is at least c. Another interesting question concerning the Tits boundary is whether the lengths of circles in $\partial_T X$ form a discrete set. Note B. Kleiner's theorem also implies circles in $\partial_T X$ are simple closed geodesics, hence are rectifiable.

Theorem 4.10 and Theorem 4.13. *Let X be a locally finite* CAT(0) 2*-complex with* Shape(X) *finite. Suppose the interior angles of all the closed* 2*-cells of X are rational multiples of π. Then there is a positive integer m such that:*

(i) *each topological circle in $\partial_T X$ has length an integral multiple of π/m;*

(ii) *the distance between any two branch points in $\partial_T X$ is either infinite or an integral multiple of π/m.*

Given a geodesic $c : \mathbb{R} \to X$ in a CAT(0) space X, the two points in the ideal boundary determined by $c_{|[0,\infty)}$ and $c_{|(-\infty,0]}$ are called the *endpoints* of c. As an application of Theorem 3.1 we discuss when two points $\xi, \eta \in \partial_\infty X$ are the endpoints of a geodesic in a locally compact CAT(0) space X. Recall a necessary condition is $d_T(\xi, \eta) \geq \pi$, and a sufficient condition is $d_T(\xi, \eta) > \pi$. We provide a criterion for ξ and η to be the endpoints of a geodesic in X when X is a CAT(0) 2-complex and $d_T(\xi, \eta) = \pi$.

A point in the ideal boundary of a CAT(0) space is called a *terminal point* if it does not lie in the interior of any Tits geodesic.

Theorem 4.15. *Let X be a locally finite* CAT(0) 2*-complex with* Shape(X) *finite. If $\xi, \eta \in \partial_\infty X$ are not terminal points and $d_T(\xi, \eta) \geq \pi$, then there is a geodesic in X with ξ and η as endpoints.*

As a further application we provide a sufficient condition for a group generated by two hyperbolic isometries to contain a free group of rank two. Recall that an isometry $g : X \to X$ is a *hyperbolic isometry* if there is a geodesic $c : \mathbb{R} \to X$ and a positive number l such that $g(c(t)) = c(t + l)$ for all t. The geodesic c is called an *axis* of g. Denote the two endpoints of the axis c by $g(+\infty)$ and $g(-\infty)$.

Theorem 5.3. *Let X be a locally finite* CAT(0) 2*-complex with* Shape(X) *finite, and g_1, g_2 two cellular hyperbolic isometries of X. Suppose each closed* 2*-cell of X is isometric to a convex polygon in the Euclidean plane. If $g_1(+\infty)$, $g_1(-\infty)$, $g_2(+\infty)$, $g_2(-\infty)$ are not terminal points and $d_T(\xi, \eta) \geq \pi$ for any $\xi \in \{g_1(+\infty), g_1(-\infty)\}$ and any $\eta \in \{g_2(+\infty), g_2(-\infty)\}$, then the group generated by g_1 and g_2 contains a free group of rank two.*

It is well known that a quasi-isometry between two Gromov hyperbolic spaces induces a homeomorphism between their boundaries. This property does not hold for CAT(0) spaces. C. Croke and B. Kleiner ([CK1]) constructed two quasi-isometric CAT(0) 2-complexes X_1 and X_2 such that $\partial_\infty X_1$ and $\partial_\infty X_2$ are not homeomorphic with respect to the cone topology. It is also clear from their proof that $\partial_T X_1$ and $\partial_T X_2$ are not isometric. In general, it is unclear whether the Tits boundaries of two quasi-isometric CAT(0) spaces are homeomorphic.

For any CAT(0) 2-complex X, set $\mathrm{Core}(\partial_T X) = \bigcup c$ where c varies over all the topological circles in $\partial_T X$. Let d_c be the induced path metric of d_T on $\mathrm{Core}(\partial_T X)$. For $\xi, \eta \in \mathrm{Core}(\partial_T X)$, Kleiner's theorem implies that $d_c(\xi, \eta) = d_T(\xi, \eta)$ if ξ, η lie in the same path component of $\mathrm{Core}(\partial_T X)$, and $d_c(\xi, \eta) = \infty$ otherwise.

Theorem 6.1. *For $i = 1, 2$, let X_i be a locally finite* CAT(0) 2*-complex admitting a cocompact isometric action. If X_1 and X_2 are (L, A) quasi-isometric, then* $\mathrm{Core}(\partial_T X_1)$ *and* $\mathrm{Core}(\partial_T X_2)$ *are L^2-bi-Lipschitz with respect to the metric d_c.*

The paper is organized as follows. In Section 2, we recall basic facts about CAT(0) spaces and Tits boundary. In Section 3 we use support set to prove Theorem 3.1. In Section 4 we give some applications of Theorem 3.1; in this section we first record several results concerning flat sectors and rays in CAT(0) 2-complexes, then we prove Theorems 4.10, 4.13 and 4.15. In Section 5 we discuss when a group generated by two hyperbolic isometries contains a free group (Theorem 5.3). In Section 6 we study Tits boundaries of quasi-isometric CAT(0) 2-complexes (Theorem 6.1).

2. Preliminaries

The reader is referred to [B], [BBr], [BH] and [K] for more details on the material in this section.

2.1. **CAT(κ) spaces.** Let X be a metric space. For any $x \in X$ and any $r > 0$, $B(x,r) = \{x' \in X : d(x,x') < r\}$ and $\overline{B}(x,r) = \{x' \in X : d(x,x') \leq r\}$ are respectively the open and closed metric balls with center x and radius r. For any subset $A \subset X$ and any $\epsilon > 0$, the ϵ-neighborhood of A is $N_\epsilon(A) = \{x \in X : d(x,a) \leq \epsilon \text{ for some } a \in A\}$. For any two subsets $A, B \subset X$, the Hausdorff distance between A and B is $d_H(A,B) = \inf\{\epsilon : A \subset N_\epsilon(B), B \subset N_\epsilon(A)\}$; $d_H(A,B)$ is defined to be ∞ if there is no $\epsilon > 0$ with $A \subset N_\epsilon(B)$ and $B \subset N_\epsilon(A)$.

The *Euclidean cone* over a metric space X is the metric space $C(X)$ defined as follows. As a set $C(X) = X \times [0,\infty)/X \times \{0\}$. We use tx to denote the image of (x,t). We define $d(t_1x_1, t_2x_2) = \sqrt{t_1^2 + t_2^2 - 2t_1t_2\cos(d(x_1,x_2))}$ if $d(x_1,x_2) \leq \pi$, and $d(t_1x_1, t_2x_2) = t_1 + t_2$ if $d(x_1,x_2) \geq \pi$. The point $O = X \times \{0\}$ is called the cone point of $C(X)$.

Let X be a metric space. A *geodesic* in X is a continuous map $c : I \to X$ from an interval I into X such that, for any point $t \in I$, there exists a neighborhood U of t with $d(c(s_1), c(s_2)) = |s_1 - s_2|$ for all $s_1, s_2 \in U$. If the above equality holds for all $s_1, s_2 \in I$, then we call c a *minimal geodesic*. The image of a geodesic shall also be called a geodesic. When I is a closed interval $[a,b]$, we say c is a *geodesic segment* of length $b-a$ and c connects $c(a)$ and $c(b)$. A metric space X is called a *geodesic metric space* if for any two points $x, y \in X$ there is a minimal geodesic segment connecting them.

A *triangle* in a metric space X is the union of three geodesic segments $c_i : [a_i, b_i] \to X$ $(i = 1,2,3)$ where $c_1(b_1) = c_2(a_2)$, $c_2(b_2) = c_3(a_3)$ and $c_3(b_3) = c_1(a_1)$. For any real number κ, let M_κ^2 stand for the 2-dimensional simply connected complete Riemannian manifold with constant sectional curvature κ, and $D(\kappa)$ denote the diameter of M_κ^2 ($D(\kappa) = \infty$ if $\kappa \leq 0$). Given a triangle $\Delta = c_1 \cup c_2 \cup c_3$ in X where $c_i : [a_i, b_i] \to X$ $(i = 1,2,3)$, a triangle Δ' in M_κ^2 is a *comparison triangle* for Δ if they have the same edge lengths, that is, if $\Delta' = c_1' \cup c_2' \cup c_3'$ and $c_i' : [a_i, b_i] \to M_\kappa^2$ $(i = 1,2,3)$. A point $x' \in \Delta'$ corresponds to a point $x \in \Delta$ if there is some i and some $t_i \in [a_i, b_i]$ with $x' = c_i'(t_i)$ and $x = c_i(t_i)$. We notice if the perimeter of a triangle $\Delta = c_1 \cup c_2 \cup c_3$ in X is less than $2D(\kappa)$, that is, if $\text{length}(c_1) + \text{length}(c_2) + \text{length}(c_3) < 2D(\kappa)$, then there is a unique comparison triangle (up to isometry) in M_κ^2 for Δ.

Definition 2.1. Let $\kappa \in \mathbb{R}$. A complete metric space X is called a CAT(κ) *space* if

(i) Every two points $x, y \in X$ with $d(x,y) < D(\kappa)$ are connected by a minimal geodesic segment;

(ii) For any triangle Δ in X with perimeter less than $2D(\kappa)$ and any two points $x, y \in \Delta$, the inequality $d(x,y) \leq d(x',y')$ holds, where x' and y' are the points on a comparison triangle for Δ corresponding to x and y respectively.

When X is a CAT(κ) space and $x, y \in X$ with $d(x,y) < D(\kappa)$, the definition above implies there is a unique minimal geodesic segment $c_{xy} : [0, d(x,y)] \to X$ with $c_{xy}(0) = x$, $c_{xy}(d(x,y)) = y$. We use xy to denote the image of c_{xy}.

A *metric graph* is a graph where each edge has a metric making it isometric to a nontrivial closed segment in the real line. For a metric graph G, we say Shape(G) is finite if the set of edge lengths of G is finite. We equip the graph with the induced path metric. Notice that a metric graph G with Shape(G) finite is CAT(1) if and only if it has no simple loop with length strictly less than 2π.

A geodesic metric space X is an $\mathbb{R}$-*tree* if for any triangle $\Delta = c_1 \cup c_2 \cup c_3$ in X, c_1 is contained in the union $c_2 \cup c_3$.

Definition 2.2. Let X be an $\mathbb{R}$-tree. A point $p \in X$ is called a *branch point* of X if $X - \{p\}$ has at least three components.

2.2. **Space of directions.** A *pseudo-metric* on a set X is a function $d : X \times X \to [0, \infty)$ that is symmetric and satisfies the triangle inequality.

If (X, d) is a pseudo-metric space, then we get a metric space (X^*, d^*) by letting X^* be the set of maximal zero diameter subsets and setting $d^*(S_1, S_2) := d(s_1, s_2)$ for any $s_i \in S_i$.

Let X be a CAT(κ) space. If $p, x, y \in X$ and $d(p,x) + d(x,y) + d(y,p) < 2D(\kappa)$, then there is a well-defined geodesic triangle $\triangle pxy$. The *comparison angle* of the triangle $\triangle pxy$ at p is defined to be the angle of the comparison triangle in M^2_κ for $\triangle pxy$ at the vertex corresponding to p; this angle is denoted by $\widetilde{\angle}_p(x,y)$. The CAT($\kappa$) condition implies that if $x' \in px - \{p\}$ and $y' \in py - \{p\}$, then $\widetilde{\angle}_p(x', y') \le \widetilde{\angle}_p(x,y)$. Therefore, if we let $x' \in px$, $y' \in py$ tend to p, then $\widetilde{\angle}_p(x', y')$ has a limit; we call this limit the angle between px and py at p, and denote it by $\angle_p(x,y)$. If we let $y' \in py$ tend to p, then $\widetilde{\angle}_p(x, y')$ also tends to $\angle_p(x,y)$. The function $p \to \angle_p(x,y)$ is upper semi-continuous. $\angle_p$ defines a pseudo-metric on the collection of geodesic segments leaving p. We define $\Sigma^*_p X$ to be the metric space associated to the pseudo-metric $\angle_p$. The space of directions at p is the completion of $\Sigma^*_p X$, and is denoted by $\Sigma_p X$. For any $x \in B(p, D(\kappa)) - \{p\}$, the point in $\Sigma_p X$ coming from the geodesic segment px shall be called the initial direction of px, and denoted by $\log_p(x)$. Thus we have a map $\log_p : B(p, D(\kappa)) - \{p\} \to \Sigma_p X$.

Theorem 2.3 (I. Nikolaev [N])**.** *Let X be a* CAT(κ) *space and $p \in X$. Then $\Sigma_p X$ is a* CAT(1) *space.*

2.3. **CAT(0) 2-complexes.** A 2-dimensional *CW*-complex is called a *polygonal complex* if (1) all the attaching maps are homeomorphisms; (2) the intersection of any two closed cells is either empty or exactly one closed cell.

A 0-cell is also called a vertex.

A polygonal complex is *piecewise Riemannian* if the following conditions hold:

(1) For each closed 2-cell A whose boundary contains n ($n \ge 3$) vertices $v_1, \cdots, v_n$ there is a Riemannian metric on A such that $v_i v_{i+1}$ (i mod n) is a geodesic segment in the Riemannian metric and the interior angle at v_i ($1 \le i \le n$) is $< \pi$;

(2) For any two closed 2-cells A_1 and A_2 with $A_1 \cap A_2 \ne \emptyset$, the metrics on A_1 and A_2 agree when restricted to $A_1 \cap A_2$.

Let X be a piecewise Riemannian polygonal complex and $x \in X$. The link Link(X, x) is a metric graph defined as follows. Let A be a closed 2-cell containing x. The unit tangent space $S_x A$ of A at x is isometric to the unit circle with length

2π. We first define a subset $\mathrm{Link}(A,x)$ of S_xA. For any $v \in S_xA$, $v \in \mathrm{Link}(A,x)$ if and only if the initial segment of the geodesic with initial point x and initial direction v lies in A. Then $\mathrm{Link}(A,x) = S_xA$ if x lies in the interior of A; $\mathrm{Link}(A,x)$ is a closed semicircle (with length π) if x lies in the interior of a 1-cell contained in A; and $\mathrm{Link}(A,x)$ is a closed segment with length α if x is a vertex of A and the interior angle of A at x is α. Similarly, if x is contained in a closed 1-cell B we can define S_xB and $\mathrm{Link}(B,x) \subset S_xB$. We note S_xB consists of two points at distance π apart, $\mathrm{Link}(B,x) = S_xB$ if x lies in the interior of B and $\mathrm{Link}(B,x)$ consists of a single point if x is a vertex of B. When x lies in a closed 1-cell B and B is contained in a closed 2-cell A, S_xB and $\mathrm{Link}(B,x)$ can be naturally identified with subsets of S_xA and $\mathrm{Link}(A,x)$ respectively.

We define $\mathrm{Link}(X,x) = \bigcup_A \mathrm{Link}(A,x)$, where A varies over all closed 1-cells and 2-cells containing x. Here $\mathrm{Link}(B,x)$ is identified with a subset of $\mathrm{Link}(A,x)$ as indicated in the last paragraph when x lies in a closed 1-cell B and B is contained in a closed 2-cell A. We let d_x be the induced path metric on $\mathrm{Link}(X,x)$.

The following is a corollary of Ballmann and Buyalo's Theorem ([BBr]). For a piecewise Riemannian polygonal complex X, we say $\mathrm{Shape}(X)$ is finite if X has only finitely many isometry types of cells.

Proposition 2.4. *Let X be a simply connected piecewise Riemannian polygonal complex with $\mathrm{Shape}(X)$ finite. Then X is a $\mathrm{CAT}(0)$ space if and only if the following conditions hold:*

(i) *the Gauss curvature of the open 2-cells is bounded from above by 0;*

(ii) *for every vertex v of X every simple loop in $\mathrm{Link}(X,v)$ has length at least 2π.*

In this paper a CAT(0) 2-complex shall always mean a CAT(0) piecewise Riemannian polygonal complex. When X is a CAT(0) 2-complex with $\mathrm{Shape}(X)$ finite, for any $x \in X$, there is a natural identification between $\Sigma_x^* X = \Sigma_x X$ and $\mathrm{Link}(X,x)$, and the path metric on $\Sigma_x X$ corresponds to d_x on $\mathrm{Link}(X,x)$.

Let X be a CAT(0) space. A subset $A \subset X$ is a *convex* subset if $xy \subset A$ for any $x, y \in A$. Let $A \subset X$ be a closed convex subset. The orthogonal projection onto A, $\pi_A : X \to A$ can be defined as follows: for any $x \in X$ the inequality $d(x, \pi_A(x)) \leq d(x,a)$ holds for all $a \in A$. It follows that for any $x \notin A$ and any $a \neq \pi_A(x)$ we have $\angle_{\pi_A(x)}(x,a) \geq \pi/2$. π_A is 1-Lipschitz: $d(\pi_A(x), \pi_A(y)) \leq d(x,y)$ for any $x, y \in X$.

Let X be a CAT(0) space. Then $\angle_x(y,z) + \angle_y(x,z) + \angle_z(x,y) \leq \pi$ for any three distinct points $x, y, z \in X$.

2.4. **Ideal boundary of a CAT(0) space.** Let X be a CAT(0) space. A *geodesic ray* in X is a geodesic $c : [0,\infty) \to X$. Consider the set of geodesic rays in X. Two geodesic rays c_1 and c_2 are said to be *asymptotic* if $f(t) := d(c_1(t), c_2(t))$ is a bounded function. It is easy to check that this defines an equivalence relation. The set of equivalence classes is denoted by $\partial_\infty X$ and called the *ideal boundary* of X. If $\xi \in \partial_\infty X$ and c is a geodesic ray belonging to ξ, we write $c(\infty) = \xi$. For any $\xi \in \partial_\infty X$ and any $x \in X$, there is a unique geodesic ray $c_{x\xi} : [0,\infty) \to X$ with $c_{x\xi}(0) = x$ and $c_{x\xi}(\infty) = \xi$. The image of $c_{x\xi}$ is denoted by $x\xi$.

Set $\overline{X} = X \cup \partial_\infty X$. The cone topology on $\overline{X}$ has as a basis the open sets of X together with the sets

$$U(x,\xi,R,\epsilon) = \{z \in \overline{X} | z \notin B(x,R), d(c_{xz}(R), c_{x\xi}(R)) < \epsilon\},$$

where $x \in X$, $\xi \in \partial_\infty X$ and $R > 0$, $\epsilon > 0$. The topology on X induced by the cone topology coincides with the metric topology on X.

We can also define a metric on $\partial_\infty X$. Let $c_1, c_2 : [0,\infty) \to X$ be two geodesic rays with $c_1(0) = c_2(0) = x$. For $t_1, t_2 \in (0,\infty)$, consider the comparison angle $\widetilde{\angle}_x(c_1(t_1), c_2(t_2))$. The CAT(0) condition implies that if $t_1 < t_1'$, $t_2 < t_2'$, then $\widetilde{\angle}_x(c_1(t_1), c_2(t_2)) \le \widetilde{\angle}_x(c_1(t_1'), c_2(t_2'))$. It follows that both $\lim_{t\to 0} \widetilde{\angle}_x(c_1(t), c_2(t))$ and $\lim_{t\to\infty} \widetilde{\angle}_x(c_1(t), c_2(t))$ exist. We can prove that $\lim_{t\to\infty} \widetilde{\angle}_x(c_1(t), c_2(t))$ depends only on the points $\xi_1, \xi_2 \in \partial_\infty X$ represented respectively by c_1 and c_2. We call $\angle_T(\xi_1, \xi_2) := \lim_{t\to\infty} \widetilde{\angle}_x(c_1(t), c_2(t))$ the *Tits angle* between ξ_1 and ξ_2. The Tits metric d_T on $\partial_\infty X$ is the path metric induced by $\angle_T$. We denote $\partial_T X := (\partial_\infty X, d_T)$. We shall also call $\angle_x(\xi_1, \xi_2) := \lim_{t\to 0} \widetilde{\angle}_x(c_1(t), c_2(t))$ the angle at x between ξ_1 and ξ_2. Note that $\angle_x(\xi_1, \xi_2) = \angle_x(p, q)$ for any $p \in x\xi_1$, $q \in x\xi_2$, $p, q \ne x$, where $\angle_x(p,q)$ is defined in Section 2.2. From the definition we see

$$\angle_x(\xi_1, \xi_2) \le \widetilde{\angle}_x(c_1(t_1), c_2(t_2)) \le \angle_T(\xi_1, \xi_2) \le d_T(\xi_1, \xi_2)$$

for all $t_1, t_2 \in (0, \infty)$.

It should be noted that the topology induced by d_T is in general different from the cone topology. For instance, when $X = \mathbb{H}^n$, the n-dimensional real hyperbolic space, $\partial_T X$ is discrete while $\partial_\infty X$ with the cone topology is homeomorphic to $\mathbb{S}^{n-1}$.

Here we record some basic properties of the Tits metric (see [B] or [BH]). For any geodesic $c : \mathbb{R} \to X$ in a CAT(0) space, we call the two points in $\partial_\infty X$ determined by the two rays $c_{|[0,+\infty)}$ and $c_{|(-\infty,0]}$ the endpoints of c, and denote them by $c(+\infty)$ and $c(-\infty)$ respectively.

Proposition 2.5. *Let X be a* CAT(0) *space, and ξ_1, $\xi_2 \in \partial_T X$.*

(i) *$\partial_T X$ is a* CAT(1) *space.*

(ii) *$\angle_T(\xi_1, \xi_2) = \sup_{x\in X} \angle_x(\xi_1, \xi_2)$.*

(iii) *If X is locally compact and $d_T(\xi_1, \xi_2) > \pi$, then there is a geodesic in X with ξ_1 and ξ_2 as endpoints.*

(iv) *If X is locally compact and $d_T(\xi_1, \xi_2) < \infty$, then there is a minimal geodesic segment in $\partial_T X$ connecting ξ_1 and ξ_2.*

Recall a *sector* is a closed convex subset of the Euclidean plane $\mathbb{E}^2$ whose boundary is the union of two rays emanating from the origin. We equip a sector with the induced metric. The image of an isometric embedding from a sector into a CAT(0) space X is called a *flat sector* in X, and the image of the origin is called the cone point of the flat sector. We notice if S is a flat sector in a CAT(0) space X, then $\partial_T S$ is a closed interval and isometrically embed into the Tits boundary $\partial_T X$.

Let X be a CAT(0) space and $p \in X$. For each $\xi \in \partial_\infty X$ the geodesic ray $p\xi$ gives rise to a point in $\Sigma_p X$. Thus $\log_p : X - \{p\} \to \Sigma_p X$ extends to a map $\overline{X} - \{p\} \to \Sigma_p X$, which is continuous and shall still be denoted by $\log_p$.

Proposition 2.6 ([B])**.** *Let X be a* CAT(0) *space and $p \in X$. Then the map $\log_p$ restricted to $\partial_T X$ is a 1-Lipschitz map, that is, $\angle_p(\xi_1, \xi_2) \le d_T(\xi_1, \xi_2)$ for any ξ_1, ξ_2 in $\partial_T X$. If $\xi_1, \xi_2 \in \partial_T X$ with $d_T(\xi_1, \xi_2) = \angle_p(\xi_1, \xi_2) < \pi$, then the two rays $p\xi_1$, $p\xi_2$ bound a flat sector of angle $d_T(\xi_1, \xi_2)$.*

Let Y be a CAT(κ) space. We say the *geometric dimension* of Y is ≤ 1 if $\Sigma_p Y$ is either empty or discrete for all $p \in Y$. The geometric dimension of Y is defined to be 1 if it is ≤ 1 and $\Sigma_p Y$ is nonempty for at least one $p \in Y$. The following result is a consequence of B. Kleiner's theorem (Theorem 7.1 of [K]). The reader is referred to [K] for the general result and general definition of geometric dimension.

Theorem 2.7 (B. Kleiner [K])**.** *Let X be a* CAT(0) 2*-complex. Then the geometric dimension of $\partial_T X$ is ≤ 1. Furthermore, if there exists an isometric embedding from the Euclidean plane into X, then the geometric dimension of $\partial_T X$ is* 1*.*

Let X be a CAT(0) 2-complex and $a, b, c \in \partial_T X$ three distinct points such that $d_T(a,b)$, $d_T(a,c) < \pi$. Theorem 2.7 implies that if $\log_a(b) \neq \log_a(c)$, then $\angle_a(b,c) = \pi$. It follows that $ba \cup ac$ is a geodesic segment when $\log_a(b) \neq \log_a(c)$. From this it is easy to derive the following corollary.

Corollary 2.8. *Let X be a* CAT(0) 2*-complex. Then for any $\xi \in \partial_T X$ and any $r : 0 < r < \pi/2$, the closed metric ball $\overline{B}(\xi, r)$ is an $\mathbb{R}$-tree.*

It follows from Corollary 2.8 that each embedded path in $\partial_T X$ is rectifiable and after reparameterization is a geodesic.

Corollary 2.9. *Let X be a* CAT(0) 2*-complex. Then embedded paths in $\partial_T X$ are geodesics. In particular, all the topological circles in $\partial_T X$ are simple closed geodesics.*

2.5. **Quasi-isometry and quasi-flats.**

Definition 2.10. Let $L \geq 1$, $A \geq 0$. A (not necessarily continuous) map $f : X \to Y$ between two metric spaces is called a (L, A) *quasi-isometric embedding* if the following holds for all $x_1, x_2 \in X$:

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A.$$

f is a (L, A) *quasi-isometry* if in addition $d_H(Y, f(X)) \leq A$. We call f a quasi-isometric embedding if it is a (L, A) quasi-isometric embedding for some $L \geq 1$, $A \geq 0$.

Notice if $f : X \to Y$ is a (L, A) quasi-isometry, then there is some $A' > 0$ and a (L, A') quasi-isometry $g : Y \to X$ with $d(g(f(x)), x) \leq A'$, $d(f(g(y)), y) \leq A'$ for all $x \in X$, $y \in Y$. Such a map g is called a *quasi-inverse* of f.

Let X be a CAT(0) space. The image of a quasi-isometric embedding from the Euclidean plane $\mathbb{E}^2$ into X is called a *quasi-flat.*

3. Segments in the Tits boundary

The main goal of this section is to establish the following result, which says that away from the endpoints, a segment in the Tits boundary is the Tits boundary of a flat sector. The definition of a flat sector is given in Section 2.4.

Theorem 3.1. *Let X be a* CAT(0) 2*-complex with* Shape(X) *finite, and $\gamma : [0, h] \to \partial_T X$ a geodesic segment in the Tits boundary of X with length $h \leq \pi$. Then for any $\epsilon > 0$, there exists a flat sector S in X with $\partial_T S = \gamma([\epsilon, h - \epsilon])$.*

Theorem 3.1 does not hold if we do not assume Shape(X) is finite, as shown by the following example. Let $S(\alpha)$ be a sector with angle $\alpha > 0$. We inductively define Y_i for $i \geq 1$. Let $Y_1 = S(\pi/4)$, and let $c_1 : \mathbb{R} \to Y_1$ be one of the two rays in the boundary of Y_1. Y_2 is obtained from Y_1 and $S(\frac{1}{2^2})$ by identifying one of the two rays in the boundary of $S(\frac{1}{2^2})$ with $c_1([2, +\infty))$ (via an isometry). For any $i \geq 2$, let $c_i : \mathbb{R} \to Y_i$ be the ray in the boundary of Y_i such that $c_i(1) = c_1(1)$. Y_{i+1} is obtained from Y_i and $S(\frac{1}{2^{i+1}})$ by identifying one of the two rays in the boundary of $S(\frac{1}{2^{i+1}})$ with $c_i([i+1, +\infty))$ (via an isometry). Then $Y_i \subset Y_{i+1}$ for all $i \geq 1$. Let $Y = \bigcup_{i=1}^{\infty} Y_i$. Y is a CAT(0) space and $\partial_T Y$ is a closed segment with length $\pi/4 + 1/2$. Let X be obtained from two copies of Y by identifying the two rays in their boundaries that contain $c_1(1)$ via an isometry. Then X is a CAT(0) space and $\partial_T X$ is a closed segment with length $2(\pi/4 + 1/2) = \pi/2 + 1$. The conclusion of Theorem 3.1 does not hold for this X. We see that X is homeomorphic to a sector, and $c_x := \text{Link}(X, x)$ is a circle for any x in the interior of X. There are points x_i $(i \geq 2)$ in the interior of X such that $\text{length}(c_{x_i}) > 2\pi$ and $\text{length}(c_{x_i}) \to 2\pi$ as $i \to +\infty$. It follows that X does not admit any triangulation that makes it a CAT(0) 2-complex with $\text{Shape}(X)$ finite.

3.1. **A Gauss-Bonnet type theorem.** In the proof of Theorem 3.1 we will use a Gauss-Bonnet type theorem for noncompact piecewise Riemannian surfaces, which relates the Tits distance (at infinity) and the curvature inside the surface. We now describe this result.

Let F be a piecewise Riemannian polygonal complex with the following properties:

(1) F is a CAT(0) space with the induced path metric;

(2) F is homeomorphic to a sector in the Euclidean plane, in particular, F is a manifold with boundary;

(3) the manifold boundary of F is $\partial F = c_1 \cup c_2$, where c_1 and c_2 are geodesic rays with $c_1 \cap c_2 = \{p\}$, $p \in F$.

For each point x in the interior of F, $\text{Link}(F, x)$ is a topological circle, and for each $x \in \partial F$, $\text{Link}(F, x)$ is homeomorphic to a closed interval. Let $L(x)$ be the length of $\text{Link}(F, x)$. The deficiency $k(x)$ at x is defined as follows: $k(x) := 2\pi - L(x)$ if x lies in the interior of F, and $k(x) := \pi - L(x)$ if $x \in \partial F$. For each 2-cell A of F, we let $C(A)$ be the total curvature of A. Set $e(F) = \sum_A C(A) + \sum_x k(x)$, where A varies over all 2-cells and $x \neq p$ varies over all the 0-cells of F different from p.

Let $\xi_1, \xi_2 \in \partial_T F$ be represented by c_1 and c_2 respectively.

Theorem 3.2 (K. Kawamura, F. Ohtsuka [KO])**.** *Let F and ξ_1, ξ_2 be as above. If $d_T(\xi_1, \xi_2) < \pi$, then $e(F)$ is finite and $e(F) = \angle_p(\xi_1, \xi_2) - d_T(\xi_1, \xi_2)$.*

3.2. **Reduction.** In this section we reduce the proof of Theorem 3.1 to the injectivity of a certain map.

Let X and $\gamma : [0, h] \to \partial_T X$ be as in Theorem 3.1. Recall for each $x \in X$, there is a map $\log_x : \overline{X} - \{x\} \to \text{Link}(X, x)$ which sends $\xi \in \overline{X} - \{x\}$ to the initial direction of $x\xi$ at x. For ξ_1, $\xi_2 \in \partial_T X$ with $d_T(\xi_1, \xi_2) < \pi$, let $\xi_1\xi_2 \subset \partial_T X$ be the unique geodesic segment from ξ_1 to ξ_2 and $C(\xi_1\xi_2)$ the Euclidean cone over $\xi_1\xi_2$. Notice that $C(\xi_1\xi_2)$ is a sector with angle $d_T(\xi_1, \xi_2)$. Let O be the cone point of $C(\xi_1\xi_2)$. For any $x \in X$, there is a map $\rho_x : C(\xi_1\xi_2) \to X$, where for any $\xi \in \xi_1\xi_2$ the ray $O\xi$ is isometrically mapped to the ray $x\xi$. We see that ρ_x is a 1-Lipschitz map.

For each 2-cell A of X, let $C(A)$ be the total curvature of A. Note that $C(A) \leq 0$ since A has nonpositive sectional curvature. Set

$$\epsilon_1 = \min\{-C(A) : A \text{ is a 2-cell with } C(A) \neq 0\}.$$

ϵ_1 is defined to be ∞ if there is no 2-cell A with $C(A) \neq 0$. For a finite metric graph G and an edge path c in G, we denote the length of c by $l(c)$. Since X is a CAT(0) 2-complex, for any $x \in X$ each simple loop in $\mathrm{Link}(X, x)$ has length at least 2π. Define

$$\epsilon_2 = \min\{l(c) - 2\pi : c \text{ is a simple loop in } \mathrm{Link}(X, x) \text{ with } l(c) \neq 2\pi,\ x \in X\}.$$

ϵ_2 is defined to be ∞ if there is no simple loop c in any $\mathrm{Link}(X, x)$ with $l(c) \neq 2\pi$. Since $\mathrm{Shape}(X)$ is finite, ϵ_1 and ϵ_2 are well defined and greater than 0, although they may be ∞.

Lemma 3.3. *Let $x_0 \in X$ with $\angle_{x_0}(\gamma(0), \gamma(h)) \geq d_T(\gamma(0), \gamma(h)) - \epsilon/4$, where $0 < \epsilon < \min\{\epsilon_1, \epsilon_2, h\}$. If the map $\log_{x_0|\gamma(\epsilon/2)\gamma(h-\epsilon/2)}$ is injective, and the surface $F_0 := \rho_{x_0}(C(\gamma(\epsilon/2)\gamma(h - \epsilon/2)))$ is convex in X, then there is a point $p \in X$ such that $\rho_p : C(\gamma(\epsilon)\gamma(h - \epsilon)) \to X$ is an isometric embedding.*

Proof. The injectivity of the map $\log_{x_0|\gamma(\epsilon/2)\gamma(h-\epsilon/2)}$ implies that the map

$$\rho_{x_0} : C(\gamma(\epsilon/2)\gamma(h - \epsilon/2)) \to X$$

is a topological embedding. So $F_0 = \rho_{x_0}(C(\gamma(\epsilon/2)\gamma(h - \epsilon/2)))$ is homeomorphic to a sector in the Euclidean plane. The manifold boundary ∂F_0 of F_0 is the union of two geodesic rays $x_0\gamma(\epsilon/2)$ and $x_0\gamma(h - \epsilon/2)$ with $x_0\gamma(\epsilon/2) \cap x_0\gamma(h - \epsilon/2) = \{x_0\}$. F_0 is clearly a closed subset of X.

Since F_0 is closed and convex, it is a CAT(0) space with the induced path metric. Thus we can apply Theorem 3.2 to the surface F_0:

$$e(F_0) = \angle_{x_0}(\gamma(\epsilon/2), \gamma(h - \epsilon/2)) - d_T(\gamma(\epsilon/2), \gamma(h - \epsilon/2)),$$

where $e(F_0) = \sum_A C(A) + \sum_{p \neq x_0} k(p)$. By the assumption on x_0, we see the difference between $\angle_{x_0}(\gamma(\epsilon/2), \gamma(h - \epsilon/2))$ and $d_T(\gamma(\epsilon/2), \gamma(h - \epsilon/2))$ is less than any positive $-C(A)$ for 2-cells A of X contained in F_0. It follows that $C(A) = 0$ and so A is flat for any 2-cell A of X contained in the interior of F_0. Similarly, the difference between $\angle_{x_0}(\gamma(\epsilon/2), \gamma(h - \epsilon/2))$ and $d_T(\gamma(\epsilon/2), \gamma(h - \epsilon/2))$ is less than any positive $l(c) - 2\pi$ for simple loops c in $\mathrm{Link}(X, x)$, $x \in X$. Therefore, $\mathrm{Link}(F_0, x)$ has length 2π for any x in the interior of F_0.

Since $\mathrm{Shape}(X)$ is finite, the sizes of the cells are bounded. Therefore, there exists some constant $r > 0$ such that the surface $F_1 := F_0 - N_r(\partial F_0)$ is flat, where $N_r(\partial F_0)$ denotes the r-neighborhood of $\partial F_0 \subset X$. Now, for any $p \in F_1$ with $d(p, \partial F_0)$ sufficiently large, $\rho_p(C(\gamma(\epsilon)\gamma(h - \epsilon))) \subset F_1$ and the lemma follows. □

3.3. **Segments in the Tits boundary.** Let X and $\gamma : [0, h] \to \partial_T X$ be as in Theorem 3.1, and ϵ_1 and ϵ_2 as defined in Section 3.2. We see that it suffices to prove Theorem 3.1 for $h < \pi$. Set $\epsilon_0 = \min\{\epsilon_1, \epsilon_2, h\}$ and denote $\xi = \gamma(0)$, $\eta = \gamma(h)$. For any positive number $\epsilon < \epsilon_0$, choose a point x_0 in some open 2-cell A of X such that

$$\angle_{x_0}(\xi, \eta) \geq d_T(\xi, \eta) - \epsilon/4.$$

We see that Theorem 3.1 follows from Lemma 3.3 and the following theorem.

Theorem 3.4. *Let ϵ and x_0 be as above. Then the map $\log_{x_0|\gamma(\epsilon/2)\gamma(h-\epsilon/2)}$ is injective and the surface $F_0 := \rho_{x_0}(C(\gamma(\epsilon/2)\gamma(h - \epsilon/2)))$ is convex in X.*

We shall consider the loop $c := x_0\xi \cup x_0\eta \cup \xi\eta \subset \overline{X} = X \cup \partial_\infty X$ and its support set in X. Note that c is indeed a loop in the cone topology of $\overline{X}$. Theorem 3.1 was originally proved using a different method. The use of a support set is suggested by B. Kleiner which greatly shortened the argument.

Note that if $L \subset \overline{X}$ is homeomorphic to a circle and $x \in X - L$, then $\log_x(L)$ is a loop in $\mathrm{Link}(X, x)$.

Definition 3.5. Let $L \subset \overline{X}$ be a subset that is homeomorphic to a circle. The *support set* $\mathrm{supp}(L)$ of L is the set of $x \in X - L$ such that $\log_x(L)$ represents a nontrivial class in $H_1(\mathrm{Link}(X, x))$.

We shall first show that $\mathrm{supp}(c)$ is topologically a surface.

Lemma 3.6. *Let $x \in X - c$. Then $x \in \mathrm{supp}(c)$ if and only if $\log_x(c)$ is homotopic to a simple loop of length 2π in $\mathrm{Link}(X, x)$.*

Proof. Since $\mathrm{Link}(X, x)$ is a graph, any simple loop in $\mathrm{Link}(X, x)$ represents a nontrivial class in $H_1(\mathrm{Link}(X, x))$. Thus one direction is clear. Next we assume $x \in \mathrm{supp}(c)$. Note that $\log_x(c)$ is homotopically nontrivial as it is homologically nontrivial. Since $x \notin x_0\xi$, we have $d_x(\log_x(y_1), \log_x(y_2)) < \pi$ for any $y_1, y_2 \in x_0\xi$. The fact that $\mathrm{Link}(X, x)$ is a CAT(1) space implies the path $\log_x(x_0\xi) \subset \mathrm{Link}(X, x)$ is relative homotopic to the geodesic segment from $a_1 := \log_x(x_0)$ to $b_1 := \log_x(\xi)$. Similarly, the path $\log_x(x_0\eta)$ is relative homotopic to the geodesic segment from a_1 to $c_1 := \log_x(\eta)$ and $\log_x(\xi\eta)$ is relative homotopic to the geodesic segment from b_1 to c_1. Thus $\log_x(c)$ is homotopic to the loop $a_1b_1 * b_1c_1 * c_1a_1$. By considering the two ideal triangles $\Delta(xx_0\xi)$ and $\Delta(xx_0\eta)$ we have $\angle_{x_0}(x, \xi) + d_x(a_1, b_1) = \angle_{x_0}(x, \xi) + \angle_x(x_0, \xi) \le \pi$ and $\angle_{x_0}(x, \eta) + d_x(a_1, c_1) = \angle_{x_0}(x, \eta) + \angle_x(x_0, \eta) \le \pi$. On the other hand,

$$d_T(\xi, \eta) - \epsilon/4 \le \angle_{x_0}(\xi, \eta) \le \angle_{x_0}(\xi, x) + \angle_{x_0}(x, \eta).$$

It follows that

$$\begin{aligned}
& d_x(a_1, b_1) + d_x(b_1, c_1) + d_x(c_1, a_1) \\
\le \;& d_x(a_1, b_1) + d_T(\xi, \eta) + d_x(c_1, a_1) \\
\le \;& \epsilon/4 + \angle_{x_0}(\xi, \eta) + d_x(a_1, b_1) + d_x(c_1, a_1) \\
\le \;& \epsilon/4 + \angle_{x_0}(\xi, x) + \angle_{x_0}(x, \eta) + d_x(a_1, b_1) + d_x(c_1, a_1) \\
\le \;& \epsilon/4 + 2\pi.
\end{aligned}$$

Now the loop $a_1b_1 * b_1c_1 * c_1a_1$ is homotopically nontrivial in a CAT(1) metric graph with length at most $\epsilon/4 + 2\pi$. The choice of ϵ implies there is no simple loop in $\mathrm{Link}(X, x)$ with length at most $2\pi + \epsilon/4$ but strictly greater than 2π. Therefore, $a_1b_1 * b_1c_1 * c_1a_1$ and $\log_x(c)$ are homotopic to a simple loop of length 2π. □

For any topological space Y and $r > 0$, let $C_r(Y) = Y \times [0, r]/(Y \times \{0\})$ be the cone over Y with radius r. Since c is a circle, $C_r(c)$ is homeomorphic to the closed unit disk in the plane. Let $\partial C_r(c)$ be the boundary circle of $C_r(c)$.

For any $x \in X - c$ and any r with $0 < r < d(x, x_0\xi \cup x_0\eta)$, we can define a map $f_{x,r} : C_r(c) \to \overline{B}(x, r)$ by letting $f_{x,r}(z, s)$ $(z \in c,\ 0 \le s \le r)$ be the point on the geodesic xz at distance s from x. We observe that $f_{x,r}$ represents a class in the relative homology group $H_2(X, X - \{x\})$. By using homotopy along geodesic segments we see that for r_1, r_2 with $0 < r_1, r_2 < d(x, x_0\xi \cup x_0\eta)$, f_{x,r_1} and f_{x,r_2} represent the same class in $H_2(X, X - \{x\})$.

For any $x \in X$, let $U(x)$ be the union of all the closed 2-cells containing x. Since $\mathrm{Shape}(X)$ is finite, $U(x)$ is a neighborhood of x in X and $r(x) := d(x, \partial U(x)) > 0$.

Lemma 3.7. *Let $x \in X - c$ and r with $0 < r < d(x, x_0\xi \cup x_0\eta)$. Then $x \in \mathrm{supp}(c)$ if and only if $0 \neq [f_{x,r}] \in H_2(X, X - \{x\})$.*

Proof. We may assume $0 < r < \min\{r(x), d(x, x_0\xi \cup x_0\eta)\}$ by the remark preceding the lemma. By excision we have $H_2(X, X - \{x\}) \cong H_2(\overline{B}(x,r), \overline{B}(x,r) - \{x\})$. The exact sequence for the pair $(\overline{B}(x,r), \overline{B}(x,r) - \{x\})$ implies the boundary homomorphism $H_2(\overline{B}(x,r), \overline{B}(x,r) - \{x\}) \to H_1(\overline{B}(x,r) - \{x\})$ is an isomorphism. Notice the choice of r implies the map $\log_x$ restricted to $\partial\overline{B}(x,r)$ is a homeomorphism from $\partial\overline{B}(x,r)$ to $\mathrm{Link}(X,x)$. It follows that $H_1(\partial\overline{B}(x,r)) \cong H_1(\mathrm{Link}(X,x))$. Also, note that $H_1(\overline{B}(x,r) - \{x\}) \cong H_1(\partial\overline{B}(x,r))$. The composition of these isomorphisms is an isomorphism $g : H_2(X, X - \{x\}) \to H_1(\mathrm{Link}(X,x))$. Now it is not hard to check that g maps the class $[f_{x,r}]$ to the class $[\log_x(c)] \in H_1(\mathrm{Link}(X,x))$. □

Lemma 3.6 implies that if $x \in \mathrm{supp}(c)$, then $\log_x(c)$ is homotopic to a simple loop c_x in $\mathrm{Link}(X,x)$. For each $x \in \mathrm{supp}(c)$, let $S(x)$ be the union of all closed 2-cells that give rise to the simple loop c_x. Then $S(x)$ is homeomorphic to the closed unit disk in the Euclidean plane and contains x in its interior such that $\mathrm{Link}(S(x),x) = c_x$ has length 2π.

Lemma 3.8. *For any $x \in \mathrm{supp}(c)$, there is $r > 0$ such that the following holds: $\mathrm{supp}(c) \cap \overline{B}(x,r) = S(x) \cap \overline{B}(x,r)$. In particular, $\mathrm{supp}(c)$ is a 2-dimensional manifold, and for each $x \in \mathrm{supp}(c)$ the circle $\mathrm{Link}(\mathrm{supp}(c),x)$ has length 2π.*

Proof. For $x \in \mathrm{supp}(c)$, choose r with $0 < r < \frac{1}{2}\min\{r(x), d(x, x_0\xi \cup x_0\eta)\}$. Set $K = \overline{B}(x, r/2)$. Note that $f_{x,r}$ represents a class in $H_2(X, X - K)$. It follows from the definitions of $S(x)$ and $f_{x,r}$ that $[f_{x,r}] = [S(x) \cap \overline{B}(x,r)] \in H_2(X, X-K)$.

For any $y \in \overline{B}(x, r/4)$, by using homotopy along geodesic segments we see $f_{y,r}$ and $f_{x,r}$ as maps from $(C_r(c), \partial C_r(c))$ to $(X, X-K)$ are homotopic. As a result, $[f_{y,r}] = [f_{x,r}] \in H_2(X, X-K)$. Combining with the observation from last paragraph we see $[f_{y,r}] = [S(x) \cap \overline{B}(x,r)] \in H_2(X, X-K)$. It follows that for any $y \in \overline{B}(x,r/4)$, $f_{y,r}$ represents a nontrivial class in $H_2(X, X - \{y\})$ if and only if $S(x) \cap \overline{B}(x,r)$ represents a nontrivial class in $H_2(X, X - \{y\})$. Recall $S(x) \cap \overline{B}(x,r)$ is homeomorphic to the closed unit disk in the Euclidean plane. Now it follows easily from excision that for any $y \in \overline{B}(x,r/4)$, $S(x) \cap \overline{B}(x,r)$ represents a nontrivial class in $H_2(X, X - \{y\})$ if and only if $y \in \overline{B}(x,r/4) \cap (S(x) \cap \overline{B}(x,r)) = S(x) \cap \overline{B}(x,r/4)$. □

Corollary 3.9. *$\mathrm{supp}(c)$ is locally convex in X.*

Proof. It follows from the fact that for each $x \in \mathrm{supp}(c)$ the circle $\mathrm{Link}(\mathrm{supp}(c),x)$ has length 2π. □

An argument similar to the proof of Lemma 3.8 shows the complement of $\mathrm{supp}(c)$ in $X - c$ is open:

Lemma 3.10. *$\mathrm{supp}(c)$ is a closed subset of $X - c$.*

Recall that x_0 lies in some open 2-cell A. Clearly $A - x_0\xi \cup x_0\eta$ has two components because 2-cells are convex. Let A_0 be the component of $A - x_0\xi \cup x_0\eta$ that has interior angle less than π at x_0.

Lemma 3.11. *A_0 is contained in* $\operatorname{supp}(c)$.

Proof. Let $y_0 \in \partial A_0 - c$ be the point with $\angle_{x_0}(\xi, y_0) = \angle_{x_0}(y_0, \eta)$. We claim that if $x \neq x_0$ lies on x_0y_0 and $d(x_0, x)$ is sufficiently small, then $x \in \operatorname{supp}(c)$.

Let $x \in A_0$. Then the link $\operatorname{Link}(X, x)$ is a circle with length 2π. Set $a_1 = \log_x(x_0)$, $b_1 = \log_x(\xi)$ and $c_1 = \log_x(\eta)$. Since $x \notin x_0\xi \cup x_0\eta$, we have $d_x(a_1, b_1) < \pi$ and $d_x(a_1, c_1) < \pi$. On the other hand, $d_x(b_1, c_1) \leq d_T(\xi, \eta) < \pi$. When $x \to x_0$ along the segment y_0x_0, $x\xi \to x_0\xi$ and $x\eta \to x_0\eta$. It follows that for $x \in x_0y_0$ ($x \neq x_0$) with $d(x_0, x)$ sufficiently small, we have $a_1 \notin b_1c_1$, $b_1 \notin a_1c_1$ and $c_1 \notin a_1b_1$. Thus $\operatorname{Link}(X, x) = a_1b_1 \cup b_1c_1 \cup c_1a_1$ for such x. The proof of Lemma 3.6 shows the path $\log_x(c)$ is homotopic to the path $a_1b_1 * b_1c_1 * c_1a_1 = \operatorname{Link}(X, x)$. Therefore, if $x \neq x_0$ lies on x_0y_0 and $d(x_0, x)$ is sufficiently small, then $\log_x(c)$ represents a nonzero class in $H_1(\operatorname{Link}(X, x))$ and $x \in \operatorname{supp}(c)$.

So we have $A_0 \cap \operatorname{supp}(c) \neq \emptyset$. Since A_0 is disjoint from $X^{(1)}$, $\operatorname{supp}(c)$ is a 2-dimensional manifold and is closed in $X - c$, we see that $A_0 \subset \operatorname{supp}(c)$. $\square$

Let G be a metric graph with $\operatorname{Shape}(G)$ finite and $L \subset G$ a homotopically nontrivial loop in G. Then L is freely homotopic to a closed geodesic l in G. Since the universal cover of G is a metric tree and L represents a hyperbolic isometry, we see that $l \subset L$. It follows that for any $x \in \operatorname{supp}(c)$, the circle c_x is contained in $\log_x(c)$.

Fix a positive r_0 with $r_0 < r(x_0)$. Then $\partial\overline{B}(x_0, r_0) \cap A_0$ is an open arc contained in A_0. Fix two points $y, z \in \partial\overline{B}(x_0, r_0) \cap A_0$ with $\angle_{x_0}(y, z) \geq \angle_{x_0}(\xi, \eta) - \epsilon/4$, and let σ be the closed subarc of $\partial\overline{B}(x_0, r_0) \cap A_0$ that joins y and z.

Lemma 3.12. *For each $p \in \sigma$, there exists some $\xi' \in \xi\eta$ with $p \in x_0\xi'$ and $x_0\xi' - \{x_0\} \subset \operatorname{supp}(c)$.*

Proof. Fix $p \in \sigma \subset \operatorname{supp}(c)$. We will try to extend the geodesic segment x_0p inside $\operatorname{supp}(c)$. Since $\operatorname{supp}(c)$ is a surface and $\operatorname{Link}(\operatorname{supp}(c), p)$ is a circle with length 2π, the geodesic segment x_0p can be extended beyond p in $\operatorname{supp}(c)$. There are two cases to consider: there is either a finite maximum extension

$$x_0x - \{x_0, x\} = (x_0p - \{x_0\}) \cup (px - \{x\}) \subset \operatorname{supp}(c) \text{ where } x \in X - \operatorname{supp}(c),$$

or an infinite maximal extension

$$x_0p - \{x_0\} \subset x_0\xi' - \{x_0\} \subset \operatorname{supp}(c) \text{ where } \xi' \in \partial_\infty X.$$

If the first case occurs and $x_0x - \{x_0, x\}$ ($x \notin \operatorname{supp}(c)$) is a finite maximal extension, then Lemma 3.10 implies $x \in x_0\xi \cup x_0\eta$. This is a contradiction since for any x' in $x_0\xi \cup x_0\eta$ the geodesic segment $x_0x' \subset x_0\xi \cup x_0\eta$ does not pass through p.

Therefore, the second case occurs and we have $x_0p - \{x_0\} \subset x_0\xi' - \{x_0\} \subset \operatorname{supp}(c)$ for some $\xi' \in \partial_\infty X$. Pick a sequence of points x_i on $x_0\xi'$ with $d(x_0, x_i) \to \infty$. Since $c_{x_i} \subset \log_{x_i}(c)$ and $\log_{x_i}(\xi') \in \operatorname{Link}(\operatorname{supp}(c), x_i) = c_{x_i}$, there is a point $\xi_i \in c$ with $\log_{x_i}(\xi') = \log_{x_i}(\xi_i)$. $p \notin x_0\xi \cup x_0\eta$ implies $\xi_i \notin x_0\xi \cup x_0\eta$. Therefore, $\xi_i \in \xi\eta$. By the choice of ξ_i the sequence $\{x_0\xi_i\}_{i=1}^{\infty}$ converges to the ray $x_0\xi'$. Since the sequence $\{\xi_i\}$ lies on the closed interval $\xi\eta$ we see that $\xi' \in \xi\eta$. $\square$

Since for any $x \in \operatorname{supp}(c)$ the circle $\operatorname{Link}(\operatorname{supp}(c), x)$ has length 2π, geodesics in $\operatorname{supp}(c)$ do not branch. It follows that for each $p \in \sigma$, there is a unique $\xi' \in \xi\eta$ with the property stated in Lemma 3.12. Thus we can define a map $g : \sigma \to \xi\eta$ by $g(p) = \xi'$ where ξ' is the unique point on $\xi\eta$ with $x_0p - \{x_0\} \subset x_0\xi' - \{x_0\} \subset \operatorname{supp}(c)$.

Lemma 3.13. *The map g is continuous.*

Proof. The lemma follows easily from the facts that $\xi\eta$ is compact and that geodesics in $\mathrm{supp}(c)$ do not branch. $\square$

Lemma 3.14. *The map $\log_{x_0}$ is injective on the segment $\gamma(\epsilon/2)\gamma(h-\epsilon/2)$, and the surface $F_0 = \rho_{x_0}(C(\gamma(\epsilon/2)\gamma(h-\epsilon/2)))$ is convex in X.*

Proof. Let $\xi_1 = g(y)$, $\xi_2 = g(z)$. Notice that $d_T(\xi_1, \xi_2) \geq d_T(\xi, \eta) - \epsilon/2$ holds by the choice of y, z and x_0. We observe that it suffices to prove the map $\log_{x_0}$ is injective on the segment $\xi_1\xi_2$ and the surface $\rho_{x_0}(C(\xi_1\xi_2))$ is convex in X.

Since the map $g : \sigma \to \xi\eta$ is continuous and σ is connected, the segment $\xi_1\xi_2$ is contained in $g(\sigma)$. Therefore, for each $\xi' \in \xi_1\xi_2$, $x_0\xi' - \{x_0\} \subset \mathrm{supp}(c)$ and $x_0\xi' \cap \sigma \neq \emptyset$. Now the map $\log_{x_0}$ is injective on $\xi_1\xi_2$ since geodesics in $\mathrm{supp}(c)$ do not branch. The fact that for each $x \in \mathrm{supp}(c)$ the link $\mathrm{Link}(\mathrm{supp}(c), x)$ is a circle of length 2π implies the surface $\rho_{x_0}(C(\xi_1\xi_2))$ is locally convex in X. Since $\rho_{x_0}(C(\xi_1\xi_2))$ is also closed in X, it is convex in X. $\square$

Lemma 3.14 completes the proofs of Theorems 3.4 and 3.1.

4. Applications

In this section we give several applications of Theorem 3.1. But first we record several results concerning rays and flat sectors in a CAT(0) 2-complex. These results will be used in later applications.

4.1. **Rays and flat sectors in CAT(0) 2-complexes.** Recall flat sectors are defined in Section 2.4. To simplify notation, for any geodesic ray $\alpha : [0, \infty) \to X$, we also use α to denote its image.

Proposition 4.1. *Let X be a* CAT(0) *2-complex with* $\mathrm{Shape}(X)$ *finite, $c : [0, \infty) \to X$ a geodesic ray and $S \subset X$ a flat sector. If $S \cap c = \emptyset$ and c represents an interior point of $\partial_T S$, then there is a geodesic ray $c' : [0, \infty) \to X$ asymptotic to c with $c' \subset S \cap X^{(1)}$, where $X^{(1)}$ is the 1-skeleton of X.*

Proof. Since $\mathrm{Shape}(X)$ is finite, there is some $a > 0$ with the following property: for any two 1-cells vv_1, vv_2 sharing a vertex v, if $\angle_v(v_1, v_2) < \pi$, then $d(v, v_1) + d(v, v_2) - d(v_1, v_2) > a$. The flat sector S is a closed convex subset of X. Let $\pi : X \to S$ be the orthogonal projection. Recall π is 1-Lipschitz. Let $p = c(0)$ and $q = \pi(p)$. Also let $p = p_1, p_2, \cdots, p_k = q$ be a finite sequence of points on the segment pq with $d(p_i, p_{i+1}) < a/2$. Let c_i be the geodesic ray with $c_i(0) = p_i$ and $c_i(\infty) = c(\infty)$. We see that if $c_i \cap S \neq \emptyset$, then $c_i([t_0, \infty)) \subset S$ for some $t_0 \geq 0$. Since $c_1 \cap S = c \cap S = \emptyset$ and $c_k \subset S$, there is some i so that $c_i \cap S = \emptyset$ and $c_{i+1}([t_0, \infty)) \subset S$ for some $t_0 \geq 0$. Therefore, it suffices to prove the lemma when $d(p, S) = d(p, q) < a/2$.

Now we assume $d(p, S) = d(p, q) < a/2$. As c represents a point in $\partial_T S$, the convex function $g(t) := d(c(t), S)$ is bounded from above, and is therefore non-increasing. Thus $d(c(t), S) < a/2$ for all $t \geq 0$. The assumption $S \cap c = \emptyset$ implies that for any t, $\pi(c(t))$ lies either on the manifold boundary of S or in $S \cap X^{(1)}$. Since c represents an interior point of $\partial_T S$, for t sufficiently large $\pi(c(t))$ has to lie in the interior of S and thus in $S \cap X^{(1)}$. We may assume that $\pi(c(t)) \in S \cap X^{(1)}$ for all t by considering a sub-ray of c. Then $\pi(c) \subset S \cap X^{(1)}$ is a continuous path going to infinity. Let $t_1 < t_2 < t_3$ so that $v_1 := \pi(c(t_1))$, $v_2 := \pi(c(t_2))$,

$v_3 := \pi(c(t_3))$ are 0-cells, and v_1v_2 and v_2v_3 are 1-cells. We claim $\angle_{v_2}(v_1, v_3) = \pi$. The claim clearly implies there is a geodesic ray $c' : [0,\infty) \to X$ asymptotic to c with $c' \subset \pi(c) \subset S \cap X^{(1)}$. Suppose $\angle_{v_2}(v_1, v_3) < \pi$. The choice of a implies $a + d(v_1, v_3) < d(v_1, v_2) + d(v_2, v_3)$. Triangle inequality implies

$$\begin{array}{rl} & d(c(t_1), c(t_3)) \\ \leq & d(c(t_1), \pi(c(t_1))) + d(\pi(c(t_1)), \pi(c(t_3))) + d(\pi(c(t_3)), c(t_3)) \\ \leq & a/2 + d(\pi(c(t_1)), \pi(c(t_3))) + a/2 \\ = & a + d(v_1, v_3) \\ < & d(v_1, v_2) + d(v_2, v_3), \end{array}$$

or $d(c(t_1), c(t_3)) < d(v_1, v_2) + d(v_2, v_3)$. On the other hand, since π is 1-Lipschitz, the length of $\pi \circ c_{|[t_1,t_3]}$ is less than or equal to the length of $c_{|[t_1,t_3]}$. It follows

$$\begin{array}{rl} & d(v_1, v_2) + d(v_2, v_3) \\ = & d(\pi(c(t_1)), \pi(c(t_2))) + d(\pi(c(t_2)), \pi(c(t_3))) \\ \leq & \text{length}(\pi \circ c_{|[t_1,t_2]}) + \text{length}(\pi \circ c_{|[t_2,t_3]}) \\ = & \text{length}(\pi \circ c_{|[t_1,t_3]}) \\ \leq & \text{length}(c_{|[t_1,t_3]}) \\ = & d(c(t_1), c(t_3)), \end{array}$$

or $d(v_1, v_2) + d(v_2, v_3) \leq d(c(t_1), c(t_3))$. The contradiction proves the claim. $\square$

Corollary 4.2. *Let X be a* CAT(0) 2*-complex with* Shape(X) *finite and $S_1, S_2 \subset X$ two flat sectors. If $\partial_T S_1 \cap \partial_T S_2$ is a nontrivial interval, then $S_1 \cap S_2 \neq \emptyset$.*

Proof. Suppose $S_1 \cap S_2 = \emptyset$. For each $\xi \in \text{interior}(\partial_T S_1 \cap \partial_T S_2)$, let c_ξ be a ray in S_2 asymptotic to ξ. Then $c_\xi \cap S_1 = \emptyset$. Proposition 4.1 implies there is a ray $c'_\xi \subset S_1 \cap X^{(1)}$ representing ξ. There are uncountably many points in interior$(\partial_T S_1 \cap \partial_T S_2)$, so there are uncountably many rays in $S_1 \cap X^{(1)}$ pointing to uncountably many directions. On the other hand, there are only countably many 1-cells in $S_1 \cap X^{(1)}$ and they can only give rise to countably many directions. The contradiction proves the corollary. $\square$

For a flat sector $S \subset X$ and $x \in S$, let $S(x) \subset S$ be the flat sector with cone point x and $\partial_T S(x) = \partial_T S$.

Proposition 4.3. *Let X be a* CAT(0) 2*-complex with* Shape(X) *finite and $\sigma : [0, h] \to \partial_T X$ a geodesic segment with length $h \leq \pi$. Suppose there are numbers t_i $(1 \leq i \leq 4)$ such that $0 < t_1 < t_2 < t_3 < t_4 < h$, and two flat sectors S_1 and S_2 with $\partial_T S_1 = \sigma([t_1, t_3])$ and $\partial_T S_2 = \sigma([t_2, t_4])$. Let $c_1 : [0,\infty) \to S_1$ and $c_2 : [0,\infty) \to S_2$ be rays with $c_1(+\infty) = \sigma(t'_1)$, $t_1 < t'_1 < t_2$ and $c_2(+\infty) = \sigma(t'_2)$, $t_3 < t'_2 < t_4$ respectively. Then there is some $u_0 \geq 0$ such that $S_1(c_1(u)) \cap S_2(c_2(u'))$ is a flat sector for all $u, u' \geq u_0$.*

The proof of Proposition 4.3 is divided into a few lemmas.

Lemma 4.4. *There is some $a \geq 0$ such that $c_1([a,\infty)) \cap S_2 = \emptyset$ and $c_2([a,\infty)) \cap S_1 = \emptyset$.*

Proof. Assume $c_1 \cap S_2 \neq \emptyset$. Since $c_1 \cap S_2$ is a closed convex subset of c_1, it is either a closed segment of c_1 or a subray of c_1. But $c_1(+\infty) = \sigma(t'_1) \notin \partial_T S_2$ implies $c_1 \cap S_2$ cannot be a ray. Therefore, there is some $a \geq 0$ such that $c_1([a,\infty)) \cap S_2 = \emptyset$. The proof of the second equality is similar. $\square$

Lemma 4.5. *Using the notation as in Proposition 4.3 and Lemma 4.4, there are* $u_1, u_2 \geq a$ *such that* $S_1(c_1(u_1)) \cap S_2(c_2(u_2))$ *is a flat sector with cone point* $c_1(u_1)\sigma(t_3) \cap c_2(u_2)\sigma(t_2)$.

Proof. Since $\partial_T S_1 \cap \partial_T S_2 = \sigma([t_2, t_3])$ is a nontrivial interval, Corollary 4.2 implies $S_1 \cap S_2 \neq \emptyset$. $S_1 \cap S_2$ is a closed convex subset of S_i $(i = 1, 2)$. For any point $p \in S_1 \cap S_2$, let S_p be the flat sector with cone point p and $\partial_T S_p = \sigma([t_2, t_3])$. Clearly we have $S_p \subset S_1 \cap S_2$. Fix a point $p \in S_1 \cap S_2$ and consider the subsets S_p and c_2 of S_2. It is clear that for large enough u we have $c_2(u)\sigma(t_2) \cap S_p \neq \emptyset$. It follows that $c_2(u)\sigma(t_2) \cap S_1 \neq \emptyset$ for large enough u. Fix a $u_2 \geq a$ with $c_2(u_2)\sigma(t_2) \cap S_1 \neq \emptyset$. Note that $c_2(u_2)\sigma(t_2) \cap S_1$ is a subray of $c_2(u_2)\sigma(t_2)$.

Consider the rays $c_2(u_2)\sigma(t_2) \cap S_1$ and c_1 in S_1. Since $t_1 < t_1' < t_2 < t_3$ and $c_1(+\infty) = \sigma(t_1')$, we see for large enough u, that $c_1(u)\sigma(t_3) \cap c_2(u_2)\sigma(t_2) \neq \emptyset$. Fix a $u_1 \geq a$ such that $c_1(u_1)\sigma(t_3) \cap c_2(u_2)\sigma(t_2) \neq \emptyset$. Clearly $c_1(u_1)\sigma(t_3) \cap c_2(u_2)\sigma(t_2)$ consists of a single point. Let x be this point. We see that $x \in S_1 \cap S_2$ and that $S_x \subset S := S_1(c_1(u_1)) \cap S_2(c_2(u_2))$. We claim $S_x = S$.

Suppose $S_x \neq S$. Pick $y \in S - S_x$. Note that $\{y\}$ and S_x are contained in the flat sector $S_1(c_1(u_1))$. Since x lies on the boundary ray $c_1(u_1)\sigma(t_3)$ of $S_1(c_1(u_1))$, $0 < \angle_x(y, \sigma(t_2)) < \angle_x(y, \sigma(t_3)) \leq \pi$. Similarly, by viewing $\{y\}$ and S_x as subsets of $S_2(c_2(u_2))$ we see that $0 < \angle_x(y, \sigma(t_3)) < \angle_x(y, \sigma(t_2)) \leq \pi$. A contradiction. □

Lemma 4.6. *Using the notation as in Proposition 4.3 and Lemma 4.4, if there are* $u_1, u_2 \geq a$ *such that* $S_1(c_1(u_1)) \cap S_2(c_2(u_2))$ *is a flat sector with* $c_1(u_1)\sigma(t_3) \cap c_2(u_2)\sigma(t_2)$ *as cone point, then for all* $u \geq u_1$, $u' \geq u_2$, $S_1(c_1(u)) \cap S_2(c_2(u'))$ *is a flat sector.*

Proof. Let $S = S_1(c_1(u_1)) \cap S_2(c_2(u_2))$. Then $\partial_T S = \sigma([t_2, t_3])$. Note that both S and $S_2(c_2(u'))$ $(u' \geq u_2)$ are subsectors of the flat sector $S_2(c_2(u_2))$. It is clear that $S \cap S_2(c_2(u'))$ is a flat sector with cone point $c_1(u_1)\sigma(t_3) \cap c_2(u')\sigma(t_2)$. It follows from $S_1(c_1(u_1)) \cap S_2(c_2(u')) \subset S_1(c_1(u_1)) \cap S_2(c_2(u_2)) \cap S_2(c_2(u')) = S \cap S_2(c_2(u'))$ that $S_1(c_1(u_1)) \cap S_2(c_2(u')) = S \cap S_2(c_2(u'))$ is a flat sector with $c_1(u_1)\sigma(t_3) \cap c_2(u')\sigma(t_2)$ as cone point. Now a similar argument, but fixing u' and increasing u_1, shows that for all $u \geq u_1$, $S_1(c_1(u)) \cap S_2(c_2(u'))$ is a flat sector. □

The proof of Proposition 4.3 is now complete.

4.2. **Circles in Tits boundary.** We recall that an *n-flat* in a CAT(0) space X is the image of an isometric embedding from the n-dimensional Euclidean space into X.

Theorem 4.7 (V. Schroeder [BGS], B. Leeb [L])**.** *Let* X *be a locally compact* CAT(0) *space. Suppose* $\mathbb{S}^{n-1} \subset \partial_T X$ *is a unit* $(n-1)$ *sphere in the Tits boundary that does not bound a unit hemisphere. Then there is an n-flat* $F \subset X$ *such that* $\partial_T F = \mathbb{S}^{n-1}$.

Theorems 4.7 and 2.7 imply that if X is a locally finite CAT(0) 2-complex, then any unit circle in $\partial_T X$ is the ideal boundary of a 2-flat. Theorem 3.1 enables us to generalize this result to topological circles in the Tits boundary. Recall that Corollary 2.9 implies topological circles in $\partial_T X$ are simple closed geodesics.

Let $A \subset X$ be a subset of a CAT(0) space. A point $\xi \in \partial_\infty X$ is a *limit point* of A if there is a sequence $a_i \in A$ $(i \geq 1)$ such that $\{a_i\}$ converges to ξ in the cone topology. The *limit set* $L(A) \subset \partial_\infty X$ of A is the set of limit points of A. Recall a quasi-flat in a CAT(0) 2-complex X is the image of a quasi-isometric embedding from the Euclidean plane $\mathbb{E}^2$ into X.

Proposition 4.8. *Let X be a* CAT(0) 2*-complex with* Shape(X) *finite and $C \subset \partial_T X$ a topological circle in the Tits boundary. Then there is a quasi-flat E of X with the following properties:*

(i) *E is homeomorphic to the plane with the closed unit disk removed;*

(ii) *E is flat, i.e., each point of E has a neighborhood in E which is isometric to an open subset of $\mathbb{E}^2$;*

(iii) $L(E) = C$.

Proof. By Corollary 2.9 we may assume that the circle C is a simple closed geodesic. Choose points $a_1, a_2, \cdots, a_n$ on C in cyclic order such that they divide C into intervals of equal length $l < \pi/4$. For each i (mod n) let m_i be the midpoint of $a_i a_{i+1}$. Theorem 3.1 implies that there are flat sectors S_i $(1 \leq i \leq n)$ such that $a_i a_{i+1}$ is contained in the interior of $\partial_T S_i$ and $m_j \notin \partial_T S_i$ for $j \neq i$. Let $c_i : [0, \infty) \to S_i$ be a ray in S_i with $c_i(+\infty) = m_i$. Now Proposition 4.3 implies that there is some $u_0 \geq 0$ such that for any $u_i \geq u_0$ $(1 \leq i \leq n)$, the intersection $S_i(c_i(u_i)) \cap S_{i+1}(c_{i+1}(u_{i+1}))$ (i mod n) is a flat sector. Since $\partial_T S_i \cap \partial_T S_j = \emptyset$ for any $1 \leq i, j \leq n$ with $i - j \not\equiv -1, 0, 1 \bmod n$, there is some $u_0' \geq 0$ such that for any $u_i \geq u_0'$ $(1 \leq i \leq n)$, the intersection $S_i(c_i(u_i)) \cap S_j(c_j(u_j)) = \emptyset$ for any $1 \leq i, j \leq n$ with $i - j \not\equiv -1, 0, 1 \bmod n$. Now choose $u_i \geq u_0, u_0'$ and let E_i be the interior of the flat sector $S_i(c_i(u_i))$. Set $E = \bigcup_i E_i$. Now it is easy to see that E is a quasi-flat with the desired properties. □

We call a geodesic c of the form $c : (0, \infty) \to X$ in a CAT(0) space X an *open geodesic ray.*

Corollary 4.9. *Let $C \subset \partial_T X$ and $E \subset X$ be as in Proposition 4.8. Then E admits a foliation by open geodesic rays with the following properties:*

(i) *each ray in the foliation is asymptotic to a point in C, and for each $\xi \in C$ there is at least one ray in the foliation asymptotic to ξ;*

(ii) *there is a constant $a > 0$ such that the distance between any two asymptotic rays in the foliation is at most a.*

Proof. We use the notation in the proof of Proposition 4.8. Let p_i denote the cone point of the flat sector $S_{i-1}(c_{i-1}(u_{i-1})) \cap S_i(c_i(u_i))$. For each i $(1 \leq i \leq n)$, let $S_i' \subset \overline{E}$ be the flat sector with cone point p_i and $\partial_T S_i' = m_{i-1} m_i$. As a flat sector S_i' is a union of rays issuing from its cone point. $S_i(c_i(u_i)) - S_i' \cup S_{i+1}'$ is basically a flat strip: it is a convex subset of the flat sector $S_i(c_i(u_i))$ whose boundary contains two asymptotic rays. We foliate $S_i(c_i(u_i)) - S_i' \cup S_{i+1}'$ by parallel rays. In this way we get a foliation of E by open geodesic rays with the desired properties. □

One question about circles in the Tits boundary is whether the lengths of circles in the Tits boundary form a discrete set.

Theorem 4.10. *Let X be a* CAT(0) 2*-complex with* Shape(X) *finite. If the interior angles of all the* 2*-cells of X are rational multiples of π, then there is a positive integer m so that the length of each topological circle in the Tits boundary is an integral multiple of π/m.*

The following lemma is not hard to prove.

Lemma 4.11. *Let p_1, $p_2, \cdots, p_n \in \mathbb{E}^2$ be n points in the Euclidean plane, a, b the two rays emanating from p_1, p_n respectively, and e_i ($i = 1, 2, \cdots, n-1$) the segment connecting p_i and p_{i+1}. Suppose the union $c := a \cup e_1 \cup \cdots \cup e_{n-1} \cup b$ is a simple path and A is one of the two components of $\mathbb{E}^2 - c$. Denote the interior angle of A at p_i by A_i. Then the length of the closed interval $L(A) \subset \partial_T \mathbb{E}^2$ is $\sum_{i=1}^{n} A_i - (n-1)\pi$.* □

Proof of Theorem 4.10. Since Shape(X) is finite, the assumption implies there is a positive integer m so that the interior angles of 2-cells are integral multiples of π/m.

Let C be a topological circle in $\partial_T X$ and E a quasi-flat provided by Proposition 4.8 with $L(E) = C$. Corollary 4.9 implies that E admits a foliation by open geodesic rays. Choose points $a_1, a_2, \cdots, a_n$ on C in cyclic order such that they divide C into intervals with length $< \pi/2$. For each i ($1 \leq i \leq n$), let c_i be an open geodesic ray in the foliation of E that represents a_i. Then $c_i \cap X^{(1)} \neq \emptyset$. Pick a point $x_i \in c_i \cap X^{(1)}$ as follows. If $c_i \cap X^{(1)}$ contains vertices, then let x_i be a fixed vertex in $c_i \cap X^{(1)}$; otherwise, if $c_i \cap X^{(1)}$ contains no vertex, let x_i be any fixed point in $c_i \cap X^{(1)}$.

Note that $E - \bigcup c_i$ has n components, each of which is contractible. Let E_i be the component of $E - \bigcup c_i$ that contains c_i and c_{i+1} (i mod n) on the boundary. Let $\alpha_i \subset E \cap X^{(1)}$ be a simple path from x_i to x_{i+1} with interior$(\alpha_i) \subset E_i$. Then α_i divides E_i into two components. Let D_i be the unbounded component of $E_i - \alpha_i$, $A_{i,j}$ ($j = 2, \cdots, k_i - 1$) the interior angles of D_i at vertices in the interior of the path α_i, and $A_{i,1}$ and A_{i,k_i} the interior angles of D_i at x_i and x_{i+1} respectively. Note that $A_{i,j}$ ($1 < j < k_i$) and $A_{i+1,1} + A_{i,k_i}$ (i mod n) are integral multiples of π/m. Now the theorem follows by applying Lemma 4.11 to each D_i and adding up all the equalities. □

4.3. **Branch points in Tits boundary.** Let X be a CAT(0) 2-complex. Since small metric balls in $\partial_T X$ are $\mathbb{R}$-trees (Corollary 2.8) we can talk about branch points in $\partial_T X$ (see Definition 2.2). The following proposition says a branch point in $\partial_T X$ is represented by a geodesic ray where flat sectors branch off.

Proposition 4.12. *Let X be a* CAT(0) *2-complex with* Shape(X) *finite and $\xi \in \partial_T X$ a branch point. Then there are two flat sectors S_1 and S_2 and geodesic ray $c : [0, \infty) \to X$ such that*

(i) *$c(0)$ is the common cone point of S_1 and S_2;*

(ii) *$c(+\infty) = \xi$ lies in the interior of the segment $\partial_T S_i$ ($i = 1, 2$);*

(iii) *$S_1 \cup S_2 - c$ has three components and the closure of each component is a flat sector; in particular, $c \subset X^{(1)}$.*

Proof. Since $\xi \in \partial_T X$ is a branch point, there are points $\xi_1, \xi_2, \xi_3 \in \partial_T X$ such that $0 < d_T(\xi, \xi_i) < \pi/4$ and $\xi_1\xi_2 \cap \xi_1\xi_3 = \xi_1\xi$. By Theorem 3.1 we may assume there are flat sectors S_1' and S_2' such that $\partial_T S_1' = \xi_1\xi_2$, $\partial_T S_2' = \xi_1\xi_3$. Corollary 4.2 implies $S_1' \cap S_2' \neq \emptyset$. Thus $S_1' \cap S_2'$ is a nonempty closed convex subset of the flat sector S_1'. It follows that $\partial_T(S_1' \cap S_2') = \xi_1\xi$. Note that for any x in the interior of $S_1' \cap S_2'$, the intersection $x\xi_2 \cap (S_1' \cap S_2')$ is a segment. Since $x\xi_2$ lies in the interior of S_1', the intersection of $x\xi_2$ with the boundary of $S_1' \cap S_2'$ lies either in $X^{(1)}$ or on $o_2\xi_3$, where o_2 denotes the cone point of S_2'. Since $\xi_3 \notin \partial_T(S_1' \cap S_2')$, the intersection of $x\xi_2$ with the boundary of $S'_1 \cap S'_2$ cannot lie on $o_2\xi_3$ when $d(o_2, x)$ is sufficiently large. So there is an infinite path in the boundary of $S'_1 \cap S'_2$ that is contained in $X^{(1)}$. Since $\mathrm{Shape}(X)$ is finite and $S'_1 \cap S'_2$ is convex, there is a geodesic ray $c \subset X^{(1)}$ such that c lies in the boundary of $S'_1 \cap S'_2$. It is easy to see that $c(+\infty) = \xi$.

Set $S_1 = S'_1(c(0))$, $S_2 = S'_2(c(0))$. Also, let $S \subset S_1 \cap S_2$ be the flat sector with cone point $c(0)$ and $\partial_T S = \xi_1\xi$. The choice of c implies that S_1 and S_2 branch off at c and $S = S_1 \cap S_2$. □

We would like to know if there is a positive constant $c = c(X)$ such that the distance between any two branch points in $\partial_T X$ is at least c. The positive answer to the question would imply that the components of the Tits boundary are almost like simplicial metric graphs. Recall that the Tits boundary of a 2-dimensional Euclidean building is a 1-dimensional spherical building.

Theorem 4.13. *Let X be a locally finite* CAT(0) 2*-complex with* $\mathrm{Shape}(X)$ *finite. If the interior angles of all the* 2*-cells of X are rational multiples of π, then there is a positive integer m such that the distance between any two branch points in $\partial_T X$ is either infinite or an integral multiple of π/m. When X is not locally finite and if ξ and η are two branch points in $\partial_T X$ with $d_T(\xi, \eta) < \pi$, then $d_T(\xi, \eta)$ is an integral multiple of π/m.*

Proof. Let ξ and η be two branch points in $\partial_T X$ with $d_T(\xi, \eta) < \infty$ ($d_T(\xi, \eta) < \pi$ if X is not locally finite). Let $\sigma \subset \partial_T X$ be a minimal geodesic from ξ to η. Since ξ is a branch point, there are points ξ_i ($i = 1, 2, 3$) in $\partial_T X$ with $\xi_3 \in \sigma$ and $0 < d_T(\xi_i, \xi) < \pi/4$ such that $\xi_3\xi_1 \cap \xi_3\xi_2 = \xi_3\xi$. Similarly, there are points $\eta_i \in \partial_T X$ ($i = 1, 2, 3$) with $\eta_3 \in \sigma$ and $0 < d_T(\eta_i, \eta) < \pi/4$ such that $\eta_3\eta_1 \cap \eta_3\eta_2 = \eta_3\eta$. Set $\sigma_1 = \xi_1\xi \cup \sigma \cup \eta\eta_1$ and $\sigma_2 = \xi_2\xi \cup \sigma \cup \eta\eta_2$.

By Theorem 3.1 and Proposition 4.3 (see the proof of Proposition 4.8) there are flat sectors $S_1, \cdots, S_k$ such that $S_i \cap S_{i+1}$ is a flat sector, $S_i \cap S_j = \emptyset$ for $|j - i| \geq 2$ and $L(E_1) = \sigma_1$, where $E_1 = \bigcup_i S_i$. We can foliate E_1 by open geodesic rays as in Corollary 4.9. Similarly, there are flat sectors $S'_1, \cdots, S'_k$ such that $S'_i \cap S'_{i+1}$ is a flat sector, $S'_i \cap S'_j = \emptyset$ for $|j - i| \geq 2$ and $L(E_2) = \sigma_2$, where $E_2 = \bigcup_i S'_i$. The proof of Proposition 4.12 shows that there are geodesic rays $c'_1, c'_2 \subset E_1 \cap X^{(1)}$ such that $c'_1(+\infty) = \xi$, $c'_2(+\infty) = \eta$.

We see that the assumption implies there is a positive integer m so that the interior angles of 2-cells are integral multiples of π/m. Choose points $\xi = a_1, a_2, \cdots, a_n = \eta$ on σ in linear order such that they divide σ into intervals with length $< \pi/2$. Set $c_1 = c'_1$ and $c_n = c'_2$. For each i ($1 < i < n$), let c_i be an open geodesic ray in the foliation of E_1 that represents a_i. Now Lemma 4.11 and the proof of Theorem 4.10 show that $d_T(\xi, \eta)$ is an integral multiple of π/m. □

4.4. **π-Visibility.** Let X be a CAT(0) space. A *flat half-plane* in X is the image of an isometric embedding $f : \{(x, y) \in \mathbb{E}^2 : y \geq 0\} \to X$, and in this case we say the geodesic $c : \mathbb{R} \to X$, $c(t) = f(t, 0)$ bounds the flat half-plane.

Let $\xi, \eta \in \partial_T X$ with $d_T(\xi, \eta) = \pi$. If X is locally compact and there is a geodesic c in X with ξ and η as endpoints, then c bounds a flat half-plane (see [BH]). In general, there is no geodesic in X with ξ and η as endpoints.

Recall that if X is a CAT(0) 2-complex, then for any $\xi \in \partial_T X$ and any $r : 0 < r < \pi/2$, the closed metric ball $\overline{B}(\xi, r)$ is an $\mathbb{R}$-tree. It is necessary to distinguish those points in $\partial_T X$ that are "dead ends".

Definition 4.14. Let X be a CAT(0) 2-complex and $\xi \in \partial_T X$. We say ξ is a *terminal point* if ξ does not lie in the interior of any geodesic segment in $\partial_T X$.

Theorem 4.15. *Let X be a locally finite* CAT(0) *2-complex with* Shape(X) *finite. If $\xi, \eta \in \partial_T X$ are not terminal points and $d_T(\xi, \eta) \geq \pi$, then there is a geodesic in X with ξ and η as endpoints.*

Proof. By Proposition 2.5 we may assume $d_T(\xi, \eta) = \pi$. Let $\sigma : [0, \pi] \to \partial_T X$ be a minimal geodesic from ξ to η. Since $\xi, \eta \in \partial_T X$ are not terminal points and small metric balls in $\partial_T X$ are $\mathbb{R}$-trees, there is some ϵ, $0 < \epsilon < \pi/4$ such that σ extends to a locally isometric map $[-\epsilon, \pi + \epsilon] \to \partial_T X$, which we still denote by σ.

We see that $\sigma_{|[-\epsilon, \pi/2+\epsilon]}$ is a minimal geodesic in $\partial_T X$ with length less than π. By Theorem 3.1 there is a flat sector S_1 in X such that $\partial_T S_1 = \sigma([-\epsilon/2, \pi/2 + \epsilon/2])$. Similarly, there is a flat sector S_2 in X with $\partial_T S_2 = \sigma([\pi/2 - \epsilon/2, \pi + \epsilon/2])$. Since $\partial_T S_1 \cap \partial_T S_2 = \sigma([\pi/2 - \epsilon/2, \pi/2 + \epsilon/2])$ is a nontrivial interval, Corollary 4.2 implies $S_1 \cap S_2 \neq \emptyset$. Pick $x \in S_1 \cap S_2$ and let $S \subset S_1 \cap S_2$ be the subsector with cone point x and $\partial_T S = \sigma([\pi/2 - \epsilon/2, \pi/2 + \epsilon/2])$. Fix a point p in the interior of the flat sector S, and let c_1, c_2, c_3 be rays starting from p belonging to $\sigma(0)$, $\sigma(\pi)$ and $\sigma(\pi/2)$ respectively. Since c_1 and c_3 are contained in the flat sector S_1, $\angle_p(c_1(\infty), c_3(\infty)) = \pi/2$. Similarly, $\angle_p(c_2(\infty), c_3(\infty)) = \pi/2$. Since the initial segment of each c_i is contained in the flat sector S, it is clear that the angle $\angle_p(c_1(\infty), c_2(\infty)) = \pi$. It follows that $c_1 \cup c_2$ is a complete geodesic in X with endpoints ξ and η. □

Remark 4.16. The conclusion of Theorem 4.15 does not hold if X is not a CAT(0) 2-complex. For instance, the universal covers of nonpositively curved 3-dimensional graph manifolds ([BS], [CK2]) are counterexamples.

5. Free subgroups

In this section we establish a sufficient condition (Theorem 5.2) for the existence of free subgroups in a group acting isometrically and cellularly on a piecewise Euclidean CAT(0) 2-complex.

5.1. **Free subgroup criterion.** We recall the following well-known lemma.

Lemma 5.1 (Ping-Pong Lemma). *Let G be a group acting on a set X, and h_1, h_2 two elements of G. If X_1, X_2 are disjoint subsets of X and for all $n \neq 0$, $i \neq j$, $h_i^n(X_j) \subset X_i$, then the subgroup generated by h_1, h_2 is free of rank two.*

We will apply the Ping-Pong Lemma in the following setting. Let X be a CAT(0) space and g_1, g_2 two hyperbolic isometries of X (the reader is referred to the introduction for the definition of a hyperbolic isometry g and the notation $g(+\infty)$, $g(-\infty)$). Let $c_1 : \mathbb{R} \to X$ and $c_2 : \mathbb{R} \to X$ be axes of g_1 and g_2 respectively, and $\pi_1 : X \to c_1(\mathbb{R})$ and $\pi_2 : X \to c_2(\mathbb{R})$ orthogonal projections onto c_1 and c_2 respectively. For $T > 0$, let $X_1 = \pi_1^{-1}(c_1((-\infty, -T] \cup [T, \infty)))$ and $X_2 = \pi_2^{-1}(c_2((-\infty, -T] \cup [T, \infty)))$. If $X_1 \cap X_2 = \emptyset$, then the conditions in the Ping-Pong Lemma are satisfied with $h_1 = g_1^k$ and $h_2 = g_2^k$ for sufficiently large k. Consequently, g_1^k and g_2^k generate a free group of rank two.

A CAT(0) 2-complex is *piecewise Euclidean* if all its closed 2-cells are isometric to convex polygons in the Euclidean plane.

Theorem 5.2. *Let X be a piecewise Euclidean* CAT(0) 2*-complex with* Shape(X) *finite. Suppose g_1 and g_2 are two cellular hyperbolic isometries of X such that for any $\xi \in \{g_1(+\infty), g_1(-\infty)\}$ and any $\eta \in \{g_2(+\infty), g_2(-\infty)\}$, there is a geodesic in X with ξ and η as endpoints. Then the group generated by g_1 and g_2 contains a free group of rank two.*

Using the notation after the Ping-Pong Lemma, we will prove in Section 5.3 that under the assumption of Theorem 5.2 there exists $T > 0$ such that $X_1 \cap X_2 = \emptyset$.

The following result is an immediate consequence of Theorem 5.2 and Theorem 4.15.

Theorem 5.3. *Let X be a locally finite piecewise Euclidean* CAT(0) 2*-complex with* Shape(X) *finite, and g_1, g_2 two cellular hyperbolic isometries of X. If $g_1(+\infty)$, $g_1(-\infty)$, $g_2(+\infty)$, $g_2(-\infty)$ are not terminal points and $d_T(\xi, \eta) \geq \pi$ for any $\xi \in \{g_1(+\infty), g_1(-\infty)\}$ and any $\eta \in \{g_2(+\infty), g_2(-\infty)\}$, then the group generated by g_1 and g_2 contains a free group of rank two.*

Definition 5.4. A hyperbolic isometry g of a CAT(0) space X is called a *rank one isometry* if no axis of g bounds a flat half-plane in X.

If g is a rank one isometry of a locally compact CAT(0) space X, then $g(+\infty)$ and $g(-\infty)$ are isolated points in $\partial_T X$, that is, if $\xi = g(+\infty)$ or $g(-\infty)$ and $\eta \neq \xi$, then $d_T(\xi, \eta) = \infty$ (see [B]). In particular, there is a geodesic $\gamma : \mathbb{R} \to X$ with $\gamma(+\infty) = \xi$ and $\gamma(-\infty) = \eta$.

Corollary 5.5. *Let X be a piecewise Euclidean* CAT(0) 2*-complex and G a group acting on X properly and cocompactly by cellular isometries. Suppose $g_1, g_2 \in G$ are two hyperbolic isometries such that $d_T(\xi, \eta) \geq \pi$ for any $\xi \in \{g_1(+\infty), g_1(-\infty)\}$ and any $\eta \in \{g_2(+\infty), g_2(-\infty)\}$. Then the subgroup generated by g_1 and g_2 contains a free group of rank two.*

Proof. Note that in this case for any hyperbolic isometry $g \in G$ either g is a rank one isometry or $g(+\infty)$, $g(+\infty)$ are not terminal points. In any case there is a geodesic with ξ and η as endpoints, and the corollary follows from Theorem 5.2. □

5.2. **Support set of a triangle.** Let X be a piecewise Euclidean CAT(0) 2-complex with Shape(X) finite. In this section we show the closure of the support set of a triangle in X is a compact surface with boundary. Recall that support set is defined in Section 3.3.

Let $p_1, p_2, p_3 \in X$ be three distinct points. Suppose $c = p_1p_2 \cup p_2p_3 \cup p_3p_1$ is homeomorphic to a circle. For any $x \in X - c$, $\log_x(c) \subset \mathrm{Link}(X, x)$ has length at most 3π. Then the arguments in Section 3.3 show that $\mathrm{supp}(c)$ is a surface and is closed in $X - c$. It follows that $\overline{\mathrm{supp}(c)} \subset \mathrm{supp}(c) \cup c$. We view p_1, p_2, p_3 as vertices and p_1p_2, p_1p_3, p_2p_3 as part of the 1-skeleton. Then for any open 2-cell A, either $A \subset \mathrm{supp}(c)$ or $A \cap \mathrm{supp}(c) = \emptyset$; and for any open 1-cell B, if $B \cap \mathrm{supp}(c) \neq \emptyset$, then $B \subset \mathrm{supp}(c)$ and there are exactly two open 2-cells $A_1, A_2 \subset \mathrm{supp}(c)$ with $B \subset \overline{A_1}$, $B \subset \overline{A_2}$.

Since X is piecewise Euclidean with Shape(X) finite, there is some $r > 0$ such that $\overline{B}(p_1, r)$ is isometric to $\overline{B}(O, r) \subset C(\mathrm{Link}(X, p_1))$, where O is the cone point of the Euclidean cone $C(\mathrm{Link}(X, p_1))$. We may also assume $\overline{B}(p_1, r) \cap p_2p_3 = \emptyset$ by choosing a smaller r. Let $\xi_i = \log_{p_1}(p_i) \in \mathrm{Link}(X, p_1)$ ($i = 2, 3$). Then $d_{p_1}(\xi_2, \xi_3) < \pi$. Note that $C(\xi_2\xi_3)$ can be identified with a subset of $C(\mathrm{Link}(X, p_1))$.

It follows that $\overline{B}(O,r) \cap C(\xi_2\xi_3)$ can be identified with a subset D of $\overline{B}(p_1,r)$. D is homeomorphic to the closed unit disk in the plane and so its interior U is homeomorphic to the open unit disk.

Lemma 5.6. *Let c, r and U be as above. Then* $\mathrm{supp}(c) \cap B(p_1,r) = U$.

Proof. Since X is piecewise Euclidean, the proof for $U \subset \mathrm{supp}(c) \cap B(p_1,r)$ is similar to the proof of Lemma 3.11 and we omit it.

Let A be an open 2-cell such that $\overline{A}$ contains the initial segment of p_1p_2 and $A \cap U = \emptyset$. Then for any $x \in A$, $d_x(\log_x(p_1), \log_x(p_2)) < \pi$. When $x \in A$ and $d(x, p_1p_2)/d(x,p_1)$ is small enough, $\log_x(p_3)$ lies on the geodesic segment in $\mathrm{Link}(X,x)$ from $\log_x(p_1)$ to $\log_x(p_2)$. It follows that in this case $\log_x(c)$ is homotopically trivial and $x \notin \mathrm{supp}(c)$. By the observation in the second paragraph of this section, $A \cap \mathrm{supp}(c) = \emptyset$. Similarly, $A \cap \mathrm{supp}(c) = \emptyset$ if A is an open 2-cell such that $\overline{A}$ contains the initial segment of p_1p_3 and $A \cap U = \emptyset$.

Suppose $\mathrm{supp}(c) \cap B(p_1,r) - U$ is nonempty. Then the preceding paragraph and the observation in the second paragraph of this section imply that there are open 2-cells $A_1, \cdots, A_k \subset \mathrm{supp}(c)$ disjoint from U such that $C = \bigcup_{i=1}^k \overline{A_i}$ is homeomorphic to a closed disk and contains p_1 in the interior. Note that $\mathring{C} - \{p_1\} \subset \mathrm{supp}(c)$, where $\mathring{C}$ denotes the interior of the closed disk C (not the interior of C as a subset of X). C determines a simple loop $\omega \subset \mathrm{Link}(X,p_1)$. Since $\mathrm{Link}(X,p_1)$ is $CAT(1)$, there exists a point $\xi \in \omega$ with $d_{p_1}(\xi, \log_{p_1}(p_2)) \geq \pi$. Now let $x \in \mathring{C} - \{p_1\}$ with $\log_{p_1}(x) = \xi$. Then $xp_2 = xp_1 \cup p_1p_2$, $\log_x(p_1p_2)$ is a single point and $\log_x(c)$ has length $< 2\pi$. It follows that $\log_x(c)$ is homotopically trivial and $x \notin \mathrm{supp}(c)$, contradicting to $\mathring{C} - \{p_1\} \subset \mathrm{supp}(c)$. $\square$

Proposition 5.7. *Let X be a piecewise Euclidean* $\mathrm{CAT}(0)$ *2-complex with* $\mathrm{Shape}(X)$ *finite and $p_1, p_2, p_3 \in X$ three distinct points. Suppose $c = p_1p_2 \cup p_2p_3 \cup p_3p_1$ is homeomorphic to a circle. Then $\overline{\mathrm{supp}(c)} = \mathrm{supp}(c) \cup c$ and it is a compact surface with boundary c.*

Proof. Recall $\mathrm{supp}(c)$ is a topological surface and is closed in $X - c$, and $\overline{\mathrm{supp}(c)} \subset \mathrm{supp}(c) \cup c$. We shall show for each point $x \in c$ that there is some $r > 0$ such that $B(x,r) \cap \overline{\mathrm{supp}(c)}$ is homeomorphic to the closed upper half-plane with x corresponding to the origin. Lemma 5.6 implies that this is the case when $x \in \{p_1, p_2, p_3\}$.

Let x be a point in the interior of p_1p_2. Let $c' = xp_1 \cup p_1p_3 \cup p_3x$ and $c'' = xp_2 \cup p_2p_3 \cup p_3x$. We orient c' and c'' so that they have the opposite orientations on xp_3. Then c is homotopic to the loop $c' * c''$.

Let $\xi_i = \log_x(p_i) \in \mathrm{Link}(X,x)$ $(i = 1,2,3)$. Note that $d_x(\xi_3, \xi_1) < \pi$, otherwise $x \in p_1p_3$, a contradiction to the fact that c is a circle. Similarly $d_x(\xi_3,\xi_2) < \pi$. By Lemma 5.6 there is some $r > 0$ such that $B(x,r) \cap \mathrm{supp}(c') = U'$ and $B(x,r) \cap \mathrm{supp}(c'') = U''$, where U' and U'' are as defined preceding Lemma 5.6 corresponding to c' and c'' respectively. Recall under the identification of $\overline{B}(x,r)$ and $\overline{B}(O,r) \subset C(\mathrm{Link}(X,x))$, $\overline{U'}$ corresponds to $\overline{B}(O,r) \cap C(\xi_1\xi_3)$ and $\overline{U''}$ corresponds to $\overline{B}(O,r) \cap C(\xi_1\xi_2)$. Since c is homotopic to the path $c' * c''$, we conclude that $B(x,r) \cap \mathrm{supp}(c) - xp_3 \subset U' \cup U''$ and $U' - U'' \subset \mathrm{supp}(c)$, $U'' - U' \subset \mathrm{supp}(c)$.

Since $\xi_3\xi_1 \cap \xi_3\xi_2$ is a segment (possibly degenerate), $\overline{U'} \cap \overline{U''}$ is a common sector (possibly a segment) of $\overline{U'}$ and $\overline{U''}$, and $D = \overline{U' - U''} \cup \overline{U'' - U'}$ is homeomorphic to the closed unit disk. Let U be the interior of the disk D. Since $\mathrm{supp}(c)$ is a topological surface and is closed in $X - c$, $B(x,r) \cap \mathrm{supp}(c) = U$ and $\overline{B}(x,r) \cap \overline{\mathrm{supp}(c)} = D$. We have shown $\overline{\mathrm{supp}(c)}$ is a surface with boundary.

For any $y \in X$, let $C_y(c) = \bigcup_{p\in c} yp$. Note that for $z, y \in X - c$, $C_y(c)$ and $C_z(c)$ represent the same class in $H_2(X, X - \{z\})$ and $z \in \mathrm{supp}(c)$ if and only if $C_z(c)$ represents a nontrivial class in $H_2(X, X - \{z\})$. It follows that if $z \notin C_y(c)$, then $z \notin \mathrm{supp}(c)$. Consequently, $\mathrm{supp}(c) \subset C_y(c)$ and $\overline{\mathrm{supp}(c)}$ is compact. □

Since Shape(X) is finite, there is a number $\epsilon_0 > 0$ with the following property: if $x \in X$ and $\omega \subset \mathrm{Link}(X, x)$ is a simple loop with length at most $2\pi + \epsilon_0$, then it has length 2π.

Lemma 5.8. *Let $0 < \epsilon < \epsilon_0$. If $\angle_{p_1}(p_2, p_3) + \angle_{p_2}(p_3, p_1) + \angle_{p_3}(p_1, p_2) > \pi - \epsilon$, then $\mathrm{Link}(\mathrm{supp}(c), x)$ has length 2π for each $x \in \mathrm{supp}(c)$.*

Proof. Recall that the sum of the angles of a triangle is at most π and for any $p \in X$, $\angle_p$ satisfies the triangle inequality. Let $x \notin c$. By considering the three triangles $\Delta(xp_1p_2)$, $\Delta(xp_2p_3)$, $\Delta(xp_3p_1)$ and using the assumption $\angle_{p_1}(p_2, p_3)+\angle_{p_2}(p_3, p_1)+\angle_{p_3}(p_1, p_2) > \pi - \epsilon$ we see that $\angle_x(p_1, p_2) + \angle_x(p_2, p_3) + \angle_x(p_3, p_1) < 2\pi + \epsilon$. So the path $\log_x(c)$ is homotopic to a loop with length $< 2\pi + \epsilon < 2\pi + \epsilon_0$. By the remark before the lemma, $\log_x(c)$ is either homotopically trivial or homotopic to a simple loop with length 2π. When $x \in \mathrm{supp}(c)$, $\log_x(c)$ is homotopic to the simple loop $\mathrm{Link}(\mathrm{supp}(c), x)$ and so $\mathrm{Link}(\mathrm{supp}(c), x)$ has length 2π. □

5.3. **Proof of Theorem 5.2.** We recall a special case of a result of M. Bridson ([Br]):

Lemma 5.9. *Let X be a piecewise Euclidean* CAT(0) 2*-complex (not necessarily locally finite) with* Shape(X) *finite. Then there is a constant $\beta > 0$ such that every geodesic segment $\sigma \subset X$ is contained in a connected subcomplex X_σ that is the union of less than β*length$(\sigma) + 1$ *closed cells.*

Lemma 5.9 implies that for any fixed $a > 0$ the set $\{X_\sigma : \mathrm{length}(\sigma) \leq a\}$ contains only a finite number of X_σ up to isometry. Because of this we do not have to assume X is locally finite in Theorem 5.2. We will often implicitly use this observation in this section; for instance, we use it to establish the existence of $\delta, \epsilon_1, \epsilon_2$ defined below.

We first prove:

Proposition 5.10. *Let X be a piecewise Euclidean* CAT(0) 2*-complex with* Shape(X) *finite. Let g_1 be a cellular hyperbolic isometry of X with axis $c_1 : \mathbb{R} \to X$, and $\gamma : [0, +\infty) \to X$ a geodesic ray with $\gamma(+\infty) = c_1(+\infty)$. Then either $c_1 \cap \gamma$ is a ray or c_1 and γ bound a half flat strip at infinity, that is, there are $T_0 \geq 0$, $a > 0$, $u \in \mathbb{R}$ and an isometric embedding $h : [T_0, +\infty) \times [0, a] \to X$ such that $h(t, 0) = c_1(t + u)$, $h(t, a) = \gamma(t)$ for all $t \geq T_0$.*

We reparameterize c_1 so that $\gamma(0)$ and $c_1(0)$ lie on the same horosphere centered at $c_1(+\infty)$. The function $f(t) := d(\gamma(t), c_1)$ is convex and non-increasing. Note that if f is not strictly decreasing, then $f_{|[T_0,+\infty)}$ is a constant function for some $T_0 \geq 0$, and hence either $c_1 \cap \gamma$ is a ray or c_1 and γ bound a half flat strip at infinity. From now on we assume f is strictly decreasing, and we derive a contradiction from this. Since c_1 is an axis of a cellular hyperbolic isometry, there is some $t_0 \geq 0$ such that $\gamma([t_0, +\infty))$ contains no vertices of X.

For any $t_1 > t_0$ let $t_1' = \max\{t : \pi_1(\gamma(t)) = c_1(t_1)\}$, where $\pi_1 : X \to c_1$ is the orthogonal projection onto c_1. Let

$$c = \gamma(t_1')c_1(t_1) \cup c_1([t_1, +\infty)) \cup \gamma([t_1', +\infty)) \cup \{c_1(+\infty)\}.$$

Then c is a circle in $\overline{X}$. The proof of Proposition 5.7 shows that $\overline{\mathrm{supp}(c)}$ is a topological surface with boundary $c - \{c_1(+\infty)\}$. We see that $\angle_{\gamma(t)}(\pi_1(\gamma(t)), \gamma(\infty)) \to \pi/2$ as $t \to +\infty$. It follows that for any $\epsilon > 0$, we can choose a sufficiently large t_1 so that $\angle_{c_1(t_1)}(c_1(+\infty), \gamma(t_1')) + \angle_{\gamma(t_1')}(c_1(+\infty), c_1(t_1)) > \pi - \epsilon$. Then the proof of Lemma 5.8 shows that for any $x \in \mathrm{supp}(c)$, the link $\mathrm{Link}(\mathrm{supp}(c), x)$ is a circle with length 2π. For any $x \in \overline{\mathrm{supp}(c)}$, let $c_x = \mathrm{Link}(\overline{\mathrm{supp}(c)}, x)$. Note c_x is a circle when $x \in \mathrm{supp}(c)$ and is a segment when $x \in c - \{c_1(+\infty)\}$.

Lemma 5.11. *For sufficiently large t_1, $\overline{\mathrm{supp}(c)}$ is contractible, and c_x has length π if x lies on $c_1((t_1, +\infty))$ or the interior of $c_1(t_1)\gamma(t_1')$.*

Proof. Let $x \in c_1((t_1, +\infty))$ be a point such that c_x has length $> \pi$. Then x either is a vertex of X or lies in the interior of some edge e of X and $e \cap c_1 = \{x\}$. Since c_1 is an axis of the cellular hyperbolic isometry g_1, and X is a piecewise Euclidean 2-complex with $\mathrm{Shape}(X)$ finite, there are only a finite number of such x modulo g_1, and there is a positive number $\epsilon_1 > 0$ such that $\mathrm{length}(c_x) > \pi + \epsilon_1$ for all such x.

Choose a sufficiently large t_1 so that

$$\angle_{c_1(t_1)}(c_1(+\infty), \gamma(t_1')) + \angle_{\gamma(t_1')}(c_1(+\infty), c_1(t_1)) > \pi - \epsilon_1.$$

Suppose there is some $x \in c_1((t_1, +\infty))$ so that c_x has length $> \pi$. Then by the previous paragraph its length $> \pi + \epsilon_1$. There is a unique point ξ in the segment c_x such that the subsegment from $\log_x(c_1(t_1))$ to ξ has length π. We extend $c_1(t_1)x$ into $\mathrm{supp}(c)$ in the direction of ξ. The extension is unique since c_y has length 2π for any $y \in \mathrm{supp}(c)$. The extended geodesic hits $\gamma([t_1', +\infty))$ at a point y. Then $\angle_y(c_1(t_1), c_1(+\infty)) \geq \angle_{c_1(t_1)}(\gamma(t_1'), c_1(+\infty)) + \angle_{\gamma(t_1')}(c_1(t_1), c_1(+\infty)) > \pi - \epsilon_1$. By considering the ideal triangle $\Delta(xyc_1(+\infty))$ we have

$$\begin{aligned}&\angle_x(y, c_1(+\infty)) + \angle_y(x, c_1(+\infty))\\ &\qquad = (\mathrm{length}(c_x) - \pi) + \angle_y(c_1(t_1), c_1(+\infty)) > \epsilon_1 + (\pi - \epsilon_1) = \pi,\end{aligned}$$

a contradiction.

Note that for any $x \in c_1(t_1)\gamma(t_1')$, $x \neq c_1(t_1), \gamma(t_1')$, c_x has length π. Otherwise, we can extend $c_1(t_1)x$ into $\mathrm{supp}(c)$ until it hits $\gamma((t_1', +\infty))$, contradicting to the choice of t_1'.

Now consider $Y := \overline{\mathrm{supp}(c)}$ as a piecewise Euclidean 2-complex with the path metric d_Y (rather than the induced metric from X). Then (Y, d_Y) is complete and has nonpositive curvature. Suppose the surface (with boundary) Y is not contractible. Then there is a closed geodesic σ in (Y, d_Y). σ cannot be contained in $\mathrm{supp}(c)$ since any geodesic in $\mathrm{supp}(c)$ is also a geodesic in X and X is CAT(0). Since c_x has length π for any x in the interior of $c_1([t_1, +\infty))$ or of $c_1(t_1)\gamma(t_1')$, σ does not intersect $c_1([t_1, +\infty)) \cup c_1(t_1)\gamma(t_1')$. So $\sigma \cap \gamma((t_1', +\infty)) \neq \emptyset$. Let x_1x_2 be a component of $\sigma \cap \gamma((t_1', +\infty))$ with $x_1, x_2 \in \gamma((t_1', +\infty))$ and $x_1 \in \gamma(t_1')x_2$. We parameterize σ, $\sigma : [0, l] \to Y$ so that $\sigma(0) = x_1$ and $\sigma(a) = x_2$ for some a, $0 \leq a < l$. Then there is some $a' > 0$ with $\sigma((a, a + a')) \subset \mathrm{supp}(c)$. Since c_x has length 2π for any $x \in \mathrm{supp}(c)$, $\sigma((a, a + a'))$ admits a unique extension in $\mathrm{supp}(c)$

and eventually must hit $c_1([t_1, +\infty))$. This contradicts the above observation that σ does not intersect $c_1([t_1, +\infty)) \cup c_1(t_1)\gamma(t_1')$. $\square$

Lemma 5.11 implies that $\overline{\mathrm{supp}(c)}$ is homeomorphic to $\{(x, y) \in \mathbb{E}^2 : x \geq 0,\ 0 \leq y \leq 1\}$.

Lemma 5.12. *Let $x \in \gamma((t_1', +\infty))$ such that c_x has length greater than π. Then there is an edge e of X and a segment e' of e such that x lies in the interior of e', $e' \subset \overline{\mathrm{supp}(c)}$ and $e' \cap c = \{x\}$.*

Proof. Recall $\gamma([t_1', +\infty))$ does not contain any vertices. Since the length of c_x is $> \pi$, x lies in the interior of an edge e. $\mathrm{Link}(X, x)$ consists of two vertices and edges of length π connecting the two vertices. Since $x \in \gamma$, the d_x distance between the two endpoints of c_x is π. The assumption that the length of c_x is $> \pi$ now implies that the two vertices of $\mathrm{Link}(X, x)$ lie in the interior of c_x. The lemma now follows from the definition of a link. $\square$

Lemma 5.13. *There are only a finite number of points x on $\gamma([t_1', +\infty))$ such that* $\mathrm{length}(c_x) > \pi$.

Proof. For $t > t_1'$ let $\alpha_t = \angle_{\gamma(t)}(\pi_1(\gamma(t)), c_1(+\infty))$. Then $\alpha_t \leq \pi/2$ and $t \to \alpha_t$ is a non-decreasing function.

Let $x = \gamma(t)(t > t_1')$ be a point such that c_x has length $> \pi$. Then by Lemma 5.12 there is an edge e of X and a segment e' of e such that x lies in the interior of e', $e' \subset \overline{\mathrm{supp}(c)}$ and $e' \cap \gamma = \{x\}$. Since c_y has length 2π for any $y \in \mathrm{supp}(c)$, if $e \not\subset \overline{\mathrm{supp}(c)}$, then e has to exit $\overline{\mathrm{supp}(c)}$ at $c_1([t_1, +\infty)) \cup c_1(t_1)\gamma(t_1')$. Let α_e be the acute angle between $x\pi_1(x)$ and e. Lemma 5.11 implies for any $y \in e \cap \mathrm{supp}(c)$, the acute angle between e and $y\pi_1(y)$ is also α_e. On the other hand, we have $\alpha_{t'} < \alpha_e < \alpha_{t''}$ if $t' < t$ and $t'' > t$ are very close to t. Now the monotonicity of the function $t \to \alpha_t$ implies that if $x_1 \neq x_2 \in \gamma([t_1', +\infty))$ and c_{x_1}, c_{x_2} have length $> \pi$, then $\alpha_{e_1} \neq \alpha_{e_2}$, where e_1 and e_2 are the two edges of X corresponding to x_1 and x_2 respectively as given by Lemma 5.12. Since c_1 is an axis of a cellular hyperbolic isometry, there are only a finite number of such edges e, and hence there are only a finite number of points $x \in \gamma([t_1', +\infty))$ such that c_x has length $> \pi$. $\square$

By Lemma 5.13 $c_{\gamma(t)}$ has length π for sufficiently large t. Now Lemma 5.11 and the fact that c_y has length 2π for $y \in \mathrm{supp}(c)$ imply that $c_1([t, +\infty))$ and $\gamma([t, +\infty))$ bound a half flat strip, contradicting the assumption that $f(t) = d(\gamma(t), c_1)$ is strictly decreasing. The proof of Proposition 5.10 is complete.

Let g_1, g_2 be as in Theorem 5.2 and let c_1, c_2 be axes of g_1, g_2 respectively. Let $\pi_i : X \to c_i$ $(i = 1, 2)$ be the orthogonal projection onto c_i. For $T > 0$, let $X_1 = \pi_1^{-1}(c_1((-\infty, -T] \cup [T, \infty)))$ and $X_2 = \pi_2^{-1}(c_2((-\infty, -T] \cup [T, \infty)))$. We have observed in Section 5.1 that to prove Theorem 5.2 it suffices to show there exists $T > 0$ such that $X_1 \cap X_2 = \emptyset$. We shall prove there is some $T > 0$ such that $\pi_1^{-1}(c_1([T, \infty))) \cap \pi_2^{-1}(c_2([T, \infty))) = \emptyset$; the other three cases are similar.

Lemma 5.14. *There is some $T > 0$ such that $\pi_1^{-1}(c_1([T, \infty))) \cap \pi_2^{-1}(c_2([T, \infty))) = \emptyset$.*

Let $\gamma : \mathbb{R} \to X$ be a complete geodesic from $c_2(+\infty)$ to $c_1(+\infty)$, that is, $\gamma(-\infty) = c_2(+\infty)$ and $\gamma(+\infty) = c_1(+\infty)$. By Proposition 5.10 either $c_1([0, +\infty)) \cap \gamma([0, +\infty))$ is a ray or $c_1([0, +\infty))$ and $\gamma([0, +\infty))$ bound a half flat strip at infinity. Similarly, for $c_2([0, +\infty))$ and $\gamma((-\infty, 0])$. After reparameterization we may assume there is some $T_0 > 0$ such that either $c_1(t) = \gamma(t)$ for all $t \geq T_0$, or $c_1([T_0, +\infty))$ and $\gamma([T_0, +\infty))$ bound a half flat strip. Similarly, for $c_2([T_0, +\infty))$ and $\gamma((-\infty, -T_0])$. Let $a \geq 0$ ($b \geq 0$) be the distance between $c_1([T_0, +\infty))$ and $\gamma([T_0, +\infty))$ ($c_2([T_0, +\infty))$ and $\gamma((-\infty, -T_0])$). Since c_1 and c_2 are axes of cellular hyperbolic isometries and $\mathrm{Shape}(X)$ is finite, there is some $\delta > 0$ such that if some edge e intersects one of the three geodesics c_1, c_2, γ and makes a nonzero angle with it, then this angle is $> \delta$. Fix some $T > T_0$ such that in any Euclidean triangle where one side has length $\geq T - T_0$ and another side has length $l \leq \max\{a, b\}$, the angle facing the side with length l is $< \delta$.

We assume $\pi_1^{-1}(c_1([T, +\infty))) \cap \pi_2^{-1}(c_2([T, +\infty)))$ contains a point x and we derive a contradiction from this. Denote by $\pi_\gamma : X \to \gamma$ the orthogonal projection of X onto γ. Let $\gamma(t_1) = \pi_\gamma(x)$. We assume $t_1 \leq T_0$ and consider $c_1([T_0, +\infty))$ and $\gamma([T_0, +\infty))$. The case when $t_1 \geq T_0$ can be handled similarly by considering $c_2([T_0, +\infty))$ and $\gamma((-\infty, -T_0])$.

Let $c_1(t) = \pi_1(x)$. Then $t \geq T$ by the choice of x. Set $z = c_1(t)$. Note that $\pi_\gamma(z) = \gamma(t)$ and recall $\pi_\gamma(x) = \gamma(t_1)$ with $t_1 \leq T_0 < T \leq t$. Since π_γ is continuous, by replacing x with a point on xz we may assume $\pi_\gamma(x) = \gamma(T_0)$. Set $y = \gamma(T_0)$. We may assume x is the only point on xz that is mapped to y under π_γ. Note that the distance a between $c_1([T_0, +\infty))$ and $\gamma([T_0, +\infty))$ must be positive, otherwise the triangle $\Delta(xyz)$ has two angles $\geq \pi/2$, which is impossible in a CAT(0) space.

Consider the circle $c = xy \cup yz \cup zx$. By Proposition 5.7 $\overline{\mathrm{supp}(c)}$ is a compact surface with boundary c. For any $p \in \overline{\mathrm{supp}(c)}$, let $c_p = \mathrm{Link}(\overline{\mathrm{supp}(c)}, p)$. Note that $\angle_y(x, z), \angle_z(x, y) > \pi/2 - \delta$, hence $\angle_y(x, z) + \angle_z(x, y) + \angle_x(y, z) > \pi - 2\delta$. Lemma 5.8 implies that for any $p \in \mathrm{supp}(c)$, c_p has length 2π if we choose T sufficiently large and thus make δ sufficiently small.

We see that for any p in the interior of xy, c_p has length π. Otherwise, c_p has length $> \pi$ and we can extend the geodesic yp into $\mathrm{supp}(c)$ until it hits xz, contradicting our assumption that x is the only point on xz that is mapped to y under π_γ. Now an argument similar to the proof of Lemma 5.11 shows that $\overline{\mathrm{supp}(c)}$ is contractible and hence is homeomorphic to a Euclidean triangle.

Let p be the first point on zx from z to x such that c_p has length $> \pi$. We extend zp into $\mathrm{supp}(c)$ until it hits xy at a point x'. By replacing x with x', we may assume for each p in the interior of yx or of zx, c_p has length π. We shall show that $\overline{\mathrm{supp}(c)}$ is isometric to a Euclidean triangle.

Since $\mathrm{Shape}(X)$ is finite, there is some $\epsilon_2 > 0$ with the following properties:

(1) for any point x in the 1-skeleton of X and any two vertices $\xi_1, \xi_2 \in \mathrm{Link}(X, x)$, if $\pi - \epsilon_2 < d_x(\xi_1, \xi_2) < \pi + \epsilon_2$, then $d_x(\xi_1, \xi_2) = \pi$;

(2) for any x in the 1-skeleton of X and any simple loop $\omega \subset \mathrm{Link}(X, x)$, if ω has length $< 2\pi + \epsilon_2$, then it has length 2π.

Lemma 5.15. *Let $0 < \epsilon < \epsilon_2$. Let $x \in X$, $\xi, \eta \in \mathrm{Link}(X, x)$ with $d_x(\xi, \eta) = \pi$, and $\sigma \subset \mathrm{Link}(X, x)$ an injective path from ξ to η with $\pi < \mathrm{length}(\sigma) < \pi + \epsilon$. Then there is a vertex ξ_1 in the interior of σ satisfying $\mathrm{length}(\sigma) = \pi + 2\,\mathrm{length}(\xi\xi_1)$ or $\mathrm{length}(\sigma) = \pi + 2\,\mathrm{length}(\eta\xi_1)$.*

Proof. The assumption implies x lies in the 1-skeleton of X. Let $\sigma_1 \subset \mathrm{Link}(X, x)$ be a geodesic segment from ξ to η with length π. The choice of ϵ_2 implies $\sigma \cap \sigma_1$ has two components, one of them is a single point and the other one is a nondegenerate segment. The nondegenerate segment of $\sigma \cap \sigma_1$ starts at either ξ or η. Without loss of generality, we assume there is a point ξ_1 in the interior of σ_1 such that σ and σ_1 share $\xi\xi_1$ and diverge at ξ_1. Then ξ_1 must be a vertex. Let σ' (σ_1') be the subsegment of σ (σ_1) from ξ_1 to η. Then $\sigma' \cup \sigma_1'$ is a simple loop with length 2π. Now the lemma follows. □

Let S be the half flat strip bounded by $c_1([T_0, +\infty))$ and $\gamma([T_0, +\infty))$.

Lemma 5.16. $\overline{\mathrm{supp}(c)}$ *is isometric to a Euclidean triangle.*

Proof. Recall that $\angle_y(x,z) + \angle_z(x,y) + \angle_x(y,z) > \pi - 2\delta$. We may choose $\delta < \epsilon_2/2$ at the beginning. Since $\overline{\mathrm{supp}(c)}$ is homeomorphic to a Euclidean triangle, Gauss-Bonnett implies that c_p has length $< \pi + 2\delta < \pi + \epsilon_2$ for any $p \in yz$. It suffices to show c_p has length π for any p in the interior of yz.

Let $p \in yz$ be a point such that c_p has length $> \pi$. Denote $\xi = \log_p(y)$ and $\eta = \log_p(z)$. Note that $d_p(\xi, \eta) = \pi$ since yz lies in the flat strip S. Now Lemma 5.15 implies that there is a vertex $\xi_1 \in c_p$ with $\mathrm{length}(c_p) = \pi + 2\ \mathrm{length}(\xi\xi_1)$ or $\mathrm{length}(c_p) = \pi + 2\ \mathrm{length}(\eta\xi_1)$. Without loss of generality, we assume $\mathrm{length}(c_p) = \pi + 2\ \mathrm{length}(\xi\xi_1)$. Then there is an edge e of X such that $p \in e$ and e has the direction ξ_1 at p. Notice the proof of Lemma 5.15 shows $\xi_1 \in \mathrm{Link}(S, p)$. Since S is flat, by examining the links at the points in an edge, we see that $e \subset S$. Let $\xi_2 \in \mathrm{Link}(S, p)$ be the direction that is parallel to $c_1([T_0, +\infty))$ and satisfies $d_p(\xi_2, \xi) < \pi/2$. Set $\alpha = \angle_y(z, c_1(+\infty))$. Note that $d_p(\xi_2, \xi) = \alpha < \delta$. It follows that $d_p(\xi_1, \xi_2) < 2\delta < \epsilon_2$.

Since c_1 is an axis of a cellular hyperbolic isometry and $\mathrm{Shape}(X)$ is finite, we may choose $\epsilon_2 > 0$ small enough such that for any edge $e' \subset S$ of X, if e' makes an angle $< \epsilon_2$ with a segment in S parallel to $c_1([T_0, +\infty))$, then e' is parallel to $c_1([T_0, +\infty))$.

It follows that e is parallel to $c_1([T_0, +\infty))$ and $d_p(\xi_1, \xi) = \alpha$. We extend zp into $\mathrm{supp}(c)$ in the direction of ξ_1 until it hits xy at a point y_1. Then $\angle_{y_1}(x,p) \geq \angle_y(x,z) + \angle_p(y, y_1) = \angle_y(x,z) + \alpha = \angle_y(x,z) + \angle_y(z, c_1(+\infty)) \geq \angle_y(x, c_1(+\infty)) \geq \pi/2$. The last inequality follows from the fact that y is the orthogonal projection of x onto γ.

Note that $\mathrm{length}(c_p) = \pi + 2\alpha$. Let $\bar{\xi}_1 \in c_p$ with $d_p(\bar{\xi}_1, \eta) = \alpha$. Then $d_p(\xi_1, \bar{\xi}_1) = \pi$. We extend yp into $\mathrm{supp}(c)$ in the direction of $\bar{\xi}_1$ until it hits xz at a point z_1. Then similar to the last paragraph we have $\angle_{z_1}(x,p) \geq \pi/2$. Note that $y_1 z_1 = y_1 p \cup p z_1$. Now the triangle $x y_1 z_1$ has two angles $\geq \pi/2$, which is impossible in a CAT(0) space. □

Now S and $\overline{\mathrm{supp}(c)}$ are closed and convex in X. The intersection $S \cap \overline{\mathrm{supp}(c)}$ is a convex set containing yz. The choices of T and δ then imply that $S \cap \overline{\mathrm{supp}(c)}$ contains either the initial segment of $yc_1(+\infty)$ or the initial segment of $zc_1(T_0)$. Suppose $S \cap \overline{\mathrm{supp}(c)}$ contains the initial segment of $yc_1(+\infty)$. Then $\angle_y(x,z) = \angle_y(x, c_1(+\infty)) + \angle_y(c_1(+\infty), z) \geq \pi/2 + \alpha$. The inequality follows from the fact that $y = \pi_\gamma(x)$. By considering the triangle $\Delta(xyz)$ we see $\angle_z(x,y) < \pi/2 - \alpha$. It follows that $\angle_z(x, c_1(T_0)) \leq \angle_z(x,y) + \angle_z(y, c_1(T_0)) < (\pi/2 - \alpha) + \alpha = \pi/2$, or $\angle_z(x, c_1(T_0)) < \pi/2$, contradicting the fact that $z = \pi_1(x)$. Similarly, we obtain a contradiction when $S \cap \overline{\mathrm{supp}(c)}$ contains the initial segment of $zc_1(T_0)$. The contradiction shows that $\pi_1^{-1}(c_1([T, \infty))) \cap \pi_2^{-1}(c_2([T, \infty))) = \emptyset$.

The proofs of Lemma 5.14 and Theorem 5.2 are now complete.

6. Quasi-isometries between CAT(0)2-complexes

In this section we study how the Tits boundary behaves under quasi-isometries.

For any CAT(0) 2-complex X, set $\mathrm{Core}(\partial_T X) = \bigcup c$ where c varies over all the topological circles in $\partial_T X$. Let d_c be the induced path metric of d_T on $\mathrm{Core}(\partial_T X)$. Then $d_c \geq d_T$ always holds. Recall that by Corollary 2.8 $\overline{B}(\xi, r) \subset \partial_T X$ is an $\mathbb{R}$-tree for any $\xi \in \partial_T X$ and any r with $0 < r < \pi/2$. It follows that for $\xi, \eta \in \mathrm{Core}(\partial_T X)$, $d_c(\xi, \eta) < \infty$ if and only if ξ, η lie in the same path component of $\mathrm{Core}(\partial_T X)$ and in this case there is a minimal Tits geodesic contained in $\mathrm{Core}(\partial_T X)$ and connecting ξ and η. In particular, $d_c(\xi, \eta) = d_T(\xi, \eta)$ if $d_c(\xi, \eta) < \infty$.

Below is the main result of this section.

Theorem 6.1. *For $i = 1, 2$, let X_i be a locally finite* CAT(0) 2*-complex admitting a cocompact isometric action. If X_1 and X_2 are (L, A) quasi-isometric, then* $\mathrm{Core}(\partial_T X_1)$ *and* $\mathrm{Core}(\partial_T X_2)$ *are L^2-bi-Lipschitz with respect to the metric d_c.*

6.1. **Tits limit set.** Let X be a CAT(0) 2-complex. For any subset $I \subset \partial_T X$ and $x \in X$, the geodesic cone over I with vertex x is $C_x(I) = \bigcup_{\xi \in I} x\xi$. Let $\mathcal{C}(\partial_T X)$ be the set of topological circles in $\partial_T X$. By Corollary 2.9 each $S \in \mathcal{C}(\partial_T X)$ is a simple closed geodesic in $\partial_T X$. It is not hard to see that $C_x(S)$ is a quasi-flat. The main ingredient in the proof of Theorem 6.1 is a result of B. Kleiner concerning top dimensional quasi-flats in CAT(0) spaces. A special case of his result is as follows:

Theorem 6.2 (Kleiner [K2])**.** *Let X be a locally finite* CAT(0) 2*-complex admitting a cocompact isometric action, and $Q \subset X$ a quasi-flat. Then there is a unique $S \in \mathcal{C}(\partial_T X)$ such that for any $x \in X$,*

$$\lim_{r \to \infty} \frac{d_H(Q \cap B(x, r), C_x(S) \cap B(x, r))}{r} = 0.$$

For convenience, we introduce the following definition.

Definition 6.3. Let $B \subset X$ be a subset of a CAT(0) space X. A point $\xi \in \partial_T X$ is a *Tits limit point* of B if there is a sequence of points $b_i \in B$, $i = 1, 2, \cdots$, such that $d(b_i, x) \to \infty$ and $\lim_{i \to \infty} \frac{d(b_i, x\xi)}{d(b_i, x)} = 0$ where $x \in X$ is a fixed point. We also say $\{b_i\}_{i=1}^{\infty}$ *Tits converges* to ξ. The Tits limit set $L_T(B)$ of B is the set of Tits limit points of B.

Note that the above definition does not depend on the choice of x. It is clear that $L_T(B)$ is closed in the Tits metric, and if $d_H(B_1, B_2) < \infty$ for $B_1, B_2 \subset X$, then $L_T(B_1) = L_T(B_2)$. A Tits limit point is a limit point in the usual sense, but a limit point does not have to be a Tits limit point. Limit points are defined in terms of the cone topology, while Tits limit points are related to the Tits metric. Using this terminology, Kleiner's theorem, in particular, implies the following: $L_T(Q) \in \mathcal{C}(\partial_T X)$ for any quasi-flat $Q \subset X$. We also have $L_T(E) = C$ for the quasi-flat in Proposition 4.8.

Lemma 6.4. *Let $S \in \mathcal{C}(\partial_T X)$, $p \in X$ and $x_i \in X$ $(i = 1, 2, \cdots)$ be a sequence of points with $\lim_{i \to \infty} d(x_i, p) = \infty$ and $\lim_{i \to \infty} \frac{d(x_i, C_p(S))}{d(x_i, p)} = 0$. Then some subsequence of $\{x_i\}$ Tits converges to a point $\xi \in S$.*

Proof. Let $y_i \in C_p(S)$ with $d(x_i, y_i) = d(x_i, C_p(S))$. Then $y_i \in p\xi_i$ for some $\xi_i \in S$. Since S is a simple closed geodesic in $\partial_T X$, a subsequence $\{\xi_{i_j}\}$ of $\{\xi_i\}$ converges to some $\xi \in S$ in the Tits metric. Then $\lim_{j\to\infty} \frac{d(y_{i_j},\, p\xi)}{d(y_{i_j},\, p)} = 0$. Now triangle inequality implies $\lim_{j\to\infty} \frac{d(x_{i_j},\, p\xi)}{d(x_{i_j},\, p)} = 0$ and $\{x_{i_j}\}$ Tits converges to ξ. $\square$

Definition 6.5. Let $L \geq 1$, $A > 0$. A metric space M is (L, A) *quasi-connected at infinity* if there is some point $a \in M$ and some $r_0 > 0$ with the following property: for any two points $x, y \in M$ with $d(x, a), d(y, a) > r_0$ there is a sequence of points $x = x_0, x_1, \cdots, x_k = y$ so that:

(i) $d(x_i, x_{i+1}) \leq A$ for $i = 0, 1, \cdots, k-1$;

(ii) $d(x_i, a) \geq \min\{d(x, a), d(y, a)\}/L$ for all i.

A metric space M is quasi-connected at infinity if it is (L, A) quasi-connected at infinity for some $L \geq 1$, $A > 0$.

Note that if two metric spaces M_1 and M_2 are quasi-isometric, then M_1 is quasi-connected at infinity if and only if M_2 is.

Lemma 6.6. *Let X be a* CAT(0) 2*-complex and $Q \subset X$ a quasi-flat. If $B \subset Q$ is quasi-connected at infinity, then $L_T(B) \subset L_T(Q)$ is path-connected.*

Proof. Suppose B is (L, A) quasi-connected at infinity. Set $S = L_T(Q)$. $L_T(B) \subset S$ is clear since $B \subset Q$. Let $\xi \neq \eta \in L_T(B)$.

There are two sequences $\{a_i\} \subset B$, $\{b_i\} \subset B$ with $d(a_i, a) \to \infty$, $\frac{d(a_i, a\xi)}{d(a_i, a)} \to 0$ and $d(b_i, a) \to \infty$, $\frac{d(b_i, a\eta)}{d(b_i, a)} \to 0$ as $i \to \infty$, where a is the base point of B in the definition of quasi-connectedness. The quasi-connectedness of B implies that for each i there is a sequence $a_i = x_i^1, x_i^2, \cdots, x_i^{k_i} = b_i$, $x_i^j \in B$ with $d(x_i^j, a) \geq \min\{d(a_i, a), d(b_i, a)\}/L$ and $d(x_i^j, x_i^{j+1}) \leq A$. Let $\tilde{x}_i^j \in C_a(S)$ with $d(x_i^j, \tilde{x}_i^j) = d(x_i^j, C_a(S))$. We also let $\tilde{x}_i^0 \in a\xi$, $\tilde{x}_i^{k_i+1} \in a\eta$ with $d(a_i, \tilde{x}_i^0) = d(a_i, a\xi)$ and $d(b_i, \tilde{x}_i^{k_i+1}) = d(b_i, a\eta)$. Then $\tilde{x}_i^j \in a\xi_i^j$ for some $\xi_i^j \in S$. We may choose $\xi_i^0 = \xi$, $\xi_i^{k_i+1} = \eta$. The facts $d(x_i^j, x_i^{j+1}) \leq A$ and $d_H(Q \cap B(a, r), C_a(S) \cap B(a, r))/r \to 0$ as $r \to \infty$ imply $\frac{d(\tilde{x}_i^j, \tilde{x}_i^{j+1})}{d(\tilde{x}_i^j, a)} \to 0$ as $i \to \infty$. Since S is compact in $\partial_T X$ it is not hard to see that there are positive numbers ϵ_i with $\epsilon_i \to 0$ as $i \to \infty$ and $d_T(\xi_i^j, \xi_i^{j+1}) < \epsilon_i$.

ξ and η divide S into two closed intervals I_1 and I_2. After possibly passing to a subsequence and relabeling I_1 and I_2 we have $N_{\epsilon_i}(\{\xi_i^0, \cdots, \xi_i^{k_i+1}\}) \supset I_1$. Now it follows from the definition of Tits limit points that $I_1 \subset L_T(B)$ and ξ, η can be connected by a path in $L_T(B)$. $\square$

6.2. **Induced map between sets of branch points.** Let $f : X_1 \to X_2$ be a (L, A) quasi-isometry as in Theorem 6.1. Then f induces a bijection $g : \mathcal{C}(\partial_T X_1) \to \mathcal{C}(\partial_T X_2)$ as follows. For each circle $S \in \mathcal{C}(\partial_T X_1)$, define $g(S) = L_T(f(C_x(S)))$ for any $x \in X_1$. It is clear that $g(S)$ does not depend on x.

To prove Theorem 6.1, we first show that the bijection g preserves the intersection pattern of circles. This implies there is a bijection between the set of branch points of $\mathrm{Core}(\partial_T X_1)$ and that of $\mathrm{Core}(\partial_T X_2)$. We then show that this bijection between branch points is actually induced by the quasi-isometry f, so it is a local bi-Lipschitz map (see Definition 6.14). Finally, we extend this map to $\mathrm{Core}(\partial_T X_1)$.

First we look at the intersection of two circles in $\partial_T X$.

Lemma 6.7. *Let X be a* CAT(0) 2*-complex and $S_1, S_2 \in \mathcal{C}(\partial_T X)$. Then $S_1 \cap S_2 \subset S_i$ is a disjoint union of finitely many points and finitely many closed intervals.*

Proof. Write $S_1 = \bigcup_i U_i$ as a finite union of closed intervals each of which has length $< \pi$. Similarly, $S_2 = \bigcup_j V_j$. Then $S_1 \cap S_2 = \bigcup_{i,j}(U_i \cap V_j)$. Since $\partial_T X$ is a CAT(1) space, and U_i and V_j are closed intervals of length $< \pi$, $U_i \cap V_j$ is empty, a single point or a closed interval. Thus $S_1 \cap S_2 = \bigcup_{i,j}(U_i \cap V_j) \subset S_i$ is a union of finitely many points and finitely many closed intervals. □

For any topological space Y, $\pi_0(Y)$ denotes the set of path components of Y.

Lemma 6.8. *Let $S_1, S_2 \in \mathcal{C}(\partial_T X_1)$. Then there is a unique bijective map $k : \pi_0(S_1 \cap S_2) \to \pi_0(g(S_1) \cap g(S_2))$ such that $L_T(f(C_x(I))) = k(I)$ for all $I \in \pi_0(S_1 \cap S_2)$, where $x \in X_1$.*

Proof. Let $x \in X_1$ and $I \in \pi_0(S_1 \cap S_2)$. $C_x(I)$ is quasi-connected at infinity since by Lemma 6.7 I is a single point or a closed interval. f is a quasi-isometry implies $f(C_x(I))$ is also quasi-connected at infinity. Since $C_x(I) \subset C_x(S_i)$ $(i = 1, 2)$, we have $f(C_x(I)) \subset f(C_x(S_i))$ and $L_T(f(C_x(I))) \subset L_T(f(C_x(S_i))) = g(S_i)$. It follows from Lemma 6.6 that $L_T(f(C_x(I))) \subset g(S_1) \cap g(S_2)$ is path connected. Let $k(I)$ be the component of $g(S_1) \cap g(S_2)$ that contains $L_T(f(C_x(I)))$. By considering a quasi-inverse of f and using Lemma 6.4, we see that $L_T(f(C_x(I))) = k(I)$ and that k is bijective. k is clearly unique. □

Lemma 6.9. *Let $I \in \pi_0(S_1 \cap S_2)$. Then I is a single point component if and only if $k(I)$ is.*

Proof. Suppose the lemma is false. By possibly replacing f with a quasi-inverse, we may assume $g(I) = \{\eta\}$ $(\eta \in \partial_T X_2)$ for some nontrivial closed interval component I of $S_1 \cap S_2$. By Lemma 6.8 $L_T(f(C_x(I))) = \{\eta\}$, where $x \in X_1$ is a fixed base point. Then there are $\xi_1 \neq \xi_2 \in I$ with $L_T(f(x\xi_1)) = L_T(f(x\xi_2)) = \{\eta\}$. Let $x_i \in x\xi_1$ and $y_i \in x\xi_2$ $(i \geq 1)$ with $d(x_i, x) = d(y_i, x) = i$. Then both $\{f(x_i)\}$ and $\{f(y_i)\}$ Tits converge to η. Let $x'_i, y'_i \in f(x)\eta$ $(i \geq 1)$ with $d(f(x_i), x'_i) = d(f(x_i), f(x)\eta)$ and $d(f(y_i), y'_i) = d(f(y_i), f(x)\eta)$. Then for large enough i, there is some j_i such that $d(x'_i, y'_{j_i}) \leq L + A$ (f is a (L, A) quasi-isometry). Then $\lim_{i\to\infty} \frac{d(f(x_i), f(y_{j_i}))}{d(f(x_i), f(x))} = 0$. Since f is a quasi-isometry, we have $\lim_{i\to\infty} \frac{d(x_i, y_{j_i})}{d(x_i, x)} = 0$. This contradicts the assumption that $x_i \in x\xi_1$, $y_{j_i} \in x\xi_2$ and $\xi_1 \neq \xi_2$. □

Lemma 6.10. *Let I be a nontrivial interval component of $S_1 \cap S_2$, ξ_1, ξ_2 the two endpoints of I, and η_1, η_2 the two endpoints of $k(I)$. Then for any $x \in X_1$ either $L_T(f(x\xi_1)) = \{\eta_1\}$, $L_T(f(x\xi_2)) = \{\eta_2\}$, or $L_T(f(x\xi_1)) = \{\eta_2\}$, $L_T(f(x\xi_2)) = \{\eta_1\}$.*

Proof. We claim $L_T(f(x\xi_i))$ does not contain any interior point of $k(I)$. Suppose at least one of $L_T(f(x\xi_1))$, $L_T(f(x\xi_2))$, say $L_T(f(x\xi_1))$ does contain some interior point η of $k(I)$. Let $d_0 = d_T(\eta, g(S_1) - k(I))$. Choose $\xi \in S_1 - I$ with $d_T(\xi_1, \xi)$ sufficiently small. Since $\eta \in L_T(f(x\xi_1))$, there is a sequence of points $x_i \in x\xi_1$, such that $\{f(x_i)\}$ Tits converges to η. Let $y_i \in x\xi$ with $d(x, y_i) = d(x, x_i)$. By passing to a subsequence, we may assume that $\{f(y_i)\}$ Tits converges to a point $\eta' \in g(S_1)$. Since f is a quasi-isometry, we see that $d_T(\eta', \eta) \leq d_0/2$ if $d_T(\xi_1, \xi)$ is sufficiently small. The choice of d_0 then implies $\eta' \in k(I)$. It follows that $\xi \in L_T(f^{-1}(C_y(k(I))))$, where $y \in X_2$ and $f^{-1} : X_2 \to X_1$ is a quasi-inverse of f. This is a contradiction since $L_T(f^{-1}(C_y(k(I)))) = I$.

By Lemma 6.6 $L_T(f(x\xi_i))$ is path connected. So either $L_T(f(x\xi_i)) = \{\eta_1\}$ or $L_T(f(x\xi_i)) = \{\eta_2\}$. Similarly, $L_T(f^{-1}(y\eta_i)) = \{\xi_1\}$ or $\{\xi_2\}$. Now the lemma follows easily. □

Definition 6.11. Let $f : X_1 \to X_2$ be a quasi-isometry between two CAT(0) 2-complexes and $A_1 \subset \partial_T X_1$, $A_2 \subset \partial_T X_2$. A map $h : A_1 \to A_2$ is *Tits induced* by f if it satisfies the following property: for any sequence $\{x_i\} \subset X_1$ Tits converging to $\xi \in A_1$, $\{f(x_i)\}$ Tits converges to $h(\xi)$.

Define $B_1 \subset \mathrm{Core}(\partial_T X_1)$ as follows:

$$B_1 = \{\xi \in \partial_T X_1 : \xi \text{ is an endpoint of some } I \in \pi_0(S_1 \cap S_2), S_1, S_2 \in \mathcal{C}(\partial_T X_1)\}.$$

Here ξ is an endpoint of I when $I = \{\xi\}$. Similarly, we define $B_2 \subset \mathrm{Core}(\partial_T X_2)$. Lemma 6.9 and Lemma 6.10 imply the following proposition:

Proposition 6.12. *Let $f : X_1 \to X_2$ be a quasi-isometry between two* CAT(0) 2*-complexes, and B_1, B_2 as above. Then there is a bijective map $h : B_1 \to B_2$ with $h(S \cap B_1) = g(S) \cap B_2$ for any $S \in \mathcal{C}(\partial_T X_1)$, such that h is Tits induced by f and h^{-1} is Tits induced by a quasi-inverse of f.*

By using Lemma 6.6 and considering a quasi-inverse of f it is not hard to show the following lemma.

Lemma 6.13. *Let $S \in \mathcal{C}(\partial_T X_1)$. Then $h_{|S\cap B_1} : S \cap B_1 \to g(S) \cap B_2$ preserves the order of points, that is, if $a_1, a_2, a_3, a_4 \in S \cap B_1$ are four points in cyclic order on S, then $h(a_1), h(a_2), h(a_3), h(a_4)$ are in cyclic order on $g(S)$.*

6.3. **Bi-Lipschitz map between the cores.** Let $f : X_1 \to X_2$ be a (L, A) quasi-isometry as in Theorem 6.1. In this section we shall extend the map h in Proposition 6.12 to a bi-Lipschitz map from $\mathrm{Core}(\partial_T X_1)$ to $\mathrm{Core}(\partial_T X_2)$. We first show h is a local bi-Lipschitz map.

Definition 6.14. Let $L_0 \geq 1$. A map $h : Y_1 \to Y_2$ between two metric spaces is a *local L_0-bi-Lipschitz map* if there is some $\epsilon > 0$ such that $1/L_0 \cdot d(a, b) \leq d(h(a), h(b)) \leq L_0 \cdot d(a, b)$ for all $a, b \in Y_1$ with $d(a, b) \leq \epsilon$.

The following lemma is easy to prove.

Lemma 6.15. *Let Y_1, Y_2 be two metric spaces, $A_1 \subset Y_1$, and $h : A_1 \to Y_2$ a local L_0-bi-Lipschitz map for some $L_0 \geq 1$. If Y_2 is complete, then h uniquely extends to a local L_0-bi-Lipschitz map $\bar{h} : \bar{A}_1 \to Y_2$.*

Lemma 6.16. *For any $\lambda > 1$, the map $h : B_1 \to B_2$ is a local λL^2-bi-Lipschitz map with respect to the Tits metric d_T.*

Proof. Given $\lambda > 1$, fix some $\mu > 0$ so that if $\sin t \leq \mu$, $0 < t < \pi/2$, then $t/\lambda \leq \sin t \leq t$. Set $\epsilon = \mu/L^2$. Let $\xi, \eta \in B_1$ with $d_T(\xi, \eta) \leq \epsilon$. Fix a point $p \in X_1$ and let $x_i = \gamma_{p\xi}(i)$, $y_i = \gamma_{p\eta}(i)$, $i \geq 1$. Then $\lim_{i\to\infty} \frac{d(x_i, y_i)}{d(p, y_i)} = 2 \sin \frac{d_T(\xi,\eta)}{2}$. As h is Tits induced by f, $\{f(x_i)\}$ Tits converges to $h(\xi)$ and $\{f(y_i)\}$ Tits converges to $h(\eta)$. Therefore, $\widetilde{\angle}_{f(p)}(f(x_i), f(y_i)) \to d_T(h(\xi), h(\eta))$ as $i \to \infty$. On the other hand, by considering the comparison triangle of $\triangle f(p)f(x_i)f(y_i)$ we see that

$\sin\widetilde{\angle_{f(p)}}(f(x_i), f(y_i)) \le \frac{d(f(x_i),f(y_i))}{d(f(p),f(y_i))}$. It follows that

$$\begin{aligned}
&\sin[d_T(h(\xi), h(\eta))]\\
&= \sin[\lim_{i\to\infty} \widetilde{\angle_{f(p)}}(f(x_i), f(y_i))] = \lim_{i\to\infty} \sin[\widetilde{\angle_{f(p)}}(f(x_i), f(y_i))]\\
&\le \limsup_{i\to\infty} \tfrac{d(f(x_i),f(y_i))}{d(f(p),f(y_i))} \le \limsup_{i\to\infty} \tfrac{Ld(x_i,y_i)+A}{\frac{d(p,y_i)}{L}-A}\\
&= \limsup_{i\to\infty} L^2 \tfrac{d(x_i,y_i)}{d(p,y_i)} = \lim_{i\to\infty} L^2 \tfrac{d(x_i,y_i)}{d(p,y_i)} = 2L^2 \sin \tfrac{d_T(\xi,\eta)}{2}\\
&\le 2L^2 \tfrac{d_T(\xi,\eta)}{2} = L^2 d_T(\xi,\eta) \le L^2\epsilon = L^2 \tfrac{\mu}{L^2} = \mu.
\end{aligned}$$

The choice of μ now implies $d_T(h(\xi), h(\eta)) \le \lambda \sin[d_T(h(\xi), h(\eta))] \le \lambda L^2 d_T(\xi, \eta)$. Similarly, we have $d_T(\xi, \eta) \le \lambda L^2 d_T(h(\xi), h(\eta))$ by considering a quasi-inverse of f. $\square$

Let $\overline{B}_i \subset \partial_T X_i$ $(i = 1, 2)$ be the closure of B_i in $\partial_T X_i$ with respect to the Tits metric. The completeness of $\partial_T X_2$ and Lemma 6.15 imply that h uniquely extends to a local bi-Lipschitz map from $\overline{B}_1$ to $\partial_T X_2$, which we still denote by h. Since $\mathrm{Core}(\partial_T X_2)$ may not be closed in $\partial_T X_2$, $h(\overline{B}_1)$ may not lie in $\mathrm{Core}(\partial_T X_2)$.

Lemma 6.17. *$h : \overline{B}_1 \to \partial_T X_2$ is Tits induced by f and $h(\overline{B}_1 \cap \mathrm{Core}(\partial_T X_1)) \subset \mathrm{Core}(\partial_T X_2)$.*

Proof. It is easy to see that h is Tits induced by f. Let $\xi \in S \cap \overline{B}_1 - B_1$ with $S \in \mathcal{C}(\partial_T X_1)$. Fix some $p \in X_1$ and let $x_i = \gamma_{p\xi}(i)$. Since $\{x_i\}$ Tits converges to ξ, $\{f(x_i)\}$ Tits converges to $h(\xi)$. On the other hand, $\{f(x_i)\} \subset f(C_p(S))$ implies $L_T(\{f(x_i)\}) \subset L_T(f(C_p(S))) = g(S)$. It follows that $h(\xi) \in L_T(\{f(x_i)\}) \subset g(S)$. $\square$

Lemma 6.17 implies $h(\overline{B}_1 \cap \mathrm{Core}(\partial_T X_1)) \subset \overline{B}_2 \cap \mathrm{Core}(\partial_T X_2)$. By considering a quasi-inverse of f we see that $h : \overline{B}_1 \cap \mathrm{Core}(\partial_T X_1) \to \overline{B}_2 \cap \mathrm{Core}(\partial_T X_2)$ is a bijective map and its inverse is Tits induced by a quasi-inverse of f. The proof of Lemma 6.17 shows that $h(S \cap \overline{B}_1) = g(S) \cap \overline{B}_2$ for any $S \in \mathcal{C}(\partial_T X_1)$. As $h_{|S\cap\overline{B}_1} : S \cap \overline{B}_1 \to g(S) \cap \overline{B}_2$ is Tits induced by f, it preserves the order of points.

Lemma 6.18. *Let $S \in \mathcal{C}(\partial_T X_1)$ with $S\cap\overline{B}_1 \ne \emptyset$ and let I be a component of $S - \overline{B}_1$. Denote the two endpoints of I by a, b. Then $L_T(f(C_x(\bar{I}))) \subset g(S)$ $(x \in X_1)$ is one of the two closed segments in $g(S)$ with endpoints $h(a)$ and $h(b)$. Furthermore, the interior of $L_T(f(C_x(\bar{I})))$ is a component of $g(S) - \overline{B}_2$.*

Proof. Let I_1 and I_2 be the two components of $g(S) - \{h(a), h(b)\}$. Since $a, b \in \bar{I}$ we have $h(a), h(b) \in L_T(f(C_x(\bar{I})))$. By Lemma 6.6 $L_T(f(C_x(\bar{I})))$ is path connected. Hence $L_T(f(C_x(\bar{I}))) \supset \bar{I}_1$ or $L_T(f(C_x(\bar{I}))) \supset \bar{I}_2$. Without loss of generality, we assume that $L_T(f(C_x(\bar{I}))) \supset \bar{I}_1$. Since h^{-1} is Tits induced by a quasi-inverse of f and $\bar{I} \cap \overline{B}_1 = \{a, b\}$, we conclude that $L_T(f(C_x(\bar{I}))) \cap \overline{B}_2 = \{h(a), h(b)\}$. It follows that I_1 is a component of $g(S) - \overline{B}_2$. We will show that $L_T(f(C_x(\bar{I}))) = \bar{I}_1$.

Suppose $L_T(f(C_x(\bar{I}))) \ne \bar{I}_1$. Fix some $\eta \in L_T(f(C_x(\bar{I}))) - \bar{I}_1$. Then there is a sequence $\{x_i\} \subset C_x(\bar{I})$ Tits converging to some $\xi \in I$ so that $\{f(x_i)\}$ Tits converges to η. Let $[\eta, h(a)]$ be the closed subinterval of $\bar{I}_2$ with endpoints η and $h(a)$. Similarly, define $[\eta, h(b)] \subset \bar{I}_2$, $[\xi, a], [\xi, b] \subset \bar{I}$. Note that $L_T(f(C_x([\xi, a]))) \supset [\eta, h(a)]$ and $L_T(f(C_x([\xi, b]))) \supset [\eta, h(b)]$. Now $L_T(f(C_x(\bar{I}))) \supset \bar{I}_1$ implies $L_T(f(C_x(\bar{I}))) \supset g(S)$, a contradiction. $\square$

Lemma 6.19. *The map* $h : \overline{B}_1 \cap \mathrm{Core}(\partial_T X_1) \to \overline{B}_2 \cap \mathrm{Core}(\partial_T X_2)$ *extends to a bijective map* $h : \mathrm{Core}(\partial_T X_1) \to \mathrm{Core}(\partial_T X_2)$.

Proof. Let I be a component of $\mathrm{Core}(\partial_T X_1) - \overline{B}_1$. If $I = S$ is a circle, then $S \cap \overline{B}_1 = \emptyset$. Therefore, $g(S) \cap \overline{B}_2 = \emptyset$ and $g(S)$ is a component of $\mathrm{Core}(\partial_T X_2)$. We define $h_{|S}$ to be a similarity between S and $g(S)$.

Suppose $I = (a, b) \subset S$ is a proper subset of a circle S. Let us use the notation in the proof of Lemma 6.18. In this case we define $h_{|\bar{I}}$ to be the unique similarity from $\bar{I}$ to $\bar{I}_1$ extending $h_{|\{a,b\}}$. The map $h : \mathrm{Core}(\partial_T X_1) \to \mathrm{Core}(\partial_T X_2)$ is clearly bijective. $\square$

Lemma 6.20. *Let* I *be a connected component of* $\mathrm{Core}(\partial_T X_1) - \overline{B}_1$. *Then the following holds:* $\mathrm{length}(I)/L^2 \leq \mathrm{length}(h(I)) \leq L^2 \mathrm{length}(I)$.

Proof. By considering a quasi-inverse of f, it is sufficient to prove $\mathrm{length}(h(I)) \leq \lambda L^2 \mathrm{length}(I)$ for any $\lambda > 1$. For any fixed $\lambda > 1$, choose μ and ϵ as in the proof of Lemma 6.16. We consider two cases depending on whether I is a circle.

First suppose $I = (a, b) \subset S$ is a proper subset of a circle $S \subset \partial_T X_1$. Let $a = \xi_0$, $\xi_1, \cdots, \xi_n = b$ divide $\bar{I}$ into subintervals of equal length $< \epsilon$. Fix a point $x \in X_1$ and let $x_i^j = \gamma_{x\xi_i}(t_j)$ $(0 \leq i \leq n)$ with $t_j \to \infty$. After passing to a subsequence we may assume $\{f(x_i^j)\}_{j=1}^{\infty}$ Tits converges to some $\eta_i \in L_T(f(C_x(\bar{I}))) = h(\bar{I}) \subset g(S)$. We see that $\eta_0 = h(a)$, $\eta_n = h(b)$ since $h_{|\overline{B}_1 \cap \mathrm{Core}(\partial_T X_1)}$ is Tits induced by f. Since $d_T(\xi_i, \xi_{i+1}) \leq \epsilon$, the proof of Lemma 6.16 shows $d_T(\eta_i, \eta_{i+1}) \leq \lambda L^2 d_T(\xi_i, \xi_{i+1})$. Thus $\mathrm{length}(h(I)) \leq \sum_{i=0}^{n-1} d_T(\eta_i, \eta_{i+1}) \leq \sum_{i=0}^{n-1} \lambda L^2 d_T(\xi_i, \xi_{i+1}) = \lambda L^2 \mathrm{length}(I)$.

Now assume $I = S$ is a circle. As above let $\xi_0, \xi_1, \cdots, \xi_n = \xi_0$ divide S into subintervals of equal length $l < \epsilon$. The same proof yields $\eta_i \in g(S) (\eta_n = \eta_0)$ with $d_T(\eta_i, \eta_{i+1}) \leq \lambda L^2 l$. Fix some $y \in X_2$. For any $\eta \in g(S)$, let $y^j = \gamma_{y\eta}(t_j)$ with $t_j \to \infty$. We may assume $\{f^{-1}(y^j)\}$ Tits converges to some $\xi \in S$, where f^{-1} is a quasi-inverse of f. There is some i with $d_T(\xi, \xi_i) \leq l/2$. Then $d_T(\eta, \eta_i) \leq \lambda L^2 d_T(\xi, \xi_i) \leq \lambda L^2 \times l/2$. It follows that the n closed intervals centered at η_i $(i = 0, 1, \cdots, n-1)$ with length $l\lambda L^2$ cover the circle $g(S)$. Therefore, $\mathrm{length}(g(S)) \leq nl\lambda L^2 = \lambda L^2 \mathrm{length}(S)$. $\square$

Lemma 6.21. *The bijective map* $h : \mathrm{Core}(\partial_T X_1) \to \mathrm{Core}(\partial_T X_2)$ *is* L^2*-bi-Lipschitz with respect to the metric* d_c.

Proof. It suffices to show that h is λL^2-Lipschitz with respect to d_c for all $\lambda > 1$. Fix $\lambda > 1$. Choose μ and ϵ as in the proof of Lemma 6.16. We claim that for any $\xi_1, \xi_2 \in \mathrm{Core}(\partial_T X_1)$, if $d_c(\xi_1, \xi_2) < \epsilon$, then $d_T(h(\xi_1), h(\xi_2)) \leq \lambda L^2 d_c(\xi_1, \xi_2)$. Let us assume the claim for a while. For any $\xi, \eta \in \mathrm{Core}(\partial_T X_1)$ with $d_c(\xi, \eta) < \infty$, let $\alpha : [a, b] \to \mathrm{Core}(\partial_T X_1)$ be a minimal Tits geodesic contained in $\mathrm{Core}(\partial_T X_1)$ and connecting ξ and η. The claim implies the map $h \circ \alpha$ is rectifiable with respect to the Tits metric d_T and $\mathrm{length}(h \circ \alpha) \leq \lambda L^2 \mathrm{length}(\alpha) = \lambda L^2 d_c(\xi, \eta)$, hence $d_c(h(\xi), h(\eta)) \leq \lambda L^2 d_c(\xi, \eta)$.

Now we prove the claim.

Case 1. $\xi_1, \xi_2 \in \overline{B}_1$. In this case the claim follows from the definition of h and the proof of Lemma 6.16 since $d_T(\xi_1, \xi_2) = d_c(\xi_1, \xi_2)$ when $d_c(\xi_1, \xi_2) < \infty$.

Case 2. Exactly one of ξ_1, ξ_2 lies in $\overline{B}_1$, say $\xi_1 \in \overline{B}_1$ and $\xi_2 \notin \overline{B}_1$. $d_c(\xi_1, \xi_2) < \epsilon$ implies $\xi_1\xi_2 \subset \mathrm{Core}(\partial_T X_1)$. Since $\xi_1\xi_2 \cap \overline{B}_1$ is closed in $\xi_1\xi_2$ and $\xi_2 \notin \overline{B}_1$, there is a point $a \in \xi_1\xi_2$ with $d_T(a, \xi_2) > 0$ and $a\xi_2 \cap \overline{B}_1 = \{a\}$. Then there is a component I of $\mathrm{Core}(\partial_T X_1) - \overline{B}_1$ with $a\xi_2 \subset \overline{I}$. We also have $d_c(\xi_1, \xi_2) = d_c(\xi_1, a) + d_c(a, \xi_2)$. Since the map $h_{|I}$ is a similarity and $\mathrm{length}(h(I)) \leq L^2 \mathrm{length}(I)$, we have $d_T(h(a), h(\xi_2)) \leq L^2 d_T(a, \xi_2)$. On the other hand, $d_T(h(\xi_1), h(a)) \leq \lambda L^2 d_T(\xi_1, a)$ by Case 1. Therefore,

$$\begin{aligned} d_T(h(\xi_1), h(\xi_2)) &\leq d_T(h(\xi_1), h(a)) + d_T(h(a), h(\xi_2)) \\ &\leq \lambda L^2 d_T(\xi_1, a) + L^2 d_T(a, \xi_2) \leq \lambda L^2 (d_T(\xi_1, a) + d_T(a, \xi_2)) \\ &= \lambda L^2 (d_c(\xi_1, a) + d_c(a, \xi_2)) = \lambda L^2 d_c(\xi_1, \xi_2). \end{aligned}$$

Case 3. $\xi_1, \xi_2 \notin \overline{B}_1$. The proof in this case is similar to that in Case 2. □

Lemma 6.21 completes the proof of Theorem 6.1.

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