TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 4, Pages 1601–1626
S 0002-9947(04)03722-5
Article electronically published on November 29, 2004

STABLE BRANCHING RULES FOR CLASSICAL SYMMETRIC PAIRS

ROGER HOWE, ENG-CHYE TAN, AND JEB F. WILLENBRING

Abstract. We approach the problem of obtaining branching rules from the point of view of dual reductive pairs. Specifically, we obtain a stable branching rule for each of 10 classical families of symmetric pairs. In each case, the branching multiplicities are expressed in terms of Littlewood-Richardson coefficients. Some of the formulas are classical and include, for example, Littlewood's restriction rule as a special case.

1. Introduction

Given completely reducible representations, V and W of complex algebraic groups G and H respectively, together with an embedding $H \hookrightarrow G$, we let $[V, W] = \dim \operatorname{Hom}_H(W, V)$, where V is regarded as a representation of H by restriction. If W is irreducible, then $[V, W]$ is the multiplicity of W in V. This number may of course be infinite if V or W is infinite dimensional. A description of the numbers $[V, W]$ is referred in the mathematics and physics literature as a *branching rule*.

The context of this paper has its origins in the work of D. Littlewood. In [Li2], Littlewood describes two classical branching rules from a combinatorial perspective (see also [Li1]). Specifically, Littlewood's results are branching multiplicities for GL_n to O_n and GL_{2n} to Sp_{2n}. These pairs of groups are significant in that they are examples of symmetric pairs. A symmetric pair is a pair of groups (H, G) such that G is a reductive algebraic group and H is the fixed point set of a regular involution defined on G. It follows that H is a closed, reductive algebraic subgroup of G.

The goal of this paper is to put the formula into the context of the first-named author's theory of dual reductive pairs. The advantage of this point of view is that it relates branching from one symmetric pair to another and as a consequence Littlewood's formula may be generalized to all classical symmetric pairs.

Littlewood's result provides an expression for the branching multiplicities in terms of the classical Littlewood-Richardson coefficients (to be defined later) when the highest weight of the representation of the general linear group lies in a certain stable range.

The point of this paper is to show how when the problem of determining branching multiplicities is put in the context of dual pairs, a Littlewood-like formula results

Received by the editors November 11, 2003.
2000 *Mathematics Subject Classification.* Primary 22E46.

for any classical symmetric pair. To be precise, we consider 10 families of symmetric pairs which we group into subsets determined by the embedding of H in G (see Table I in §3).

1.1. **Parametrization of representations.** Let G be a classical reductive algebraic group over $\mathbb{C}$: $G = GL_n(\mathbb{C}) = GL_n$, the general linear group; or $G = O_n(\mathbb{C}) = O_n$, the orthogonal group; or $G = Sp_{2n}(\mathbb{C}) = Sp_{2n}$, the symplectic group. We shall explain our notations on irreducible representations of G using integer partitions. In each of these cases, we select a Borel subalgebra of the classical Lie algebra as is done in [GW]. Consequently, all highest weights are parameterized in the standard way (see [GW]).

A non-negative integer partition λ, with k parts, is an integer sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$. Sometimes we may refer to partitions as Young or Ferrers diagrams. We use the same notation for partitions as is done in [Ma]. For example, we write $\ell(\lambda)$ to denote the length (or depth) of a partition, $|\lambda|$ for the size of a partition (i.e., $|\lambda| = \sum_i \lambda_i$). Also, λ' denotes the transpose (or conjugate) of λ (i.e., $(\lambda')_i = |\{\lambda_j : \lambda_j \geq i\}|$). A partition where all parts are even is called an *even* partition, and we shall denote an even partition $2\delta_1 \geq 2\delta_2 \geq \ldots \geq 2\delta_k$ simply by 2δ.

$\mathbf{GL_n}$ **representations:** Given non-negative integers p and q such that $n \geq p + q$ and non-negative integer partitions λ^+ and λ^- with p and q parts respectively, let $F_{(n)}^{(\lambda^+,\lambda^-)}$ denote the irreducible rational representation of GL_n with highest weight given by the n-tuple

$$(\lambda^+, \lambda^-) \quad = \quad \underbrace{\left(\lambda_1^+, \lambda_2^+, \cdots, \lambda_p^+, 0, \cdots, 0, -\lambda_q^-, \cdots, -\lambda_1^-\right)}_{n}.$$

If $\lambda^- = (0)$, then we will write $F_{(n)}^{\lambda^+}$ for $F_{(n)}^{(\lambda^+,\lambda^-)}$. Note that if $\lambda^+ = (0)$, then $\left(F_{(n)}^{\lambda^-}\right)^*$ is equivalent to $F_{(n)}^{(\lambda^+,\lambda^-)}$.

$\mathbf{O_n}$ **representations:** The complex (or real) orthogonal group has two connected components. Because the group is disconnected we cannot index irreducible representation by highest weights. There is however an analog of Schur-Weyl duality for the case of O_n in which each irreducible rational representation is indexed uniquely by a non-negative integer partition ν such that $(\nu')_1 + (\nu')_2 \leq n$. That is, the sum of the first two columns of the Young diagram of ν is at most n. (See [GW], Chapter 10, for details.) Let $E_{(n)}^{\nu}$ denote the irreducible representation of O_n indexed by ν in this way.

The irreducible rational representations of SO_n may be indexed by their highest weight, since the group is a connected reductive linear algebraic group. In [GW], Section 5.2.2, the irreducible representations of O_n are determined in terms of their restrictions to SO_n (which is a normal subgroup having index 2). See [GW], Sections 10.2.4 and 10.2.5, for the correspondence between this parametrization and the above parametrization by partitions.

$\mathbf{Sp_{2n}}$ **representations:** For a non-negative integer partition ν with p parts where $p \leq n$, let $V_{(2n)}^{\nu}$ denote the irreducible rational representation of Sp_{2n}, where the highest weight indexed by the partition ν is given by the n tuple

$$\underbrace{(\nu_1, \nu_2, \cdots, \nu_p, 0, \cdots, 0)}_{n} .$$

1.2. **Littlewood-Richardson coefficients.** Fix a positive integer n_0. Let λ, μ and ν denote non-negative integer partitions with at most n_0 parts. For any $n \geq n_0$ we have

$$\left[F^{\mu}_{(n)} \otimes F^{\nu}_{(n)}, F^{\lambda}_{(n)}\right] = \left[F^{\mu}_{(n_0)} \otimes F^{\nu}_{(n_0)}, F^{\lambda}_{(n_0)}\right] .$$

And so we define

$$c^{\lambda}_{\mu\,\nu} := \left[F^{\mu}_{(n)} \otimes F^{\nu}_{(n)}, F^{\lambda}_{(n)}\right]$$

for some (indeed any) $n \geq n_0$.

The numbers $c^{\lambda}_{\mu\,\nu}$ are known as the Littlewood-Richardson coefficients and are extensively studied in the algebraic combinatorics literature. Many treatments are defined from wildly different points of view. See [BKW], [CGR], [Fu], [GW], [JK], [Kn1], [Ma], [Sa], [St3] and [Su] for examples.

1.3. **Stability and the Littlewood restriction rules.** We now state the Littlewood restriction rules.

Theorem 1.1 ($O_n \subseteq GL_n$)**.** *Given λ such that $\ell(\lambda) \leq \frac{n}{2}$ and μ such that $(\mu')_1 + (\mu')_2 \leq n$, then*

$$[F^{\lambda}_{(n)}, E^{\mu}_{(n)}] = \sum_{2\delta} c^{\lambda}_{\mu\ 2\delta}, \tag{1.1}$$

where the sum is over all non-negative even integer partitions 2δ.

Theorem 1.2 ($Sp_{2n} \subseteq GL_{2n}$)**.** *Given λ such that $\ell(\lambda) \leq n$ and μ such that $\ell(\mu) \leq n$, then*

$$[F^{\lambda}_{(2n)}, V^{\mu}_{(2n)}] = \sum_{2\delta} c^{\lambda}_{\mu\ (2\delta)'}, \tag{1.2}$$

where the sum is over all non-negative integer partitions with even columns $(2\delta)'$.

Notice that the hypotheses of the above two theorems do not include an arbitrary parameter for the representation of the general linear group. The parameters which fall within this range are said to be in the *stable range*. These hypotheses are necessary but for certain μ it is possible to weaken them considerably; see [EW1] and [EW2].

One purpose of this paper is to make the first steps toward a uniform stable range valid for all symmetric pairs. In the situation presented here we approach the stable range on a case-by-case basis. Within the stable range, one can express the branching multiplicity in terms of the Littlewood-Richardson coefficients. These kinds of branching rules will later be combined with the rich combinatorics literature on the Littlewood-Richardson coefficients to provide more algebraic structure to branching rules.

2. Statement of the results

We now state our main theorem. It addresses 10 families of symmetric pairs, which we state in 10 parts. The parts are grouped into 4 subsets named: Diagonal, Direct sum, Polarization and Bilinear form. These names describe the embedding of H into G (see Table I of §3).

Main Theorem.

2.1. **Diagonal.**

2.1.1. $\mathbf{GL_n} \subset \mathbf{GL_n} \times \mathbf{GL_n}$. Given non-negative integers, p, q, r and s with $n \geq p+q+r+s$. Let λ^+, μ^+, ν^+, λ^-, μ^-, ν^- be non-negative integer partitions. If $\ell(\lambda^+) \leq p+r$, $\ell(\lambda^-) \leq q+s$, $\ell(\mu^+) \leq p$, $\ell(\mu^-) \leq q$, $\ell(\nu^+) \leq r$ and $\ell(\nu^-) \leq s$, then

$$\left[F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(n)}^{(\nu^+,\nu^-)}, F_{(n)}^{(\lambda^+,\lambda^-)}\right] = \sum c_{\alpha_2\,\alpha_1}^{\lambda^+} c_{\alpha_1\,\gamma_1}^{\mu^+} c_{\gamma_1\,\beta_2}^{\nu^-} c_{\beta_2\,\beta_1}^{\lambda^-} c_{\beta_1\,\gamma_2}^{\mu^-} c_{\gamma_2\,\alpha_2}^{\nu^+},$$

where the sum is over non-negative integer partitions α_1, α_2, β_1, β_2, γ_1 and γ_2.

2.1.2. $\mathbf{O_n} \subset \mathbf{O_n} \times \mathbf{O_n}$. Given non-negative integer partitions λ, μ and ν such that $\ell(\lambda) \leq \lfloor n/2 \rfloor$ and $\ell(\mu) + \ell(\nu) \leq \lfloor n/2 \rfloor$, then

$$\left[E_{(n)}^{\mu} \otimes E_{(n)}^{\nu}, E_{(n)}^{\lambda}\right] = \sum c_{\alpha\,\beta}^{\lambda} c_{\alpha\,\gamma}^{\mu} c_{\beta\,\gamma}^{\nu},$$

where the sum is over all non-negative integer partitions α, β, γ.

2.1.3. $\mathbf{Sp_{2n}} \subset \mathbf{Sp_{2n}} \times \mathbf{Sp_{2n}}$. Given non-negative integer partitions λ, μ and ν such that $\ell(\lambda) \leq n$ and $\ell(\mu) + \ell(\nu) \leq n$, then

$$\left[V_{(2n)}^{\mu} \otimes V_{(2n)}^{\nu}, V_{(2n)}^{\lambda}\right] = \sum c_{\alpha\,\beta}^{\lambda} c_{\alpha\,\gamma}^{\mu} c_{\beta\,\gamma}^{\nu},$$

where the sum is over all non-negative integer partitions α, β, γ.

2.2. **Direct sum.**

2.2.1. $\mathbf{GL_n} \times \mathbf{GL_m} \subset \mathbf{GL_{n+m}}$. Let p and q be non-negative integers such that $p+q \leq \min(n,m)$. Let λ^+, μ^+, ν^+ and λ^-, μ^-, ν^- be non-negative integer partitions. If $\ell(\lambda^+)$, $\ell(\mu^+)$, $\ell(\nu^+) \leq p$ and $\ell(\lambda^-), \ell(\mu^-), \ell(\nu^-) \leq q$, then

$$\left[F_{(n+m)}^{(\lambda^+,\lambda^-)}, F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(m)}^{(\nu^+,\nu^-)}\right] = \sum c_{\mu^+\,\nu^+}^{\gamma^+} c_{\mu^-\,\nu^-}^{\gamma^-} c_{\gamma^+\,\delta}^{\lambda^+} c_{\gamma^-\,\delta}^{\lambda^-},$$

where the sum is over all non-negative integer partitions γ^+, γ^-, δ.

2.2.2. $\mathbf{O_n} \times \mathbf{O_m} \subset \mathbf{O_{n+m}}$. Let λ, μ and ν be non-negative integer partitions such that $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \frac{1}{2}\min(n,m)$. Then

$$\left[E_{(n+m)}^{\lambda}, E_{(n)}^{\mu} \otimes E_{(m)}^{\nu}\right] = \sum c_{\mu\,\nu}^{\gamma} c_{\gamma\,2\delta}^{\lambda},$$

where the sum is over all non-negative integer partitions δ and γ.

2.2.3. $\mathbf{Sp_{2n}} \times \mathbf{Sp_{2m}} \subset \mathbf{Sp_{2(n+m)}}$. Let λ, μ and ν be non-negative integer partitions such that $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \min(n,m)$. Then

$$\left[V_{(2(n+m))}^{\lambda}, V_{(2n)}^{\mu} \otimes V_{(2m)}^{\nu}\right] = \sum c_{\mu\,\nu}^{\gamma} c_{\gamma\,(2\delta)'}^{\lambda},$$

where the sum is over all non-negative integer partitions δ and γ.

2.3. **Polarization.**

2.3.1. $\mathbf{GL_n} \subset \mathbf{O_{2n}}$. Let μ^+, μ^- and λ be non-negative integer partitions with at most $\lfloor n/2 \rfloor$ parts. Then

$$\left[E^{\lambda}_{(2n)}, F^{(\mu^+,\mu^-)}_{(n)}\right] = \sum c^{\gamma}_{\mu^+\,\mu^-} c^{\lambda}_{\gamma\,(2\delta)'},$$

where the sum is over all non-negative integer partitions δ and γ.

2.3.2. $\mathbf{GL_n} \subset \mathbf{Sp_{2n}}$. Let μ^+, μ^- and λ be non-negative integer partitions with at most $\lfloor n/2 \rfloor$ parts. Then

$$\left[V^{\lambda}_{(2n)}, F^{(\mu^+,\mu^-)}_{(n)}\right] = \sum c^{\gamma}_{\mu^+\,\mu^-} c^{\lambda}_{\gamma\,2\delta},$$

where the sum is over all non-negative integer partitions δ and γ.

2.4. **Bilinear form.**

2.4.1. $\mathbf{O_n} \subset \mathbf{GL_n}$. Let λ^+, λ^- and μ denote non-negative integer partitions with at most $\lfloor n/2 \rfloor$ parts. Then

$$\left[F^{(\lambda^+,\lambda^-)}_{(n)}, E^{\mu}_{(n)}\right] = \sum c^{\mu}_{\alpha\,\beta} c^{\lambda^+}_{\alpha\,2\gamma} c^{\lambda^-}_{\beta\,2\delta},$$

where the sum is over all non-negative integer partitions α, β, γ and δ.

2.4.2. $\mathbf{Sp_{2n}} \subset \mathbf{GL_{2n}}$. Let λ^+, λ^- and μ denote non-negative integer partitions with at most n parts. Then

$$\left[F^{(\lambda^+,\lambda^-)}_{(2n)}, V^{\mu}_{(2n)}\right] = \sum c^{\mu}_{\alpha\,\beta} c^{\lambda^+}_{\alpha\,(2\gamma)'} c^{\lambda^-}_{\beta\,(2\delta)'},$$

where the sum is over all non-negative integer partitions α, β, γ and δ.

2.5. **Remarks.** Although a thorough survey is beyond our present goals, we wish to record here many previous works on branching rules which in many cases overlap with ours. We are grateful to the referee who has given us an extensive list of references with comments on related works by experts. We shall briefly summarize related works as follows:

(a) **Diagonal:** The first rule 2.1.1 appears as (4.6) with (4.15) in King's paper [Ki2]. The branching rules 2.1.2 and 2.1.3 for orthogonal and symplectic groups goes back to Newell [Ne] and Littlewood [Li3]. A more rigorous account of the Diagonal rules also appears in [Ki4], along with a treatment of rational representations of GL_n. See Theorem 4.5 and Theorem 4.1 of [Ki4] and the references therein. Our methods are also cast in this same generality. Further, 2.1.2 and 2.1.3 are beautifully presented in Sundaram's survey [Su] with references to the proofs in [BKW].

(b) **Direct sum:** Rule 2.2.1 appears as (5.8) with (4.16) in one of the earlier works of King [Ki1], which derives from a conjecture in the Ph.D. thesis by Abramsky [Ab]. These branching rules are also addressed in [Ko] and [KT]. Specifically, 2.2.1 can be found in Proposition 2.6 of [Ko], and 2.2.2 and 2.2.3 can be found in Theorem 2.5 and Corollary 2.6 of [KT]. An account of the Direct Sum rules also appears in [Ki4] (see (2.1.6) and the references therein).

(c) **Polarization:** The polarization branching rules 2.3.1 and 2.3.2 are stated as (4.21) and (4.22), respectively, in [Ki3], and also as Theorem A1 of [KT].

(d) **Bilinear form:** The Littlewood restriction rule is a special case of formulas, 2.4.1 and 2.4.2 (see [Li1] and [Li2]). These two formulas can be viewed as a generalization of Littlewood's restriction rule. Besides the Diagonal branching formulas, [Su] also presents a thorough treatment of the classical Littlewood restriction rules. However, in the most general form, rules 2.4.1 and 2.4.2 appear as (5.7) with (4.19), and (5.8) with (4.23) respectively in [Ki2].

Most of the results have been sufficiently well known by experts. For a well-presented survey of the representation theory of the classical groups from a combinatorial point of view we refer the reader to [Su]. Also, the late Wybourne and his students have even incorporated these results in the software package SCHUR downloadable at http://smc.vnet.net/Schur.html. This package implements all the modification rules given in [Ki2] and [BKW] that allow the stable branching rules to be generalized so as to cover all possible non-stable cases as well.

From our point of view, it is striking that the theory of dual pairs leads to proofs of all 10 of these formulas in such a unified manner. We feel that this unifying theme should be brought out in the literature more systematically than it has been.

In [Su], Theorem 5.4, it is shown how the Littlewood Richardson rule for branching from GL_{2n} to Sp_{2n} may be modified to obtain a version of the Littlewood restriction rule which is valid outside the stable range. Removing the stability condition for Littlewood's restriction rules is a delicate problem, which was also addressed in [EW1] and [EW2]. Classically, Newell [Ne] presents modification rules to the Littlewood restriction rules to solve the branching problem outside of the stable range (see [Su] and [Ki2]). For some recent remarks on the literature of branching rules we refer the reader to [Ki3], [Kn2], [Kn3], [Kn4] and [Pr]. The discussions in [Kn2] are relevant to our approach. Some of the results in [Kn2] are important special cases of the results in [GK].

While we only require decompositions of tensor products of infinite-dimensional holomorphic discrete series as in [Re], we also wish to note the numerous works in more generality which can be found in [RWB], [KW], [TTW], [OZ] and [KTW], among many others. It is interesting to note that the papers [RWB], [KW], [TTW] and [KTW] have exploited the duality correspondence (see §3.1) to relate the multiplicities of tensor products of infinite-dimensional representations to multiplicities of tensor products of finite-dimensional representations in the same spirit of [Ho1].

3. Dual pairs and reciprocity

The formulation of classical invariant theory in terms of dual pairs [Ho2] allows one to realize branching properties for classical symmetric pairs by considering concrete realizations of representations on algebras of polynomials on vector spaces.

3.1. **Dual pairs and duality correspondence.** Let $W \simeq \mathbb{R}^{2m}$ be a $2m$-dimensional real vector space with symplectic form $\langle\cdot,\cdot\rangle$. Let $Sp(W) = Sp_{2m}(\mathbb{R})$ denote the isometry group of the form $\langle\cdot,\cdot\rangle$. A pair of subgroups (G, G') of $Sp_{2m}(\mathbb{R})$ is called a *reductive dual pair* (in $Sp_{2m}(\mathbb{R})$) if

(a) G is the centralizer of G' in $Sp_{2m}(\mathbb{R})$ and vice versa, and
(b) both G and G' act reductively on W.

The fundamental group of $Sp_{2m}(\mathbb{R})$ is the fundamental group of U_m, its maximal compact subgroup, and is isomorphic to $\mathbb{Z}$. Let $\widetilde{Sp}_{2m}(\mathbb{R})$ denote a choice of a double cover of $Sp_{2m}(\mathbb{R})$. We will refer to this as the *metaplectic group*. Also let $\widetilde{U}_m$ denote the pull-back of the covering map on U_m. Shale-Weil constructed a distinguished representation ω of $\widetilde{Sp}_{2m}(\mathbb{R})$, which we shall refer to as the *oscillator representation*. This is a unitary representation and one realization is on the space of holomorphic functions on $\mathbb{C}^m$, commonly referred to as the Fock space. In this realization, the $\widetilde{U}_m$-finite functions appear as polynomials on $\mathbb{C}^m$ which we denote as $\mathcal{P}(\mathbb{C}^m)$. A vector $v \in \mathcal{P}(\mathbb{C}^m)$ is $\widetilde{U}_m$-finite if the span of $\widetilde{U}_m \cdot v$ in $\mathcal{P}(\mathbb{C}^m)$ is finite dimensional.

Choose $z_1, z_2, \ldots, z_m$ as a system of coordinates on $\mathbb{C}^m$. The Lie algebra action of $\mathfrak{sp}_{2m}$ (the complexified Lie algebra of $Sp_{2m}(\mathbb{R})$) on $\mathcal{P}(\mathbb{C}^m)$ can be described by the following operators:

$$\omega(\mathfrak{sp}_{2m}) = \mathfrak{sp}_{2m}^{(1,1)} \oplus \mathfrak{sp}_{2m}^{(2,0)} \oplus \mathfrak{sp}_{2m}^{(0,2)}, \tag{3.1}$$

where

$$\begin{aligned} \mathfrak{sp}_{2m}^{(1,1)} &= \text{ Span } \left\{ \frac{1}{2}\left(z_i \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_i} z_j \right) \right\}, \\ \mathfrak{sp}_{2m}^{(2,0)} &= \text{ Span } \{ z_i z_j \}, \\ \mathfrak{sp}_{2m}^{(0,2)} &= \text{ Span } \left\{ \frac{\partial^2}{\partial z_i \partial z_j} \right\}. \end{aligned} \tag{3.2}$$

The decomposition (3.1) is an instance of the complexified Cartan decomposition

$$\mathfrak{sp}_{2m} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-, \tag{3.3}$$

where $\mathfrak{sp}_{2m}^{(1,1)} \simeq \omega(\mathfrak{k})$, $\mathfrak{sp}_{2m}^{(2,0)} \simeq \omega(\mathfrak{p}^+)$ and $\mathfrak{sp}_{2m}^{(0,2)} \simeq \omega(\mathfrak{p}^-)$. If $\mathcal{P}(\mathbb{C}^m) = \sum_{s \geq 0} \mathcal{P}^s(\mathbb{C}^m)$ is the natural grading on $\mathcal{P}(\mathbb{C}^m)$, it is immediate that $\mathfrak{sp}_{2m}^{(i,j)}$ brings $\mathcal{P}^s(\mathbb{C}^m)$ to $\mathcal{P}^{s+i-j}(\mathbb{C}^m)$.

Let us restrict our dual pairs to the following:

$$(O_n(\mathbb{R}), Sp_{2k}(\mathbb{R})), \qquad (U_n, U_{p,q}), \qquad (Sp(n), O^*_{2k}). \tag{3.4}$$

Observe that the first member is compact, and these pairs are usually loosely referred to as *compact pairs*.

To avoid technicalities involving covering groups, instead of the real groups (G_0, G'_0), we shall discuss in the context of pairs $(G, \mathfrak{g}')$, where G is a complexification of G_0 and $\mathfrak{g}'$ is a complexification of the Lie algebra of G'_0. The use of the phrase "up to a central character" in the statements (a) to (c) below basically suppresses the technicalities involving covering groups. Each of these pairs can be conveniently realized as follows:

(a) $(O_n(\mathbb{R}), Sp_{2k}(\mathbb{R})) \subset Sp_{2nk}(\mathbb{R})$:
Let $\mathbb{C}^n \otimes \mathbb{C}^k$ be the space of n by k complex matrices. The complexified pair $(O_n, \mathfrak{sp}_{2k})$ acts on $\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^k)$ which are the $\widetilde{U}_{nk}$-finite functions. The group O_n acts by left multiplication on $\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^k)$ and can be identified with the holomorphic extension of the $O_n(\mathbb{R})$ action on the Fock space. The action of the subalgebra $\mathfrak{gl}_k$ of $\mathfrak{sp}_{2k}$ is (up to a central character) the derived action coming from the natural right action of multiplication by GL_k.

(b) $(U_n, U_{p,q}) \subset Sp_{2n(p+q)}(\mathbb{R})$:
For this pair, we may identify the $\widetilde{U}_{n(p+q)}$-finite functions with the polynomial ring $\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^p \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^q)$. The complexified pair is $(GL_n, \mathfrak{gl}_{p,q})$. There is a natural action of GL_n and $GL_p \times GL_q$ on this polynomial ring as follows:

$$(g, h_1, h_2) \cdot F(X, Y) = F(g^{-1}Xh_1, g^tYh_2),$$

where $X \in \mathbb{C}^n \otimes \mathbb{C}^p$, $Y \in (\mathbb{C}^n)^* \otimes \mathbb{C}^q$, $g \in GL_n$, $h_1 \in GL_p$ and $h_2 \in GL_q$. Obviously both left and right actions commute. Here $\mathfrak{gl}_{p,q} \simeq \mathfrak{gl}_{p+q}$, but we choose to differentiate the two because of the role of the subalgebra $\mathfrak{gl}_p \oplus \mathfrak{gl}_q$, which acts by (up to a central character) the derived action of $GL_p \times GL_q$ on the polynomial ring.

(c) $(Sp(n), O_{2k}^*) \subset Sp_{4nk}(\mathbb{R})$:
In this case, $\mathcal{P}(\mathbb{C}^{2n} \otimes \mathbb{C}^k)$ are the $\widetilde{U}_{2nk}$-finite functions, with natural left and right actions by Sp_{2n} and GL_k, respectively. The complexified pair is $(Sp_{2n}, \mathfrak{o}_{2k})$, where the subalgebra $\mathfrak{gl}_k$ of $\mathfrak{o}_{2k}$ acts by (up to a central character) the derived right action of GL_k.

With the realizations of these compact pairs $(G, \mathfrak{g}') \subset Sp_{2m}(\mathbb{R})$, let us look at the representations that appear. Form

$$\mathfrak{g}'^{(i,j)} = \mathfrak{sp}_{2m}^{(i,j)} \cap \omega(\mathfrak{g}')$$

to get

$$\omega(\mathfrak{g}') = \mathfrak{g}'^{(1,1)} \oplus \mathfrak{g}'^{(2,0)} \oplus \mathfrak{g}'^{(0,2)}. \tag{3.5}$$

Observe that G_0' is Hermitian symmetric in all three cases, and the decomposition above is an instance of the complexified Cartan decomposition

$$\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'^{+} \oplus \mathfrak{p}'^{-}, \tag{3.6}$$

where $\mathfrak{g}'^{(1,1)} \simeq \omega(\mathfrak{k}')$, $\mathfrak{g}'^{(2,0)} \simeq \omega(\mathfrak{p}'^{+})$ and $\mathfrak{g}'^{(0,2)} \simeq \omega(\mathfrak{p}'^{-})$. In particular, $\mathfrak{k}'$ has a one-dimensional center and $\mathfrak{p}'^{\pm}$ are the $\pm i$ eigenspaces of this center. Each $\mathfrak{p}'^{\pm}$ is an abelian Lie algebra. Note, in particular, that

$$[\mathfrak{g}'^{(1,1)}, \mathfrak{g}'^{(2,0)}] \subset \mathfrak{g}'^{(2,0)} \quad \text{and} \quad [\mathfrak{g}'^{(1,1)}, \mathfrak{g}'^{(0,2)}] \subset \mathfrak{g}'^{(0,2)}. \tag{3.7}$$

A representation (ρ, V_ρ) of $\mathfrak{g}'$ is *holomorphic* if there is a non-zero vector $v_0 \in V_\rho$ killed by $\rho(\mathfrak{p}'^{-})$. The following are the key properties of holomorphic representations:

(a) There is a non-trivial subspace

$$(V_\rho)_0 = \ker \rho(\mathfrak{p}'^{-}) = \{v \in V_\rho \mid \rho(Y) \cdot v = 0 \text{ for all } Y \in \mathfrak{p}'^{-}\}$$

which is $\mathfrak{k}'$ irreducible. This is known as the *lowest* $\mathfrak{k}'$*-type* of ρ.

(b) V_ρ is generated by $(V_\rho)_0$, more precisely,

$$V_\rho = \mathcal{U}(\mathfrak{p}'^{+}) \cdot V_0 = \mathcal{S}(\mathfrak{p}'^{+}) \cdot V_0.$$

The second equality results because $\mathfrak{p}'^{+}$ is abelian.

Now one of the key features in the formalism of dual pairs is the branching decomposition of the oscillator representation. The branching property for compact pairs alluded to is (see [Ho2] and the references therein)

$$\mathcal{P}(\mathbb{C}^m)\mid_{G\times\mathfrak{g}'} = \bigoplus_{\tau\in S\subset\widehat{G}} \tau\otimes V_{\tau'}, \tag{3.8}$$

where S is a subset of the set of irreducible representations of G, denoted by $\widehat{G}$. The representations $V_{\tau'}$ (written to emphasize the correspondence $\tau\leftrightarrow\tau'$ and the dependence on $\tau\in\widehat{G}$) are irreducible holomorphic representations of $\mathfrak{g}'$. They are known to be derived modules of irreducible unitary representations of some appropriate cover of G'_0 ([Ho3]). The key feature of this branching is the uniqueness of the correspondence, i.e., a representation of G appearing uniquely determines the representation of the $\mathfrak{g}'$ module that appears and vice-versa. We refer to this as the *duality correspondence.*

This duality is subjugated to another correspondence in the space of harmonics.

Theorem 3.1 ([Ho2], [KV])**.** *Let $\mathcal{H}=\ker{\mathfrak{g}'}^{(0,2)}$ be the space of harmonics. Then $\mathcal{H}$ is a $G\times K'$ module, and it admits a multiplicity-free $G\times K'$ (hence $G\times\mathfrak{k}'$) decomposition*

$$\mathcal{H} = \bigoplus_{\tau\in S\subset\widehat{G}} \tau\otimes\ker\rho_{\tau'}({\mathfrak{g}'}^{(0,2)}). \tag{3.9}$$

We also have the separation of variables theorem providing the following $G\times\mathfrak{g}'$ decomposition:

$$\begin{aligned}\mathcal{P}(\mathbb{C}^m) &= \mathcal{H}\cdot\mathcal{S}({\mathfrak{g}'}^{(2,0)}) = \left\{\bigoplus_{\tau\in S\subset\widehat{G'}} \tau\otimes\ker\rho_{\tau'}({\mathfrak{g}'}^{(0,2)})\right\}\cdot\mathcal{S}({\mathfrak{g}'}^{(2,0)})\\ &= \bigoplus_{\tau\in S\subset\widehat{G}} \tau\otimes\left\{\mathcal{S}({\mathfrak{g}'}^{(2,0)})\cdot\ker\rho_{\tau'}({\mathfrak{g}'}^{(0,2)})\right\} = \bigoplus_{\tau\in S\subset\widehat{G}} \tau\otimes V_{\tau'}.\end{aligned} \tag{3.10}$$

The structure of $V_{\tau'}$ is even nicer in certain category of pairs, which we will refer to as the *stable range*. The stable range refers to the following:

(a) $(O_n(\mathbb{R}), Sp_{2k}(\mathbb{R}))$ for $n\geq 2k$;
(b) $(U_n, U_{p,q})$ for $n\geq p+q$;
(c) $(Sp(n), O^*_{2k})$ for $n\geq k$.

In the stable range, the holomorphic representations of $\mathfrak{g}'$ that occur have $\mathfrak{k}'$-structure which are nicer ([HC], [Sc1], [Sc2]); namely,

$$V_{\tau'} = \mathcal{S}({\mathfrak{g}'}^{(2,0)})\otimes\ker\tau'({\mathfrak{g}'}^{(0,2)}).$$

They are known as *holomorphic discrete series* or *limits of holomorphic discrete series* (in some limiting cases of the parameters determining τ') of the appropriate covering group of G'_0. It is these representations that will feature prominently in this paper.

Let us conclude by describing the duality correspondence for the compact dual pairs in the stable range. Parts of the following well-known result can be found in several places; see [EHW], [HC], [Ho2], [Ho3], [Sc1], [Sc2] for example.

Theorem 3.2. (a) $(O_n(\mathbb{R}), Sp_{2k}(\mathbb{R}))$*: The duality correspondence for* $O_n \times \mathfrak{sp}_{2k}$ *is*

$$\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^k) = \bigoplus_{\lambda} E^{\lambda}_{(n)} \otimes \widetilde{E}^{\lambda}_{(2k)}, \tag{3.11}$$

where λ *runs through the set of all non-negative integer partitions such that* $l(\lambda) \leq k$ *and* $(\lambda')_1 + (\lambda')_2 \leq n$*. The space* $\widetilde{E}^{\lambda}_{(2k)}$ *is an irreducible holomorphic representation of* $\mathfrak{sp}_{2k}$ *of lowest* $\mathfrak{gl}_k$*-type* $F^{\lambda}_{(k)}$*. In the stable range* $n \geq 2k$*,*

$$\widetilde{E}^{\lambda}_{(2k)} \simeq \mathcal{S}(\mathfrak{sp}^{(2,0)}_{2k}) \otimes F^{\lambda}_{(k)} \simeq \mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^{\lambda}_{(k)}.$$

(b) $(U_n, U_{p,q})$*: The duality correspondence for* $GL_n \times \mathfrak{gl}_{p,q}$ *is*

$$\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^p \otimes (\mathbb{C}^n)^* \otimes \mathbb{C}^q) = \bigoplus_{\lambda^+,\lambda^-} F^{(\lambda^+,\lambda^-)}_{(n)} \otimes \widetilde{F}^{(\lambda^+,\lambda^-)}_{(p,q)}, \tag{3.12}$$

where the sum is over all non-negative integer partitions λ^+ *and* λ^- *such that* $l(\lambda^+) \leq p$*,* $l(\lambda^-) \leq q$ *and* $l(\lambda^+) + l(\lambda^-) \leq n$*. The space* $\widetilde{F}^{(\lambda^+,\lambda^-)}_{(p,q)}$ *is an irreducible holomorphic representation of* $\mathfrak{gl}_{p,q}$ *with lowest* $\mathfrak{gl}_p \oplus \mathfrak{gl}_q$*-type* $F^{\lambda^+}_{(p)} \otimes F^{\lambda^-}_{(q)}$*. In the stable range* $n \geq p+q$*,*

$$\widetilde{F}^{(\lambda^+,\lambda^-)}_{(p,q)} \simeq \mathcal{S}(\mathfrak{gl}^{(2,0)}_{p,q}) \otimes F^{\lambda^+}_{(p)} \otimes F^{\lambda^-}_{(q)} \simeq \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda^+}_{(p)} \otimes F^{\lambda^-}_{(q)}.$$

The degenerate case when $q = 0$ *is particularly interesting. This is the* $GL_n \times GL_p$ *duality:*

$$\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^p) = \bigoplus_{\lambda} F^{\lambda}_{(n)} \otimes F^{\lambda}_{(p)}, \tag{3.13}$$

where the sum is over all integer partitions λ *such that* $l(\lambda) \leq \min(n,p)$*.*

(c) $(Sp(n), O^*_{2k})$*: The duality correspondence for* $Sp_{2n} \times \mathfrak{so}_{2k}$ *is*

$$\mathcal{P}(\mathbb{C}^{2n} \otimes \mathbb{C}^k) = \bigoplus_{\lambda} V^{\lambda}_{(2n)} \otimes \widetilde{V}^{\lambda}_{(2k)}, \tag{3.14}$$

where λ *runs through the set of all non-negative integer partitions such that* $l(\lambda) \leq \min(n,k)$*. The space* $\widetilde{V}^{\lambda}_{(2k)}$ *is an irreducible holomorphic representation of* $\mathfrak{so}_{2k}$ *with lowest* $\mathfrak{gl}_k$*-type* $F^{\lambda}_{(k)}$*. In the stable range* $n \geq k$*,*

$$\widetilde{V}^{\lambda}_{(2k)} \simeq \mathcal{S}(\mathfrak{so}^{(2,0)}_{2k}) \otimes F^{\lambda}_{(k)} \simeq \mathcal{S}(\wedge^2\mathbb{C}^k) \otimes F^{\lambda}_{(k)}.$$

3.2. **Symmetric pairs and reciprocity pairs.** In the context of dual pairs, we would like to understand the branching of irreducible representations from G to H, for symmetric pairs (H, G). Table I lists the symmetric pairs which we will cover in this paper.

If G is a classical group over $\mathbb{C}$, then G can be embedded as one member of a dual pair in the symplectic group as described in [Ho2]. The resulting pairs of groups are (GL_n, GL_m) or (O_n, Sp_{2m}), each inside Sp_{2nm}, and are called *irreducible* dual pairs. In general, a dual pair of reductive groups in Sp_{2r} is a product of such pairs.

Proposition 3.1. *Let* G *be a classical group, or a product of two copies of a classical group. Let* G *belong to a dual pair* (G, G') *in a symplectic group* Sp_{2m}*. Let* $H \subset G$ *be a symmetric subgroup, and let* H' *be the centralizer of* H *in* Sp_{2m}*. Then* (H, H') *is also a dual pair in* Sp_{2m}*, and* G' *is a symmetric subgroup inside* H'*.*

TABLE I. Classical Symmetric Pairs

Description	**H**	**G**
Diagonal	GL_n	$GL_n \times GL_n$
Diagonal	O_n	$O_n \times O_n$
Diagonal	Sp_{2n}	$Sp_{2n} \times Sp_{2n}$
Direct Sum	$GL_n \times GL_m$	GL_{n+m}
Direct Sum	$O_n \times O_m$	O_{n+m}
Direct Sum	$Sp_{2n} \times Sp_{2m}$	$Sp_{2(n+m)}$
Polarization	GL_n	O_{2n}
Polarization	GL_n	Sp_{2n}
Bilinear Form	O_n	GL_n
Bilinear Form	Sp_{2n}	GL_{2n}

TABLE II. Reciprocity Pairs

Symmetric Pair $(\mathbf{H}, \mathbf{G})$	$(\mathbf{H}, \mathfrak{h}')$	$(\mathbf{G}, \mathfrak{g}')$
$(GL_n, GL_n \times GL_n)$	$(GL_n, \mathfrak{gl}_{m+\ell})$	$(GL_n \times GL_n, \mathfrak{gl}_m \times \mathfrak{gl}_\ell)$
$(O_n, O_n \times O_n)$	$(O_n, \mathfrak{sp}_{2(m+\ell)})$	$(O_n \times O_n, \mathfrak{sp}_{2m} \times \mathfrak{sp}_{2\ell})$
$(Sp_{2n}, Sp_{2n} \times Sp_{2n})$	$(Sp_{2n}, \mathfrak{so}_{2(m+\ell)})$	$(Sp_{2n} \times Sp_{2n}, \mathfrak{so}_{2m} \oplus \mathfrak{so}_{2\ell})$
$(GL_n \times GL_m, GL_{n+m})$	$(GL_n \times GL_m, \mathfrak{gl}_\ell \times \mathfrak{gl}_\ell)$	$(GL_{n+m}, \mathfrak{gl}_\ell)$
$(O_n \times O_m, O_{n+m})$	$(O_n \times O_m, \mathfrak{sp}_{2\ell} \oplus \mathfrak{sp}_{2\ell})$	$(O_{n+m}, \mathfrak{sp}_{2\ell})$
$(Sp_{2n} \times Sp_{2m}, Sp_{2(n+m)})$	$(Sp_{2n} \times Sp_{2m}, \mathfrak{so}_{2\ell} \oplus \mathfrak{so}_{2\ell})$	$(Sp_{2(n+m)}, \mathfrak{so}_{2\ell})$
(GL_n, O_{2n})	$(GL_n, \mathfrak{gl}_{2m})$	$(O_{2n}, \mathfrak{sp}_{2m})$
(GL_n, Sp_{2n})	$(GL_n, \mathfrak{gl}_{2m})$	$(Sp_{2n}, \mathfrak{so}_{2m})$
$(O_n, \mathfrak{gl}_n)$	$(O_n, \mathfrak{sp}_{2m})$	$(GL_n, \mathfrak{gl}_m)$
(Sp_{2n}, GL_{2n})	$(Sp_{2n}, \mathfrak{so}_{2m})$	$(GL_{2n}, \mathfrak{gl}_m)$

Proof. This can be shown by fairly easy case-by-case checking. These are shown in Table II. We call these pair of pairs *reciprocity pairs*. These are special cases of see-saw pairs [Ku]. $\square$

Consider the dual pairs (G, G') and (H, H') in Sp_{2m}. We illustrate them in the see-saw manner as follows:

$$\begin{array}{ccc} G & - & G' \\ \cup & & \cap \\ H & - & H' \end{array}$$

Recall the duality correspondence for $G \times \mathfrak{g}'$ and $H \times \mathfrak{h}'$ on the space $\mathcal{P}(\mathbb{C}^m)$:

$$\mathcal{P}(\mathbb{C}^m)\,|_{G\times\mathfrak{g}'} = \bigoplus_{\sigma\in S\subset\widehat{G}} \sigma \otimes V_{\sigma'} = \bigoplus_{\sigma\in S\subset\widehat{G}} \mathcal{P}(\mathbb{C}^m)_{\sigma\otimes\sigma'},$$

$$\mathcal{P}(\mathbb{C}^m)\,|_{H\times\mathfrak{h}'} = \bigoplus_{\tau\in T\subset\widehat{H}} \tau \otimes W_{\tau'} = \bigoplus_{\tau\in T\subset\widehat{H}} \mathcal{P}(\mathbb{C}^m)_{\tau\otimes\tau'},$$

where we have written $\mathcal{P}(\mathbb{C}^m)_{\sigma\otimes\sigma'}$ and $\mathcal{P}(\mathbb{C}^m)_{\tau\otimes\tau'}$ as the $\sigma\otimes\sigma'$-isotypic component and $\tau\otimes\tau'$-isotypic component in $\mathcal{P}(\mathbb{C}^m)$, respectively. Given σ and τ', we can seek the $\sigma\otimes\tau'$-isotypic component in $\mathcal{P}(\mathbb{C}^m)$ in two ways as follows:

$$\mathcal{P}(\mathbb{C}^m)_{\sigma\otimes\tau'} \simeq (\sigma\,|_H)_\tau \otimes V_{\sigma'} \simeq \tau \otimes (W_{\tau'}\,|_{\mathfrak{h}'})_{\sigma'}. \tag{3.15}$$

In other words, we have the equality of multiplicities (as pointed out in [Ho1])

$$[\sigma, \tau] = [W_{\tau'}, V_{\sigma'}], \tag{3.16}$$

that is, the multiplicity of τ in $\sigma\mid_H$ is equal to the multiplicity of $V_{\sigma'}$ in $W_{\tau'}\mid_{\mathfrak{h}'}$. This is good enough for our purposes in this paper. However, this equality of multiplicities is just a feature of some deeper phenomenon—an isomorphism of certain branching algebras which captures the respective branching properties.

3.3. **Branching algebras.** One approach to branching problems exploits the fact that the representations have a natural product structure, embodied by the algebra of regular functions on the flag manifold of the group. For a reductive complex algebraic G, let N_G be a maximal unipotent subgroup of G. The group N_G is determined up to conjugacy in G. Let A_G denote a maximal torus which normalizes N_G, so that $B_G = A_G \cdot N_G$ is a Borel subgroup of G. Let $\widehat{A}_G^+$ be the set of dominant characters of A_G—the semigroup of highest weights of irreducible representations of G. It is well known (see for instance, [Ho4]) that the space of regular functions on the coset space G/N_G decomposes (under the action of G by left translations) as a direct sum of one copy of each irreducible representation V_ψ, of highest weight ψ, of G:

$$\mathcal{R}(G/N_G) \simeq \bigoplus_{\psi \in \widehat{A}_G^+} V_\psi.$$

We note that $\mathcal{R}(G/N_G)$ has the structure of an $\widehat{A}_G^+$-graded algebra, for which the V_ψ are the graded components. Let $H \subset G$ be a reductive subgroup and $A_H = A_G \cap H$ be a maximal torus of H normalizing N_H, a maximal unipotent subgroup of H, so that $B_H = A_H \cdot N_H$ is a Borel subgroup of H. We consider the algebra $\mathcal{R}(G/N_G)^{N_H}$ of functions on G/N_G which are invariant under left translations by N_H. This is an $(\widehat{A}_G^+ \times \widehat{A}_H^+)$-graded algebra. Knowledge of $\mathcal{R}(G/N_G)^{N_H}$ as a $(\widehat{A}_G^+ \times \widehat{A}_H^+)$-graded algebra tell us how representations of G decompose when restricted to H, in other words, it describes the branching rule from G to H. We will call $\mathcal{R}(G/N_G)^{N_H}$ the (G, H) *branching algebra.* When $G \simeq H \times H$, and H is embedded diagonally in G, the branching algebra describes the decomposition of tensor products of representations of H, and we then call it the *tensor product algebra* for H.

Let us briefly explain how branching algebras, dual pairs and reciprocity are related. For a reciprocity pair $(G, \mathfrak{g}')$, $(H, \mathfrak{h}')$, the duality correspondences are subjugated to a correspondence in the space of harmonics $\mathcal{H}$ (see Theorem 3.1). Branching from holomorphic discrete series of $\mathfrak{h}'$ to $\mathfrak{g}'$ behaves very much like finite-dimensional representations in relation to their highest weights and is captured entirely by the branching from the lowest $K_{H'}$-type to $K_{G'}$. Although $\mathcal{H}$ is not an algebra, it can still be identified as a quotient algebra of $\mathcal{P}(\mathbb{C}^m)$. With the $G \times K_{G'}$ as well as $H \times K_{H'}$ multiplicity-free decomposition of $\mathcal{H}$, one allows $\mathcal{H}^{N_H \times N_{K_{G'}}}$ to be interpreted as a branching algebra from $K_{H'}$ to $K_{G'}$ as well as a branching algebra from G to H. This double interpretation solves two related branching problems simultaneously. Classical invariant theory also provides a flexible approach which allows an inductive approach to the computation of branching algebras, and makes evident natural connections between different branching algebras. We refer to readers to [HTW] for more details.

4. Proofs

4.1. **Proofs of the tensor product formulas.**

4.1.1. $\mathbf{GL_n} \subset \mathbf{GL_n} \times \mathbf{GL_n}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccc} U_n \times U_n & - & \mathfrak{u}_{p,q} \oplus \mathfrak{u}_{r,s} & \text{Complexified} & GL_n \times GL_n & - & \mathfrak{gl}_{p,q} \oplus \mathfrak{gl}_{r,s} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ U_n & - & \mathfrak{u}_{p+r,q+s} & & GL_n & - & \mathfrak{gl}_{p+r,q+s} \end{array}$$

Regarding the dual pair $(GL_n \times GL_n, \mathfrak{gl}_{p,q} \oplus \mathfrak{gl}_{r,s})$, Theorem 3.2 gives the decomposition

$$\begin{aligned} &\mathcal{P}\left(\left(\mathbb{C}^n \otimes \mathbb{C}^p \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^q\right) \oplus \left(\mathbb{C}^n \otimes \mathbb{C}^r \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^s\right)\right) \\ &\cong \bigoplus \left(F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(n)}^{(\nu^+,\nu^-)}\right) \otimes \left(\widetilde{F}_{(p,q)}^{(\mu^+,\mu^-)} \otimes \widetilde{F}_{(r,s)}^{(\nu^+,\nu^-)}\right), \end{aligned}$$

where the sum is over non-negative integer partitions μ^+, ν^+, μ^-, and ν^- such that

$$\begin{array}{ll} \ell(\mu^+) \le p, & \ell(\mu^-) \le q, \\ \ell(\nu^+) \le r, & \ell(\nu^-) \le s, \\ \ell(\mu^+) + \ell(\mu^-) \le n, & \ell(\nu^+) + \ell(\nu^-) \le n. \end{array}$$

Regarding the dual pair $(GL_n, \mathfrak{gl}_{p+r,q+s})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^{p+r} \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^{q+s}\right) \cong \bigoplus F_{(n)}^{(\lambda^+,\lambda^-)} \otimes \widetilde{F}_{(p+r,q+s)}^{(\lambda^+,\lambda^-)},$$

where the sum is over all non-negative integer partitions λ^+ and λ^- such that $\ell(\lambda^+) + \ell(\lambda^-) \le n$, $\ell(\lambda^+) \le p + r$ and $\ell(\lambda^-) \le q + s$.

We assume that we are in the stable range: $n \ge p + q + r + s$, so that as a $GL_{p+r} \times GL_{q+s}$ representation (see Theorem 3.2),

$$\widetilde{F}_{(p+r,q+s)}^{(\lambda^+,\lambda^-)} \cong \mathcal{S}(\mathbb{C}^{p+r} \otimes \mathbb{C}^{q+s}) \otimes F_{(p+r)}^{\lambda^+} \otimes F_{(q+s)}^{\lambda^-}.$$

As a $GL_p \times GL_q \times GL_r \times GL_s$-representation, $\widetilde{F}_{(p+r,q+s)}^{(\lambda^+,\lambda^-)}$ is equivalent to

$$\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes \mathcal{S}(\mathbb{C}^r \otimes \mathbb{C}^s) \otimes \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^s) \otimes \mathcal{S}(\mathbb{C}^r \otimes \mathbb{C}^q) \otimes F_{(p+r)}^{\lambda^+} \otimes F_{(q+s)}^{\lambda^-}.$$

Note that $n \ge p + q + r + s$ implies that $n \ge p + q$ and $n \ge r + s$, so that (see Theorem 3.2)

$$\widetilde{F}_{(p,q)}^{(\mu^+,\mu^-)} \cong \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p)}^{\mu^+} \otimes F_{(q)}^{\mu^-}$$

and

$$\widetilde{F}_{(r,s)}^{(\nu^+,\nu^-)} \cong \mathcal{S}(\mathbb{C}^r \otimes \mathbb{C}^s) \otimes F_{(r)}^{\nu^+} \otimes F_{(s)}^{\nu^-}.$$

Our see-saw pair implies (see (3.15))

$$\left[F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(n)}^{(\nu^+,\nu^-)}, F_{(n)}^{(\lambda^+,\lambda^-)}\right] = \left[\widetilde{F}_{(p+r,q+s)}^{(\lambda^+,\lambda^-)}, \widetilde{F}_{(p,q)}^{(\mu^+,\mu^-)} \otimes \widetilde{F}_{(r,s)}^{(\nu^+,\nu^-)}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned} &\left[F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(n)}^{(\nu^+,\nu^-)}, F_{(n)}^{(\lambda^+,\lambda^-)}\right] \\ &= \left[\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^s) \otimes \mathcal{S}(\mathbb{C}^r \otimes \mathbb{C}^q) \otimes F_{(p+r)}^{\lambda^+} \otimes F_{(q+s)}^{\lambda^-}, F_{(p)}^{\mu^+} \otimes F_{(q)}^{\mu^-} \otimes F_{(r)}^{\nu^+} \otimes F_{(s)}^{\nu^-}\right]. \end{aligned}$$

Next we will combine the standard decompositions

$$F_{(p+r)}^{\lambda^+} \cong \bigoplus c_{\alpha_1\,\alpha_2}^{\lambda^+} F_{(p)}^{\alpha_1} \otimes F_{(r)}^{\alpha_2},$$
$$F_{(q+s)}^{\lambda^-} \cong \bigoplus c_{\beta_1\,\beta_2}^{\lambda^-} F_{(q)}^{\beta_1} \otimes F_{(s)}^{\beta_2}$$

with the multiplicity-free decompositions (see (3.12))

$$\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^s) \cong \bigoplus F_{(p)}^{\gamma_1} \otimes F_{(s)}^{\gamma_1},$$
$$\mathcal{S}(\mathbb{C}^r \otimes \mathbb{C}^q) \cong \bigoplus F_{(r)}^{\gamma_2} \otimes F_{(q)}^{\gamma_2}.$$

This implies the result

$$\left[F_{(n)}^{(\mu^+,\mu^-)} \otimes F_{(n)}^{(\nu^+,\nu^-)}, F_{(n)}^{(\lambda^+,\lambda^-)}\right] = \sum c_{\alpha_2\,\alpha_1}^{\lambda^+} c_{\alpha_1\,\gamma_1}^{\mu^+} c_{\gamma_1\,\beta_2}^{\nu^-} c_{\beta_2\,\beta_1}^{\lambda^-} c_{\beta_1\,\gamma_2}^{\mu^-} c_{\gamma_2\,\alpha_2}^{\nu^+}.$$

4.1.2. $\mathbf{O_n} \subset \mathbf{O_n} \times \mathbf{O_n}$. We consider the following see-saw pair and its complexification:

$$\begin{array}{ccccccc} O_n(\mathbb{R}) \times O_n(\mathbb{R}) & - & sp_{2p}(\mathbb{R}) \oplus sp_{2q}(\mathbb{R}) & \text{Complexified} & O_n \times O_n & - & \mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ O_n(\mathbb{R}) & - & sp_{2(p+q)}(\mathbb{R}) & & O_n & - & \mathfrak{sp}_{2(p+q)} \end{array}$$

Regarding the dual pair $(O_n \times O_n, \mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^p \oplus \mathbb{C}^n \otimes \mathbb{C}^q\right) \cong \bigoplus \left(E_{(n)}^{\mu} \otimes E_{(n)}^{\nu}\right) \otimes \left(\widetilde{E}_{(2p)}^{\mu} \otimes \widetilde{E}_{(2q)}^{\nu}\right),$$

where the sum is over non-negative integer partitions μ and ν such that

$$\begin{array}{ll} \ell(\mu) \le p, & (\mu')_1 + (\mu')_2 \le n, \\ \ell(\nu) \le q, & (\nu')_1 + (\nu')_2 \le n. \end{array}$$

Regarding the dual pair $(O_n, \mathfrak{sp}_{2(p+q)})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^{p+q}\right) \cong \bigoplus E_{(n)}^{\lambda} \otimes \widetilde{E}_{(2(p+q))}^{\lambda},$$

where the sum is over all non-negative integer partitions λ such that $\ell(\lambda) \le p+q$, and $(\lambda')_1 + (\lambda')_2 \le n$.

We assume that we are in the stable range: $n \ge 2(p+q)$, so that as a GL_{p+q} representation (see Theorem 3.2),

$$\widetilde{E}_{(2(p+q))}^{\lambda} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^{p+q}) \otimes F_{(p+q)}^{\lambda}.$$

As a $GL_p \times GL_q$-representation, $\widetilde{E}_{(2(p+q))}^{\lambda}$ is equivalent to

$$\mathcal{S}(\mathcal{S}^2\mathbb{C}^p) \otimes \mathcal{S}(\mathcal{S}^2\mathbb{C}^q) \otimes \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p+q)}^{\lambda}.$$

Note that $n \ge 2(p+q)$ implies that $n \ge 2p$ and $n \ge 2q$, so that (see Theorem 3.2)

$$\widetilde{E}_{(2p)}^{\mu} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^p) \otimes F_{(p)}^{\mu}$$

and

$$\widetilde{E}_{(2q)}^{\nu} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^q) \otimes F_{(q)}^{\nu}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[E_{(n)}^{\mu} \otimes E_{(n)}^{\nu}, E_{(n)}^{\lambda}\right] = \left[\widetilde{E}_{(2(p+q))}^{\lambda}, \widetilde{E}_{(2p)}^{\mu} \otimes \widetilde{E}_{(2q)}^{\nu}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned}&\left[\widetilde{E}^{\lambda}_{(2(p+q))}, \widetilde{E}^{\mu}_{(2p)} \otimes \widetilde{E}^{\nu}_{(2q)}\right]\\ &= \left[\mathcal{S}(\mathcal{S}^2\mathbb{C}^p) \otimes \mathcal{S}(\mathcal{S}^2\mathbb{C}^q) \otimes \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda}_{(p+q)}, \mathcal{S}(\mathcal{S}^2\mathbb{C}^p) \otimes F^{\mu}_{(p)} \ \otimes \ \mathcal{S}(\mathcal{S}^2\mathbb{C}^q) \otimes F^{\nu}_{(q)}\right]\\ &= \left[\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda}_{(p+q)}, F^{\mu}_{(p)} \otimes F^{\nu}_{(q)}\right].\end{aligned}$$

Next we will combine the decomposition

$$F^{\lambda}_{(p+q)} \cong \bigoplus c^{\lambda}_{\alpha\,\beta} F^{\alpha}_{(p)} \otimes F^{\beta}_{(q)}$$

with the multiplicity-free decomposition (see (3.12))

$$\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \cong \bigoplus F^{\gamma}_{(p)} \otimes F^{\gamma}_{(q)}$$

to obtain the result, but first note that in the above decompositions α, β, and γ range over all non-negative integer partitions such that $\ell(\alpha) \leq p$, $\ell(\beta) \leq q$ and $\ell(\gamma) \leq \min(p, q)$. So we obtain

$$\left[E^{\mu}_{(n)} \otimes E^{\nu}_{(n)}, E^{\lambda}_{(n)}\right] = \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha\,\beta} c^{\mu}_{\alpha\,\gamma} c^{\nu}_{\beta\,\gamma}.$$

The above sum is over all non-negative integer partitions α, β, γ such that $\ell(\alpha) \leq p$, $\ell(\beta) \leq q$ and $\ell(\gamma) \leq \min(p, q)$, however, the support of the Littlewood-Richardson coefficients is contained inside the set of such (α, β, γ) when we choose p and q such that $\ell(\lambda) \leq \lfloor n/2 \rfloor := p + q$, with $\ell(\mu) := p$ and $\ell(\nu) := q$.

4.1.3. $\mathbf{Sp_{2n}} \subset \mathbf{Sp_{2n}} \times \mathbf{Sp_{2n}}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccccc} Sp(n) \times Sp(n) & - & \mathfrak{so}^*_{2p} \oplus \mathfrak{so}^*_{2q} & \text{Complexified} & Sp_{2n} \times Sp_{2n} & - & \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2q} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ Sp(n) & - & so^*_{2(p+q)} & & Sp_{2n} & - & \mathfrak{so}_{2(p+q)} \end{array}$$

Regarding the dual pair $(Sp_{2n} \times Sp_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2q})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2n} \otimes \mathbb{C}^p \oplus \mathbb{C}^{2n} \otimes \mathbb{C}^q\right) \cong \bigoplus \left(V^{\mu}_{(2n)} \otimes V^{\nu}_{(2n)}\right) \otimes \left(\widetilde{V}^{\mu}_{(2p)} \otimes \widetilde{V}^{\nu}_{(2q)}\right),$$

where the sum is over non-negative integer partitions μ and ν such that

$$\ell(\mu) \leq \min(n, p), \quad \ell(\nu) \leq \min(n, q).$$

Regarding the dual pair $(Sp_{2n}, \mathfrak{so}_{2(p+q)})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2n} \otimes \mathbb{C}^{p+q}\right) \cong \bigoplus V^{\lambda}_{(2n)} \otimes \widetilde{V}^{\lambda}_{(2(p+q))},$$

where the sum is over all non-negative integer partitions λ such that $\ell(\lambda) \leq \min(n, p+q)$.

We assume that we are in the stable range: $n \geq p + q$, so that as a GL_{p+q} representation (see Theorem 3.2),

$$\widetilde{V}^{\lambda}_{(2(p+q))} \cong \mathcal{S}(\wedge^2\mathbb{C}^{p+q}) \otimes F^{\lambda}_{(p+q)}.$$

As a $GL_p \times GL_q$-representation, $\widetilde{V}^{\lambda}_{(2(p+q))}$ is equivalent to

$$\mathcal{S}(\wedge^2\mathbb{C}^p) \otimes \mathcal{S}(\wedge^2\mathbb{C}^q) \otimes \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda}_{(p+q)}.$$

Note that $n \geq p+q$ implies that $n \geq p$ and $n \geq q$, so that (see Theorem 3.2)

$$\widetilde{V}^{\mu}_{(2p)} \cong \mathcal{S}(\wedge^2\mathbb{C}^p) \otimes F^{\mu}_{(p)}$$

and

$$\widetilde{V}^{\nu}_{(2q)} \cong \mathcal{S}(\wedge^2\mathbb{C}^q) \otimes F^{\nu}_{(q)}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[V^{\mu}_{(2n)} \otimes V^{\nu}_{(2n)}, V^{\lambda}_{(2n)}\right] = \left[\widetilde{V}^{\lambda}_{(2(p+q))}, \widetilde{V}^{\mu}_{(2p)} \otimes \widetilde{V}^{\nu}_{(2q)}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned}
&\left[\widetilde{V}^{\lambda}_{(2(p+q))}, \widetilde{V}^{\mu}_{(2p)} \otimes \widetilde{V}^{\nu}_{(2q)}\right] \\
&= \left[\mathcal{S}(\wedge^2\mathbb{C}^p) \otimes \mathcal{S}(\wedge^2\mathbb{C}^q) \otimes \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda}_{(p+q)}, \mathcal{S}(\wedge^2\mathbb{C}^p) \otimes F^{\mu}_{(p)} \ \otimes \ \mathcal{S}(\wedge^2\mathbb{C}^q) \otimes F^{\nu}_{(q)}\right] \\
&= \left[\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda}_{(p+q)}, F^{\mu}_{(p)} \otimes F^{\nu}_{(q)}\right].
\end{aligned}$$

Next we will combine the decomposition

$$F^{\lambda}_{(p+q)} \cong \bigoplus c^{\lambda}_{\alpha\,\beta} F^{\alpha}_{(p)} \otimes F^{\beta}_{(q)}$$

with the multiplicity-free decomposition (see (3.12))

$$\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \cong \bigoplus F^{\gamma}_{(p)} \otimes F^{\gamma}_{(q)}$$

to obtain the result, but first note that in the above decompositions α, β, and γ range over all non-negative integer partitions such that $\ell(\alpha) \leq p$, $\ell(\beta) \leq q$ and $\ell(\gamma) \leq \min(p,q)$. So we obtain

$$\left[V^{\mu}_{(2n)} \otimes V^{\nu}_{(2n)}, V^{\lambda}_{(2n)}\right] = \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha\,\beta} c^{\mu}_{\alpha\,\gamma} c^{\nu}_{\beta\,\gamma}.$$

The above sum is over all non-negative integer partitions α, β, γ such that $\ell(\alpha) \leq p$, $\ell(\beta) \leq q$ and $\ell(\gamma) \leq \min(p,q)$. However, the support of the Littlewood-Richardson coefficients is contained inside the set of such (α,β,γ) when we choose p and q such that $\ell(\lambda) \leq \lfloor n/2 \rfloor := p+q$, with $\ell(\mu) := p$ and $\ell(\nu) := q$.

4.2. **Proofs of the direct sum branching rules.**

4.2.1. $\mathbf{GL_n} \times \mathbf{GL_m} \subset \mathbf{GL_{n+m}}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccc}
U_{n+m} & - & u_{p,q} & \text{Complexified} & GL_{n+m} & - & \mathfrak{gl}_{p,q} \\
\cup & & \cap & \rightsquigarrow & \cup & & \cap \\
U_n \times U_m & - & u_{p,q} \oplus u_{p,q} & & GL_n \times GL_m & - & \mathfrak{gl}_{p,q} \oplus \mathfrak{gl}_{p,q}
\end{array}$$

Regarding the dual pair $(GL_{n+m}, \mathfrak{gl}_{p+q})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{n+m} \otimes \mathbb{C}^p \oplus \left(\mathbb{C}^{n+m}\right)^* \otimes \mathbb{C}^q\right) \cong \bigoplus F^{(\lambda^+,\lambda^-)}_{(n+m)} \otimes \widetilde{F}^{(\lambda^+,\lambda^-)}_{(p,q)},$$

where the sum is over non-negative integer partitions λ^+ and λ^- such that $\ell(\lambda^+) \leq p$, $\ell(\lambda^-) \leq q$ and $\ell(\lambda^+) + \ell(\lambda^-) \leq n+m$. Regarding the dual pair $(GL_n \times GL_m, \mathfrak{gl}_{p+q} \oplus \mathfrak{gl}_{p+q})$, Theorem 3.2 gives the decomposition

$$\begin{aligned}
&\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^p \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^q \oplus \mathbb{C}^m \otimes \mathbb{C}^p \oplus (\mathbb{C}^m)^* \otimes \mathbb{C}^q\right) \\
&\cong \bigoplus \left(F^{(\mu^+,\mu^-)}_{(n)} \otimes F^{(\nu^+,\nu^-)}_{(m)}\right) \otimes \left(\widetilde{F}^{(\mu^+,\mu^-)}_{(p,q)} \otimes \widetilde{F}^{(\nu^+,\nu^-)}_{(p,q)}\right),
\end{aligned}$$

where the sum is over all non-negative integer partitions μ^+, μ^-, ν^+ and ν^- such that

$$\begin{array}{ll} \ell(\mu^+) + \ell(\mu^-) \leq n, & \ell(\nu^+) + \ell(\nu^-) \leq m, \\ \ell(\mu^+) \leq p, & \ell(\mu^-) \leq q, \\ \ell(\nu^+) \leq p, & \ell(\nu^-) \leq q. \end{array}$$

We assume that we are in the stable range: $\min(n, m) \geq p + q$, so that as a $GL_p \times GL_q$ representation (see Theorem 3.2),

$$\begin{aligned} \widetilde{F}^{(\mu^+,\mu^-)}_{(p,q)} &\cong \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\mu^+}_{(p)} \otimes F^{\mu^-}_{(q)}, \\ \widetilde{F}^{(\nu^+,\nu^-)}_{(p,q)} &\cong \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\nu^+}_{(p)} \otimes F^{\nu^-}_{(q)}. \end{aligned}$$

Note that $\min(n, m) \geq p + q$ implies that $n + m \geq p + q$, so that (see Theorem 3.2)

$$\widetilde{F}^{(\lambda^+,\lambda^-)}_{(p,q)} \cong \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda^+}_{(p)} \otimes F^{\lambda^-}_{(q)}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[\widetilde{F}^{(\mu^+,\mu^-)}_{(p,q)} \otimes \widetilde{F}^{(\nu^+,\nu^-)}_{(p,q)}, \widetilde{F}^{(\lambda^+,\lambda^-)}_{(p,q)}\right] = \left[F^{(\lambda^+,\lambda^-)}_{(n+m)}, F^{(\mu^+,\mu^-)}_{(n)} \otimes F^{(\nu^+,\nu^-)}_{(m)}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned} &\left[\widetilde{F}^{(\mu^+,\mu^-)}_{(p,q)} \otimes \widetilde{F}^{(\nu^+,\nu^-)}_{(p,q)}, \widetilde{F}^{(\lambda^+,\lambda^-)}_{(p,q)}\right] \\ &= \Big[\left(\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\mu^+}_{(p)} \otimes F^{\mu^-}_{(q)}\right) \otimes \left(\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\nu^+}_{(p)} \otimes F^{\nu^-}_{(q)}\right), \\ &\qquad\qquad\qquad\qquad \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\lambda^+}_{(p)} \otimes F^{\lambda^-}_{(q)}\Big] \\ &= \left[\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F^{\mu^+}_{(p)} \otimes F^{\mu^-}_{(q)} \otimes F^{\nu^+}_{(p)} \otimes F^{\nu^-}_{(q)}, F^{\lambda^+}_{(p)} \otimes F^{\lambda^-}_{(q)}\right]. \end{aligned}$$

Next combine the above decomposition with (see (3.12))

$$\mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \cong \bigoplus F^{\delta}_{(p)} \otimes F^{\delta}_{(q)},$$

where the sum is over all non-negative integer partitions δ with at most $\min(p, q)$ parts. We then obtain

$$\begin{aligned} &\left[F^{(\lambda^+,\lambda^-)}_{(n+m)}, F^{(\mu^+,\mu^-)}_{(n)} \otimes F^{(\nu^+,\nu^-)}_{(m)}\right] \\ &= \left[\left(\bigoplus\nolimits_\delta F^{\delta}_{(p)} \otimes F^{\delta}_{(q)}\right) \otimes \left(F^{\mu^+}_{(p)} \otimes F^{\mu^-}_{(q)}\right) \otimes \left(F^{\nu^+}_{(p)} \otimes F^{\nu^-}_{(q)}\right), F^{\lambda^+}_{(p)} \otimes F^{\lambda^-}_{(q)}\right]. \end{aligned}$$

We combine this fact with the following two tensor product decompositions:

$$F^{\mu^+}_{(p)} \otimes F^{\nu^+}_{(p)} \quad\cong\quad \bigoplus c^{\gamma^+}_{\mu^+\,\nu^+} F^{\gamma^+}_{(p)} \quad \text{and} \quad F^{\mu^-}_{(q)} \otimes F^{\nu^-}_{(q)} \quad\cong\quad \bigoplus c^{\gamma^-}_{\mu^-\,\nu^-} F^{\gamma^-}_{(q)}$$

(where γ^+ and γ^- have at most p and q parts respectively) and then tensor the constituents with $F^{\delta}_{(p)} \otimes F^{\delta}_{(q)}$,

$$F^{\gamma^+}_{(p)} \otimes F^{\delta}_{(p)} \quad\cong\quad \bigoplus c^{\lambda^+}_{\gamma^+\,\delta} F^{\lambda^+}_{(p)} \quad \text{and} \quad F^{\gamma^-}_{(q)} \otimes F^{\delta}_{(q)} \quad\cong\quad \bigoplus c^{\lambda^-}_{\gamma^-\,\delta} F^{\lambda^-}_{(q)}$$

to obtain the result.

4.2.2. $\mathbf{O_n} \times \mathbf{O_m} \subset \mathbf{O_{n+m}}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccc} O_{n+m}(\mathbb{R}) & - & \mathfrak{sp}_{2k}(\mathbb{R}) & \text{Complexified} & O_{n+m} & - & \mathfrak{sp}_{2k} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ O_n(\mathbb{R}) \times O_m(\mathbb{R}) & - & \mathfrak{sp}_{2k}(\mathbb{R}) \oplus \mathfrak{sp}_{2k}(\mathbb{R}) & & O_n \times O_m & - & \mathfrak{sp}_{2k} \oplus \mathfrak{sp}_{2k} \end{array}$$

Regarding the dual pair $(O_{n+m}, \mathfrak{sp}_{2k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{n+m} \otimes \mathbb{C}^k\right) \cong \bigoplus E^\lambda_{(n+m)} \otimes \widetilde{E}^\lambda_{(2k)},$$

where the sum is over non-negative integer partitions λ such that $\ell(\lambda) \leq k$ and $(\lambda')_1 + (\lambda')_2 \leq n+m$. Regarding the dual pair $(O_n \times O_m, \mathfrak{sp}_{2k} \oplus \mathfrak{sp}_{2k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^k \oplus \mathbb{C}^m \otimes \mathbb{C}^k\right) \cong \bigoplus \left(E^\mu_{(n)} \otimes E^\nu_{(m)}\right) \otimes \left(\widetilde{E}^\mu_{(2k)} \otimes \widetilde{E}^\nu_{(2k)}\right),$$

where the sum is over all non-negative integer partitions μ and ν such that $\ell(\mu) \leq k$, $\ell(\nu) \leq k$, $(\mu')_1 + (\mu')_2 \leq n$ and $(\nu')_1 + (\nu')_2 \leq m$.

We assume that we are in the stable range: $\min(n,m) \geq 2k$, so that as GL_k representations (see Theorem 3.2),

$$\widetilde{E}^\mu_{(2k)} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\mu_{(k)} \quad \text{and} \quad \widetilde{E}^\nu_{(2k)} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\nu_{(k)}.$$

Note that $\min(n,m) \geq 2k$ implies that $n+m \geq 2k$, so that (see Theorem 3.2)

$$\widetilde{E}^\lambda_{(2k)} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\lambda_{(k)}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[\widetilde{E}^\mu_{(2k)} \otimes \widetilde{E}^\nu_{(2k)}, \widetilde{E}^\lambda_{(2k)}\right] = \left[E^\lambda_{(n+m)}, E^\mu_{(n)} \otimes F^\nu_{(m)}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned}\left[\widetilde{E}^\mu_{(2k)} \otimes \widetilde{E}^\nu_{(2k)}, \widetilde{E}^\lambda_{(2k)}\right] &= \left[\left(\mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\mu_{(k)}\right) \otimes \left(\mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\nu_{(k)}\right), \mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\lambda_{(k)}\right] \\ &= \left[\mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\mu_{(k)} \otimes F^\nu_{(k)}, F^\lambda_{(k)}\right].\end{aligned}$$

Next combine with the well-known multiplicity-free decomposition (see for instance, Theorem 3.1 of [Ho4] on page 32)

$$\mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \cong \bigoplus F^{2\delta}_{(k)},$$

where the sum is over all non-negative integer partitions δ with at most k parts. We then obtain

$$\left[E^\lambda_{(n+m)}, E^\mu_{(n)} \otimes E^\nu_{(m)}\right] = \left[\left(\bigoplus_\delta F^{2\delta}_{(k)}\right) \otimes F^\mu_{(k)} \otimes F^\nu_{(k)}, F^\lambda_{(k)}\right].$$

Combine this fact with the following two tensor product decompositions:

$$F^\mu_{(k)} \otimes F^\nu_{(k)} \cong \bigoplus c^\gamma_{\mu\,\nu} F^\gamma_{(k)} \quad \text{and} \quad F^\gamma_{(k)} \otimes F^{2\delta}_{(k)} \cong \bigoplus c^\lambda_{\gamma\,2\delta} F^\lambda_{(k)}$$

(where γ is a non-negative integer partition with at most k parts) and the result follows.

4.2.3. $\mathbf{Sp_{2n}} \times \mathbf{Sp_{2m}} \subset \mathbf{Sp_{2(n+m)}}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccc} Sp(n+m) & - & \mathfrak{so}^*_{2k} & \text{Complexified} & Sp_{2(n+m)} & - & \mathfrak{so}_{2k} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ Sp(n)\times Sp(m) & - & \mathfrak{so}^*_{2k}\oplus\mathfrak{so}^*_{2k} & & Sp_{2n}\times Sp_{2m} & - & \mathfrak{so}_{2k}\oplus\mathfrak{so}_{2k} \end{array}$$

Regarding the dual pair $(Sp_{2(n+m)}, \mathfrak{so}_{2k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2(n+m)} \otimes \mathbb{C}^k\right) \cong \bigoplus V^\lambda_{(2(n+m))} \otimes \widetilde{V}^\lambda_{(2k)},$$

where the sum is over non-negative integer partitions λ such that $\ell(\lambda) \le \min(n+m, k)$. Regarding the dual pair $(Sp_{2n}\times Sp_{2m}, \mathfrak{so}_{2k}\oplus\mathfrak{so}_{2k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2n}\otimes\mathbb{C}^k \oplus \mathbb{C}^{2m}\otimes\mathbb{C}^k\right) \cong \bigoplus \left(V^\mu_{(2n)}\otimes V^\nu_{(2m)}\right)\otimes\left(\widetilde{V}^\mu_{(2k)}\otimes\widetilde{V}^\nu_{(2k)}\right),$$

where the sum is over all non-negative integer partitions μ and ν such that $\ell(\mu) \le \min(n,k)$, $\ell(\nu) \le \min(n,k)$.

We assume that we are in the stable range: $\min(n,m) \ge k$, so that as GL_k representations (see Theorem 3.2),

$$\widetilde{V}^\mu_{(2k)} \cong \mathcal{S}(\wedge^2\mathbb{C}^k)\otimes F^\mu_{(k)} \quad \text{and} \quad \widetilde{V}^\nu_{(2k)} \cong \mathcal{S}(\wedge^2\mathbb{C}^k)\otimes F^\nu_{(k)}.$$

Note that $\min(n,m) \ge k$ implies that $n+m \ge k$, so that (see Theorem 3.2)

$$\widetilde{V}^\lambda_{(2k)} \cong \mathcal{S}(\wedge^2\mathbb{C}^k)\otimes F^\lambda_{(k)}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[\widetilde{V}^\mu_{(2k)}\otimes\widetilde{V}^\nu_{(2k)}, \widetilde{V}^\lambda_{(2k)}\right] = \left[V^\lambda_{(n+m)}, V^\mu_{(n)}\otimes V^\nu_{(m)}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned}\left[\widetilde{V}^\mu_{(2k)}\otimes\widetilde{V}^\nu_{(2k)}, \widetilde{V}^\lambda_{(2k)}\right] &= \left[\left(\mathcal{S}(\wedge^2\mathbb{C}^k)\otimes F^\mu_{(k)}\right)\otimes\left(\mathcal{S}(\wedge^2\mathbb{C}^k)\otimes F^\nu_{(k)}\right), \mathcal{S}(\wedge^2\mathbb{C}^k)\otimes F^\lambda_{(k)}\right]\\ &= \left[\mathcal{S}(\wedge^2\mathbb{C}^k)\otimes F^\mu_{(k)}\otimes F^\nu_{(k)}, F^\lambda_{(k)}\right].\end{aligned}$$

Next combine with the well-known multiplicity-free decomposition (see for instance, Theorem 3.8.1 of [Ho4] on page 44)

$$\mathcal{S}(\wedge^2\mathbb{C}^k) \cong \bigoplus F^{(2\delta)'}_{(k)},$$

where the sum is over all non-negative integer partitions δ such that $(2\delta)'$ has at most k parts. We then obtain

$$\left[V^\lambda_{(2(n+m))}, V^\mu_{(2n)}\otimes V^\nu_{(2m)}\right] = \left[\left(\bigoplus_\delta F^{(2\delta)'}_{(k)}\right)\otimes F^\mu_{(k)}\otimes F^\nu_{(k)}, F^\lambda_{(k)}\right].$$

Combine this fact with the following two tensor product decompositions:

$$F^\mu_{(k)}\otimes F^\nu_{(k)} \;\cong\; \bigoplus c^\gamma_{\mu\,\nu} F^\gamma_{(k)} \quad \text{and} \quad F^\gamma_{(k)}\otimes F^{(2\delta)'}_{(k)} \;\cong\; \bigoplus c^\lambda_{\gamma\,(2\delta)'} F^\lambda_{(k)}$$

(where γ is a non-negative integer partition with at most k parts) and the result follows.

4.3. **Proofs of the polarization branching rules.**

4.3.1. $\mathbf{GL_n} \subset \mathbf{O_{2n}}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccccc} O_{2n}(\mathbb{R}) & - & \mathfrak{sp}_{2k}(\mathbb{R}) & \text{Complexified} & O_{2n} & - & \mathfrak{sp}_{2k} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ U(n) & - & u_{k,k} & & GL_n & - & \mathfrak{gl}_{k,k} \end{array}$$

Regarding the dual pair $(O_{2n}, \mathfrak{sp}_{2k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2n} \otimes \mathbb{C}^k\right) \cong \bigoplus E^\lambda_{(2n)} \otimes \widetilde{E}^\lambda_{(2k)},$$

where the sum is over all non-negative integer partitions λ such that $\ell(\lambda) \le k$ and $(\lambda')_1 + (\lambda')_2 \le 2n$. Since the standard O_{2n} representation $\mathbb{C}^{2n} \simeq \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ is a GL_n representation, regarding the dual pair $(GL_n, \mathfrak{gl}_{k,k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^k \otimes (\mathbb{C}^n)^* \otimes \mathbb{C}^k\right) \cong \bigoplus F^{(\mu^+,\mu^-)}_{(n)} \otimes \widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)},$$

where the sum is over non-negative integer partitions μ^+ and μ^- with at most k parts such that $\ell(\mu^+) + \ell(\mu^-) \le n$.

We assume that we are in the stable range: $n \ge 2k$, so that as a $GL_k \times GL_k$ representation (see Theorem 3.2),

$$\widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)} \cong \mathcal{S}(\mathbb{C}^k \otimes \mathbb{C}^k) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}.$$

Note that $n \ge 2k$ implies that $n \ge k$, so that as GL_k representations (see Theorem 3.2)

$$\widetilde{E}^\lambda_{(2k)} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\lambda_{(k)}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[\widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)}, \widetilde{E}^\lambda_{(2k)}\right] = \left[E^\lambda_{(2n)}, F^{(\mu^+,\mu^-)}_{(n)}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned} \left[\widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)}, \widetilde{E}^\lambda_{(2k)}\right] &= \left[\mathcal{S}(\mathbb{C}^k \otimes \mathbb{C}^k) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}, \mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^\lambda_{(k)}\right] \\ &= \left[\mathcal{S}(\wedge^2\mathbb{C}^k) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}, F^\lambda_{(k)}\right]. \end{aligned}$$

Note that in the above we used the fact that as a GL_k-representation, $\otimes^2\mathbb{C}^k \cong \mathcal{S}^2\mathbb{C}^k \oplus \wedge^2\mathbb{C}^k$.

Next combine with the well-known multiplicity-free decomposition

$$\mathcal{S}(\wedge^2\mathbb{C}^k) \cong \bigoplus F^{(2\delta)'}_{(k)},$$

where the sum is over all non-negative integer partitions δ such that $(2\delta)'$ has at most k parts. We then obtain

$$\left[E^\lambda_{(2n)}, F^{(\mu^+,\mu^-)}_{(n)}\right] = \left[\left(\bigoplus_\delta F^{(2\delta)'}_{(k)}\right) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}, F^\lambda_{(k)}\right].$$

Combine this fact with the following two tensor product decompositions:

$$F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)} \cong \bigoplus c^\gamma_{\mu^+\,\mu^-} F^\gamma_{(k)} \quad \text{and} \quad F^\gamma_{(k)} \otimes F^{(2\delta)'}_{(k)} \cong \bigoplus c^\lambda_{\gamma\,(2\delta)'} F^\lambda_{(k)}$$

(where γ is a non-negative integer partition with at most k parts) and the result follows.

4.3.2. $\mathbf{GL_n} \subset \mathbf{Sp_{2n}}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccc} Sp(n) & - & \mathfrak{so}^*_{2k} & \text{Complexified} & Sp_{2n} & - & \mathfrak{so}_{2k} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ U(n) & - & u_{k,k} & & GL_n & - & \mathfrak{gl}_{k,k} \end{array}$$

Regarding the dual pair $(Sp_{2n}, \mathfrak{so}_{2k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2n} \otimes \mathbb{C}^k\right) \cong \bigoplus V^\lambda_{(2n)} \otimes \widetilde{V}^\lambda_{(2k)},$$

where the sum is over all non-negative integer partitions λ such that $\ell(\lambda) \leq \min(n,k)$. Since the standard Sp_{2n} representation $\mathbb{C}^{2n} \simeq \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ is a GL_n representation, regarding the dual pair $(GL_n, \mathfrak{gl}_{k,k})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^k \otimes (\mathbb{C}^n)^* \otimes \mathbb{C}^k\right) \cong \bigoplus F^{(\mu^+,\mu^-)}_{(n)} \otimes \widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)},$$

where the sum is over non-negative integer partitions μ^+ and μ^- with at most k parts such that $\ell(\mu^+) + \ell(\mu^-) \leq n$.

We assume that we are in the stable range: $n \geq 2k$, so that as $GL_k \times GL_k$ representations (see Theorem 3.2),

$$\widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)} \cong \mathcal{S}(\mathbb{C}^k \otimes \mathbb{C}^k) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}.$$

Note that $n \geq 2k$ implies that $n \geq k$, so that as a GL_k representation (see Theorem 3.2),

$$\widetilde{V}^\lambda_{(2k)} \cong \mathcal{S}(\wedge^2\mathbb{C}^k) \otimes F^\lambda_{(k)}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[\widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)}, \widetilde{V}^\lambda_{(2k)}\right] = \left[V^\lambda_{(2n)}, F^{(\mu^+,\mu^-)}_{(n)}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned} \left[\widetilde{F}^{(\mu^+,\mu^-)}_{(k,k)}, \widetilde{E}^\lambda_{(2k)}\right] &= \left[\mathcal{S}(\mathbb{C}^k \otimes \mathbb{C}^k) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}, \mathcal{S}(\wedge^2\mathbb{C}^k) \otimes F^\lambda_{(k)}\right] \\ &= \left[\mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}, F^\lambda_{(k)}\right]. \end{aligned}$$

Note that in the above we used the fact that as a GL_k-representation, $\otimes^2\mathbb{C}^k \cong \mathcal{S}^2\mathbb{C}^k \oplus \wedge^2\mathbb{C}^k$.

Next combine with the well-known multiplicity-free decomposition

$$\mathcal{S}(\mathcal{S}^2\mathbb{C}^k) \cong \bigoplus F^{2\delta}_{(k)},$$

where the sum is over all non-negative integer partitions δ with at most k parts. We then obtain

$$\left[V^\lambda_{(2n)}, F^{(\mu^+,\mu^-)}_{(n)}\right] = \left[\left(\bigoplus_\delta F^{2\delta}_{(k)}\right) \otimes F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)}, F^\lambda_{(k)}\right].$$

Combine this fact with the following two tensor product decompositions:

$$F^{\mu^+}_{(k)} \otimes F^{\mu^-}_{(k)} \cong \bigoplus c^\gamma_{\mu^+\,\mu^-} F^\gamma_{(k)} \quad \text{and} \quad F^\gamma_{(k)} \otimes F^{2\delta}_{(k)} \cong \bigoplus c^\lambda_{\gamma\,2\delta} F^\lambda_{(k)}$$

(where γ is a non-negative integer partition with at most k parts) and the result follows.

4.4. **Proofs of the bilinear form branching rules.**

4.4.1. $\mathbf{O_n} \subset \mathbf{GL_n}$. We consider the following see-saw pair and its complexificiation:

$$\begin{array}{ccccccc} GL_n(\mathbb{R}) & - & GL_{p+q}(\mathbb{R}) & \text{Complexified} & GL_n & - & \mathfrak{gl}_{p,q} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ O_n(\mathbb{R}) & - & \mathfrak{sp}_{2(p+q)}(\mathbb{R}) & & O_n & - & \mathfrak{sp}_{2(p+q)} \end{array}$$

Regarding the dual pair $(GL_n, \mathfrak{gl}_{p,q})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^p \otimes (\mathbb{C}^n)^* \otimes \mathbb{C}^q\right) \cong \bigoplus F_{(n)}^{(\lambda^+,\lambda^-)} \otimes \widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)},$$

where the sum is over non-negative integer partitions λ^+ and λ^- with at most p and q parts, respectively, and such that $\ell(\lambda^+)+\ell(\lambda^-) \leq n$. Noting that $\mathbb{C}^n \simeq (\mathbb{C}^n)^*$ is an O_n representation, regarding the dual pair $(O_n, \mathfrak{sp}_{2(p+q)})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^n \otimes \mathbb{C}^{p+q}\right) \cong \bigoplus E_{(n)}^{\lambda} \otimes \widetilde{E}_{(2(p+q))}^{\lambda},$$

where the sum is over all non-negative integer partitions λ such that $\ell(\lambda) \leq p+q$ and $(\lambda')_1 + (\lambda')_2 \leq n$.

We assume that we are in the stable range: $n \geq 2(p+q)$, so that as GL_{p+q} representations (see Theorem 3.2),

$$\widetilde{E}_{(2(p+q))}^{\mu} \cong \mathcal{S}(\mathcal{S}^2\mathbb{C}^{p+q}) \otimes F_{(p+q)}^{\mu}.$$

Note that $n \geq 2(p+q)$ implies that $n \geq p+q$, so that as $GL_p \times GL_q$ representations (see Theorem 3.2),

$$\widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)} \cong \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}.$$

Our see-saw pair implies (see (3.15) that

$$\left[\widetilde{E}_{(2(p+q))}^{\mu}, \widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)}\right] = \left[F_{(n)}^{(\lambda^+,\lambda^-)}, E_{(n)}^{\mu}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned} &\left[\widetilde{E}_{(2(p+q))}^{\mu}, \widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)}\right] \\ &= \left[\mathcal{S}(\mathcal{S}^2\mathbb{C}^{p+q}) \otimes F_{(p+q)}^{\mu}, \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right] \\ &= \left[\mathcal{S}(\mathcal{S}^2\mathbb{C}^p \oplus \mathcal{S}^2\mathbb{C}^q \oplus \mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p+q)}^{\mu}, \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right] \\ &= \left[\mathcal{S}(\mathcal{S}^2\mathbb{C}^p) \otimes \mathcal{S}(\mathcal{S}^2\mathbb{C}^q) \otimes F_{(p+q)}^{\mu}, F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right]. \end{aligned}$$

Next we combine the decompositions

$$F_{(p+q)}^{\mu} \cong \bigoplus c_{\alpha\,\beta}^{\mu} F_{(p)}^{\alpha} \otimes F_{(q)}^{\beta}$$

with the multiplicity-free decompositions

$$\mathcal{S}(\mathcal{S}^2\mathbb{C}^p) \cong \bigoplus F_{(p)}^{2\gamma} \qquad \text{and} \qquad \mathcal{S}(\mathcal{S}^2\mathbb{C}^q) \cong \bigoplus F_{(q)}^{2\delta},$$

where the sums are over all non-negative integer partitions γ and δ with at most p and q parts respectively. We then obtain

$$\left[F_{(n)}^{(\lambda^+,\lambda^-)}, E_{(n)}^{\mu}\right] = \left[\left(\bigoplus F_{(p)}^{2\gamma}\right) \otimes \left(\bigoplus F_{(q)}^{2\delta}\right) \otimes \left(\bigoplus c_{\alpha\,\beta}^{\mu} F_{(p)}^{\alpha} \otimes F_{(q)}^{\beta}\right), F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right].$$

Combine this fact with the following two tensor product decompositions:

$$F_{(p)}^{\alpha} \otimes F_{(p)}^{2\gamma} \quad \cong \quad \bigoplus c_{\alpha\,2\gamma}^{\lambda^+} F_{(p)}^{\lambda^+} \quad \text{and} \quad F_{(q)}^{\beta} \otimes F_{(q)}^{2\delta} \quad \cong \quad \bigoplus c_{\beta\,2\delta}^{\lambda^-} F_{(q)}^{\lambda^-},$$

and the result follows.

4.4.2. $\mathbf{Sp_{2n}} \subset \mathbf{GL_{2n}}$. We consider the following see-saw pair and its complexifici-ation:

$$\begin{array}{ccccccc} U_{2n} & - & \mathfrak{u}_{p,q} & \text{Complexified} & GL_{2n} & - & \mathfrak{gl}_{p,q} \\ \cup & & \cap & \rightsquigarrow & \cup & & \cap \\ Sp(n) & - & \mathfrak{so}^*_{2(p+q)} & & Sp_{2n} & - & \mathfrak{so}_{2(p+q)} \end{array}$$

Regarding the dual pair $(GL_{2n}, \mathfrak{gl}_{p,q})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2n} \otimes \mathbb{C}^p \otimes \left(\mathbb{C}^{2n}\right)^* \otimes \mathbb{C}^q\right) \cong \bigoplus F_{(2n)}^{(\lambda^+,\lambda^-)} \otimes \widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)},$$

where the sum is over non-negative integer partitions λ^+ and λ^- with at most p and q parts, respectively, and such that $\ell(\lambda^+) + \ell(\lambda^-) \leq 2n$. Noting that $(\mathbb{C}^{2n})^* \simeq \mathbb{C}^{2n}$ as Sp_{2n} modules, regarding the dual pair $(Sp_{2n}, \mathfrak{so}_{2(p+q)})$, Theorem 3.2 gives the decomposition

$$\mathcal{P}\left(\mathbb{C}^{2n} \otimes \mathbb{C}^{p+q}\right) \cong \bigoplus V_{(2n)}^{\lambda} \otimes \widetilde{V}_{(2(p+q))}^{\lambda},$$

where the sum is over all non-negative integer partitions λ such that $\ell(\lambda) \leq \min(2n, p+q)$.

We assume that we are in the stable range: $n \geq p+q$, so that as GL_{p+q} representations (see Theorem 3.2)

$$\widetilde{V}_{(2(p+q))}^{\mu} \cong \mathcal{S}(\wedge^2 \mathbb{C}^{p+q}) \otimes F_{(p+q)}^{\mu}.$$

Note that $n \geq p+q$ implies that $2n \geq p+q$, so that as $GL_p \times GL_q$ representations (see Theorem 3.2),

$$\widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)} \cong \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}.$$

Our see-saw pair implies (see (3.15)) that

$$\left[\widetilde{V}_{(2(p+q))}^{\mu}, \widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)}\right] = \left[F_{(2n)}^{(\lambda^+,\lambda^-)}, V_{(2n)}^{\mu}\right].$$

Using the fact that we are in the stable range,

$$\begin{aligned} &\left[\widetilde{V}_{(2(p+q))}^{\mu}, \widetilde{F}_{(p,q)}^{(\lambda^+,\lambda^-)}\right] \\ &= \left[\mathcal{S}(\wedge^2 \mathbb{C}^{p+q}) \otimes F_{(p+q)}^{\mu}, \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right] \\ &= \left[\mathcal{S}(\wedge^2 \mathbb{C}^p \oplus \wedge^2 \mathbb{C}^q \oplus \mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p+q)}^{\mu}, \mathcal{S}(\mathbb{C}^p \otimes \mathbb{C}^q) \otimes F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right] \\ &= \left[\mathcal{S}(\wedge^2 \mathbb{C}^p) \otimes \mathcal{S}(\wedge^2 \mathbb{C}^q) \otimes F_{(p+q)}^{\mu}, F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right]. \end{aligned}$$

Next we combine the decompositions

$$F_{(p+q)}^{\mu} \cong \bigoplus c_{\alpha\,\beta}^{\mu} F_{(p)}^{\alpha} \otimes F_{(q)}^{\beta}$$

with the multiplicity-free decompositions

$$\mathcal{S}(\wedge^2\mathbb{C}^p) \cong \bigoplus F_{(p)}^{(2\gamma)'} \qquad \text{and} \qquad \mathcal{S}(\wedge^2\mathbb{C}^q) \cong \bigoplus F_{(q)}^{(2\delta)'}$$

where the sums are over all non-negative integer partitions γ and δ such that $(2\gamma)'$ and $(2\delta)'$ have at most p and q parts, respectively. We then obtain

$$\left[F_{(2n)}^{(\lambda^+,\lambda^-)}, V_{(2n)}^{\mu}\right] = \left[\left(\bigoplus F_{(p)}^{(2\gamma)'}\right) \otimes \left(\bigoplus F_{(q)}^{(2\delta)'}\right) \otimes \left(\bigoplus c_{\alpha\,\beta}^{\mu} F_{(p)}^{\alpha} \otimes F_{(q)}^{\beta}\right), F_{(p)}^{\lambda^+} \otimes F_{(q)}^{\lambda^-}\right].$$

Combine with the following two tensor product decompositions:

$$F_{(p)}^{\alpha} \otimes F_{(p)}^{(2\gamma)'} \;\cong\; \bigoplus c_{\alpha\,(2\gamma)'}^{\lambda^+} F_{(p)}^{\lambda^+} \quad \text{and} \quad F_{(q)}^{\beta} \otimes F_{(q)}^{(2\delta)'} \;\cong\; \bigoplus c_{\beta\,(2\delta)'}^{\lambda^-} F_{(q)}^{\lambda^-},$$

and the result follows.

Acknowledgement

We are grateful to the referee for his elaborate and in-depth comments. Most notably, for the vast number of references and comments on related works for branching rules and tensor products of infinite-dimensional representations.

References

[Ab] J. Abramsky, Ph.D. thesis, The University of Southampton, England, 1969.

[BKW] G. Black, R. King, B. Wybourne, *Kronecker products for compact semisimple Lie groups*, J. Phys. A 16 (1983), no. 8, 1555 – 1589. MR0708193 (85e:22020)

[CGR] Y. Chen, A. Garsia and J. Remmel, *Algorithms for plethysm in combinatorics and algebra*, Contemp. Math. 24, AMS, Providence, R.I. (1984), 109 – 153. MR0777698 (86f:05010)

[EHW] T. Enright, R. Howe, and N. Wallach, *Classification of unitary highest weight modules*, in Representation Theory of Reductive Groups (P. Trombi, Ed.), pp. 97 – 144, Birkhäuser, Boston, 1983. MR0733809 (86c:22028)

[EW1] T. Enright and J. Willenbring, *Hilbert series, Howe duality, and branching rules*, Proc. Nat. Acad. Sciences 100 (2003), No. 2, 434 – 437. MR1950645

[EW2] T. Enright and J. Willenbring, *Hilbert series, Howe duality, and branching rules for classical groups*, Ann. of Math. 159 (2004), No. 1, 337 – 375. MR2052357

[Fu] W. Fulton, *Young Tableaux*, Cambridge University Press, Cambridge, UK, 1997. MR1464693 (99f:05119)

[GW] R. Goodman and N. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge U. Press, 1998. MR1606831 (99b:20073)

[GK] K. Gross and R. Kunze, *Finite-dimensional induction and new results on invariants for classical groups, I*, Amer. J. Math. 106 (1984), 893 – 974. MR0749261 (86h:22027)

[HC] Harish-Chandra, *Representations of semisimple Lie groups IV/V/VI*, Amer J. Math. 77 (1955), 743 – 777; Amer. J. Math. 78 (1956), 1 – 41; Amer J. Math. 78 (1956), 564 – 628. MR0072427 (17:282c); MR0082055 (18:490c); MR0082056 (18:490d)

[Ho1] R. Howe, *Reciprocity laws in the theory of dual pairs*, in Representation Theory of Reductive Groups, Prog. in Math. 40, P. Trombi, ed., Birkhäuser, Boston (1983), 159 – 175. MR0733812 (85k:22033)

[Ho2] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. 313 (1989), 539 – 570. MR0986027 (90h:22015a)

[Ho3] R. Howe, *Transcending classical invariant theory*, J. Amer. Math. Soc. 2 (1989), 535 – 552. MR0985172 (90k:22016)

[Ho4] R. Howe, *Perspectives on Invariant Theory*, The Schur Lectures, I. Piatetski-Shapiro and S. Gelbart (eds.), Israel Mathematical Conference Proceedings, 1995, 1 – 182. MR1321638 (96e:13006)

[HTW] R. Howe, E-C. Tan and J. Willenbring, *Reciprocity algebras and branching for classical symmetric pairs*, in preparation.

[JK] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Enc. of Math. and App. 16, Addison-Wesley, Reading, MA., 1981. MR0644144 (83k:20003)

[KV] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Inv. Math. 44 (1978), 1 – 47. MR0463359 (57:3311)

[Ki1] R. King, *Generalized Young tableaux and the general linear group*, J. Math. Phys. 11 (1970), 280 – 293. MR0251972 (40:5197)

[Ki2] R. King, *Modification rules and products of irreducible representations for the unitary, orthogonal, and symplectic groups*, J. Math. Phys. 12 (1971), 1588 – 1598. MR0287816 (44:5019)

[Ki3] R. King, *Branching rules for classical Lie groups using tensor and spinor methods*, J. Phys. A 8 (1975), 429 – 449. MR0411400 (53:15136)

[Ki4] R. King, *S-Functions and characters of Lie algebras and superalgebras*, in Invariant Theory and Tableaux (D. Stanton, Ed.), IMA Vol. Math. Appl. 19, Springer, New York, 1990, 227 – 261. MR1035497 (90k:17014)

[KTW] R. King, F. Toumazet and B. G. Wybourne, *Products and symmetrized powers of irreducible representations of $SO^*(2n)$*, J. Phys. A 31 (1998), 6691 – 6705. MR1643812 (2000i:81043)

[KW] R. King and B. Wybourne, *Holomorphic discrete series and harmonic series unitary representations of non-compact Lie groups: $Sp(2n,\mathbb{R})$, $U(p,q)$ and $SO^*(2n)$*, J. Phys. A 18 (1985), 3113 – 3139. MR0812427 (87b:22032)

[Kn1] A. Knapp, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, Princeton University Press, Princeton, NJ, 1986. MR0855239 (87j:22022)

[Kn2] A. Knapp, *Branching theorems for compact symmetric spaces*, Represent. Theory 5 (2001), 404 – 436. MR1870596 (2002i:20065)

[Kn3] A. Knapp, *Lie Groups Beyond An Introduction*, Second edition, Progress in Mathematics 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR1920389 (2003c:22001)

[Kn4] A. Knapp, *Geometric interpretations of two branching theorems of D. E. Littlewood.*, J. Algebra 270 (2003), No. 2, 728 – 754. MR2019637

[Ko] Kazuhiko Koike, *On the decomposition of tensor products of the representations of the classical groups*, Adv. Math. 74 (1989), No. 1, 57 – 86. MR0991410 (90j:22014)

[KT] K. Koike and I. Terada, *Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank*, Adv. Math. 79 (1990), No. 1, 104 – 135. MR1031827 (91a:22013)

[Ku] S. Kudla, *Seesaw dual reductive pairs*, in Automorphic Forms of Several Variables, Taniguchi Symposium, Katata, 1983, Birkhäuser, Boston (1983), 244 – 268. MR0763017 (86b:22032)

[Li1] D. Littlewood, *On invariant theory under restricted groups*, Philos. Trans. Roy. Soc. London. Ser. A., 239 (1944), 387 – 417. MR0012299 (7:6e)

[Li2] D. Littlewood, *Theory of Group Characters*, Clarendon Press, Oxford, 1945.

[Li3] D. Littlewood, *Products and plethysms of characters with orthogonal, symplectic and symmetric groups*, Can. Jour. Math. 10 (1958), 17 – 32. MR0095209 (20:1715)

[Ma] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979. MR0553598 (84g:05003)

[Ne] M. Newell, *Modification rules for the orthogonal and symplectic groups*, Proc. Roy. Irish Acad. Sect. A. 54 (1951), 153 – 163. MR0043093 (13:204e)

[OZ] B. Orsted and G. Zhang, *Tensor products of analytic continuations of holomorphic discrete series*, Can. J. Math. 49 (1997), No. 6, 1224 – 1241. MR1611656 (2000b:22012)

[Pr] R. Proctor, *Young tableaux, Gelfand patterns and branching rules for classical groups*, J. Alg., Vol. 164 (1994), 299 – 360. MR1271242 (96e:05180)

[Re] J. Repka, *Tensor product of holomorphic discrete series representations*, Can. J. Math. 31, No. 4 (1979), 863 – 844. MR0540911 (82c:22017)

[RWB] D. Rowe, B. Wybourne and P. Butler, *Unitary representations, branching rules and matrix elements for the non-compact symplectic group*, J. Phys. A 18 (1985), 939 – 953. MR0792219 (87b:22044)

[Sa] B. Sagan, *The Symmetric Group*, Wadsworth and Cole, Pacific Grove, CA, 1991. MR1093239 (93f:05102)

[Sc1] W. Schmid, *Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen*, Inv. Math. 9 (1969), 61 – 80. MR0259164 (41:3806)

[Sc2] W. Schmid, *On the characters of the discrete series (the Hermitian symmetric case)*, Inv. Math. 30 (1975), 47 – 199. MR0396854 (53:714)

[St1] R. Stanley, *The stable behaviour of some characters of $SL(n, \mathbb{C})$*, Lin. and Mult. Alg., Vol. 16 (1984), 3 – 27. MR0768993 (86e:22025)

[St2] R. Stanley, *Enumerative Combinatorics*, Vol. 1, Wadsworth and Cole, Monterey, CA, 1986. MR0847717 (87j:05003)

[St3] R. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press, Cambridge, UK, 1999. MR1676282 (2000k:05026)

[Su] S. Sundaram, *Tableaux in the representation theory of the classical Lie groups*, in Invariant Theory and Tableaux (D. Stanton, Ed.), IMA Vol. Math. Appl. 19, Springer, New York, 1990, 191 – 225. MR1035496 (91e:22022)

[TTW] J-Y Thibon, F. Toumazet and B. Wybourne, *Products and plethysms for the fundamental harmonic series representations of $U(p, q)$*, J. Phys. A 30 (1997), 4851 – 4856. MR1479810 (98k:81093)

[We] H. Weyl, *The Classical Groups*, Princeton Univ. Press, Princeton, 1946. MR0000255 (1:42c)

[Zh] D. Zhelobenko, *Compact Lie Groups and their Representations*, Transl. of Math. Mono. 40, AMS, Providence, R.I., 1973. MR0473098 (57:12776b)

Department of Mathematics, Yale University, New Haven, Connecticut 06520-8283

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53211-3029