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UPPER BOUNDS FOR THE NUMBER OF SOLUTIONS OF A DIOPHANTINE EQUATION

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Abstract. We give upper bound estimates for the number of solutions of a certain diophantine equation. Our results can be applied to obtain new lower bound estimates for the L_1-norm of certain exponential sums.

1. Introduction

Throughout the text the following notations will be used:

$A \ll B$ means $|A| \le cB$ with some absolute constant c.

$A \underset{a,b,\ldots}{\ll} B$ means $|A| \le cB$ with some constant c which may depend only on $a, b, \ldots$.

N is an integer parameter, $N \ge 3$. We also assume that $f(n)$ is an integer for integers $n, 1 \le n \le N$.

In 1981 S. V. Konyagin [7] and O. C. McGehee, L. Pigno and B. Smith [9] proved the Littlewood conjecture which states that

$$\int_0^1 \left| \sum_{n=1}^{N} \exp(2\pi i \alpha f(n)) \right| d\alpha \gg \log N. \tag{1}$$

The relation [11, p. 67]

$$\int_0^1 \left| \sum_{n=1}^{N} \exp(2\pi i \alpha n) \right| d\alpha = \frac{4}{\pi^2} \log N + O(1)$$

shows that the order $\log N$ in (1) is sharp. However, for a wide class of sequences $f(n)$ estimate (1) can be improved.

Let $f(1) < f(2) < \ldots < f(N)$ and $J = J(N)$ be the number of solutions of the equation

$$f(x) + f(y) = f(u) + f(v), \quad 1 \le x, y, u, v \le N.$$

Theorem (A. A. Karatsuba [6])**.** *For any coefficients $\gamma_n, |\gamma_n| = 1$, the inequality*

$$I = I(N) = I(N, f) := \int_0^1 \left| \sum_{n=1}^{N} \gamma_n \exp(2\pi i \alpha f(n)) \right| d\alpha \ge \left(\frac{N^3}{J} \right)^{1/2}$$

is valid.

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If $f(n)$ is a polynomial with integer coefficients, $\deg f > 1$, then $J \ll N^{2+\varepsilon}$ for any $\varepsilon > 0$ and $N > N_0(\varepsilon, f)$. Therefore, Karatsuba's theorem gives the estimate $I(N) \gg N^{\frac{1}{2}-\varepsilon}$ for $N > N_0(\varepsilon, f)$. Note that on the other hand we have

$$I(N) \le \int_0^1 \left| \sum_{n=1}^{N} \gamma_n \exp(2\pi i \alpha f(n)) \right|^2 d\alpha = N^{\frac{1}{2}}.$$

Another example is when $f(n)$ is a very fast increasing sequence, say $f(n) = a^n$ with integer $a > 1$. Then the theorem implies

$$N^{\frac{1}{2}} \ll I(N) \ll N^{\frac{1}{2}}.$$

It should be pointed out that in this case A. A. Karatsuba carried out another approach and obtained an asymptotic formula for $I(N, f)$, and later M. A. Korolev for any positive number p obtained an asymptotic formula for the L_p-norm of the sum (none of these results is published).

S. V. Konyagin [8], using the work of G. Elekes, M. B. Nathanson and I. Z. Ruzsa [2], proved that if

$$0 < f(2) - f(1) < f(3) - f(2) < \dots < f(N) - f(N-1), \tag{2}$$

then $J \ll N^{\frac{5}{2}}$, and thus $I \gg N^{\frac{1}{4}}$. In particular, for Bochkarev's sequence [1]

$$f(n) = [2^{(\log n)^{\beta}}]$$

the estimate $I \gg N^{\frac{1}{4}}$ holds for any fixed $\beta > 1$ and $N \ge N_0(\beta)$. The same holds for the more general sequence [6]

$$f(n) = [e^{A(\log n)^{\beta}}],$$

where $A > 0, \beta > 1$ are fixed numbers.

For a new proof of Konyagin's result see our work [3].

In the present paper we give a new upper bound for $J(N)$. In particular, we slightly improve (see Corollary 2) one of the main results of [5], give another proof of Konyagin's estimate, and slightly improve it for Bochkarev's sequence.

2. The results

For a given integer l, $1 \le l < N$, let $J_l = J_l(N)$ denote the number of solutions of the equation

$$f(x) + f(y) = f(x+l) + f(z), \quad 1 \le x < x+l \le z < y \le N. \tag{3}$$

Theorem 1. *For any real number ε, $0 < \varepsilon < 1$, we have*

$$J \ll N^{2+\varepsilon} + N^{\varepsilon} \sum_{1 \le l \le N^{1-\varepsilon}} J_l.$$

Note that if $f(n)$ satisfies condition (2), then $J_l < lN$. Indeed, it would follow from (3) that $z < y < z + l$. For z we have at most N possibilities, and once z is fixed we have less than l possibilities for y. Besides, for fixed y, z, we have at most one solution of (3) in variable x. Therefore, $J_l < lN$. Taking $\varepsilon = \frac{1}{2}$ and applying the Karatsuba theorem we obtain Konyagin's estimate:

Corollary 1. *For the sequence* (2) *we have* $I \gg N^{\frac{1}{4}}$.

Corollary 2. *Let $f(x) = [F(x)]$, where the real valued function $F(x)$ is three times continuously differentiable on the segment* [1,N], $F'(x) > 0$, $F''(x) > 0, F'''(x) < 0$. *Then*

$$J \ll \left(F'(1)^{-1} + 1\right) N^{5/2} + N^2 F''(N)^{-1}.$$

In particular, if $F'(1) \geq 1$, then

$$I \gg \min\left(N^{\frac{1}{4}},\ N^{\frac{1}{2}} F''(N)^{\frac{1}{2}}\right).$$

In the case $F(x) = Ax^{\alpha}$, $A > 0$, $1 < \alpha \leq 3/2$, Corollary 2 gives

$$J \underset{\alpha,A}{\ll} N^{4-\alpha}.$$

Note that for $1 < \alpha < \frac{3}{2}$ this estimate was established and applied to the Waring-Goldbach problem by I. I. Piatetski-Shapiro [10].

In order to prove Corollary 2, we use a result from [5] which states that if $F'(1) \geq 1$, then

$$J_l \leq \frac{6N}{F''(N)} + 3lN.$$

Taking $\varepsilon = \frac{1}{2}$ and applying Theorem 1 we obtain Corollary 2 in this case. If $F'(1) < 1$, then we reduce it to the first case as it was done in [5].

Conjecture. *Let $f(x) = [Ax^{\alpha}]$, $A > 0, 1 < \alpha < 2$. Then*

$$J \underset{\alpha,A}{\ll} N^{4-\alpha}.$$

The validity of the conjecture would also have an important application to the Waring-Goldbach problem with a small non-integer exponent. For more details we refer the reader to [4].

Theorem 2. *Let $A > 0, \beta > 1$, and $f(n) = [e^{A(\log n)^{\beta}}]$. Then*

$$J \underset{A,\beta}{\ll} N^{\frac{5}{2}} (\log N)^{\frac{3(1-\beta)}{2}} \log\log N.$$

Corollary 3. *Under the assumption of Theorem 2 the estimate*

$$I \underset{A,\beta}{\gg} N^{\frac{1}{4}} (\log N)^{\frac{3(\beta-1)}{4}} (\log\log N)^{-\frac{1}{2}}$$

holds for all $N > N_0(A, \beta)$.

It is interesting to investigate J for more general rapidly increasing sequences, in particular for $f(n) = [e^{An^{\beta}}]$, where $A > 0, 0 < \beta < 1$. In this connection we would like to stress an unpublished work of S. V. Konyagin, where for $\beta > \frac{1}{3}$ he obtains an asymptotic formula $J \sim 2N^2$.

3. Proof of Theorem 1

Denote

$$S(\alpha) := \sum_{1 \leq n \leq N} e^{2\pi i \alpha f(n)}.$$

For a given integer $s, 1 \leq s \leq [N^{\varepsilon}]$, put

$$I_s = \{n \in Z : (s-1)N^{1-\varepsilon} < n \leq sN^{1-\varepsilon}\},$$

and for $s = [N^\varepsilon] + 1$ put

$$I_s = \{n \in Z : (s-1)N^{1-\varepsilon} < n \le N\}.$$

Then

$$|S(\alpha)|^4 = \left| \sum_{1\le n\le N} e^{4\pi i\alpha f(n)} + 2 \sum_{1\le n<m\le N} e^{2\pi i\alpha(f(n)+f(m))} \right|^2$$
$$\ll N^2 + \left| \sum_{1\le n<m\le N} e^{2\pi i\alpha(f(n)+f(m))} \right|^2$$
$$\ll N^2 + \left| \sum_{s\le 1+N^\varepsilon} \sum_{n\in I_s} \sum_{n<m\le N} e^{2\pi i\alpha(f(n)+f(m))} \right|^2,$$

whence

$$|S(\alpha)|^4 \ll N^2 + N^\varepsilon \sum_{s\le 1+N^\varepsilon} \left| \sum_{n\in I_s} \sum_{n<m\le N} e^{2\pi i\alpha(f(n)+f(m))} \right|^2.$$

Therefore,

$$J = \int_0^1 |S(\alpha)^4| d\alpha \ll N^2 + N^\varepsilon \sum_{s\le 1+N^\varepsilon} \sideset{}{'}\sum_{n,n_1,m,m_1} 1 \ll N^2 + N^\varepsilon J',$$

where the prime means that the inside summation is taken over the integers n, n_1, m, m_1 with conditions

$$f(n) + f(m) = f(n_1) + f(m_1),\ n \in I_s,\ n_1 \in I_s,\ n < m \le N,\ n_1 < m_1 \le N,$$

and J' is the number of solutions of the equation

$$f(n) + f(m) = f(n_1) + f(m_1),\ |n - n_1| \le N^{1-\varepsilon},\ n < m \le N,\ n_1 < m_1 \le N.$$

Theorem 1 now follows from

$$J' \ll N^2 + \sum_{l\le N^{1-\varepsilon}} J_l.$$

4. Proof of Theorem 2

Obviously $J \le 8J_1$, where J_1 is the number of solutions of the equation

$$f(x) + f(y) = f(u) + f(v),\ 1 \le x, y, u, v \le N,\ x \ge y,\ u \ge v,\ x \ge u.$$

From

$$e^{A(\log x)^\beta} \le e^{A(\log u)^\beta + 2}$$

it follows that

$$0 \le x - u \le cN(\log N)^{1-\beta}$$

with some $c = c(A, \beta) > 0$. Hence $J \le 8J_2$ where J_2 is the number of solutions of the equation

$$f(x) - f(u) = f(v) - f(y),\ 1 \le x, y, u, v \le N,\ 0 \le x - u \le N_1.$$

Here $N_1 = cN(\log N)^{1-\beta}$.

Let $T_1(n)$ denote the number of solutions of the equation

$$f(x) - f(u) = n,\ 1 \le x, u \le N,\ 0 \le x - u \le N_1,$$

and let $T_2(n)$ denote the number of solutions of the equation

$$f(y) - f(v) = n, \ 1 \le y, v \le N.$$

Then

$$J^2 \le 64J_2^2 = 64\left(\sum_n T_1(n)T_2(n)\right)^2 \le 64\sum_n T_1^2(n)\sum_n T_2^2(n) = 64J_3J,$$

where $J_3 := \sum_n T_1^2(n)$ is equal to the number of solutions of the equation

$$f(x) - f(u) = f(v) - f(y)$$

subject to

$$1 \le x, y, u, v \le N, \ 0 \le x - u, \ v - y \le N_1.$$

Therefore $J \le 64N^2 + 64J_4$, where J_4 denotes the number of solutions of the same equation subject to

$$1 \le x, y, u, v \le N, \ 1 \le x - u, \ v - y \le N_1.$$

From the inequality $|a + b|^2 \le 2|a|^2 + 2|b|^2$ and the relation

$$J_4 = \int_0^1 \left| \sum_{\substack{1 \le x - u \le N_1 \\ x, u \le N}} e^{2\pi i\alpha(f(x)-f(u))} \right|^2 d\alpha$$

it follows that

$$J \le 64N^2 + 64J_4 \le 64N^2 + 128J_5 + 128J_6, \tag{4}$$

where

$$J_5 = \int_0^1 \left| \sum_{\substack{1 \le x - u \le N_2 \\ x, u \le N}} e^{2\pi i\alpha(f(x)-f(u))} \right|^2 d\alpha, \tag{5}$$

$$J_6 = \int_0^1 \left| \sum_{\substack{N_2 < x - u \le N_1 \\ x, u \le N}} e^{2\pi i\alpha(f(x)-f(u))} \right|^2 d\alpha, \tag{6}$$

and

$$N_2 = N_1(\log N)^{-10\beta} = cN(\log N)^{1-11\beta}. \tag{7}$$

To prove Theorem 2, we obtain upper bounds for J_5 and J_6.

Estimate of J_5. From (5) it follows that J_5 is equal to the number of solutions of the corresponding diophantine equation. Therefore, the wider the range of summation over x and u, the larger the value of the integral on the right-hand side of (5). Hence,

$$J_5 \le \int_0^1 \left| \sum_{l \ll NN_2^{-1}} \sum_{\substack{(l-1)N_2 < x \le lN_2 \\ 1 \le x - u \le N_2}} e^{2\pi i\alpha(f(x)-f(u))} \right|^2 d\alpha,$$

whence, by Cauchy inequality,

$$J_5 \ll NN_2^{-1} \sum_{l \ll NN_2^{-1}} \int_0^1 \left| \sum_{\substack{(l-1)N_2 < x \le lN_2 \\ 1 \le x-u \le N_2}} e^{2\pi i\alpha(f(x)-f(u))} \right|^2 d\alpha.$$

For a fixed l the integral on the right-hand side is not greater than the number of solutions of the equation

$$f(x_1 + (l-1)N_2)) - f(u_1 + (l-1)N_2) = f(v_1 + (l-1)N_2)) - f(y_1 + (l-1)N_2)$$

with

$$1 \le x_1, y_1, u_1, v_1 \le 2N_2.$$

This number, according to Konyagin's estimate, is $\ll_{A,\beta} N_2^{5/2}$. Therefore,

$$J_5 \ll_{A,\beta} N^2 N_2^{-2} N_2^{5/2},$$

whence, by (7),

$$J_5 \ll_{A,\beta} N^{5/2} (\log N)^{\frac{3(1-\beta)}{2}} \log\log N. \tag{8}$$

Estimate of J_6. Note that J_6, defined by (6), is equal to the number of solutions of the equation

$$f(x) - f(u) = f(v) - f(y) \tag{9}$$

subject to the condition

$$1 \le x, y, u, v \le N,\ N_2 < x - u \le N_1,\ \ N_2 < v - y \le N_1. \tag{10}$$

Let us prove that $|x - v| \ll_{A,\beta} N_1 \log\log N$. Without loss of generality we may suppose $x \ge v$.

If $u \le v$, then $x - v \le x - u$ and we are done in this case. Otherwise, from (9), we have

$$e^{A(\log x)^\beta} - e^{A(\log u)^\beta} < 2e^{A(\log v)^\beta},$$

whence

$$e^{A(\log x)^\beta - A(\log v)^\beta} - e^{A(\log u)^\beta - A(\log v)^\beta} \le 2.$$

Therefore,

$$e^{A(\log u)^\beta - A(\log v)^\beta} \left(e^{A(\log x)^\beta - A(\log u)^\beta} - 1 \right) \le 2.$$

From $e^r \ge 1 + r$ it follows that

$$e^{A(\log u)^\beta - A(\log v)^\beta} \left(A(\log x)^\beta - A(\log u)^\beta \right) \le 2. \tag{11}$$

On the other hand, for some $t \in (u, x)$ we have

$$A(\log x)^\beta - A(\log u)^\beta = A\beta(x-u)t^{-1}(\log t)^{\beta-1},$$

which, in view of (10), is $\gg_{A,\beta} N_2 N^{-1} (\log N)^{\beta-1}$. Therefore, by (7),

$$A(\log x)^\beta - A(\log u)^\beta \gg_{A,\beta} (\log N)^{-10\beta}.$$

Hence, from (11),

$$e^{A(\log u)^\beta - A(\log v)^\beta} \ll_{A,\beta} (\log N)^{10\beta},$$

whence

$$(\log u)^\beta - (\log v)^\beta \underset{A,\beta}{\ll} \log\log N.$$

Then, for some real $t \in (v, u)$, we have

$$(u - v)t^{-1}(\log t)^{\beta-1} \underset{A,\beta}{\ll} \log\log N.$$

Therefore,

$$u - v \underset{A,\beta}{\ll} N(\log N)^{1-\beta} \log\log N \underset{A,\beta}{\ll} N_1 \log\log N.$$

Together with $0 < x - u \le N_1$ we conclude that $x - v \underset{A,\beta}{\ll} N_1 \log\log N$.

Thus, for some $c_1 = c_1(A, \beta) > 0$, we have

$$|x - v| \le c_1 N_1 \log\log N. \tag{12}$$

We can split the range of variation of x into intervals of length at most $c_1 N_1 \log\log N$. The number of such intervals is

$$\underset{A,\beta}{\ll} N(N_1 \log\log N)^{-1} = (\log N)^{\beta-1}(\log\log N)^{-1}.$$

It then follows that there exists $l, 1 \le l \ll (\log N)^{\beta-1}(\log\log N)^{-1}$, such that

$$J_6 \underset{A,\beta}{\ll} (\log N)^{\beta-1}(\log\log N)^{-1} J_7, \tag{13}$$

where J_7 denotes the number of solutions of (9) subject to conditions (10), (12) and

$$(l - 1)c_1 N_1 \log\log N < x \le l c_1 N_1 \log\log N.$$

Hence, in view of (12),

$$(k - 1)c_1 N_1 \log\log N < x \le (k + 1)c_1 N_1 \log\log N$$

and

$$(k - 1)c_1 N_1 \log\log N < v \le (k + 1)c_1 N_1 \log\log N,$$

where $k = \max(1, l - 1)$. Then, taking (10) into account, we have

$$J_7 \underset{A,\beta}{\ll} \int_0^1 \left| \sum_{\substack{(k-1)L < x \le (k+1)L \\ 1 \le x - u \le N_1}} e^{2\pi i\alpha(f(x) - f(u))} \right|^2 d\alpha,$$

where $L = c_1 N_1 \log\log N$.

We again use the fact that the wider the range of summation over x and u the larger the value of the integral. This gives

$$J_7 \underset{A,\beta}{\ll} \int_0^1 \left| \sum_{n \ll \log\log N} \sum_{\substack{(k-1)L + (n-1)N_1 < x \le (k-1)L + nN_1 \\ 1 \le x - u \le N_1}} e^{2\pi i\alpha(f(x) - f(u))} \right|^2 d\alpha.$$

By Cauchy inequality

$$J_7 \underset{A,\beta}{\ll} \log\log N \sum_{n \ll \log\log N} \int_0^1 \left| \sum_{\substack{(k-1)L + (n-1)N_1 < x \le (k-1)L + nN_1 \\ 1 \le x - u \le N_1}} e^{2\pi i\alpha(f(x) - f(u))} \right|^2 d\alpha.$$

It then follows that for some fixed $n = n_0 \ll \log\log N$ we have

$$J_7 \underset{A,\beta}{\ll} (\log\log N)^2 \int_0^1 \left| \sum_{\substack{(k-1)L+(n_0-1)N_1<x\le(k-1)L+n_0N_1 \\ 1\le x-u\le N_1}} e^{2\pi i\alpha(f(x)-f(u))} \right|^2 d\alpha.$$

The latter integral does not exceed the number of solutions of the equation

$$f_1(x_1) - f_1(u_1) = f_1(v_1) - f_1(y_1)$$

with $1 \le x_1, y_1, u_1, v_1 \le 2N_1$ and $f_1(z) = f((k-1)L + (n_0-1)N_1 + z)$. From Konyagin's estimate we conclude that this integral is $\underset{A,\beta}{\ll} N_1^{5/2}$. Hence

$$J_7 \underset{A,\beta}{\ll} (\log\log N)^2 N_1^{5/2}.$$

Therefore, by (13) and (7), we obtain

$$J_6 \underset{A,\beta}{\ll} (\log N)^{\beta-1}(\log\log N)^{-1} J_7 \underset{A,\beta}{\ll} N^{\frac{5}{2}} (\log N)^{\frac{3(1-\beta)}{2}} \log\log N.$$

This estimate, by virtue of (8) and (4), proves Theorem 2.

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