

POSITIVITY PRESERVING TRANSFORMATIONS FOR q -BINOMIAL COEFFICIENTS

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ABSTRACT. Several new transformations for q -binomial coefficients are found, which have the special feature that the kernel is a polynomial with nonnegative coefficients. By studying the group-like properties of these positivity preserving transformations, as well as their connection with the Bailey lemma, many new summation and transformation formulas for basic hypergeometric series are found. The new q -binomial transformations are also applied to obtain multisum Rogers–Ramanujan identities, to find new representations for the Rogers–Szegő polynomials, and to make some progress on Bressoud’s generalized Borwein conjecture. For the original Borwein conjecture we formulate a refinement based on new triple sum representations of the Borwein polynomials.

1. INTRODUCTION

1.1. **q -binomial transformations.** In the literature on q -series one finds numerous transformations of type

$$(1.1) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \frac{q^{\frac{1}{4}r^2}(q; q)_L}{(q; q)_{\frac{1}{2}(L-r)}(q; q)_r} \left[\begin{matrix} r \\ \frac{1}{2}(r-j) \end{matrix} \right] = q^{\frac{1}{4}j^2} \left[\begin{matrix} L \\ \frac{1}{2}(L-j) \end{matrix} \right]$$

and

$$(1.2) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \frac{q^{\frac{1}{8}r^2}(q; q)_L}{(q, -q^{\frac{1}{2}(r+1)}; q)_{\frac{1}{2}(L-r)}(q; q)_r} \left[\begin{matrix} r \\ \frac{1}{2}(r-j) \end{matrix} \right] = q^{\frac{1}{8}j^2} \left[\begin{matrix} L \\ \frac{1}{2}(L-j) \end{matrix} \right],$$

where j and L are integers such that $j \equiv L \pmod{2}$. (Throughout this paper the notation $a \equiv b \pmod{c}$ instead of $a \equiv b \pmod{c}$ will be used in equations for brevity.) Here

$$\left[\begin{matrix} L \\ a \end{matrix} \right]_q = \left[\begin{matrix} L \\ a \end{matrix} \right] = \begin{cases} \frac{(q; q)_L}{(q; q)_a (q; q)_{L-a}} & \text{for } a \in \{0, \dots, L\}, \\ 0 & \text{otherwise} \end{cases}$$

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is a q -binomial coefficient,

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$$

is a q -shifted factorial, and

$$(a_1, \dots, a_k; q)_n = \prod_{j=1}^k (a_j; q)_n.$$

Important features of (1.1) and (1.2) are (i) the sum over a q -binomial coefficient multiplied by a simple factor again yields a q -binomial coefficient, (ii) only the lower entries of the q -binomial coefficients and a simple exponential factor on the right depend on j , (iii) they can readily be iterated.

As an example of this last point let us consider the simple q -binomial identity

$$(1.3) \quad \sum_{j=-L}^L (-1)^j q^{\binom{j}{2}} \begin{bmatrix} 2L \\ L-j \end{bmatrix} = \delta_{L,0},$$

which is a special case of the finite form of Jacobi's triple product identity [3, Ch. 3, Example 1] (see also (7.7)). Replacing L by r , multiplying both sides by

$$\frac{q^{r^2} (q; q)_{2L}}{(q; q)_{L-r} (q; q)_{2r}}$$

and summing over r using (1.1) with $L \rightarrow 2L$, $j \rightarrow 2j$ and $r \rightarrow 2r$, yields

$$(1.4) \quad \sum_{j=-L}^L (-1)^j q^{j^2 + \binom{j}{2}} \begin{bmatrix} 2L \\ L-j \end{bmatrix} = (q^{L+1}; q)_L.$$

This bounded version of Euler's pentagonal number theorem [42] is of the same form as (1.3) and we may repeat the above procedure to find the well-known bounded analogue of the first Rogers–Ramanujan identity [2, 58]

$$(1.5) \quad \sum_{j=-L}^L (-1)^j q^{2j^2 + \binom{j}{2}} \begin{bmatrix} 2L \\ L-j \end{bmatrix} = (q^{L+1}; q)_L \sum_{r=0}^L q^{r^2} \begin{bmatrix} L \\ r \end{bmatrix}.$$

Some known q -binomial transformations similar to (1.1) and (1.2), but in which the base of the q -binomial coefficient is changed from q to q^2 or q^3 , are given by

$$(1.6) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \frac{q^{\frac{1}{4}r^2} (q; q)_L}{(q^2; q^2)_{\frac{1}{2}(L-r)} (q; q)_r} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix} = q^{\frac{1}{4}j^2} \begin{bmatrix} L \\ \frac{1}{2}(L-j) \end{bmatrix}_{q^2},$$

$$(1.7) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \frac{q^{\frac{1}{8}r^2} (q^{\frac{1}{2}(r+1)}; q)_{L-r} (q; q)_L}{(q^2, q^{r+1}; q^2)_{\frac{1}{2}(L-r)} (q; q)_r} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix} = q^{\frac{1}{8}j^2} \begin{bmatrix} L \\ \frac{1}{2}(L-j) \end{bmatrix}_{q^2},$$

and

$$(1.8) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \frac{q^{\frac{1}{4}r^2} (q; q)_{\frac{1}{2}(3L-r)}}{(q^3; q^3)_{\frac{1}{2}(L-r)} (q; q)_r} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix} = q^{\frac{1}{4}j^2} \begin{bmatrix} L \\ \frac{1}{2}(L-j) \end{bmatrix}_{q^3},$$

assuming once again that $j \equiv L \pmod{2}$.

All of the above transformations are of the form

$$(1.9) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{\frac{1}{8}\gamma r^2} f_{L,r}(q) \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix} = q^{\frac{1}{8}\gamma j^2} \begin{bmatrix} L \\ \frac{1}{2}(L-j) \end{bmatrix}_{q^k}$$

with $f_{L,r}(q)$ a polynomial in q or $q^{1/2}$, which for $0 \leq r < L$ has both positive and negative coefficients.

The issue of positivity of coefficients in polynomial expressions of the type given by the left-hand sides of (1.3), (1.4) and (1.5) has recently received considerable attention in relation to conjectures of Borwein [8] and Bressoud [19]. For this reason it is important to find q -binomial transformations à la (1.9) with $f_{L,r}(q)$ a polynomial with nonnegative coefficients. We will refer to such transformations as positivity preserving. Indeed, applying a positivity preserving transformation to an identity like (1.3) — with a polynomial with nonnegative coefficients on the right — results in a new identity which again has a polynomial with nonnegative coefficients on the right.

1.2. Outline. In the next section five new, positivity preserving q -binomial transformations plus two related, rational transformations are proved. In order to establish the positivity of one of our results we generalize nonnegativity theorems of Andrews for q -binomial coefficients and of Haiman for principally specialized Schur functions.

Group-like relations among our q -binomial transformations and those listed in the Introduction are investigated in Section 3. This will give rise to numerous new transformation formulas for balanced and ‘almost’ balanced basic hypergeometric series.

The inverses of the transformations for q -binomial coefficients are established in Section 4. Again this will lead to several elegant new summation formulas.

The relation between our q -binomial transformations and the Bailey lemma is the subject of Sections 5 and 6. The reader may indeed have recognized (1.1) and (1.2) as special cases of the ordinary Bailey lemma in its version due to Andrews [6] and Paule [41], and (1.6)–(1.8) as special cases of base-changing extensions of the Bailey lemma discovered by Bressoud, Ismail and Stanton [20]. In Section 5 we show that our new transformations correspond to new types of base-changing Bailey lemmas. In Section 6 this is exploited to yield some new (and old) transformations for basic hypergeometric series.

Sections 7 and 8 deal with simple applications of the q -binomial transformations of section 2. In Section 7 new single and multisum identities of the Rogers–Ramanujan identities are proved and in Section 8 we obtain a remarkable new representation of the Rogers–Szegő polynomials.

Finally, in Section 9, we use the positivity preserving nature of our results to make some progress on Bressoud’s generalized Borwein conjecture. In the last section we also prove new triple-sum representation for the Borwein polynomials and use this to formulate a new conjecture that implies the original Borwein conjecture.

2. POSITIVITY PRESERVING q -BINOMIAL TRANSFORMATIONS

The reason that none of the transformations of the previous section preserves positivity is not a very deep one. Setting $q = 1$ in (1.9) yields

$$\sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L f_{L,r}(1) \binom{r}{\frac{1}{2}(r-j)} = \binom{L}{\frac{1}{2}(L-j)},$$

which has the unique solution $f_{L,r}(1) = \delta_{L,r}$. Hence the only polynomial solution to (1.9) that preserves positivity is the less-than-exciting $f_{L,r}(q) = \delta_{L,r}$ for $k = 1$ and $\gamma = 0$. To get around this problem we need to modify (1.9), and in the following we look for polynomials $f_{L,r}(q)$ with nonnegative coefficients that satisfy

$$(2.1) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{\frac{1}{4}\gamma r^2} f_{L,r}(q) \left[\binom{r}{\frac{1}{2}(r-j)} \right]_{q^k} = q^{\frac{1}{4}\gamma j^2} \left[\binom{2L}{L-j} \right], \quad k \geq 1,$$

or small variations hereof (see (2.18) below).

To see that from a positivity point of view (2.1) is indeed more promising than (1.9), let us again set $q = 1$. Multiplying both sides by x^{2j} and summing over j using the binomial theorem gives

$$\sum_{r=0}^L f_{L,r}(1)(x^2 + x^{-2})^r = (x + x^{-1})^{2L}.$$

This is readily solved to yield

$$(2.2) \quad f_{L,r}(1) = 2^{L-r} \binom{L}{r},$$

a solution that may well have q -analogues free of minus signs.

In the remainder of the paper we will make extensive use of basic hypergeometric series, and before presenting our solutions to (2.1) we need to introduce some further notation [30]. First,

$$\begin{aligned} {}_r\phi_s \left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right] &= {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_r; q, z) \\ &= \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{s-r+1} z^k. \end{aligned}$$

Here it is assumed that the b_i are such that none of the factors in the denominator is zero, $q \neq 0$ if $r > s + 1$ and $|q| < 1$ whenever the ${}_r\phi_s$ is nonterminating. Moreover, if the series does not terminate, then $r \leq s + 1$ with $|z| < 1$ if $r = s + 1$. If it does however terminate, one can reverse the order of summation as discussed in [30, Exercise 1.4]. An ${}_{r+1}\phi_r$ series is called balanced if $z = q$ and $a_1 \cdots a_{r+1}q = b_1 \cdots b_r$, well-poised if $qa_1 = a_2b_1 = \cdots = a_{r+1}b_r$ and very-well-poised if it is well-poised and $a_2 = -a_3 = a_1^{1/2}q$. We will always abbreviate such very-well-poised series by ${}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z)$. Whenever one of the numerator parameters in a q -hypergeometric series is q^{-n} we assume n to be a nonnegative integer. (Hence, provided the base of the series is q (or $q^{1/2}$, $q^{1/3}$ etc.), the series will terminate.) After these definitions we return to (2.1). For $k = 1$ it is not hard to see that there are no factorizable solutions (two nonfactorizable or non- q -hypergeometric solutions are given in Section 9), and all our results will involve a change of base.

There is of course ample precedent for base-changing transformations; see, e.g., [1, 13, 20, 28, 29, 30, 52, 53, 56].

Our first result is of a quadratic nature assuming $k = \gamma = 2$.

Lemma 2.1. *For L and j integers there holds*

$$(2.3) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{\frac{1}{2}r^2} (-q; q)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix}_{q^2} = q^{\frac{1}{2}j^2} \begin{bmatrix} 2L \\ L-j \end{bmatrix}.$$

This corresponds to

$$(2.4) \quad f_{L,r}(q) = (-q; q)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2},$$

which is about the simplest imaginable q -analogue of (2.2). Since the q -binomial coefficient on the right is a polynomial with nonnegative coefficients [3], so is $f_{L,r}(q)$.

By the substitution $q \rightarrow 1/q$ and the simple identities

$$\begin{bmatrix} L \\ a \end{bmatrix}_{q^{-1}} = q^{-a(L-a)} \begin{bmatrix} L \\ a \end{bmatrix}_q \quad \text{and} \quad (a; q^{-1})_n = (-1)^n a^n q^{-\binom{n}{2}} (a^{-1}; q)_n$$

we obtain the following corollary of Lemma 2.1.

Corollary 2.1. *For L and j integers there holds*

$$(2.5) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{\binom{L-r}{2}} (-q; q)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix}_{q^2} = \begin{bmatrix} 2L \\ L-j \end{bmatrix}.$$

This corresponds to (2.1) with $k = 2$ and $\gamma = 0$.

Proof of Lemma 2.1. Without loss of generality we may assume that $0 \leq j \leq L$. After shifting $r \rightarrow 2r + j$ the identity (2.3) corresponds to

$$(2.6) \quad {}_2\phi_1(q^{-n}, q^{1-n}; aq; q^2, aq^{2n}) = \frac{(a; q^2)_n}{(a; q)_n}$$

with $(a, n) \rightarrow (q^{2j+1}, L - j)$. (Throughout this paper we denote the simultaneous variable changes $a_1 \rightarrow b_1, \dots, a_k \rightarrow b_k$ by $(a_1, \dots, a_k) \rightarrow (b_1, \dots, b_k)$.) Equation (2.6) readily follows from the q -Gauss sum [30, Eq. (II.8)]

$$(2.7) \quad {}_2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}. \quad \square$$

Our next result is a somewhat more complicated quadratic transformation, in accordance with (2.1) for $k = 2$ and $\gamma = 1$.

Lemma 2.2. *For L and j integers there holds*

$$(2.8) \quad (1 + q^L) \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{\frac{1}{4}r^2} (-q^{r+2}; q^2)_{L-r-1} \begin{bmatrix} L \\ r \end{bmatrix} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix}_{q^2} = q^{\frac{1}{4}j^2} \begin{bmatrix} 2L \\ L-j \end{bmatrix}.$$

To make sense of the above lemma we need to extend our earlier definition of the q -shifted factorial, and for nonnegative n we set $(a; q)_{-n} = 1/(aq^{-n}; q)_n$. Note that this implies that $1/(q)_{-n} = 0$. With this definition it is once again clear that the corresponding polynomial $f_{L,r}(q)$ has nonnegative coefficients.

Before proving (2.8) we state a variation that is not of the form (2.1).

Lemma 2.3. *For L and j integers there holds*

$$(2.9) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \frac{q^{\frac{1}{4}r(r+2)}}{1+q^r} (-q^{r+1}; q^2)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix}_{q^2} = \frac{q^{\frac{1}{4}j(j+2)}}{1+q^j} \begin{bmatrix} 2L \\ L-j \end{bmatrix}.$$

Proof of Lemmas (2.2) and (2.3). Without loss of generality we may assume that $0 \leq j \leq L$. Shifting $r \rightarrow 2r + j$ the summations (2.8) and (2.9) correspond to

$$(2.10) \quad {}_3\phi_2 \left[\begin{matrix} a/b, q^{-n}, q^{1-n} \\ aq, q^{2-2n}/b \end{matrix}; q^2, q^2 \right] = \frac{(b; q)_n (a; q^2)_n}{(a; q)_n (b; q^2)_n}$$

with $(a, b, n) \rightarrow (q^{2j+1}, -q^j, L-j)$ and $(a, b, n) \rightarrow (q^{2j+1}, -q^{j+1}, L-j)$, respectively. Equation (2.10) follows from the $q \rightarrow q^2$ case of the q -Pfaff-Saalschütz sum [30, Eq. (II.24)] written in the form

$$(2.11) \quad {}_3\phi_2 \left[\begin{matrix} a, b, c \\ d, abcq/d \end{matrix}; q, q \right] = \frac{(q/d, abq/d, acq/d, bcq/d; q)_\infty}{(aq/d, bq/d, cq/d, abcq/d; q)_\infty},$$

provided the ${}_3\phi_2$ terminates. □

Our final solution to (2.1) provides a positivity-preserving transformation of a quartic nature.

Lemma 2.4. *For L and j integers there holds*

$$(2.12) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{L-r} (-q^{-1}; q^2)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix}_{q^4} = \begin{bmatrix} 2L \\ L-j \end{bmatrix}.$$

Once again we state a variation that is not of the form (2.1).

Lemma 2.5. *For L and j integers there holds*

$$(2.13) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \frac{q^r}{1+q^{2r}} (-q; q^2)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} r \\ \frac{1}{2}(r-j) \end{bmatrix}_{q^4} = \frac{q^j}{1+q^{2j}} \begin{bmatrix} 2L \\ L-j \end{bmatrix}.$$

Proof of Lemma 2.4. Without loss of generality we may assume that $0 \leq j \leq L$. After shifting $r \rightarrow 2r + j$ the identity (2.12) corresponds to

$$(2.14) \quad {}_4\phi_3 \left[\begin{matrix} aq, aq^3, q^{-2n}, q^{2-2n} \\ a^2q^2, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right] = q^{-n} \frac{(-q; q)_n (-a; q^2)_n}{(-q^{-1}; q^2)_n (-a; q)_n}$$

with $(a, n) \rightarrow (-q^{2j+1}, L-j)$. The above equation follows from [20, Eq. (2.1)] by the substitution $(C, D, m, q) \rightarrow (-q^{1-2n}, aq, \lfloor n/2 \rfloor, q^2)$. Unfortunately, the proof of [20, Eq. (2.1)] as stated in [20] appears to be incomplete, and below we provide the full details of the derivation of (2.14).

First recall Sears' ${}_4\phi_3$ transformation [30, Eq. (III.15)], which we write in the form

$$(2.15) \quad {}_4\phi_3 \left[\begin{matrix} a, b, c, d \\ e, f, abcdq/ef \end{matrix}; q, q \right] = \frac{(q/f, abq/f, acdq/ef, bcdq/ef; q)_\infty}{(aq/f, bq/f, cdq/ef, abcdq/ef; q)_\infty} {}_4\phi_3 \left[\begin{matrix} a, b, e/c, e/d \\ e, abq/f, ef/cd \end{matrix}; q, q \right],$$

provided both series terminate. Letting

$$(a, b, c, d, e, f, q) \rightarrow (q^{-2n}, q^{2-2n}, aq, aq^3, a^2q^2, -q^{3-2n}, q^4)$$

(2.14) can be written as

$$(2.16) \quad {}_4\phi_3 \left[\begin{matrix} aq^{-1}, aq, q^{-2n}, q^{2-2n} \\ a^2q^2, -q^{1-2n}, -q^{3-2n}; q^4, q^4 \end{matrix} \right] = \frac{(-q; q)_n (-a; q^2)_n}{(-q; q^2)_n (-a; q)_n}.$$

At first sight it may appear that little progress has been made, but upon closer inspection one may note that the parameters in this new ${}_4\phi_3$ series are tuned to allow the application of Singh’s quadratic transformation [30, Eq. (III.21)]

$$(2.17) \quad {}_4\phi_3 \left[\begin{matrix} a^2, b^2, c, d \\ abq^{1/2}, -abq^{1/2}, -cd; q, q \end{matrix} \right] = {}_4\phi_3 \left[\begin{matrix} a^2, b^2, c^2, d^2 \\ a^2b^2q, -cd, -cdq; q^2, q^2 \end{matrix} \right],$$

true provided both series terminate. Indeed, utilizing this transform with

$$(a, b, c, d, q) \rightarrow ((a/q)^{1/2}, (aq)^{1/2}, q^{-n}, q^{1-n}, q^2)$$

we arrive at (2.10) with $(a, b) \rightarrow (-a, -q)$.

Equation (2.16) may also be derived from the summation [1, Eq. (4.3) with $b = 1$] (rediscovered in [20, Eq. (2.2)]) by making the substitutions $(a, b, w, m, q) \rightarrow (aq^{-1}, 1, -aq^{2\lfloor(n+1)/2\rfloor}, \lfloor n/2 \rfloor, q^2)$. \square

Proof of Lemma 2.5. Without loss of generality we may assume that $0 \leq j \leq L$. After shifting $r \rightarrow 2r + j$ the sum (2.13) corresponds to (2.16) with $(a, n) \rightarrow (-q^{2j+1}, L - j)$. \square

Our final transformation for q -binomial coefficients takes a form that is slightly different from (2.1).

Lemma 2.6. *For L and j integers such that $j \equiv L \pmod{2}$ there holds*

$$(2.18) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^{\lfloor L/3 \rfloor} \frac{q^{\frac{3}{4}r^2} (q^3; q^3)_{\frac{1}{2}(L-r-2)} (1 - q^L)}{(q^3; q^3)_r (q; q)_{\frac{1}{2}(L-3r)}} \left[\begin{matrix} r \\ \frac{1}{2}(r - j) \end{matrix} \right]_{q^3} = q^{\frac{3}{4}j^2} \left[\begin{matrix} L \\ \frac{1}{2}(L - 3j) \end{matrix} \right].$$

When $r = L = 0$ the factor multiplying the q -binomial coefficient in the summand on the left should be taken to be 1.

Proof of Lemma 2.6. Shifting $r \rightarrow 2r + j$ and defining $n = (L - 3j)/2$ we arrive at the $(a, b, c, d, q) \rightarrow (q^{-n}, q^{1-n}, q^{2-n}, q^{3j+3}, q^3)$ instance of (2.11). \square

Again an important question is whether the polynomial

$$(2.19) \quad f_{L,r}(q) = \frac{(q^3; q^3)_{\frac{1}{2}(L-r-2)} (1 - q^L)}{(q^3; q^3)_r (q; q)_{\frac{1}{2}(L-3r)}}$$

for $0 \leq 3r \leq L$ and $r \equiv L \pmod{2}$ has nonnegative coefficients. To answer this is not entirely trivial, and we need a generalization of a result of Andrews [10] that arose in connection with a monotonicity conjecture of Friedman, Joichi and Stanton [26].

Theorem 2.1. *Let k and n be positive integers, $j \in \{0, \dots, n\}$ and $g = \gcd(n, j)$. Then*

$$A_{n,j,k}(q) = \frac{1 - q^k}{1 - q^n} \left[\begin{matrix} n \\ j \end{matrix} \right]$$

is a reciprocal polynomial of degree $j(n - j) + k - n$ with nonnegative coefficients if $k \equiv 0 \pmod{g}$.

For $k = 1$ this is Andrews' result [10, Thm. 2].

Assuming the theorem it is not difficult to show that $f_{L,r}(q)$ given by (2.19) is a polynomial with nonnegative coefficients. First we note that for $r = 0$ or $3r = L$ this is obvious; $f_{3r,r}(q) = 1$ and $f_{2L,0}(q) = (1 + q^L)(q^3; q^3)_{L-1}/(q; q)_{L-1}$ ($L > 0$), where the positivity of the second polynomial follows from $(1 - q^3)/(1 - q) = 1 + q + q^2$. In the following we may therefore assume $0 < 3r < L$, which implies that $k := \gcd((L - r)/2, r) \leq (L - 3r)/2$ as follows. There holds $r = uk$ and $(L - r)/2 = vk$ with $v > u$ and $\gcd(u, v) = 1$. Hence $(L - 3r)/2 = (v - u)k$ so that $k \leq (L - 3r)/2$. Next we observe the decomposition

$$f_{L,r}(q) = \frac{(1 - q^k)(q^3; q^3)_{(L-3r)/2}}{(1 - q^{3k})(q; q)_{(L-3r)/2}} \times \frac{1 - q^L}{1 - q^k} \times A_{(L-r)/2,r,k}(q^3),$$

where all three factors on the right are polynomials with nonnegative coefficients. The first term is because $k \leq (L - 3r)/2$ so that

$$\frac{(1 - q^k)(q^3; q^3)_{(L-3r)/2}}{(1 - q^{3k})(q; q)_{(L-3r)/2}} = \prod_{\substack{j=1 \\ j \neq k}}^{(L-3r)/2} (1 + q^j + q^{2j}),$$

the second term is because $k \mid L$, and the last term is because of Theorem 2.1 with $k = g$.

It is possible to arrive at Theorem 2.1 by modifying Andrews' proof for $k = 1$. Instead, however, we will establish a more general theorem generalizing results of Haiman [33, §2.5] that he used to show polynomiality and nonnegativity of a conjectured expression for a specialization of the Frobenius series $\mathcal{F}(q, t)$ of 'diagonal harmonics'. For most of the terminology and notation used below we refer to [40, 49].

Let s_λ be the Schur function labelled by the partition λ and define

$$B_{\lambda,d,k}(q) = \frac{1 - q^k}{1 - q^d} s_\lambda(1, q, \dots, q^{d-1}).$$

Theorem 2.2. *Let d and k be positive integers, and let λ be a partition such that $l(\lambda) \leq d$. Set $g = \gcd(d, |\lambda|)$. Then $B_{\lambda,d,k}(q)$ is a reciprocal polynomial of degree $k - d + \sum_{i=1}^{l(\lambda)} (d - i)\lambda_i$ with nonnegative coefficients for every λ if $k \equiv 0 \pmod{g}$.*

For $k = 1$ this is due to Haiman.

Before proving the theorem let us show that it includes the previous theorem as a special case. For notational convenience we set $q^\delta = (q^{d-1}, \dots, q, 1)$ ($\delta = (d - 1, \dots, 1, 0)$) so that for f a symmetric function $f(1, q, \dots, q^{d-1})$ may be written as $f(q^\delta)$. Now we choose $\lambda = (j)$ and use that [40, Ch. 1.3, Example 1], [49, Prop. 2.19.12]

$$s_{(j)}(q^\delta) = \begin{bmatrix} j + d - 1 \\ j \end{bmatrix}.$$

Therefore

$$B_{(j),n-j,k}(q) = \frac{1 - q^k}{1 - q^{n-j}} \begin{bmatrix} n - 1 \\ j \end{bmatrix} = \frac{1 - q^k}{1 - q^n} \begin{bmatrix} n \\ j \end{bmatrix} = A_{n,j,k}(q).$$

By Theorem 2.2 the statement of Theorem 2.1 now follows, be it that $j \in \{0, \dots, n - 1\}$ and $g = \gcd(j, n - j)$. Since $\gcd(n - j, j) = \gcd(n, j)$ and since Theorem 2.1 is trivially true for $j = n$, this completes our derivation.

Proof of Theorem 2.2. Let λ' be the conjugate of the partition $\lambda = (\lambda_1, \dots, \lambda_d)$. Then we have [40, Ch. 1.3, Example 1], [49, Thm. 7.21.2]

$$(2.20) \quad s_\lambda(q^\delta) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{d+c(x)}}{1 - q^{h(x)}}.$$

Here for each $x = (i, j) \in \lambda$ (a partition and its diagram are identified) the hook-length and content of x are given by $h(x) = \lambda_i + \lambda'_j - i - j + 1$ and $c(x) = j - i$, respectively, and $n(\lambda) = \sum_{i=1}^d (i - 1)\lambda_i$. To proceed further we need the following lemma, communicated to us by Richard Stanley.

Lemma 2.7. *Let $i \mid d$, and let ω_i be an i th primitive root of unity. Then for $l(\lambda) \leq d$, $s_\lambda(\omega_i^\delta) = 0$ iff λ has a non-empty i -core.*

To prove this we note that $i \mid d$ and (2.20) imply that $s_\lambda(\omega_i^\delta) = 0$ iff the number of hook-lengths $h(x)$ divisible by i is strictly less than the number of contents $c(x)$ divisible by i . Next we recall that the i -core of λ is obtained from λ by repeated removal of border strips of length i from the diagram of λ until no further strips of length i can be removed [40, Ch. 1.1, Example 8(c)], [49, Exercise 7.59.d]. It is straightforward to verify that each time a border strip is removed, the number of hook-lengths and the number of contents divisible by i is decreased by one. When we finally reach the i -core of λ the number of hook-lengths divisible by i becomes zero. On the other hand, unless the i -core is empty, there will still be a content divisible by i , for example, $c(1, 1) = 0$. This completes the proof of the lemma.

Remark 2.1. If $i \nmid |\lambda|$, then λ has a non-empty i -core. If $i \mid |\lambda|$ and either λ or λ' consists of a single row, then λ has an empty i -core. However, in general, the i -core of λ is not necessarily empty when $i \mid |\lambda|$.

Next, since $s_\lambda(q^\delta)$ is a polynomial, the only potential poles of $B_{\lambda,d,k}(q)$ are poles of $R_{k,d}(q) := (1 - q^k)/(1 - q^d)$. Clearly, $R_{k,d}(q)$ has first order poles at each i th primitive root of unity ω_i , provided $i > 1$, $i \mid d$, but $i \nmid k$. Now, if $k \equiv 0 \pmod{g}$, then $i \nmid |\lambda|$ and, as a result, the i -core of λ is not empty. Hence, by Lemma 2.7, $s_\lambda(\omega_i^\delta) = 0$. Thus, if $k \equiv 0 \pmod{g}$, every pole of $R_{k,d}(q)$ is cancelled by a zero of $s_\lambda(q^\delta)$, and consequently $B_{\lambda,d,k}(q)$ is polynomial if $k \equiv 0 \pmod{g}$.

In the remainder we assume that $k \equiv 0 \pmod{g}$.

The degree of $B_{\lambda,d,k}(q)$ immediately follows from the degree of $s_\lambda(q^\delta)$ given in [40, Ch. 1.3, Example 1]. To show that the polynomial $B_{\lambda,d,k}(q)$ has nonnegative coefficients and is reciprocal, we use that $s_\lambda(q^\delta)$ is a reciprocal, unimodal polynomial with nonnegative coefficients [40, Ch. 1.3, Example 1, Ch. 1.8, Example 4], [49, Exercise 7.75.c]. This immediately implies the reciprocity of $B_{\lambda,d,k}(q)$. To see that it also implies nonnegativity we denote the degree of $s_\lambda(q^\delta)$ by D and note that it suffices to show positivity for $k = g$ thanks to $1 - q^k = (1 - q^g)(1 + q^g + \dots + q^{k-g})$ for $k = mg$. Now, by the unimodality and nonnegativity of $s_\lambda(q^\delta)$, it follows that $(1 - q)s_\lambda(q^\delta)$ is a polynomial of degree $D + 1$ with nonnegative coefficients up to the coefficient of $q^{\lfloor (D+1)/2 \rfloor}$. Hence

$$B_{\lambda,d,g}(q) = \frac{1 + q + \dots + q^{g-1}}{1 - q^d} (1 - q)s_\lambda(q^\delta)$$

is a polynomial of degree $D + g - d$ with nonnegative coefficients up to the coefficient of $q^{\lfloor (D+1)/2 \rfloor}$. But by its reciprocity and by the fact that $\lfloor (D + 1)/2 \rfloor \geq \lfloor (D + g - d)/2 \rfloor$, it follows that all its coefficients must be nonnegative. \square

We conclude this section with the following remarks.

Remark 2.2. Lemma 2.7 is closely related to [40, Ch. 1.3, Example 17(a)]. It is also a straightforward corollary of [51, Lem. 2].

Remark 2.3. It is important to realize that $B_{\lambda,d,k}(q)$ can be a polynomial in q for $k \not\equiv 0 \pmod{g}$. Indeed, the argument given above suggests that $B_{\lambda,d,k}(q)$ is a polynomial as long as the i -core of λ is not empty for any i that divides d but not k . For example, consider the 5-core partition $\mu = (5, 2, 2, 1)$. Then

$$B_{\mu,d,k}(q) = q^9 \frac{1 - q^5}{1 - q} \frac{1 - q^7}{1 - q} \frac{1 - q^k}{1 - q} \frac{1 - q^4}{1 - q^2} \frac{1 - q^9}{1 - q^3}$$

is a polynomial for any positive k . Note, however, that when $\lambda = (j)$, λ cannot have a nonempty i -core if $i \mid j, i > 1$. Hence, $B_{(j),d,k}(q) = A_{d+j,j,k}(q)$ is a polynomial in q iff $k \equiv 0 \pmod{g}$. For $k = 1$ this is due to Andrews [10, Thm. 2].

3. GROUP-LIKE RELATIONS

3.1. Preliminaries. Not all of the q -binomial transformations of the previous two sections are independent, and many relations of various degree of complexity can be found. Such relations are important because they often imply new summation or transformation formulas. For the results of Section 1 the occurrence of relations was first investigated by Bressoud *et al.* [20] and later studied in more detail by Stanton [50] who introduced the notion of the Bailey–Rogers–Ramanujan group.

For notational reasons we write $q^{\frac{1}{8}\gamma r^2} f_{L,r}(q)$ in (1.9) as $F_{L,r}(q)$ and add as a superscript the relevant equation number. For example,

$$F_{L,r}^{(1.1)}(q) = \frac{q^{\frac{1}{4}r^2}(q; q)_L}{(q; q)_{(L-r)/2}(q; q)_r}.$$

Likewise we write $q^{\frac{1}{4}\gamma r^2} f_{L,r}(q)$ in (2.1) as $F_{L,r}(q)$ and again add equation numbers, and we write $F_{L,r}^{(2.18)}(q)$ for the kernel of (2.18). For instance,

$$F_{L,r}^{(2.3)}(q) = q^{\frac{1}{2}r^2} (-q; q)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2}.$$

With this notation we quote from [20, 50]:

$$(3.1a) \quad \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.2)}(q) F_{s,r}^{(1.2)}(q) = F_{L,r}^{(1.1)}(q),$$

$$(3.1b) \quad \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.7)}(q) F_{s,r}^{(1.2)}(q) = F_{L,r}^{(1.6)}(q),$$

and the more complicated

$$(3.2a) \quad \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.1)}(q)F_{s,r}^{(1.2)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.2)}(q)F_{s,r}^{(1.1)}(q),$$

$$(3.2b) \quad \begin{aligned} \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.7)}(q)F_{s,r}^{(1.1)}(q) &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.6)}(q)F_{s,r}^{(1.2)}(q) \\ &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.2)}(q^2)F_{s,r}^{(1.7)}(q), \end{aligned}$$

$$(3.2c) \quad \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.6)}(q)F_{s,r}^{(1.1)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.2)}(q^2)F_{s,r}^{(1.6)}(q),$$

$$(3.2d) \quad \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.7)}(q^3)F_{s,r}^{(1.8)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.8)}(q^2)F_{s,r}^{(1.7)}(q).$$

(It seems that (3.2a), (3.2b), and (3.2d) are actually missing in [20, 50].) The relations in equation (3.1) correspond to summations and the relations in (3.2) to transformations for basic hypergeometric series. For example, after shifting $s \rightarrow 2s + r$ and replacing $(L - r)/2$ by n , then using a polynomial argument to replace $q^{(r+1)/2}$ by the indeterminate a , and finally using a polynomial argument to replace q^{-n} by b , (3.2d) becomes the balanced transformation

$$(3.3) \quad {}_5\phi_4 \left[\begin{matrix} a, aq, b^2, b^2\omega, b^2\omega^2 \\ a^2, -a^2, -a^2q, b^6q^2/a^4 \end{matrix}; q^2, q^2 \right] \\ = \frac{(a^4/b^6; q^2)_\infty (a^3; q^3)_\infty (a^6q^3; q^6)_\infty}{(a^4; q^2)_\infty (a^3/b^6; q^3)_\infty (a^6q^3/b^6; q^6)_\infty} \\ \times {}_5\phi_4 \left[\begin{matrix} a^2, a^2q, a^2q^2, b^3, -b^3 \\ a^3, a^3q^{3/2}, -a^3q^{3/2}, b^6q^3/a^3 \end{matrix}; q^3, q^3 \right],$$

provided both series terminate, i.e., provided a or b is of the form q^{-n} . Here $\omega = \exp(2\pi i/3)$. For a simple proof of (3.1) and (3.2), and hence for a proof of the above new transformation we refer to the next (sub)section.

In the following we extend the analysis of [20, 50] and present two sets of relations, one of the type $\sum FF = F$ as in (3.1) and one of the type $\sum FF = \sum FF$ as in (3.2). The transformations implied by the second set are especially interesting, as many appear to be new.

3.2. **Relations of the type $\sum FF = F$.** Our first set of results, which should be read as five different ways to decompose $F_{L,r}^{(2.3)}(q)$, is given by

$$\begin{aligned}
 (3.4) \quad F_{L,r}^{(2.3)}(q^2) &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.12)}(q^2)F_{s,r}^{(1.6)}(q^4) = \sum_{s=r}^L F_{2L,2s}^{(1.6)}(q)F_{s,r}^{(2.12)}(q) \\
 &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.5)}(q^2)F_{s,r}^{(1.1)}(q^4) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.8)}(q^2)F_{s,r}^{(1.2)}(q^4) \\
 &= \sum_{s=r}^L F_{2L,2s}^{(1.2)}(q^2)F_{s,r}^{(2.5)}(q^2).
 \end{aligned}$$

Similarly, there are three different decompositions of $F_{L,r}^{(2.8)}(q)$,

$$\begin{aligned}
 (3.5) \quad F_{L,r}^{(2.8)}(q^2) &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.12)}(q^2)F_{s,r}^{(1.7)}(q^4) = \sum_{s=r}^L F_{2L,2s}^{(1.7)}(q)F_{s,r}^{(2.12)}(q) \\
 &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.5)}(q^2)F_{s,r}^{(1.2)}(q^4).
 \end{aligned}$$

Proof. Since all of the above eight results (and those of Sections 3.1 and 3.3) arise in similar fashion, we only show how to prove the very first relation.

First take (1.6) and make the substitution $(L, q) \rightarrow (s, q^2)$. Then multiply this by $F_{L,s}^{(2.12)}(q)$ and sum over s to arrive at

$$\sum_{\substack{s=0 \\ s \equiv j \pmod{2}}}^L \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^s F_{L,s}^{(2.12)}(q)F_{s,r}^{(1.6)}(q^2) \left[\begin{matrix} r \\ \frac{1}{2}(r-j) \end{matrix} \right]_{q^2} = q^{\frac{1}{2}j^2} \sum_{\substack{s=0 \\ s \equiv j \pmod{2}}}^L F_{L,s}^{(2.12)}(q) \left[\begin{matrix} s \\ \frac{1}{2}(s-j) \end{matrix} \right]_{q^4}.$$

Now change the order of summation on the left and apply (2.12) on the right to get

$$\sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \left[\begin{matrix} r \\ \frac{1}{2}(r-j) \end{matrix} \right]_{q^2} \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.12)}(q)F_{s,r}^{(1.6)}(q^2) = q^{\frac{1}{2}j^2} \left[\begin{matrix} 2L \\ L-j \end{matrix} \right].$$

Comparing this with (2.3) yields

$$(3.6) \quad \sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \left[\begin{matrix} r \\ \frac{1}{2}(r-j) \end{matrix} \right]_{q^2} \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L \left[F_{L,s}^{(2.12)}(q)F_{s,r}^{(1.6)}(q^2) - F_{L,r}^{(2.3)}(q) \right] = 0,$$

which should hold for all integers L and j such that $0 \leq |j| \leq L$.

The above equation is of the form

$$\sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L \left[\begin{matrix} r \\ \frac{1}{2}(r-j) \end{matrix} \right] h_{L,r}(q) = 0,$$

where, without loss of generality, it may be assumed that $0 \leq j \leq L$. Hence the lower bound on the sum may be replaced by j . Recursively it can be seen that $h_{L,r}(q) = 0$ is the unique solution. Indeed, by taking $j = L$ and $j = L - 1$ it

follows that $h_{L,L}(q) = h_{L,L-1}(q) = 0$. Next taking $j = L - 2$ and $j = L - 3$ it in turn follows that $h_{L,L-2}(q) = h_{L,L-3}(q) = 0$. Repeatedly decreasing j by 2 it thus follows after $\lfloor L/2 \rfloor + 1$ steps that all $h_{L,r}(q)$ for $0 \leq r \leq L$ must vanish. Applying this reasoning to (3.6) yields the desired $F_{L,s}^{(2.12)}(q)F_{s,r}^{(1.6)}(q^2) - F_{L,r}^{(2.3)}(q) = 0$. \square

Like (3.1), the relations of (3.4) and (3.5) (which should all be read as the left-hand side being equal to one of the right-hand side expressions) imply summation formulas. The only one of these that is possibly new corresponds to the second relation of (3.5). After the replacement $(q^{r+1/2}, L - r) \rightarrow (a, n)$ this sum can be stated as

$$(3.7) \quad {}_4\phi_3 \left[\begin{matrix} iq^{-1/2}, -iq^{-1/2}, q^{-n}, -q^{-n} \\ -q, a, q^{1-2n}/a \end{matrix}; q, q^2 \right] = \frac{1 + a^2q^{2n-1}}{1 + a^2q^{-1}} \frac{(-a^2q^{-1}; q^4)_n}{(a; q)_{2n}}.$$

Presumably this follows by contiguity from the $b = iq^{1/2}$ case of the easily established

$${}_4\phi_3 \left[\begin{matrix} b, q/b, q^{-n}, -q^{-n} \\ -q, a, q^{1-2n}/a \end{matrix}; q, q \right] = \frac{(ab, aq/b; q^2)_n}{(a; q)_{2n}}$$

or from (3.17) with $bq = -1$. In Section 6.1 we rederive (3.7) from the more general transformation formula (6.7).

3.3. Relations of the type $\sum FF = \sum FF$. This time there are a rather large number of results, all of which can be proved using the method detailed in Section 3.2. Those relations that imply base-changing transformations from q to q^k for fixed k have been grouped together. Here k will be an element of the set $\{1, 4/3, 3/2, 2, 4, 6, 9, 12\}$.

3.3.1. Linear transformations. There are just two linear relations

$$(3.8a) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.5)}(q)F_{s,r}^{(2.3)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.8)}(q)F_{s,r}^{(2.5)}(q^2),$$

$$(3.8b) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.3)}(q)F_{s,r}^{(2.5)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.8)}(q)F_{s,r}^{(2.3)}(q^2),$$

which are dual in the sense of $q \leftrightarrow 1/q$. The corresponding q -hypergeometric transformations are nothing but specializations of the identity obtained by equating the right-hand sides of the Jackson transformations [30, Eq. (III.4)] and [30, Eq. (III.5)].

3.3.2. Transformations from q to $q^{4/3}$. Much more interesting than (3.8) are the generalized commutation relations

$$\sum_{s=r}^{\lfloor L/3 \rfloor} F_{2L,2s}^{(2.18)}(q)F_{s,r}^{(2.13)}(q^3) = \sum_{\substack{s=3r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.13)}(q)F_{s,r}^{(2.18)}(q^4),$$

$$\sum_{s=r}^{\lfloor L/3 \rfloor} F_{2L,2s}^{(2.18)}(q)F_{s,r}^{(2.12)}(q^3) = \sum_{\substack{s=3r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.12)}(q)F_{s,r}^{(2.18)}(q^4).$$

Making the variable change $s \rightarrow s + r$ on the left and $s \rightarrow 2s + 3r$ on the right and then substituting $(q^{6r}, L - 3r) \rightarrow (a, n)$, the above relations imply the balanced and ‘almost’ balanced formulas

$$\begin{aligned} & {}_5\phi_4 \left[\begin{matrix} iq^{3/2}, -iq^{3/2}, q^{-n}, q^{1-n}, q^{2-n} \\ -q^3, a^{1/2}q^{3/2}, -a^{1/2}q^{3/2}, q^{3-3n}/a \end{matrix}; q^3, q^3 \right] \\ &= \frac{(-q, a; q^2)_n}{(-q; q)_n (a; q^3)_n} {}_5\phi_4 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, q^{-2n}, q^{2-2n} \\ a, aq^2, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right] \end{aligned}$$

and

$$\begin{aligned} & {}_5\phi_4 \left[\begin{matrix} iq^{-3/2}, -iq^{-3/2}, q^{-n}, q^{1-n}, q^{2-n} \\ -q^3, a^{1/2}q^{3/2}, -a^{1/2}q^{3/2}, q^{3-3n}/a \end{matrix}; q^3, q^6 \right] \\ &= q^n \frac{(-q^{-1}, a; q^2)_n}{(-q; q)_n (a; q^3)_n} {}_6\phi_5 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, -aq^4, q^{-2n}, q^{2-2n} \\ a, -a, aq^2, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right], \end{aligned}$$

respectively. To the best of our knowledge these are the first examples of a transformations relating base q^3 and q^4 .

3.3.3. *Transformations from q to $q^{3/2}$.* Again there are two results, not dissimilar to the previous pair:

$$\begin{aligned} \sum_{s=r}^{\lfloor L/3 \rfloor} F_{2L, 2s}^{(2.18)}(q) F_{s,r}^{(2.8)}(q^3) &= \sum_{\substack{s=3r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.8)}(q) F_{s,r}^{(2.18)}(q^2), \\ \sum_{s=r}^{\lfloor L/3 \rfloor} F_{2L, 2s}^{(2.18)}(q) F_{s,r}^{(2.9)}(q^3) &= \sum_{\substack{s=3r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.9)}(q) F_{s,r}^{(2.18)}(q^2). \end{aligned}$$

Making the same variable change as above and then substituting $(-q^{3r}, L - 3r) \rightarrow (a, n)$, this yields

$$\begin{aligned} & {}_5\phi_4 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, q^{-n}, q^{1-n} \\ a, -a, -aq, q^{2-2n}/a \end{matrix}; q^2, q^2 \right] \\ &= \frac{(a^2; q^3)_n}{(a; q^2)_n (-a; q)_n} {}_5\phi_4 \left[\begin{matrix} a^{1/2}, -a^{1/2}, q^{-n}, q^{1-n}, q^{2-n} \\ a, aq^{3/2}, -aq^{3/2}, q^{3-3n}/a^2 \end{matrix}; q^3, q^3 \right] \end{aligned}$$

and

$$\begin{aligned} & {}_5\phi_4 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, q^{-n}, q^{1-n} \\ aq, -aq, -a, q^{1-2n}/a \end{matrix}; q^2, q^2 \right] \\ &= \frac{1 - a^2 q^{2n}}{1 - a^2} \frac{(a^2; q^3)_n}{(aq; q^2)_n (-aq; q)_n} \\ &\quad \times {}_5\phi_4 \left[\begin{matrix} a^{1/2} q^{3/2}, -a^{1/2} q^{3/2}, q^{-n}, q^{1-n}, q^{2-n} \\ aq^3, aq^{3/2}, -aq^{3/2}, q^{3-3n}/a^2 \end{matrix}; q^3, q^3 \right]. \end{aligned}$$

Both of these results should be compared with (3.3).

3.3.4. *Quadratic transformations.* There are quite a number of different relations of a quadratic nature. First,

$$(3.9a) \quad \sum_{s=r}^L F_{2L,2s}^{(1.1)}(q)F_{s,r}^{(2.5)}(q) = \sum_{s=r}^L F_{2L,2s}^{(1.2)}(q)F_{s,r}^{(2.3)}(q) \\ = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.3)}(q)F_{s,r}^{(1.1)}(q^2),$$

$$(3.9b) \quad \sum_{s=r}^L F_{2L,2s}^{(1.2)}(q)F_{s,r}^{(2.8)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.8)}(q)F_{s,r}^{(1.1)}(q^2) \\ = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.3)}(q)F_{s,r}^{(1.2)}(q^2).$$

The first equality in (3.9a) corresponds to a specialization of the transformation [30, Eq. (III.4)], and the second equality implies the $(a, b, c, n) \rightarrow (q^{r+1/2}, \infty, 0, L - r)$ specialization of

$$(3.10) \quad {}_4\phi_3 \left[\begin{matrix} b, c, -c, q^{-n} \\ a, c^2, -bq^{1-n}/a \end{matrix}; q, q \right] \\ = \frac{(a^2/b; q)_n (c^2; q^2)_n}{(a, c^2, -a/b; q)_n} {}_4\phi_3 \left[\begin{matrix} a^2/b^2, a^2/c^2, q^{-n}, q^{1-n} \\ a^2/b, a^2q/b, q^{2-2n}/c^2 \end{matrix}; q^2, q^2 \right],$$

proved in Section 6.2. Similarly, the second equality in (3.9b) corresponds to a specialization of the transformation [30, Eq. (III.12)], and the first equality implies the $(a, b, c, n) \rightarrow (q^{r+1/2}, \infty, iq^{r/2}, L - r)$ specialization of (3.10).

Next are the four closely related results

$$(3.11a) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.3)}(q)F_{s,r}^{(2.13)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.13)}(q)F_{s,r}^{(2.3)}(q^4),$$

$$(3.11b) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.5)}(q)F_{s,r}^{(2.13)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.13)}(q)F_{s,r}^{(2.5)}(q^4),$$

$$(3.11c) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.3)}(q)F_{s,r}^{(2.12)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.12)}(q)F_{s,r}^{(2.3)}(q^4),$$

$$(3.11d) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.5)}(q)F_{s,r}^{(2.12)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.12)}(q)F_{s,r}^{(2.5)}(q^4),$$

The first as well as the last two relations are dual in the sense of $q \leftrightarrow 1/q$. After the substitution $(q^{4r+2}, L - 2r) \rightarrow (a, n)$, (3.11a) implies

$${}_4\phi_3 \left[\begin{matrix} iq, -iq, q^{-n}, q^{1-n} \\ -q^2, a^{1/2}, -a^{1/2} \end{matrix}; q^2, aa^{2n-1} \right] = \frac{(-q; q^2)_n}{(-q; q)_n} {}_4\phi_3 \left[\begin{matrix} 0, -aq^{-2}, q^{-2n}, q^{2-2n} \\ a, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right]$$

and (3.11c) implies

$$\begin{aligned} {}_4\phi_3 \left[\begin{matrix} iq^{-1}, -iq^{-1}, q^{-n}, q^{1-n} \\ -q^2, a^{1/2}, -a^{1/2} \end{matrix}; q^2, aq^{2n+1} \right] \\ = q^n \frac{(-q^{-1}; q^2)_n}{(-q; q)_n} {}_4\phi_3 \left[\begin{matrix} 0, -aq^2, q^{-2n}, q^{2-2n} \\ a, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right]. \end{aligned}$$

Finally there holds

$$(3.12a) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.8)}(q) F_{s,r}^{(2.13)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.13)}(q) F_{s,r}^{(2.8)}(q^4),$$

$$(3.12b) \quad \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.8)}(q) F_{s,r}^{(2.12)}(q^2) = \sum_{s=r}^{\lfloor L/2 \rfloor} F_{L,2s}^{(2.12)}(q) F_{s,r}^{(2.8)}(q^4).$$

After the replacement $(q^{2r}, L - 2r) \rightarrow (a, n)$ these yield

$$(3.13) \quad {}_4\phi_3 \left[\begin{matrix} iq, -iq, q^{-n}, q^{1-n} \\ -q^2, aq, -q^{2-2n}/a \end{matrix}; q^2, q^2 \right] \\ = \frac{(-a; q)_n (-q; q^2)_n}{(-q; q)_n (-a; q^2)_n} {}_4\phi_3 \left[\begin{matrix} ia, -ia, q^{-2n}, q^{2-2n} \\ a^2q^2, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right]$$

and

$$(3.14) \quad {}_4\phi_3 \left[\begin{matrix} iq^{-1}, -iq^{-1}, q^{-n}, q^{1-n} \\ -q^2, aq, -q^{2-2n}/a \end{matrix}; q^2, q^4 \right] \\ = q^n \frac{(-q^{-1}; q^2)_n (-a; q)_n}{(-q; q)_n (-a; q^2)_n} {}_5\phi_4 \left[\begin{matrix} ia, -ia, -a^2q^4, q^{-2n}, q^{2-2n} \\ -a^2, a^2q^2, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right],$$

respectively. It is not hard to see that (3.13) is a special case of

$$(3.15) \quad {}_4\phi_3 \left[\begin{matrix} b, q^2/b, q^{-n}, q^{1-n} \\ -q^2, aq, -q^{2-2n}/a \end{matrix}; q^2, q^2 \right] \\ = \frac{(-a; q)_n (-q; q^2)_n}{(-q; q)_n (-a; q^2)_n} {}_4\phi_3 \left[\begin{matrix} aq/b, ab/q, q^{-2n}, q^{2-2n} \\ a^2q^2, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right],$$

which generalizes (2.16) and follows by first applying Singh's quadratic transformation (2.17) to the right side and then using Sears' ${}_4\phi_3$ transformation (2.15). (Equation (3.15) also follows from [1, Eq. (4.3)] by a single use of Sears' transform.) Because of the ${}_5\phi_4$ series on the right, it is unclear whether (3.14) admits a similar kind of generalization.

3.3.5. *Quartic transformations.* Our list of quartic relations begins with

$$(3.16a) \quad \sum_{s=r}^L F_{2L,2s}^{(1.6)}(q)F_{s,r}^{(2.5)}(q) = \sum_{s=r}^L F_{2L,2s}^{(1.7)}(q)F_{s,r}^{(2.3)}(q) \\ = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.8)}(q^2)F_{s,r}^{(1.6)}(q^2),$$

$$(3.16b) \quad \sum_{s=r}^L F_{2L,2s}^{(1.7)}(q)F_{s,r}^{(2.8)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.8)}(q^2)F_{s,r}^{(1.7)}(q^2).$$

The first equality in (3.16a) once again corresponds to a specialization of [30, Eq. (III.12)]. More interesting are the second equality in (3.16a) and the generalized commutation relation (3.16b). These prove the $(a, b, n) \rightarrow (q^{r+1/2}, 0, L - r)$, respectively, $(a, b, n) \rightarrow (q^{r+1/2}, -q^r, L - r)$ case of

$$(3.17) \quad {}_4\phi_3 \left[\begin{matrix} b^{1/2}, -b^{1/2}, q^{-n}, -q^{-n} \\ a, b, q^{1-2n}/a \end{matrix}; q, q \right] \\ = \frac{1 + a^2q^{2n-1}(-a^2q^{-1}; q^4)_n}{1 + a^2q^{-1}(a; q)_{2n}} {}_4\phi_3 \left[\begin{matrix} -bq, -bq^3, q^{-2n}, q^{2-2n} \\ -a^2q^3, b^2q^2, -q^{5-4n}/a^2 \end{matrix}; q^4, q^4 \right],$$

established in Section 6.2. As a variation on the above there also holds

$$(3.18a) \quad \sum_{s=r}^L F_{2L,2s}^{(1.6)}(q)F_{s,r}^{(2.3)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.3)}(q^2)F_{s,r}^{(1.6)}(q^2),$$

$$(3.18b) \quad \sum_{s=r}^L F_{2L,2s}^{(1.6)}(q)F_{s,r}^{(2.8)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.3)}(q^2)F_{s,r}^{(1.7)}(q^2).$$

Here (3.18a), respectively, (3.18b) imply the $(a, b, n) \rightarrow (q^{r+1/2}, 0, L - r)$ and $(a, b, n) \rightarrow (q^{r+1/2}, -q^r, L - r)$ instances of

$$(3.19) \quad {}_4\phi_3 \left[\begin{matrix} b^{1/2}, -b^{1/2}, q^{-n}, -q^{-n} \\ a, -a, b \end{matrix}; q, -a^2q^{2n} \right] \\ = \frac{(-a^2q; q^2)_n}{(a^2; q^2)_n} {}_4\phi_3 \left[\begin{matrix} -bq, -bq^3, q^{-2n}, q^{2-2n} \\ -a^2q, -a^2q^3, b^2q^2 \end{matrix}; q^4, a^4q^{4n} \right].$$

Once again this is proved in Section 6.2. Then by making the substitution $(q^{r+1/2}, L - r) \rightarrow (a, n)$ the relation

$$(3.20) \quad \sum_{s=r}^L F_{2L,2s}^{(1.7)}(q)F_{s,r}^{(2.5)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.5)}(q^2)F_{s,r}^{(1.6)}(q^2)$$

yields the $b \rightarrow \infty$ limit of the quartic transformation

$$(3.21) \quad {}_4\phi_3 \left[\begin{matrix} b^{1/2}, -b^{1/2}, q^{-n}, -q^{-n} \\ a, b, q^{1-2n}/a \end{matrix}; q, q \right] \\ = \frac{(-a^2q; q^2)_n (b^2; q^4)_n}{(a; q)_{2n} (b^2; q^2)_n} {}_4\phi_3 \left[\begin{matrix} a^2/b, a^2q^2/b, q^{-2n}, q^{2-2n} \\ -a^2q, -a^2q^3, q^{4-4n}/b^2 \end{matrix}; q^4, q^4 \right].$$

It is not hard to prove this identity by applying Sears' ${}_4\phi_3$ transformation (2.15) with

$$(a, b, c, d, e, f, q) \rightarrow (q^{-2n}, q^{2-2n}, -bq, -bq^3, -a^2q^3, b^2q^2, q^4)$$

to the right-hand side of (3.17).

Next is the pair

$$\begin{aligned} \sum_{s=r}^L F_{2L,2s}^{(1.2)}(q)F_{s,r}^{(2.13)}(q) &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.13)}(q)F_{s,r}^{(1.2)}(q^4), \\ \sum_{s=r}^L F_{2L,2s}^{(1.2)}(q)F_{s,r}^{(2.12)}(q) &= \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.12)}(q)F_{s,r}^{(1.2)}(q^4). \end{aligned}$$

After the substitutions $(q^{r+1/2}, L-r) \rightarrow (a, n)$ these lead to

$$(3.22) \quad {}_3\phi_2 \left[\begin{matrix} iq^{1/2}, -iq^{1/2}, q^{-n} \\ -q, a \end{matrix}; q, -aq^n \right] = \frac{(-q; q^2)_n}{(-q, a; q)_n} {}_3\phi_2 \left[\begin{matrix} -a^2q^{-1}, q^{-2n}, q^{2-2n} \\ -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right]$$

and

$$(3.23) \quad {}_3\phi_2 \left[\begin{matrix} iq^{-1/2}, -iq^{-1/2}, q^{-n} \\ -q, a \end{matrix}; q, -aq^{n+1} \right] = q^n \frac{(-q^{-1}; q^2)_n}{(-q, a; q)_n} {}_3\phi_2 \left[\begin{matrix} -a^2q^3, q^{-2n}, q^{2-2n} \\ -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right],$$

which we failed to generalize to the level of ${}_4\phi_3$ (or ${}_5\phi_4$) series. It is however not hard to see that by applying Singh's transformation (2.17) to the right-hand side, (3.22) becomes the $(b, c) \rightarrow (\infty, iq^{1/2})$ limit of (3.10). It is also possible to arrive at (3.22) and (3.23) (with A replaced by a) by taking the $a, b \rightarrow \infty$ limit in (3.25) and (3.26) such that $A = -bq^{1-n}/a$ is fixed, and by then transforming the resulting ${}_3\phi_2$ series on the right using [30, Eq. (III.13)].

Our last two quartic commutation relations are rather interesting,

$$(3.24a) \quad \sum_{s=r}^L F_{2L,2s}^{(1.1)}(q)F_{s,r}^{(2.13)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.13)}(q)F_{s,r}^{(1.1)}(q^4),$$

$$(3.24b) \quad \sum_{s=r}^L F_{2L,2s}^{(1.1)}(q)F_{s,r}^{(2.12)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.12)}(q)F_{s,r}^{(1.1)}(q^4).$$

Equation (3.24a) implies the $(a, b, n) \rightarrow (q^{2r+1}, \infty, L-r)$ instance of

$$(3.25) \quad {}_5\phi_4 \left[\begin{matrix} iq^{1/2}, -iq^{1/2}, b^{1/2}, -b^{1/2}, q^{-n} \\ -q, a^{1/2}, -a^{1/2}, -bq^{1-n}/a \end{matrix}; q, q \right] = \frac{(-q, a^2/b; q^2)_n}{(-q, -a/b; q)_n (a; q^2)_n} {}_5\phi_4 \left[\begin{matrix} a^2/b^2, -aq^{-1}, -aq, q^{-2n}, q^{2-2n} \\ a^2/b, a^2q^2/b, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right].$$

This result, which will be proved in Section 6.2, simplifies to (2.16) for $b = 1$. Similarly, (3.24b) corresponds to the $(a, b, n) \rightarrow (q^{2r+1}, \infty, L - r)$ case of

$$(3.26) \quad {}_5\phi_4 \left[\begin{matrix} iq^{-1/2}, -iq^{-1/2}, b^{1/2}, -b^{1/2}, q^{-n} \\ -q, a^{1/2}, -a^{1/2}, -bq^{1-n}/a \end{matrix}; q, q^2 \right] \\ = q^n \frac{(-q^{-1}, a^2/b; q^2)_n}{(-q, -a/b; q)_n (a; q^2)_n} {}_5\phi_4 \left[\begin{matrix} a^2/b^2, -aq, -aq^3, q^{-2n}, q^{2-2n} \\ a^2/b, a^2q^2/b, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right].$$

When $b = 1$ this simplifies to (2.14) and when $aq = -1$ (and $b \rightarrow b^2$) to

$$(3.27) \quad {}_3\phi_2(b, -b, q^{-n}; -q, b^2q^{2-n}; q, q^2) = q^n \frac{(q^{-2}/b^2; q^2)_n}{(-q, q^{-1}/b^2; q)_n}$$

needed shortly. The proof of (3.26) can again be found in Section 6.2.

Both (3.25) and (3.26) may be further manipulated into new quadratic transformations as follows. The left-hand side of (3.25) simplifies to a ${}_3\phi_2$ series by the $(b, x, y) \rightarrow (i(bq/a)^{1/2}, (a/b)^{1/2}, i(a/q)^{1/2})$ case of [30, Eq (3.5.2); $a \rightarrow q^{-n}$]

$$(3.28) \quad {}_5\phi_4 \left[\begin{matrix} bx, -bx, by, -by, q^{-n} \\ -q, bxy, -bxy, b^2q^{-n} \end{matrix}; q, q \right] = \frac{(q^2/b^2; q^2)_n}{(-q, q/b^2; q)_n} {}_3\phi_2 \left[\begin{matrix} x^2, y^2, q^{-2n} \\ b^2x^2y^2, b^2q^{-2n} \end{matrix}; q^2, b^2q^3 \right].$$

After the further substitution $(a, b, q) \rightarrow (aq, aq/b, -q)$ this leads to

$$(3.29) \quad {}_5\phi_4 \left[\begin{matrix} a, aq^2, b^2, q^{-2n}, q^{2-2n} \\ abq, abq^3, q^{1-2n}, q^{3-2n} \end{matrix}; q^4, q^4 \right] = \frac{(aq, bq; q^2)_n}{(q, abq; q^2)_n} {}_3\phi_2 \left[\begin{matrix} a, b, q^{-2n} \\ aq, q^{1-2n}/b \end{matrix}; q^2, -\frac{q^2}{b} \right].$$

When $(b, q) \rightarrow (-aq, -q)$ the left is summable by (2.16) and we infer the further identity

$$(3.30) \quad {}_2\phi_1(a, q^{-2n}; q^{-2n}/a; q^2, q/a) = \frac{(-q, aq; q)_n}{(aq^2; q^2)_n},$$

which also follows from [30, Exercise 1.8]. By the usual polynomial argument (3.29) may also be stated as

$${}_5\phi_4 \left[\begin{matrix} a, aq^2, b^2, c, cq^2 \\ abq, abq^3, cq, cq^3 \end{matrix}; q^4, q^4 \right] = \frac{(q, cq/a, cq/b, q/ab; q^2)_\infty}{(cq, q/a, q/b, cq/ab; q^2)_\infty} {}_3\phi_2 \left[\begin{matrix} a, b, c \\ aq, cq/b \end{matrix}; q^2, -\frac{q^2}{b} \right],$$

provided both series terminate. For $c = aq$ the ${}_3\phi_2$ series on the right becomes a ${}_2\phi_1$ which precisely takes the form of the sum side of the Bailey-Daum summation [30, Eq. (II.9)].

Remark 3.1. When $a = q^{2j}$ with $j \geq 1$, (3.30) may be put in the form

$$(3.31) \quad \sum_{k=0}^n q^k \begin{bmatrix} k+j-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-k+j \\ j \end{bmatrix}_{q^2} = \begin{bmatrix} n+2j \\ n \end{bmatrix}_{q^2}.$$

This has the following elegant partition theoretic interpretation. The expression

$$q^k \begin{bmatrix} k+j-1 \\ k \end{bmatrix}_{q^2}$$

is the generating function of partitions of exactly k parts, with all parts being odd and no parts exceeding $2j - 1$. The expression

$$\begin{bmatrix} n-k+j \\ j \end{bmatrix}_{q^2}$$

is the generating function of partitions of at most $n - k$ parts, with all parts being even and no parts exceeding $2j$. Hence the summand on the left of (3.31) is the generating function of partitions of at most n parts, with no parts exceeding $2j$ and exactly k odd parts. When summed over the number of odd parts, this gives the generating function of partitions of at most n parts with no parts exceeding $2j$, in accordance with the right-hand side of (3.31).

To also rewrite (3.26) as a quadratic transformation requires a bit more work. Indeed, in order to trade the ${}_5\phi_4$ on the left for a ${}_3\phi_2$ we need to prove the following companion to (3.28):

$$(3.32) \quad {}_5\phi_4 \left[\begin{matrix} bx, -bx, by, -by, q^{-n} \\ -q, bxy, -bxy, b^2q^{2-n}; q, q^2 \end{matrix} \right] = q^n \frac{(q^{-2}/b^2; q^2)_n}{(-q, q^{-1}/b^2; q)_n} {}_3\phi_2 \left[\begin{matrix} x^2, y^2, q^{-2n} \\ b^2x^2y^2, b^2q^{4-2n}; q^2, b^2q^3 \end{matrix} \right].$$

Using this with $(b, x, y) \rightarrow (i(b/aq)^{1/2}, (a/b)^{1/2}, i(aq)^{1/2})$ and making the further substitution $(a, b, q) \rightarrow (aq^{-1}, aq^{-1}/b, -q)$ yields

$$(3.33) \quad {}_5\phi_4 \left[\begin{matrix} a, aq^2, b^2, q^{-2n}, q^{2-2n} \\ abq^{-1}, abq, q^{3-2n}, q^{5-2n}; q^4, q^4 \end{matrix} \right] = \frac{(aq^{-1}, bq^{-1}; q^2)_n}{(q^{-1}, abq^{-1}; q^2)_n} {}_3\phi_2 \left[\begin{matrix} a, b, q^{-2n} \\ aq^{-1}, q^{3-2n}/b; q^2, -\frac{q^2}{b} \end{matrix} \right].$$

When $(a, b, q) \rightarrow (aq, -a, -q)$ the sum on the left can be carried out by (2.14) leading to

$$(3.34) \quad {}_2\phi_1(aq, q^{-2n}; q^{3-2n}/a; q^2, q^2/a) = q^{-n} \frac{(-q, a; q)_n}{(a/q; q^2)_n}.$$

This sum, which is in fact (3.30) with order of summation reversed, will be needed in Section 8. Again we may replace q^{-2n} in (3.33) by c to find

$${}_5\phi_4 \left[\begin{matrix} a, aq^2, b^2, c, cq^2 \\ abq^{-1}, abq, cq^3, cq^5; q^4, q^4 \end{matrix} \right] = \frac{(q^3, cq^3/a, cq^3/b, q^3/ab; q^2)_\infty}{(cq^3, q^3/a, q^3/b, cq^3/ab; q^2)_\infty} {}_3\phi_2 \left[\begin{matrix} a, b, c \\ aq^{-1}, cq^3/b; q^2, -\frac{q^2}{b} \end{matrix} \right],$$

provided both series terminate.

To the best of our knowledge (3.29) and (3.33) are new, and the result closest to these transformations that we were able to obtain using just elementary results from [30] is

$${}_4\phi_3 \left[\begin{matrix} a, aq, q^{-n}, q^{1-n} \\ b^2q, -aq^{1-n}/b, -aq^{2-n}/b; q^2, q^2 \end{matrix} \right] = \frac{(b^2; q^2)_n}{(b^2, -b/a; q)_n} {}_2\phi_1 \left[\begin{matrix} b, q^{-n} \\ q^{1-n}/b; q, \frac{q}{a} \end{matrix} \right].$$

This generalizes [30, Exercise 1.6 (i)] obtained when a tends to 0, and follows from [30, Exercise 3.4] and [30, Eq. (III.8)].

Proof of (3.32). Take (3.27) and let j be the summation variable in the ${}_3\phi_2$ series. Replace n by $n - k$, shift $j \rightarrow j - k$ and multiply both sides by

$$\frac{(x^2, y^2; q^2)_k}{(q^2, b^2x^2y^2; q^2)_k} \frac{(q^{-n}; q)_k}{(b^2q^{2-n}; q)_k} (bq)^{2k}.$$

Next sum k from 0 to n and interchange the order of the sums over j and k on the left. This gives, after some tedious but elementary manipulations involving q -shifted factorials,

$$\begin{aligned} \sum_{j=0}^n {}_3\phi_2 \left[\begin{matrix} x^2, y^2, q^{-2j} \\ b^2 x^2 y^2, q^{2-2j}/b^2 \end{matrix}; q^2, q^2 \right] & \frac{(b^2; q^2)_j (q^{-n}; q)_j}{(q^2; q^2)_j (b^2 q^{2-n}; q)_j} q^{2j} \\ & = q^n \frac{(q^{-2}/b^2; q^2)_n}{(-q, q^{-1}/b^2; q)_n} {}_3\phi_2 \left[\begin{matrix} x^2, y^2, q^{-2n} \\ b^2 x^2 y^2, b^2 q^{4-2n} \end{matrix}; q^2, b^2 q^3 \right]. \end{aligned}$$

The ${}_3\phi_2$ can be summed by (2.11) resulting in (3.32). □

3.3.6. *Sextic transformations.* Both our results take the form of generalized commutation relations. First,

$$\sum_{s=r}^L F_{2L,2s}^{(1.8)}(q) F_{s,r}^{(2.8)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.8)}(q^3) F_{s,r}^{(1.8)}(q^2),$$

which, by the substitution $(-q^r, L-r) \rightarrow (a, n)$, yields

$$\begin{aligned} {}_5\phi_4 \left[\begin{matrix} a^{1/2}, -a^{1/2}, q^{-n}, \omega q^{-n}, \omega^2 q^{-n} \\ a, aq^{1/2}, -aq^{1/2}, q^{-3n}/a^2 \end{matrix}; q, q \right] \\ = \frac{1 - a^3 q^{3n}}{1 - a^3} \frac{(a^3; q^6)_n (-a^3 q^3; q^3)_n}{(a^2 q; q)_{3n}} \\ \times {}_5\phi_4 \left[\begin{matrix} a^2 q^2, a^2 q^4, a^2 q^6, q^{-3n}, q^{3-3n} \\ -a^3 q^3, a^3 q^6, -a^3 q^6, q^{6-6n}/a^3 \end{matrix}; q^6, q^6 \right]. \end{aligned}$$

Second,

$$\sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^{\lfloor L/3 \rfloor} F_{L,s}^{(2.18)}(q^2) F_{s,r}^{(1.7)}(q^3) = \sum_{\substack{s=3r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.7)}(q) F_{s,r}^{(2.18)}(q),$$

which, by the substitution $(q^{3r}, (L-3r)/2) \rightarrow (a^2, n)$, yields

$$\begin{aligned} {}_5\phi_4 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, q^{-n}, -q^{-n} \\ a, -a, aq^{1/2}, q^{1/2-2n}/a \end{matrix}; q, q \right] \\ = \frac{1 - a^4 q^{6n}}{1 - a^4} \frac{(aq^{1/2}; q)_{2n} (a^2 q^2; q^2)_n}{(a^4; q^6)_n} \\ \times {}_5\phi_4 \left[\begin{matrix} aq^{3/2}, aq^{9/2}, q^{-2n}, q^{2-2n}, q^{4-2n} \\ a^2 q^3, -a^2 q^3, -a^2 q^6, q^{6-6n}/a^4 \end{matrix}; q^6, q^6 \right]. \end{aligned}$$

3.3.7. *Transformation from q to q^9 .* As our second-to-last relation there holds

$$\sum_{\substack{s=3r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(1.8)}(q) F_{s,r}^{(2.18)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^{\lfloor L/3 \rfloor} F_{L,s}^{(2.18)}(q^3) F_{s,r}^{(1.8)}(q^3).$$

After replacing $(q^{3r}, (L - 3r)/2) \rightarrow (a^2, n)$ this becomes

$$\begin{aligned} & {}_6\phi_5 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, q^{-n}, \omega q^{-n}, \omega^2 q^{-n} \\ a, -a, aq^{1/2}, -aq^{1/2}, q^{-3n}/a^2 \end{matrix} ; q, q \right] \\ &= \frac{1 - a^6 q^{6n}}{1 - a^6} \frac{(a^6; q^6)_n}{(a^2 q; q)_{3n}} {}_6\phi_5 \left[\begin{matrix} a^2 q^3, a^2 q^6, a^2 q^9, q^{-3n}, q^{3-3n}, q^{6-3n} \\ a^3 q^{9/2}, -a^3 q^{9/2}, a^3 q^9, -a^3 q^9, q^{9-9n}/a^6 \end{matrix} ; q^9, q^9 \right]. \end{aligned}$$

To the best of our knowledge this is the first transformation between the bases q and q^9 .

3.3.8. *Transformation from q to q^{12} .* Also our very last relation is an isolated result because $F^{(2.13)}$ commutes with all but $F^{(1.8)}$:

$$\sum_{s=r}^L F_{2L, 2s}^{(1.8)}(q) F_{s,r}^{(2.12)}(q) = \sum_{\substack{s=r \\ s \equiv r \pmod{2}}}^L F_{L,s}^{(2.12)}(q^3) F_{s,r}^{(1.8)}(q^4).$$

Making the replacement $(q^{2r+1}, L - r) \rightarrow (a, n)$ this corresponds to

$$\begin{aligned} & {}_5\phi_4 \left[\begin{matrix} iq^{-1/2}, -iq^{-1/2}, q^{-n}, \omega q^{-n}, \omega^2 q^{-n} \\ -q, a^{1/2}, -a^{1/2}, q^{1-3n}/a \end{matrix} ; q, q^2 \right] \\ &= q^{3n} \frac{(-q^{-3}, a^3 q^3; q^6)_n}{(-q^3; q^3)_n (a; q)_{3n}} {}_5\phi_4 \left[\begin{matrix} a^2 q^2, a^2 q^6, a^2 q^{10}, q^{-6n}, q^{6-6n} \\ a^3 q^3, a^3 q^9, -q^{9-6n}, -q^{15-6n} \end{matrix} ; q^{12}, q^{12} \right]. \end{aligned}$$

We believe this to be the first example of a transformation relating base q to base q^{12} .

4. INVERSE TRANSFORMATIONS

4.1. **Main results.** When iterating any of the transformations of Section 2 it is often important to start with an as simple as possible q -binomial identity as seed. One possible way to determine whether a potential seed can actually be reduced is by applying the inverses of the transformations of Lemmas 2.1–2.6.

For a transformation of the type (2.1) we consider a formula of the form

$$q^{-\frac{1}{4}\gamma L^2} \sum_{r=0}^L \tilde{f}_{L,r}(q) \left[\begin{matrix} 2r \\ r-j \end{matrix} \right] = q^{-\frac{1}{4}\gamma j^2} \left[\begin{matrix} L \\ \frac{1}{2}(L-j) \end{matrix} \right]_{q^k} \chi(L \equiv j \pmod{2})$$

as its inverse. Here χ is the truth function: $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Indeed, replacing $(L, r) \rightarrow (r, s)$ in (2.1) and then using this to eliminate the q -binomial coefficient in the above summand yields

$$\sum_{\substack{s=0 \\ s \equiv j \pmod{2}}}^L q^{\frac{1}{4}\gamma(s^2 - L^2)} \left[\begin{matrix} s \\ \frac{1}{2}(s-j) \end{matrix} \right]_{q^k} \sum_{r=s}^L \tilde{f}_{L,r}(q) f_{r,s}(q) = \left[\begin{matrix} L \\ \frac{1}{2}(L-j) \end{matrix} \right]_{q^k} \chi(L \equiv j \pmod{2}).$$

This is obviously satisfied if the inverse relations

$$(4.1a) \quad \sum_{r=s}^L \tilde{f}_{L,r}(q) f_{r,s}(q) = \delta_{L,s},$$

$$(4.1b) \quad \sum_{r=s}^L f_{L,r}(q) \tilde{f}_{r,s}(q) = \delta_{L,s}$$

hold. Here the second equation follows from the first and the fact that $f_{L,r}(q)$ is nonzero if and only if $0 \leq r \leq L$. Inverse relations such as (4.1) have been much studied in the theory of basic hypergeometric series. Most importantly, they are related to the Bailey transform [5, 9, 13, 17, 18, 54], the problem of q -Lagrange inversion [31, 32] and summations and transformations of q -hypergeometric series [1, 22, 23, 24, 27, 38, 44].

The first inverse is that of Lemma 2.1.

Lemma 4.1. *For L and j integers there holds*

$$q^{-\frac{1}{2}L^2} \sum_{r=0}^L (-1)^{r+L} q^{\binom{L-r}{2}} (-q; q)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2r \\ r-j \end{bmatrix} = q^{-\frac{1}{2}j^2} \begin{bmatrix} L \\ \frac{1}{2}(L-j) \end{bmatrix}_{q^2} \chi(L \equiv j \pmod{2}).$$

Proof. All we need to do is show that

$$\tilde{f}_{L,r}(q) = (-1)^{r+L} q^{\binom{L-r}{2}} (-q; q)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2}$$

and $f_{L,r}(q)$ as given by (2.4) satisfy (4.1a). Shifting $r \rightarrow r + s$ this becomes the $n \rightarrow L - s$ case of ${}_1\phi_0(q^{-n}; -; q, q) = \delta_{n,0}$, which follows from the q -binomial theorem [30, Eq. (II.4)]

$$(4.2) \quad {}_1\phi_0(q^{-n}; -; q, z) = (zq^{-n}; q)_n.$$

Alternatively we can prove Lemma 4.1 without resorting to inverse relations. Assuming $0 \leq j \leq L$ and shifting $r \rightarrow r + j$ the identity of the lemma becomes the $(a, c, n) \rightarrow (0, q^{j+1/2}, L - j)$ instance of [30, Eq. (II.17)]

$$(4.3) \quad {}_4\phi_3 \left[\begin{matrix} a^2q, c, -c, q^{-n} \\ c^2, aq^{1-n/2}, -aq^{1-n/2}; q, q \end{matrix} \right] = \frac{(q, c^2/a^2; q^2)_{n/2}}{(c^2q, 1/a^2; q^2)_{n/2}} \chi(n \equiv 0 \pmod{2}),$$

due to Andrews [4]. □

Next is the inverse of Lemma 2.2.

Lemma 4.2. *For L and j integers there holds*

$$q^{-\frac{1}{4}L^2} \sum_{r=0}^L (-1)^{r+L} q^{\binom{L-r}{2}} (-q^{2r-L+2}; q^2)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix} \begin{bmatrix} 2r \\ r-j \end{bmatrix} = q^{-\frac{1}{4}j^2} \begin{bmatrix} L \\ \frac{1}{2}(L-j) \end{bmatrix}_{q^2} \chi(L \equiv j \pmod{2}).$$

Proof. Using that (4.1) remains unchanged if we multiply $f_{L,r}(q)$ by $x_r(q)y_L(q)$ and divide $\tilde{f}_{L,r}(q)$ by $x_L(q)y_r(q)$ ($x_r(q) \neq 0, y_L(q) \neq 0$) we this time need to show that

$$f_{L,r}(q) = (1 + q^L) \frac{(-q^{r+2}; q^2)_{L-r-1}}{(q; q)_{L-r}},$$

$$\tilde{f}_{L,r}(q) = (-1)^{r+L} q^{\binom{L-r}{2}} \frac{(-q^{2r-L+2}; q^2)_{L-r}}{(q; q)_{L-r}}$$

satisfies (4.1a). Shifting $r \rightarrow r + s$ this is (4.3) with $c = a$ and $(a^2, n) \rightarrow (-q^s, L - s)$.

Alternatively, we may assume $0 \leq j \leq L$ and shift $r \rightarrow r + j$ to find that Lemma 4.2 is (4.3) with $(a^2, c, n) \rightarrow (-q^j, q^{j+1/2}, L - j)$. \square

The following lemma, corresponding to the inverse of (2.9) is (literally) the odd one out as the sum on the left does not vanish when $L - j$ is odd.

Lemma 4.3. *For L and j integers such that $j \equiv L \pmod{2}$ there holds*

$$q^{-\frac{1}{4}L(L+2)}(1 + q^L) \sum_{r=0}^L (-1)^{r+L} q^{\binom{L-r}{2}} (-q^{2r-L+3}; q^2)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix} \begin{bmatrix} 2r \\ r-j \end{bmatrix} \\ = q^{-\frac{1}{4}j(j+2)}(1 + q^j) \left[\begin{matrix} L \\ \frac{1}{2}(L-j) \end{matrix} \right]_{q^2}.$$

Proof. The difference with the previous two cases is that the pair

$$f_{L,r}(q) = \frac{(-q^{r+1}; q^2)_{L-r}}{(q; q)_{L-r}}, \\ \tilde{f}_{L,r}(q) = (-1)^{r+L} q^{\binom{L-r}{2}} \frac{(-q^{2r-L+3}; q^2)_{L-r}}{(q; q)_{L-r}}$$

only satisfies (4.1) for $s \equiv L \pmod{2}$. Indeed, shifting $r \rightarrow r + s$ and substituting the above, (4.1a) becomes

$$(4.4) \quad {}_5\phi_4 \left[\begin{matrix} a^2, bq, c, -c, q^{-n} \\ c^2, b, aq^{1-n/2}, -aq^{1-n/2} \end{matrix}; q, q \right] \\ = \begin{cases} \frac{1 - a^2}{1 - a^2 q^n} \frac{1 - bq^n}{1 - b} \frac{(q, c^2/a^2; q^2)_{n/2}}{(c^2q, 1/a^2; q^2)_{n/2}} & \text{if } n \text{ is even,} \\ \frac{1 - q^n}{1 - a^2 q^n} \frac{a^2 - b}{1 - b} \frac{(q, c^2q/a^2; q^2)_{(n-1)/2}}{(c^2q, q/a^2; q^2)_{(n-1)/2}} & \text{if } n \text{ is odd,} \end{cases}$$

with $c = a, b = 0$ and $(a^2, n) \rightarrow (-q^{s+1}, L - s (\equiv 0 \pmod{2}))$. Note in particular that for this choice of a, b and c the right side of (4.4) only trivializes to $\delta_{n,0}$ for even values of n , explaining why $L - s$ must be even. The proof of (4.4) is given in the next subsection.

Also the direct proof of the lemma relies on a special case of (4.4). Assuming $0 \leq j \leq L$ and shifting $r \rightarrow r + j$ Lemma 4.3 is (4.4) with $(a^2, b, c, n) \rightarrow (-q^{j+1}, 0, q^{j+1/2}, L - j (\in 2\mathbb{Z}))$. \square

The inverses of the two quartic transforms (2.12) and (2.13) are as follows.

Lemma 4.4. *For L and j integers there holds*

$$\sum_{r=0}^L (-1)^{r+L} (-q; q^2)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2r \\ r-j \end{bmatrix} = \left[\begin{matrix} L \\ \frac{1}{2}(L-j) \end{matrix} \right]_{q^4} \chi(L \equiv j \pmod{2}).$$

Lemma 4.5. *For L and j integers there holds*

$$(1 + q^{2L}) \sum_{r=0}^L (-1)^{r+L} q^{-r} (-q^{-1}; q^2)_{L-r} \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2r \\ r-j \end{bmatrix} \\ = q^{-j} (1 + q^{2j}) \left[\begin{matrix} L \\ \frac{1}{2}(L-j) \end{matrix} \right]_{q^4} \chi(L \equiv j \pmod{2}).$$

Proof. Lemmas 4.4 and 4.5 follow from (2.12) and (2.13) and the $a = -q^{-1}$ and $a = -q$ instances of the inverse pair

$$(4.5a) \quad f_{L,r}(q) = a^r \frac{(a; q^2)_{L-r}}{(q^2; q^2)_{L-r}},$$

$$(4.5b) \quad \tilde{f}_{L,r}(q) = (1/a)^r \frac{(1/a; q^2)_{L-r}}{(q^2; q^2)_{L-r}}.$$

Shifting $r \rightarrow r + s$ in (4.1) this follows from the $n \rightarrow L - s$ case of

$$(4.6) \quad {}_2\phi_1(a, q^{-2n}; aq^{2-2n}, q^2, q^2) = \delta_{n,0},$$

which is a specialization of (2.7).

The direct proof of Lemmas 4.4 and 4.5 is only interesting for the latter. Namely, if we assume that $0 \leq j \leq L$ and shift $r \rightarrow r + j$, then Lemma 4.4 is (4.3) with $(a^2, c, n) \rightarrow (-q^{j-L-1}, q^{j+1/2}, L - j)$, but Lemma 4.5 is

$$(4.7) \quad {}_5\phi_4 \left[\begin{matrix} c, -c, bq, q^{-n}, -q^{-n} \\ c^2q, b, iq^{3/2-n}, -iq^{3/2-n}; q, q^2 \end{matrix} \right] = \begin{cases} \frac{(q^2; q^4)_{n/2}(-c^2q^{-1}; q^2)_n}{(c^4q^2; q^4)_{n/2}(-q^{-1}; q^2)_n} & \text{if } n \text{ is even,} \\ \frac{1-q}{1-c^2q} \frac{c^2-b}{1-b} \frac{(q^6; q^4)_{(n-1)/2}(-c^2q; q^2)_{n-1}}{(c^4q^6; q^4)_{(n-1)/2}(-q; q^2)_{n-1}} & \text{if } n \text{ is odd,} \end{cases}$$

with $b = c^2$ and $(c, n) \rightarrow (q^{j+1/2}, L - j)$. The identity (4.7) will be proven in Section 4.2. □

Remark 4.1. By (2.7) it can also be shown that (4.1) with (4.5) (normalized) is the $b = 1/a$ case of $M(a)M(b) = M(ab)$, with $M(a)$ the infinite-dimensional, lower-triangular matrix $M(a) = (M_{i,j}(a))_{i,j \geq 0}$ whose entries are given by

$$M_{i,j}(a) = a^j (a; q^2)_{i-j} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2}.$$

Finally we state the ‘inverse’ of the cubic transformation of Lemma 2.6.

Lemma 4.6. *For L and j integers such that $j \equiv L \pmod{2}$ there holds*

$$q^{-\frac{3}{4}L^2} \sum_{\substack{r=L \\ r \equiv j \pmod{2}}}^{3L} (-1)^{\frac{1}{2}(r+L)} q^{\binom{\frac{1}{2}(3L-r)}{2}} \frac{(q^{\frac{3}{2}(r-L+2)}; q^3)_{\frac{1}{2}(3L-r)}}{(q; q)_{\frac{1}{2}(3L-r)}} \begin{bmatrix} r \\ \frac{1}{2}(r-3j) \end{bmatrix} = q^{-\frac{3}{4}j^2} \begin{bmatrix} L \\ \frac{1}{2}(L-j) \end{bmatrix}_{q^3}.$$

Proof. This case is quite different from the previous ones in that $f_{L,r}(q)$ corresponding to (2.18) is nonzero if and only if $0 \leq 3r \leq L$. As a consequence only a

left-inverse exists, and we claim that

$$f_{L,r}(q) = \frac{(aq^3; q^3)_{\frac{1}{2}(L-r-2)}(1 - aq^L)}{(aq^3; q^3)_r(q; q)_{\frac{1}{2}(L-3r)}},$$

$$\tilde{f}_{L,r}(q) = (-1)^{\frac{1}{2}(r+L)} q^{\binom{\frac{1}{2}(3L-r)}{2}} \frac{(aq^{\frac{3}{2}(r-L+2)}; q^3)_{\frac{1}{2}(3L-r)}}{(q; q)_{\frac{1}{2}(3L-r)}}$$

with $r \equiv L \pmod{2}$ satisfies

$$(4.8) \quad \sum_{\substack{r=3s \\ r \equiv 0 \pmod{2}}}^{3L} \tilde{f}_{L,r}(q) f_{r,s}(q) = \delta_{L,s}$$

for $s \equiv L \pmod{2}$. Note that this suffices to conclude Lemma 4.6 from (2.18) by taking $a = 1$. To prove that (4.8) indeed holds we replace $r \rightarrow 2r + 3s$ to arrive at the $(b, n) \rightarrow (aq^{3s}, 3(L - s)/2 (\equiv 0 \pmod{3}))$ case of

$$(4.9) \quad \sum_{r=0}^n \frac{1 - bq^{2r}}{1 - b} \frac{(b; q^3)_r (q^{-n}; q)_r q^r}{(q; q)_r (bq^{3-n}; q^3)_r} = \delta_{n,0},$$

which is [30, Eq. (3.6.17); $p \rightarrow q^3, a \rightarrow 0$] due to Bressoud [18], Gasper [27] and Krattenthaler [38].

For a direct proof of the lemma we shift $r \rightarrow 2r + j$ to obtain the ‘singular case’ $(c, n) \rightarrow (q^{3j+1}, 3(L - j)/2 (\equiv 0 \pmod{3}))$ of

$$(4.10) \quad \sum_{r=0}^n \frac{(c; q)_{2r} (q^{-n}; q)_r q^r}{(q, c; q)_r (cq^{2-n}; q^3)_r} = \frac{(q, q^2; q^3)_{\frac{1}{3}n}}{(q/c, cq^2; q^3)_{\frac{1}{3}n}} \chi(n \equiv 0 \pmod{3}),$$

which is [31, Eq. (4.32); $k \rightarrow n - k, A \rightarrow q^{1-2n}/c$] of Gessel and Stanton. □

4.2. Proofs of (4.4) and (4.7). Before proving the what-we-believe-to-be new balanced ${}_5\phi_4$ sum (4.4) we note that Andrews’ identity (4.3) arises as the case $b = a^2$ (or $b = q^{-n}$). Since (4.3) provides a q -analogue of Watson’s ${}_3F_2$ summation, (4.4) also provides a generalization of Watson’s sum. Specifically, replacing $(a, b, c) \rightarrow (q^{a/2}, q^b, q^c)$ in (4.4) and then letting q tend to one we find

$$(4.11) \quad {}_4F_3 \left[\begin{matrix} a, b + 1, c, -n \\ 2c, b, \frac{1}{2}(a - n + 2) \end{matrix}; 1 \right] = \begin{cases} \frac{a(b+n)}{b(a+n)} \frac{(\frac{1}{2}, c - \frac{1}{2}a)_{n/2}}{(c + \frac{1}{2}, -\frac{1}{2}a)_{n/2}} & \text{if } n \text{ is even,} \\ \frac{n(b-a)}{b(a+n)} \frac{(\frac{1}{2}, c - \frac{1}{2}a + \frac{1}{2})_{(n-1)/2}}{(c + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}a)_{(n-1)/2}} & \text{if } n \text{ is odd,} \end{cases}$$

where we employ standard notation for hypergeometric series [11, 30, 48]. For $b = a$ this yields Watson’s (terminating) ${}_3F_2$ sum. (Whipple extended Watson’s result to nonterminating series, but at the ${}_4F_3$ level this no longer appears to be possible.) At the end of this section another extension of Watson’s sum is to be given.

Proof of (4.4). It is not hard to establish (4.4) by application of the contiguous relation [37, Eq. (3.8)]

$$(4.12) \quad {}_r\phi_s \left[\begin{matrix} aq, b, c, (A) \\ (B) \end{matrix}; q, z \right] = \frac{(1-b)(a-c)}{(1-a)(b-c)} {}_r\phi_s \left[\begin{matrix} a, bq, c, (A) \\ (B) \end{matrix}; q, z \right] \\ - \frac{(1-c)(a-b)}{(1-a)(b-c)} {}_r\phi_s \left[\begin{matrix} a, b, cq, (Aq) \\ (B) \end{matrix}; q, z \right].$$

Here (A) , (B) and (Aq) are shorthand notations for a_1, \dots, a_{r-3} and b_1, \dots, b_s and $a_1q, \dots, a_{r-3}q$, respectively. Utilizing (4.12) with $(a, b, c) \rightarrow (b, a^2, q^{-n})$, the left-hand side of (4.4) transforms into the sum of two b -independent ${}_4\phi_3$ series. Both are summable by (4.3) to yield the desired right-hand side. \square

Proof of (4.7). To show (4.7) we split its left-hand side by (4.12) with $(a, b, c) \rightarrow (b, -q^{-n}, q^{-n})$ so that

$$\text{LHS}(4.7) = \frac{(1+q^n)(1-bq^n)}{2q^n(1-b)} {}_4\phi_3 \left[\begin{matrix} c, -c, q^{-n}, -q^{1-n} \\ c^2q, iq^{3/2-n}, -iq^{3/2-n} \end{matrix}; q, q^2 \right] \\ - \frac{(1-q^n)(1+bq^n)}{2q^n(1-b)} {}_4\phi_3 \left[\begin{matrix} c, -c, q^{1-n}, -q^{-n} \\ c^2q, iq^{3/2-n}, -iq^{3/2-n} \end{matrix}; q, q^2 \right].$$

Both the ${}_4\phi_3$ series on the right are summable by

$$(4.13) \quad {}_4\phi_3 \left[\begin{matrix} a^2, c, -c, q^{-n} \\ c^2q, aq^{1-n/2}, -aq^{1-n/2} \end{matrix}; q, q^2 \right] \\ = \begin{cases} \frac{c^2 - a^2q^n (q; q^2)_{n/2} (c^2q^2/a^2; q^2)_{n/2-1}}{1 - a^2q^n (c^2q; q^2)_{n/2} (q^2/a^2; q^2)_{n/2-1}} & \text{if } n \text{ is even,} \\ \frac{1 - a^2c^2q^n (q; q^2)_{(n+1)/2} (c^2q/a^2; q^2)_{(n-1)/2}}{1 - a^2q^n (c^2q; q^2)_{(n+1)/2} (q/a^2; q^2)_{(n-1)/2}} & \text{if } n \text{ is odd,} \end{cases}$$

leading to the right side of (4.7). To complete the proof we need to deal with (4.13). By [37, Eq. (2.3)]

$${}_{r+1}\phi_r \left[\begin{matrix} (A) \\ aq, (B) \end{matrix}; q, z \right] = {}_{r+1}\phi_r \left[\begin{matrix} (A) \\ a, (B) \end{matrix}; q, z \right] \\ - \frac{az}{(1-a)(1-aq)} \frac{\prod_{i=1}^{r+1} (1-A_i)}{\prod_{i=1}^{r-1} (1-B_i)} {}_{r+1}\phi_r \left[\begin{matrix} (Aq) \\ aq^2, (B) \end{matrix}; q, z \right]$$

with $a \rightarrow c^2q$, the left side of (4.13) can be written as the sum of two ${}_4\phi_3$ series, both of which can be summed by the $b \rightarrow \infty$ limit of (4.4). This results in the right side of (4.13). \square

To conclude this section we wish to point out that (4.4) is certainly not the only generalization of (4.3) that may be obtained using contiguous relations. For example, by (4.3) and [37, Eq. (3.3)]

$${}_{r+1}\phi_r \left[\begin{matrix} a, (A) \\ b, (B) \end{matrix}; q, z \right] = {}_{r+1}\phi_r \left[\begin{matrix} a/q, (A) \\ b/q, (B) \end{matrix}; q, z \right] \\ + \frac{z(a-b)}{(q-b)(1-b)} \frac{\prod_{i=1}^r (1-A_i)}{\prod_{i=1}^{r-1} (1-B_i)} {}_{r+1}\phi_r \left[\begin{matrix} a, (Aq) \\ bq, (Bq) \end{matrix}; q, z \right]$$

with $(a, b, (A), (B)) \rightarrow (bq, c^2q, (a^2q, c, -c, q^{-n}), (b, aq^{1-n/2}, -aq^{1-n/2}))$ it follows that

$$(4.14) \quad {}_5\phi_4 \left[\begin{matrix} a^2q, bq, c, -c, q^{-n} \\ c^2q, b, aq^{1-n/2}, -aq^{1-n/2} \end{matrix}; q, q \right] = \begin{cases} \frac{(q, c^2/a^2; q^2)_{n/2}}{(c^2q, 1/a^2; q^2)_{n/2}} & \text{if } n \text{ is even,} \\ \frac{c^2 - b}{1 - b} \frac{1 - a^2q}{c^2 - a^2q} \frac{(q, c^2q^{-1}/a^2; q^2)_{(n+1)/2}}{(c^2q, q^{-1}/a^2; q^2)_{(n+1)/2}} & \text{if } n \text{ is odd.} \end{cases}$$

For $b = c^2$ this simplifies to (4.3) and for $(a, b, c) \rightarrow (q^{a/2-1/2}, q^b, q^c)$ together with $q \rightarrow 1$ it yields

$${}_4F_3 \left[\begin{matrix} a, b + 1, c, -n \\ 2c + 1, b, \frac{1}{2}(a - n + 1) \end{matrix}; 1 \right] = \begin{cases} \frac{(\frac{1}{2}, c - \frac{1}{2}a + \frac{1}{2})_{n/2}}{(c + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}a)_{n/2}} & \text{if } n \text{ is even,} \\ \frac{a(b - 2c)}{b(a - 2c)} \frac{(\frac{1}{2}, c - \frac{1}{2}a)_{(n+1)/2}}{(c + \frac{1}{2}, -\frac{1}{2}a)_{(n+1)/2}} & \text{if } n \text{ is odd.} \end{cases}$$

This is to be compared with (4.11). For $b = 2c$ this is again Watson’s ${}_3F_2$ sum.

Finally we remark that balanced ${}_4\phi_3$ summations other than (4.3) follow from (4.4) and (4.14). Taking $b = c^2/q$ in (4.4) and $b = a^2q$ in (4.14) leads to two more such results. Especially the latter is appealing as some factors on the left of (4.14) nicely cancel leading to

$${}_4\phi_3 \left[\begin{matrix} a^2, c, -c, q^{-n} \\ c^2q, aq^{-n/2}, -aq^{-n/2} \end{matrix}; q, q \right] = \begin{cases} \frac{(q, c^2q^2/a^2; q^2)_{n/2}}{(c^2q, q^2/a^2; q^2)_{n/2}} & \text{if } n \text{ is even,} \\ \frac{(q, c^2q/a^2; q^2)_{(n+1)/2}}{(c^2q, q/a^2; q^2)_{(n+1)/2}} & \text{if } n \text{ is odd,} \end{cases}$$

where we have also replaced a by a/q .

5. THE BAILEY LEMMA

As alluded to in the Introduction, the q -binomial transformations of the first two sections are closely related to Bailey’s lemma. Presently we will make this more precise and restate our results in terms of transformations on Bailey pairs.

First we recall the definition of a Bailey pair [16]. If $\alpha(a; q) = \{\alpha_L(a; q)\}_{L \geq 0}$ and $\beta(a; q) = \{\beta_L(a; q)\}_{L \geq 0}$ are sequences such that

$$\beta_L(a; q) = \sum_{r=0}^L \frac{\alpha_r(a; q)}{(q; q)_{L-r} (aq; q)_{L+r}},$$

then $(\alpha(a; q), \beta(a; q))$ is called a Bailey pair relative to a and q . The Bailey lemma is the following powerful mechanism for generating new Bailey pairs [6, 7, 9, 13, 41, 54].

Lemma 5.1. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then so is $(\alpha'(a; q), \beta'(a; q))$ given by*

$$(5.1a) \quad \alpha'_L(a; q) = \frac{(b, c; q)_L (aq/bc)^L}{(aq/b, aq/c; q)_L} \alpha_L(a; q),$$

$$(5.1b) \quad \beta'_L(a; q) = \frac{(aq/bc; q)_L}{(q, aq/b, aq/c; q)_L} \sum_{r=0}^L \frac{(b, c, q^{-L}; q)_r q^r}{(bcq^{-L}/a; q)_r} \beta_r(a; q).$$

For $(b, c) \rightarrow (\infty, \infty)$ and $(b, c) \rightarrow (\infty, -(aq)^{1/2})$ this is equivalent to (1.1) and (1.2), respectively.

Before we state such similar results arising from the transformations of Section 2 we recall the base-changing Bailey-pair transformations of Bressoud *et al.* [20]. The first result is (equivalent to) [20, Thm. 2.2].

Lemma 5.2. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then the pair $(\alpha'(a^2; q^2), \beta'(a^2; q^2))$ given by*

$$(5.2a) \quad \alpha'_L(a^2; q^2) = \frac{(b; q)_L}{(aq/b; q)_L} \left(\frac{aq}{b}\right)^L (-1)^L q^{\binom{L}{2}} \alpha_L(a; q),$$

$$(5.2b) \quad \beta'_L(a^2; q^2) = \frac{(-aq/b; q)_{2L}}{(-aq; q)_{2L} (q^2, a^2 q^2/b^2; q^2)_L} \sum_{r=0}^L \frac{(b; q)_r (q^{-2L}; q^2)_r q^r}{(-bq^{-2L}/a; q)_r} \beta_r(a; q)$$

forms a Bailey pair relative to a^2 and q^2 .

For $b \rightarrow 0$, $b \rightarrow \infty$ and $b \rightarrow -(aq)^{1/2}$ this yields the equations (E1), (E2) and (E3) of [50]. By some simple variable changes, (E2) and (E3) can be seen to be equivalent to (1.6) and (1.7).

The next result is (equivalent to) [20, Thm. 2.3], [50, Eq. (T1)] and (1.8).

Lemma 5.3. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then the pair $(\alpha'(a^3; q^3), \beta'(a^3; q^3))$ given by*

$$(5.3a) \quad \alpha'_L(a^3; q^3) = a^L q^{L^2} \alpha_L(a; q),$$

$$(5.3b) \quad \beta'_L(a^3; q^3) = \frac{(aq; q)_{3L}}{(q^3; q^3)_L (a^3 q^3; q^3)_{2L}} \sum_{r=0}^L \frac{(q^{-3L}; q^3)_r q^r}{(q^{-3L}/a; q)_r} \beta_r(a; q)$$

forms a Bailey pair relative to a^3 and q^3 .

To the above three lemmas we now add several new base-changing Bailey lemmas. First is a Bailey-type lemma of a quadratic nature.

Lemma 5.4. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then so is $(\alpha'(a; q), \beta'(a; q))$ given by*

$$(5.4a) \quad \alpha'_{2L}(a; q) = (-1)^L b^L q^{L^2} \frac{(aq/b; q^2)_L}{(bq; q^2)_L} \alpha_L(a; q^2), \quad \alpha'_{2L+1}(a; q) = 0,$$

$$(5.4b) \quad \beta'_L(a; q) = \frac{(b; q^2)_L}{(q, b; q)_L (aq; q^2)_L} \sum_{r=0}^{\lfloor L/2 \rfloor} \frac{(aq/b; q^2)_r (q^{-L}; q)_{2r} q^{2r}}{(q^{2-2L}/b; q^2)_r} \beta_r(a; q^2).$$

For $b \rightarrow 0$, $b \rightarrow \infty$, $b \rightarrow -a^{1/2}$ and $b \rightarrow -a^{1/2}q$ this corresponds to the even j case of (2.3), (2.5), (2.8) and (2.9).

Proof. Writing the nontrivial part of (5.4) as

$$\alpha'_{2L}(a; q) = h_L(a, b)\alpha_L(a; q^2),$$

$$\beta'_L(a; q) = \sum_{r=0}^{\lfloor L/2 \rfloor} f_{L,r}(a, b)\beta_r(a; q^2),$$

the claim of the lemma boils down to showing that

$$\sum_{r=s}^{\lfloor L/2 \rfloor} \frac{f_{L,r}(a, b)}{(q^2; q^2)_{r-s}(aq^2; q^2)_{r+s}} = \frac{h_s(a, b)}{(q; q)_{L-2s}(aq; q)_{L+2s}}.$$

After shifting $r \rightarrow r + s$ this follows from (2.10) with

$$(a, b, n) \rightarrow (aq^{4s+1}, bq^{2s}, L - 2s).$$

□

Next, (2.12) and (2.13) for even j correspond to the following two quartic Bailey lemmas.

Lemma 5.5. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then so is $(\alpha'(a; q), \beta'(a; q))$ given by*

$$(5.5a) \quad \alpha'_{2L}(a; q) = \alpha_L(a^2; q^4), \quad \alpha'_{2L+1}(a; q) = 0,$$

$$(5.5b) \quad \beta'_L(a; q) = \frac{q^L(-q^{-1}; q^2)_L}{(q^2, aq; q^2)_L} \sum_{r=0}^{\lfloor L/2 \rfloor} \frac{(-aq^2, q^{-2L}; q^2)_{2r}q^{4r}}{(-q^{3-2L}; q^2)_{2r}} \beta_r(a^2; q^4).$$

Proof. Copying the proof of Lemma 5.4 this follows from (2.14) with $(a, n) \rightarrow (-aq^{4s+1}, L - 2s)$. □

Lemma 5.6. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then so is $(\alpha'(a; q), \beta'(a; q))$ given by*

$$(5.6a) \quad \alpha'_{2L}(a; q) = q^{2L} \frac{1+a}{1+aq^{4L}} \alpha_L(a^2; q^4), \quad \alpha'_{2L+1}(a; q) = 0,$$

$$(5.6b) \quad \beta'_L(a; q) = \frac{(-q; q^2)_L}{(q^2, aq; q^2)_L} \sum_{r=0}^{\lfloor L/2 \rfloor} \frac{(-a, q^{-2L}; q^2)_{2r}q^{4r}}{(-q^{1-2L}; q^2)_{2r}} \beta_r(a^2; q^4).$$

Proof. This follows from (2.16) with $(a, n) \rightarrow (-aq^{4s+1}, L - 2s)$. □

Finally there is cubic Bailey lemma corresponding to the even j case of the transformation (2.18).

Lemma 5.7. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then so is $(\alpha'(a; q), \beta'(a; q))$ given by*

$$(5.7a) \quad \alpha'_{3L}(a; q) = a^L q^{3L^2} \alpha_L(a; q^3), \quad \alpha'_{3L\pm 1}(a; q) = 0,$$

$$(5.7b) \quad \beta'_L(a; q) = \frac{(a; q^3)_L}{(q; q)_L(a; q)_{2L}} \sum_{r=0}^{\lfloor L/3 \rfloor} \frac{(q^{-L}; q)_{3r}q^{3r}}{(q^{3-3L}/a; q^3)_r} \beta_r(a; q^3).$$

Proof. Writing the nontrivial part of (5.7) as

$$\begin{aligned} \alpha'_{3L}(a; q) &= h_L(a, b)\alpha_L(a; q^3), \\ \beta'_L(a; q) &= \sum_{r=0}^{\lfloor L/3 \rfloor} f_{L,r}(a, b)\beta_r(a; q^3), \end{aligned}$$

we need to show that

$$\sum_{r=s}^{\lfloor L/3 \rfloor} \frac{f_{L,r}(a, b)}{(q^3; q^3)_{r-s}(aq^3; q^3)_{r+s}} = \frac{h_s(a, b)}{(q; q)_{L-3s}(aq; q)_{L+3s}}.$$

Replacing $r \rightarrow r + s$ and defining $n = L - 3s$, this follows from (2.11) with $(a, b, c, d, q) \rightarrow (q^{-n}, q^{1-n}, q^{2-n}, aq^{6s+3}, q^3)$. \square

Remark 5.1. Lemmas 5.4–5.7 correspond to the even j instances of the q -binomial transformations of Section 2. Equally well can one find Bailey-type lemmas corresponding to j being odd. Since we will not use these in the remainder of the paper we only state the result related to the odd case of (2.3), (2.5), (2.8) and (2.9).

Lemma 5.8. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then so is $(\alpha'(a; q), \beta'(a; q))$ given by*

$$\begin{aligned} \alpha'_{2L+1}(a; q) &= (-1)^L b^L q^{L^2} \frac{(aq^3/b; q^2)_L}{(bq; q^2)_L} \alpha_L(aq^2; q^2), \quad \alpha'_{2L}(a; q) = 0, \\ \beta'_{L+1}(a; q) &= \frac{(b; q^2)_L}{(q, b; q)_L (aq^3; q^2)_L} \sum_{r=0}^{\lfloor L/2 \rfloor} \frac{(aq^3/b; q^2)_r (q^{-L}; q)_{2r} q^{2r}}{(aq; q)_2 (q^{2-2L}/b; q^2)_r} \beta_r(aq^2; q^2). \end{aligned}$$

Again it is important to also find the inverses of the transformations (5.4)–(5.7). Since all are of the form

$$\begin{aligned} \alpha'_{nL}(a; q) &= g_L(a, q)\alpha_L(a^k; q^l), \quad \alpha'_{nL+m}(a; q) = 0, \\ \beta'_L(a; q) &= \sum_{r=0}^{\lfloor L/n \rfloor} f_{L,r}(a, q)\beta_r(a^k; q^l), \end{aligned}$$

with $n \in \{2, 3\}$ and $m \in \{1, \dots, n - 1\}$, we can only find left-inverses, defined as

$$\begin{aligned} \alpha'_L(a^k; q^l) &= \frac{\alpha_{nL}(a; q)}{g_L(a, q)}, \\ \beta'_L(a^k; q^l) &= \sum_{r=0}^{nL} \tilde{f}_{L,r}(a, q)\beta_r(a; q), \end{aligned}$$

with $\tilde{f}_{L,r}(a, q)$ given by

$$(5.8) \quad \sum_{r=ns}^{nL} \tilde{f}_{L,r}(a, q) f_{r,s}(a, q) = \delta_{L,s}.$$

Such an $\tilde{f}_{L,r}(a, q)$ is obviously not unique, but guided by our inverse relations of the previous section, it is not hard to find an $\tilde{f}_{L,r}(a, q)$ that can be expressed in simple closed form.

First we state a left-inverse of (5.4).

Lemma 5.9. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then the pair $(\alpha'(a; q^2), \beta'(a; q^2))$ given by*

$$(5.9a) \quad \alpha'_L(a; q^2) = (-1)^L b^{-L} q^{-L^2} \frac{(bq; q^2)_L}{(aq/b; q^2)_L} \alpha_{2L}(a; q),$$

$$(5.9b) \quad \beta'_L(a; q^2) = \frac{(1/b; q^2)_L}{(q; q)_{2L} (aq/b; q^2)_L} \sum_{r=0}^{2L} \frac{(aq; q^2)_r (bq, q^{-2L}; q)_r q^r}{(bq^{2-2L}; q^2)_r} \beta_r(a; q)$$

yields a Bailey pair relative to a and q^2 .

Proof. Reading off $f_{L,r}(a, q)$ and $\tilde{f}_{L,r}(a, q)$ from (5.4b) and (5.9b), respectively, the inverse relation (5.8) (with $n = 2$) can be verified by (4.3) with $c = a$ and $(a, n) \rightarrow (b^{1/2}q^s, 2L - 2s)$. \square

Next we have left-inverses of the two quartic transformations (5.5) and (5.6).

Lemma 5.10. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then the pair $(\alpha'(a^2; q^4), \beta'(a^2; q^4))$ given by*

$$(5.10a) \quad \alpha'_L(a^2; q^4) = \alpha_{2L}(a; q),$$

$$(5.10b) \quad \beta'_L(a^2; q^4) = \frac{(-q; q^2)_{2L}}{(q^2, -aq^2; q^2)_{2L}} \sum_{r=0}^{2L} \frac{(aq, q^{-4L}; q^2)_r q^r}{(-q^{1-4L}; q^2)_r} \beta_r(a; q)$$

yields a Bailey pair relative to a^2 and q^4 .

Proof. Reading off $f_{L,r}(a, q)$ and $\tilde{f}_{L,r}(a, q)$ from (5.5b) and (5.10b), (5.8) (with $n = 2$) follows from (4.6) with $(a, n) \rightarrow (-q, 2L - 2s)$. \square

Lemma 5.11. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then the pair $(\alpha'(a^2; q^4), \beta'(a^2; q^4))$ given by*

$$(5.11a) \quad \alpha'_L(a^2; q^4) = q^{-2L} \frac{1 + aq^{4L}}{1 + a} \alpha_{2L}(a; q),$$

$$(5.11b) \quad \beta'_L(a^2; q^4) = \frac{(-q^{-1}; q^2)_{2L}}{(q^2, -a; q^2)_{2L}} \sum_{r=0}^{2L} \frac{(aq, q^{-4L}; q^2)_r q^{2r}}{(-q^{3-4L}; q^2)_r} \beta_r(a; q)$$

yields a Bailey pair relative to a^2 and q^4 .

Proof. Reading off $f_{L,r}(a, q)$ and $\tilde{f}_{L,r}(a, q)$ from (5.6b) and (5.11b), (5.8) (with $n = 2$) is (4.6) with $(a, n) \rightarrow (-q^{-1}, 2L - 2s)$. \square

Finally we give a left-inverse of (5.7).

Lemma 5.12. *If $(\alpha(a; q), \beta(a; q))$ is a Bailey pair relative to a and q , then the pair $(\alpha'(a; q^3), \beta'(a; q^3))$ given by*

$$(5.12a) \quad \alpha'_L(a; q^3) = a^{-L} q^{-3L^2} \alpha_{3L}(a; q),$$

$$(5.12b) \quad \beta'_L(a; q^3) = \frac{(1/a; q^3)_L}{(q; q)_{3L}} \sum_{r=0}^{3L} \frac{(aq; q)_{2r} (q^{-3L}; q)_r q^r}{(aq^{3-3L}; q^3)_r} \beta_r(a; q)$$

yields a Bailey pair relative to a and q^3 .

Proof. Reading off $f_{L,r}(a, q)$ and $\tilde{f}_{L,r}(a, q)$ from (5.7b) and (5.12b), equation (5.8) (with $n = 3$) follows from (4.9) with $(b, n) \rightarrow (aq^{6s}, 3L - 3s)$. \square

6. q -HYPERGEOMETRIC TRANSFORMATIONS

6.1. Applications of base-changing Bailey lemmas. In the following we have compiled a list of quadratic, cubic and quartic transformation formulas obtained by twice iterating the unit Bailey pair [6]

$$(6.1) \quad \alpha_L = (-1)^L q^{\binom{L}{2}} \frac{1 - aq^{2L}}{1 - a} \frac{(a; q)_L}{(q; q)_L}, \quad \beta_L = \delta_{L,0}$$

using Lemmas 5.1–5.7. Before stating the resulting transformations two remarks are in order.

First we note that (5.6) applied to the unit Bailey pair yields

$$\alpha'_{2L}(a; q) = (-1)^L q^{2L^2} \frac{1 - aq^{4L}}{1 - a} \frac{(a^2; q^4)_L}{(q^4; q^4)_L}, \quad \beta'_L(a; q) = \frac{(-q; q^2)_L}{(q^2, aq; q^2)_L},$$

whereas (5.4) applied to the unit Bailey pair leads to

$$\alpha'_{2L}(a; q) = b^L q^{2L^2 - 1} \frac{1 - aq^{4L}}{1 - a} \frac{(a, aq/b; q^2)_L}{(q^2, bq; q^2)_L}, \quad \beta'_L(a; q) = \frac{(b; q^2)_L}{(q, b; q)_L (aq; q^2)_L},$$

where in both cases $\alpha'_{2L+1}(a; q) = 0$. Since the second result includes the first as the special case $b = -q$, we need not consider those identities obtained by first applying (5.6) to the unit Bailey pair.

A second remark is that taking the unit Bailey pair and applying the transformation (5.i) followed by (5.1) and then using a standard polynomial argument yields a result that implies the identity obtained by applying (5.i) followed by (5.k). Here $i \in \{1, \dots, 7\}$ and $k \in \{2, 3, 4, 7\}$. So, for example, we will not consider the identity obtained by successive application of (5.3) and (5.7) because, modulo a polynomial argument, it is implied by the application of (5.3) followed by (5.1).

Taking both of the above comments into account we will derive 18 different results, obtained by first applying (5.i) with $i \in \{1, 2, 3, 4, 5, 7\}$ and then (5.k) with $k \in \{1, 5, 6\}$. Five of the six identities that arise by application of (5.i) followed by (5.1) are not new. We nevertheless have chosen to state these known results, as they will be needed later to prove several of the claims made in Section 3.3.

6.1.1. *Transformation (5.i) followed by (5.1).* Applying (5.1) twice to the unit Bailey pair and replacing q^{-n} by f we find Watson’s transformation [30, Eq. (III.17)]

$$(6.2) \quad {}_8W_7(a; b, c, d, e, f; q, a^2q^2/bcdef) \\ = \frac{(aq, aq/de, aq/df, aq/ef; q)_\infty}{(aq/d, aq/e, aq/f, aq/def; q)_\infty} \\ \times {}_4\phi_3 \left[\begin{matrix} aq/bc, d, e, f \\ aq/b, aq/c, def/a \end{matrix}; q, q \right],$$

provided both series terminate. (Watson's transformation actually holds under slightly weaker conditions, but these do not follow from the above derivation.) The derivation of (6.2) using Bailey's lemma is of course well known; see e.g., [6].

Applying (5.2) and then (5.1) to the unit Bailey pair and replacing q^{-2n} by e we obtain a quadratic transformation due to Verma and Jain [52, Eq. (1.3)]:

$$(6.3) \quad {}_{10}W_9(a; b, c^{1/2}, -c^{1/2}, d^{1/2}, -d^{1/2}, e^{1/2}, -e^{1/2}; q, -a^3 q^3 / bcde) \\ = \frac{(a^2 q^2, a^2 q^2 / cd, a^2 q^2 / ce, a^2 q^2 / de; q^2)_\infty}{(a^2 q^2 / c, a^2 q^2 / d, a^2 q^2 / e, a^2 q^2 / cde; q^2)_\infty} \\ \times {}_5\phi_4 \left[\begin{matrix} -aq/b, -aq^2/b, c, d, e \\ -aq, -aq^2, a^2 q^2 / b^2, cde/a^2 \end{matrix}; q^2, q^2 \right],$$

provided both series terminate. We remark that Verma and Jain stated the above identity for $e = q^{-2n}$ only. In the calculations of Section 6.2, the above, slightly more general, form will however be crucial. Similar remarks apply to all the subsequent identities of Verma and Jain.

Applying (5.3) and then (5.1) to the unit Bailey pair and replacing q^{-3n} by d we obtain the following cubic transformation of Verma and Jain [52, Eq. (1.5)]:

$$(6.4) \quad {}_{12}W_{11}(a; b^{1/3}, b^{1/3}\omega, b^{1/3}\omega^2, c^{1/3}, c^{1/3}\omega, c^{1/3}\omega^2, d^{1/3}, d^{1/3}\omega, d^{1/3}\omega^2; q, a^4 q^4 / bcd) \\ = \frac{(a^3 q^3, a^3 q^3 / bc, a^3 q^3 / bd, a^3 q^3 / cd; q^3)_\infty}{(a^3 q^3 / b, a^3 q^3 / c, a^3 q^3 / d, a^3 q^3 / bcd; q^3)_\infty} \\ \times {}_6\phi_5 \left[\begin{matrix} aq, aq^2, aq^3, b, c, d \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2} q^3, -a^{3/2} q^3, bcd/a^3 \end{matrix}; q^3, q^3 \right],$$

provided both series terminate.

Applying (5.4) and then (5.1) to the unit Bailey pair and replacing q^{-n} by e we find a second quadratic transformation of Verma and Jain [52, Eq. (1.4)]:

$$(6.5) \quad {}_{10}W_9(a; aq/b, c, cq, d, dq, e, eq; q^2, a^2 b q^2 / c^2 d^2 e^2) \\ = \frac{(aq, aq/cd, aq/ce, aq/de; q)_\infty}{(aq/c, aq/d, aq/e, aq/cde; q)_\infty} {}_5\phi_4 \left[\begin{matrix} b^{1/2}, -b^{1/2}, c, d, e \\ (aq)^{1/2}, -(aq)^{1/2}, b, cde/a \end{matrix}; q, q \right],$$

provided both series terminate.

Next, (5.7) followed by (5.1) yields a second cubic transformation of Verma and Jain [52, Eq. (1.6)]:

$$(6.6) \quad {}_{12}W_{11}(a; b, bq, bq^2, c, cq, cq^2, d, dq, dq^2; q^3, a^4 q^3 / b^3 c^3 d^3) \\ = \frac{(aq, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty} \\ \times {}_6\phi_5 \left[\begin{matrix} a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2, b, c, d \\ a^{1/2}, -a^{1/2}, (aq)^{1/2}, -(aq)^{1/2}, bcd/a \end{matrix}; q, q \right],$$

provided both series terminate.

Finally, the identity obtained from (5.5) followed by (5.1) appears to be new:

$$(6.7) \quad \sum_{k=0}^{\infty} \frac{1 - a^2 q^{8k}}{1 - a^2} \frac{(a^2; q^4)_k}{(q^4; q^4)_k} \frac{(b, c, d; q)_{2k}}{(aq/b, aq/c, aq/d; q)_{2k}} \left(-\frac{a^2 q}{b^2 c^2 d^2}\right)^k$$

$$= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_{\infty}}{(aq/b, aq/c, aq/d, aq/bcd; q)_{\infty}} {}_5\phi_4 \left[\begin{matrix} iq^{-1/2}, -iq^{-1/2}, b, c, d \\ -q, (aq)^{1/2}, -(aq)^{1/2}, bcd/a \end{matrix}; q, q^2 \right],$$

provided both series terminate.

Interesting summations occur by making ‘singular’ specializations in the above six identities. To illustrate the idea, consider (6.2) and put the prefactor of the ${}_4\phi_3$ series to the left-hand side. If k is the summation variable of the ${}_8W_7$ series, then the summand on the left contains

$$\frac{(aq/e; q)_{\infty} (f; q)_k}{(aq/e; q)_k} = (aq^{k+1}/e; q)_{\infty} (f; q)_k$$

as a factor. By taking $e = aq^n$ and $f = q^{-n}$, this becomes

$$(q^{k-n+1}; q)_{\infty} (q^{-n}; q)_k$$

which vanishes unless $k = n$. The resulting identity is the q -Pfaff–Saalschütz formula (2.11) with $(a, b, c, d) \rightarrow (aq/bc, aq^n, q^{-n}, aq/b)$. Similarly, by taking $d = a^2 q^{2n}$ and $e = q^{-2n}$ in (6.3) and then negating a we find [20, Eq. (2.1)]

$${}_4\phi_3 \left[\begin{matrix} aq/b, aq^2/b, a^2 q^{2n}, q^{-2n} \\ aq, aq^2, a^2 q^2/b^2 \end{matrix}; q^2, q^2 \right] = \left(\frac{aq}{b}\right)^n \frac{1 - a}{1 - aq^{2n}} \frac{(-q, b; q)_n}{(a, -aq/b; q)_n},$$

and by taking $c = a^3 q^{3n}$ and $d = q^{-3n}$ in (6.4) we get

$${}_5\phi_4 \left[\begin{matrix} aq, aq^2, aq^3, a^3 q^{3n}, q^{-3n} \\ (aq)^{3/2}, -(aq)^{3/2}, a^{3/2} q^3, -a^{3/2} q^3 \end{matrix}; q^3, q^3 \right]$$

$$= (aq)^n \frac{1 - aq^{2n}}{1 - a^3 q^{6n}} \frac{(q^3; q^3)_n (aq; q)_{n-1}}{(q; q)_n (a^3 q^3; q^3)_{n-1}}.$$

Next consider (6.5) and (6.7) and again put the prefactor of the ${}_5\phi_4$ series to the other side to obtain the factor

$$(aq^{2k+1}/c; q)_{\infty} (d; q)_{2k}$$

in the summand on the left. By specializing $c = aq^n$ and $d = q^{-n}$ this vanishes unless $2k = n$. The two ensuing identities are (4.3) and, after the change $a \rightarrow a^2 q^{1-n}$,

$${}_4\phi_3 \left[\begin{matrix} iq^{-1/2}, -iq^{-1/2}, a^2 q, q^{-n} \\ -q, aq^{1-n/2}, -aq^{1-n/2} \end{matrix}; q, q^2 \right] = \frac{1 + a^2 q^{n+1}}{a^2 q + q^n} \frac{(q, -q/a^2; q^2)_{n/2}}{(-q^2, 1/a^2; q^2)_{n/2}} \chi(n \equiv 0 \pmod{2}),$$

respectively. Finally, removing the prefactor on the right and specializing $c = q^{-n}$ and $d = aq^n$ in (6.6), the summand on the left will contain the factor

$$(q^{3k-n+1}; q)_{\infty} (q^{-n}; q)_{3k}$$

so that the only nonvanishing contribution comes from $3k = n$. The resulting identity is (4.10) with order of summation reversed.

Equation (6.7) has another noteworthy specialization. Taking $b = -(aq)^{1/2}$ and $c = -d = q^{-n}$ the left simplifies to ${}_5W_4(a^2; q^{-2n}, q^{2-2n}; q^4, -aq^{4n})$ which sums to $(a^2 q^2; q^2)_n (-a; q^4)_n / ((a^2 q^2; q^4)_n (-a; q^2)_n)$ by Rogers’ q -Dougall sum [30, Eq. (II.20)]. The resulting identity is (3.7) with $a \rightarrow (aq)^{1/2}$.

6.1.2. *Transformation* (5.i) followed by (5.5) or (5.6). Applying (5.1) and then (5.5) leads to

$$(6.8) \quad \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1 - a^2 q^{8k}}{1 - a^2} \frac{(a^2, b, c; q^4)_k}{(q^4, a^2 q^4/b, a^2 q^4/c; q^4)_k} \frac{(q^{-n}; q)_{2k}}{(aq^{n+1}; q)_{2k}} \left(-\frac{a^2 q^{2n+3}}{bc} \right)^k \\ = q^n \frac{(-q^{-1}; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} a^2 q^4/bc, -aq^2, -aq^4, q^{-2n}, q^{2-2n} \\ a^2 q^4/b, a^2 q^4/c, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right].$$

For $b = 1$ this simplifies to (2.14). Likewise, applying (5.1) and then (5.6) generalizes (2.16):

$$(6.9) \quad {}_{10}W_9(a; -a, b^{1/2}, -b^{1/2}, c^{1/2}, -c^{1/2}, q^{-n}, q^{1-n}; q^2, -a^2 q^{2n+5}/bc) \\ = \frac{(-q; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} a^2 q^4/bc, -a, -aq^2, q^{-2n}, q^{2-2n} \\ a^2 q^4/b, a^2 q^4/c, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right].$$

This transformation seems to be a hybrid of (6.3) and (6.5).

From (5.2) followed by (5.5) we obtain a ${}_6\phi_5$ to ${}_4\phi_3$ transformation. Summing the ${}_6\phi_5$ by [30, Eq. (II.20)] we recover (2.14). A similar kind of situation, but with a much happier outcome, arises if we apply (5.2) followed by (5.6). In the first instance this yields

$${}_8W_7(a; ia^{1/2}, -ia^{1/2}, b, q^{-n}, q^{1-n}; q^2, aq^{2n+3}/b) \\ = \frac{(-q; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} -a, -aq^2/b, -aq^4/b, q^{-2n}, q^{2-2n} \\ a^2 q^4/b^2, -aq^4, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right],$$

but thanks to (6.2) the left-hand side may be simplified to a ${}_4\phi_3$ series. After also replacing a by $-a^2$ this gives

$${}_4\phi_3 \left[\begin{matrix} q^2, b, q^{-n}, q^{1-n} \\ aq^2, -aq^2, -bq^{1-2n}/a^2 \end{matrix}; q^2, q^2 \right] \\ = \frac{(-q; q^2)_n (-a^2 q/b; q)_n}{(-q; q)_n (-a^2 q/b; q^2)_n} {}_5\phi_4 \left[\begin{matrix} a^2, a^2 q^2/b, a^2 q^4/b, q^{-2n}, q^{2-2n} \\ a^2 q^4, a^4 q^4/b^2, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right].$$

This result, which for $b = 1$ reduces to (2.16), should be compared with (3.15).

Next, (5.3) followed by (5.5) yields

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1 - a^2 q^{8k}}{1 - a^2} \frac{(a^2; q^4)_k}{(q^4; q^4)_k} \frac{(q^{-3n}; q^3)_{2k}}{(a^3 q^{3n+3}; q^3)_{2k}} (-a^2 q^{6n+1})^k \\ = q^{3n} \frac{(-q^{-3}; q^6)_n (a^3 q^3; q^3)_n}{(-q^3; q^3)_n (a^3 q^3; q^6)_n} {}_5\phi_4 \left[\begin{matrix} a^2 q^4, a^2 q^8, a^2 q^{12}, q^{-6n}, q^{6-6n} \\ a^3 q^6, a^3 q^{12}, -q^{9-6n}, -q^{15-6n} \end{matrix}; q^{12}, q^{12} \right]$$

and (5.3) followed by (5.6) yields

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1 - a^2 q^{8k}}{1 - a^2} \frac{(a^2; q^4)_k}{(q^4; q^4)_k} \frac{(-a^3; q^{12})_k}{(-a^3 q^{12}; q^{12})_k} \frac{(q^{-3n}; q^3)_{2k}}{(a^3 q^{3n+3}; q^3)_{2k}} (-a^2 q^{6n+7})^k \\ = \frac{(-q^3; q^6)_n (a^3 q^3; q^3)_n}{(-q^3; q^3)_n (a^3 q^3; q^6)_n} {}_6\phi_5 \left[\begin{matrix} -a^3, a^2 q^4, a^2 q^8, a^2 q^{12}, q^{-6n}, q^{6-6n} \\ a^3 q^6, a^3 q^{12}, -a^3 q^{12}, -q^{3-6n}, -q^{9-6n} \end{matrix}; q^{12}, q^{12} \right].$$

From (5.4) followed by (5.5) we obtain yet another generalization of (2.14) (obtained by setting $b = a^2q^4$):

$$\begin{aligned} & \sum_{k=0}^{\lfloor n/4 \rfloor} \frac{1 - a^2q^{16k}}{1 - a^2} \frac{(a^2, a^2q^4/b; q^8)_k}{(q^8, bq^4; q^8)_k} \frac{(q^{-n}; q)_{4k}}{(aq^{n+1}; q)_{4k}} (bq^{4n-2})^k \\ &= q^n \frac{(-q^{-1}; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} b^{1/2}, -b^{1/2}, -aq^4, q^{-2n}, q^{2-2n} \\ b, aq^2, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right], \end{aligned}$$

and from (5.4) followed by (5.6) we obtain our last generalization of (2.16):

$$\begin{aligned} & {}_{10}W_9(a; -a, aq^2/b^{1/2}, -aq^2/b^{1/2}, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}; q^4, bq^{4n+2}) \\ &= \frac{(-q; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} b^{1/2}, -b^{1/2}, -a, q^{-2n}, q^{2-2n} \\ b, aq^2, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right]. \end{aligned}$$

Applying (5.7) and then (5.5) or (5.6) gives

$$\begin{aligned} & \sum_{k=0}^{\lfloor n/6 \rfloor} \frac{1 - a^2q^{24k}}{1 - a^2} \frac{(a^2; q^{12})_k}{(q^{12}; q^{12})_k} \frac{(q^{-n}; q)_{6k}}{(aq^{n+1}; q)_{6k}} (-a^2q^{6n-3})^k \\ &= q^n \frac{(-q^{-1}; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_6\phi_5 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, -aq^4, q^{-2n}, q^{2-2n} \\ a, -a, aq^2, -q^{3-2n}, -q^{5-2n} \end{matrix}; q^4, q^4 \right] \end{aligned}$$

and

$$\begin{aligned} & {}_{10}W_9(a; -a, q^{-n}, q^{1-n}, q^{2-n}, q^{3-n}, q^{4-n}, q^{5-n}; q^6, -a^2q^{6n+3}) \\ &= \frac{(-q; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} a^{2/3}, a^{2/3}\omega, a^{2/3}\omega^2, q^{-2n}, q^{2-2n} \\ a, aq^2, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, q^4 \right], \end{aligned}$$

respectively.

Finally, applying (5.5) twice leads to

$$\begin{aligned} & \sum_{k=0}^{\lfloor n/4 \rfloor} \frac{1 - a^4q^{32k}}{1 - a^4} \frac{(a^4; q^{16})_k}{(q^{16}; q^{16})_k} \frac{(q^{-n}; q)_{4k}}{(aq^{n+1}; q)_{4k}} (-q^{4n-6})^k \\ &= q^n \frac{(-q^{-1}; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} iq^{-2}, -iq^{-2}, -aq^4, q^{-2n}, q^{2-2n} \\ -q^4, aq^2, q^{3-2n}, q^{5-2n} \end{matrix}; q^4, q^8 \right] \end{aligned}$$

whereas (5.5) followed by (5.6) yields

$$\begin{aligned} & \sum_{k=0}^{\lfloor n/4 \rfloor} \frac{1 - a^4q^{32k}}{1 - a^4} \frac{(a^4; q^{16})_k}{(q^{16}; q^{16})_k} \frac{(-a; q^8)_k}{(-aq^8; q^8)_k} \frac{(q^{-n}; q)_{4k}}{(aq^{n+1}; q)_{4k}} (-q^{4n-2})^k \\ &= \frac{(-q; q^2)_n (aq; q)_n}{(-q; q)_n (aq; q^2)_n} {}_5\phi_4 \left[\begin{matrix} iq^{-2}, -iq^{-2}, -a, q^{-2n}, q^{2-2n} \\ -q^4, aq^2, q^{1-2n}, q^{3-2n} \end{matrix}; q^4, q^8 \right]. \end{aligned}$$

6.2. Generalized $\sum FF = \sum FF$ identities. Many of the results of Section 6.1 may be further exploited to yield base-changing transformations between balanced or ‘almost’ balanced series. The idea is to take two of the transformations from the previous section and to specialize the respective left-hand sides such that they coincide. As a result the corresponding right-hand sides may be equated leading to a new transformation. This way one can for example rederive all of the transformations implied by the $\sum FF = \sum FF$ relations of Section 3.3. Instead, however,

we will only prove those identities of Section 3.3 that generalize $\sum FF = \sum FF$ transformations.

As a first example we consider Watson’s transformation (6.2) and transformation (6.5) of Verma and Jain. The respective left-hand sides are given by

$${}_8W_7(a; b, c, d, e, f; q, a^2q^2/bcdef)$$

and

$${}_{10}W_9(a; aq/b, c, cq, d, dq, e, eq; q^2, a^2bq^2/c^2d^2e^2).$$

By the substitution $(b, d, e, f, q) \rightarrow (cq, aq/b, q^{-n}, q^{1-n}, q^2)$ in the first and $(d, e) \rightarrow (-aq)^{1/2}, q^{-n}$ in the second expression, both become

$${}_8W_7(a; aq/b, c, cq, q^{-n}, q^{1-n}; q^2, abq^{2n+1}/c^2).$$

Hence under these substitutions the respective right-hand sides may be equated, resulting in the quadratic transformation

$$\begin{aligned} & {}_4\phi_3 \left[\begin{matrix} b^{1/2}, -b^{1/2}, c, q^{-n} \\ (aq)^{1/2}, b, -cq^{1-n}/(aq)^{1/2} \end{matrix}; q, q \right] \\ &= \frac{(aq/c; q)_n (b; q^2)_n}{((aq)^{1/2}, b, -(aq)^{1/2}/c; q)_n} {}_4\phi_3 \left[\begin{matrix} aq/c^2, aq/b, q^{-n}, q^{1-n} \\ aq/c, aq^2/c, q^{2-2n}/b \end{matrix}; q^2, q^2 \right]. \end{aligned}$$

By the variable change $(a, b, c) \rightarrow (a^2/q, c^2, b)$ this becomes (3.10).

Next consider the pair of identities (6.3) and (6.5). If in (6.3) we let $(b, c, d, e, q) \rightarrow (aq/b, -aq^2, q^{-2n}, q^{2-2n}, q^2)$ and in (6.5) we let $(c, d, e) \rightarrow (-aq)^{1/2}, q^{-n}, -q^{-n}$, then both left-hand sides become

$$(6.10) \quad {}_8W_7(a; aq/b, q^{-n}, -q^{-n}, q^{1-n}, -q^{1-n}; q^2, abq^{4n+1}).$$

Again we conclude that under the above substitutions the right-hand sides of (6.3) and (6.5) may be equated. The resulting transformation is (3.17) with a replaced by $(aq)^{1/2}$. Equation (3.21) is found by noting that (6.10) also arises from (6.3) by letting $(b, c, d, e, q) \rightarrow (-bq, a^2q^2/b^2, q^{-2n}, q^{2-2n}, q^2)$.

In our following example we again equate appropriately specialized right-hand sides of (6.3) and (6.5), but this time $(b, c, d, e, q) \rightarrow (aq/b, \infty, q^{-2n}, q^{2-2n}, q^2)$ in (6.3) and $(c, d, e) \rightarrow (\infty, q^{-n}, -q^{-n})$ in (6.5). Since both of the left sides become

$${}_8\phi_9 \left[\begin{matrix} a, a^{1/2}q^2, -a^{1/2}q^2, aq/b, q^{-n}, -q^{-n}, q^{1-n}, -q^{1-n} \\ a^{1/2}, -a^{1/2}, bq, aq^{n+1}, -aq^{n+1}, aq^{n+2}, -aq^{n+2}, 0, 0 \end{matrix}; q^2, a^2bq^{4n+3} \right]$$

we may again equate the right sides leading to (3.19) with $a \rightarrow (aq)^{1/2}$.

Next we equate (6.5) with (6.9). In order to do so we need to choose $(b, c, d, e) \rightarrow (-q, b^{1/2}, -b^{1/2}, q^{-n})$ in the former and $c \rightarrow bq^2$ in the latter. Both left-hand sides then simplify to

$${}_{10}W_9(a; -a, b^{1/2}, -b^{1/2}, b^{1/2}q, -b^{1/2}q, q^{-n}, q^{1-n}; q^2, -a^2q^{2n+3}/b^2).$$

The corresponding identity obtained by equating the respective right-hand sides is (3.25) with $a \rightarrow aq$.

Finally we treat the pair of identities (6.7) and (6.8). In the first we let $(b, c, d) \rightarrow (b^{1/2}, -b^{1/2}, q^{-n})$ and in the second we let $c \rightarrow bq^2$. Then both the left sides become

$$\sum_{k=0}^{\infty} \frac{1 - a^2q^{8k}}{1 - a^2} \frac{(a^2; q^4)_k}{(q^4; q^4)_k} \frac{(b; q^2)_{2k}}{(a^2q^2/b; q^2)_{2k}} \frac{(q^{-n}; q)_{2k}}{(aq^{n+1}; q)_{2k}} \left(-\frac{a^2q^{2n+1}}{b^2} \right)^k.$$

Accordingly, we may equate right-hand sides to find (3.26) with $a \rightarrow aq$.

7. ROGERS–RAMANUJAN TYPE IDENTITIES

In this section the transformations of Section 2 are applied to yield identities of the Rogers–Ramanujan type.

A first remark is that most of the single-sum Rogers–Ramanujan identities that result when applying our new q -binomial transformations are well known and can nearly all be found in Slater’s compendium of 130 such identities [47]. This should come as no surprise since in the large L limit most of our transformations reduce to sums implied by the ordinary Bailey lemma (5.1). In view of recent work on a polynomial analogue of the Slater list by Sills [45], we note that our transformations give rise to rather natural polynomial versions of many of the single-sum identities, different from those in [45]. For example, all of the Rogers–Ramanujan identities in Slater’s list that have a product side of the form

$$\frac{(\pm q^a, \pm q^{b-a}, q^b; q^b)_\infty}{(q^2; q^2)_\infty} \quad \text{or} \quad (-q; q^2)_\infty \frac{(\pm q^a, \pm q^{b-a}, q^b; q^b)_\infty}{(q^2; q^2)_\infty}$$

can be given a polynomial analogue by using the transformations (2.3) and (2.8), respectively. To give just one example of this we take (1.3) as a starting point and first apply the quadratic transformation (1.6) to get a polynomial identity equivalent to G(1) in Slater’s list of Bailey pairs [46]. Then applying (2.3) or (2.8) we obtain

$$\sum_{j=-L}^L (-1)^j q^{j(7j+1)/2} \begin{bmatrix} 2L \\ L-2j \end{bmatrix} = \sum_{n=0}^\infty \frac{q^{2n^2} (q^2; q^2)_L (-q; q)_{L-2n}}{(q^2; q^2)_n (-q; q)_{2n} (q^2; q^2)_{L-2n}}$$

and

$$\sum_{j=-L}^L (-1)^j q^{j(5j+1)/2} \begin{bmatrix} 2L \\ L-2j \end{bmatrix} = (1 + q^L) \sum_{n=0}^\infty \frac{q^{n^2} (q; q)_L (-q^2; q^2)_{L-n-1}}{(q^4; q^4)_n (q; q)_{L-2n}}.$$

Assuming $|q| < 1$, taking the large L limit and using the Jacobi triple product identity [30, Eq. (II.28)]

$$(7.1) \quad \sum_{k=-\infty}^\infty (-1)^k a^k q^{\binom{k}{2}} = (a, q/a, q; q)_\infty,$$

we find the Rogers–Selberg identity

$$(7.2) \quad \sum_{n=0}^\infty \frac{q^{2n^2}}{(q^2; q^2)_n (-q; q)_{2n}} = \frac{(q^3, q^4, q^7; q^7)_\infty}{(q^2; q^2)_\infty}$$

and Rogers’

$$\sum_{n=0}^\infty \frac{q^{n^2}}{(q^4; q^4)_n} = (-q; q^2)_\infty \frac{(q^2, q^3, q^5; q^5)_\infty}{(q^2; q^2)_\infty},$$

labelled (33) and (20) in Slater’s list, respectively.

To actually find results that are new one has to work a little harder. One particularly nice example of a result that appears to be new is the following ‘perfect’ Rogers–Ramanujan identity involving the bases q, q^2, q^3 and q^6 :

$$(7.3) \quad \sum_{n=0}^\infty \frac{q^{n(n+1)/2} (q^3; q^3)_n}{(q; q)_n^2 (q^3; q^2)_n} = \frac{(q^6; q^6)_\infty}{(q; q)_\infty (q^3; q^2)_\infty}.$$

To prove this we take two polynomial identities equivalent to the Bailey pairs pairs $J(1)$ and $J(1)-J(2)$ [47]

$$(7.4) \quad \sum_{j=-\infty}^{\infty} (-1)^j q^{3j(3j+1)/2} \begin{bmatrix} 2L \\ L-3j \end{bmatrix} = \begin{cases} 1, & L = 0, \\ (1+q^L) \frac{(q^3; q^3)_{L-1}}{(q; q)_{L-1}}, & L > 0, \end{cases}$$

and

$$(7.5) \quad \sum_{j=-\infty}^{\infty} (-1)^j q^{9j(j+1)/2} \begin{bmatrix} 2L \\ L-3j-1 \end{bmatrix} = q^{L-1} \frac{(q^3; q^3)_{L-1}}{(q; q)_{L-1}} \chi(L > 0),$$

and calculate the sum (7.4) + q^{L+1} (7.5) to get

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{3j(3j+1)/2} \begin{bmatrix} 2L+1 \\ L-3j \end{bmatrix} = \frac{(q^3; q^3)_L}{(q; q)_L}.$$

Replacing $q \rightarrow q^2$ and then applying the quadratic transformation (2.8) gives

$$\begin{aligned} \sum_{j=-\infty}^{\infty} (-1)^j q^{6j(3j+1)} \begin{bmatrix} 2L \\ L-6j-1 \end{bmatrix} \\ = (1+q^L)(q; q)_L \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q^6; q^6)_n (-q; q^2)_{L-n-1}}{(q^2; q^2)_n^2 (q^2; q^4)_{n+1} (q; q)_{L-2n-1}}. \end{aligned}$$

By (7.1) this yields (7.3) with $q \rightarrow q^2$ in the large L limit.

An identity similar to (7.3) that is in Slater’s list as item (78) results if we apply (2.9) to (7.4). We include its derivation here to demonstrate that also transformations of the type (2.9) and (2.13) may be successfully exploited to derive Rogers–Ramanujan identities, despite the factor $1+q^{aj}$ in the denominator on the right. First, by (2.9) and (7.4)

$$\begin{aligned} 2 \sum_{j=-\infty}^{\infty} \frac{(-1)^j q^{6j(3j+1)}}{1+q^{6j}} \begin{bmatrix} 2L \\ L-6j \end{bmatrix} \\ = (-q; q^2)_L + 2 \sum_{n=1}^{\infty} \frac{q^{n(n+1)}(q^6; q^6)_{n-1} (-q; q^2)_{L-n}}{(-q; q^2)_n (q^2; q^2)_{n-1}} \begin{bmatrix} L \\ 2n \end{bmatrix}. \end{aligned}$$

By negating j it follows that the term $1+q^{6j}$ on the left cancels the prefactor 2. Then taking the large L limit and replacing q^2 by q yields

$$(7.6) \quad 1 + 2 \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2} (q^3; q^3)_{n-1}}{(q; q)_n (q; q^2)_n (q; q)_{n-1}} = \frac{(q^9; q^9)_{\infty} (q^9; q^{18})_{\infty}}{(q; q)_{\infty} (q; q^2)_{\infty}}.$$

For our final single-sum Rogers–Ramanujan result we first establish the truth of a family of polynomial identities obtained previously by Andrews [8, Eq. (4.5)] for $k = 3$.

Proposition 7.1. For integers $L \geq 0, k \geq 2$ and $i \in \{1, \dots, k - 1\}$ there holds

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3kj-k+2i)/2} \left[\begin{matrix} 2L \\ L - 3j \end{matrix} \right]_{q^{k/3}} = \sum_{n=0}^{\lfloor L/3 \rfloor} \frac{q^{kn^2} (q^i, q^{k-i}; q^k)_n (1 - q^{2kL/3}) (q^k; q^k)_{L-n-1}}{(q^k; q^k)_{2n} (q^{k/3}; q^{k/3})_{L-3n}}.$$

Proof. According to the finite form of Jacobi’s triple product identity [3, Ch. 3, Example 1]

$$(7.7) \quad \sum_{j=-L}^L (-1)^j a^j q^{\binom{j}{2}} \left[\begin{matrix} 2L \\ L - j \end{matrix} \right] = (a, q/a; q)_L.$$

Replacing $q \rightarrow q^k$, applying the cubic transformation (2.18) with $q \rightarrow q^{k/3}$, and specializing $a = q^i$ yields the claimed proposition. \square

In the above, (2.18) was used for even values of L and j even. Needed for the odd case is the polynomial identity

$$\sum_{j=-L}^L (-1)^j a^j q^{\binom{j+1}{2}} \left[\begin{matrix} 2L + 1 \\ L - j \end{matrix} \right] = (1 - q^L/a)(1/a, q/a; q)_L$$

which easily follows from (7.7). Mimicing the earlier proof and then taking $a = q^{-i}$ results in the odd version of Proposition 7.1.

Proposition 7.2. For integers $L \geq 0, k \geq 2$ and $i \in \{1, \dots, k - 1\}$ there holds

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3kj+3k-2i)/2} \left[\begin{matrix} 2L + 1 \\ L - 3j - 1 \end{matrix} \right]_{q^{k/3}} = \sum_{n=0}^{\lfloor (L-1)/3 \rfloor} \frac{q^{kn(n+1)} (1 - q^{kn+i}) (q^i, q^{k-i}; q^k)_n (1 - q^{2kL/3+k/3}) (q^k; q^k)_{L-n-1}}{(q^k; q^k)_{2n+1} (q^{k/3}; q^{k/3})_{L-3n-1}}.$$

Letting L tend to infinity and using the Jacobi triple product identity (7.1) gives Rogers–Ramanujan-type identities for modulus $3k$.

Corollary 7.1. For $k \geq 2$ and $i \in \{1, \dots, k - 1\}$,

$$(7.8) \quad \sum_{n=0}^{\infty} \frac{q^{kn^2} (q^i, q^{k-i}; q^k)_n}{(q^k; q^k)_{2n}} = \frac{(q^{k+i}, q^{2k-i}, q^{3k}; q^{3k})_{\infty}}{(q^k; q^k)_{\infty}}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{kn(n+1)} (1 - q^{kn+i}) (q^i, q^{k-i}; q^k)_n}{(q^k; q^k)_{2n+1}} = \frac{(q^i, q^{3k-i}, q^{3k}; q^{3k})_{\infty}}{(q^k; q^k)_{\infty}}.$$

For $k = 3$ this yields three modulus 9 identities due to Bailey [15, Eqs. (1.6)–(1.8)].

Another natural choice for a in all of the above would have been $a = -q^i$. This would for example give the following companion to (7.8):

$$\sum_{n=0}^{\infty} \frac{q^{kn^2} (-q^i, -q^{k-i}; q^k)_n}{(q^k; q^k)_{2n}} = \frac{(-q^{k+i}, -q^{2k-i}, q^{3k}; q^{3k})_{\infty}}{(q^k; q^k)_{\infty}}.$$

The power of the transformations of Section 2 in deriving new Rogers–Ramanujan-type identities becomes fully clear when considering multisum identities. Because of the iterative nature of the Lemmas 2.1–2.6 a sheer endless number of elegant new multisum identities may be obtained. In particular, by combining the results (1.1), (1.2), (1.6)–(1.8) with (2.3), (2.8) and (2.18), each seed (initial q -binomial identity) becomes the root of a octonary tree (modulo redundancies implied by the relations of Section 3) of polynomial Rogers–Ramanujan identities. As such roots one can take the identities implied by the A–M families of Bailey pairs as tabulated by Slater [46, 47] with the provision that only ‘independent’ or ‘irreducible’ Bailey pairs should be employed. For example, the root (1.3) has among its sons or successors (polynomial identities equivalent to) the pairs B(1) (by (1.1)), H(2) (by (1.2)), G(1) (by (1.6)), L(2) (by (1.7)), C(1) (by (2.3)), I(7)+I(8) (by (2.9)) and J(1) of (7.4) (by (2.18)). Moreover, depending on the fine details of the polynomial identity associated with a particular node of the tree, one may also invoke Lemmas 2.3 and 2.5 (and of course Lemma 2.4). Indeed, as we have already seen in the derivation of (7.6), the undesirable (from a Rogers–Ramanujan identities point of view) denominator terms $1 + q^{aj}$ may actually cancel, permitting the use of the Jacobi triple product identity in the large L limit. Also, a polynomial identity may arise that is well suited for further iteration. For example, if we once again take (1.3) as a starting point and apply (2.9) we get after simplification

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \begin{bmatrix} 2L \\ L - 2j \end{bmatrix} = (-q; q^2)_L,$$

which is a companion to the Bailey pair I(12). This same identity also follows by applying (2.13) to (1.3). Lastly we note that the tree can be further enhanced by noting that the polynomial identities obtained by application of (1.1), (1.2), (1.6) and (2.3) are not ‘self-dual’. That is, by replacing $q \rightarrow 1/q$ a different identity results that yet again may be further iterated. In the case of (2.3) this is equivalent to also using its companion (2.5). This way it can for example be seen that also the Bailey pairs H(4) (dual to B(1)), G(4) (dual to G(1)) and C(4) (dual to C(1)) become part of the tree rooted in (1.3).

Below we shall only give a representative sample of the multisum identities contained in the tree with (1.3) as a root, and we encourage the reader to exploit their own favourite combination of transformations. In all our Rogers–Ramanujan series we assume $|q| < 1$. Furthermore, unless stated otherwise we adopt the convention that $n_0 := L$ and $n_k := 0$.

Theorem 7.1. *For $k \geq 2$ there holds*

$$\begin{aligned} \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{n_1^2 + 2n_2^2 + \dots + 2^{k-2}n_{k-1}^2}}{(q; q)_{2n_1}} \prod_{j=2}^k (-q^{2^{j-2}}; q^{2^{j-2}})_{n_{j-1} - 2n_j} \begin{bmatrix} n_{j-1} \\ 2n_j \end{bmatrix}_{q^{2^{j-1}}} \\ = \frac{(q^{\frac{1}{2}(4^k - 2^k)}, q^{\frac{1}{2} \cdot 4^k}, q^{4^k - 2^{k-1}}; q^{4^k - 2^{k-1}})_{\infty}}{(q; q)_{\infty}}. \end{aligned}$$

For $k = 2$ this is item (61) of Slater’s list.

Proof. Let

$$(7.9) \quad G(L; \alpha, \beta, K; q) = G(L; \alpha, \beta, K) = \sum_{j=-\infty}^{\infty} (-1)^j q^{Kj((\alpha+\beta)j+\alpha-\beta)/2} \begin{bmatrix} 2L \\ L - Kj \end{bmatrix}.$$

A k -fold application of (2.3) to (1.3) yields the polynomial identity

$$G(L; 2^k - 1, 2^k, 2^k) = \sum_{n_1, \dots, n_{k-1} \geq 0} \prod_{j=1}^k q^{2^j n_j^2} (-q^{2^{j-1}}; q^{2^{j-1}})_{n_{j-1} - 2n_j} \begin{bmatrix} n_{j-1} \\ 2n_j \end{bmatrix}_{q^{2^j}}.$$

For $k = 1$ this is the Bailey pair identity C(1). Taking the large L limit and replacing q^2 by q yields the theorem thanks to Jacobi's triple product identity (7.1). \square

Theorem 7.2. *For $k \geq 2$ there holds*

$$\begin{aligned} \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{n_1^2 + 2n_2^2 + \dots + 2^{k-2}n_{k-1}^2}}{2(q; q)_{2n_1} (-q^2; q^2)_{n_1-1}} \prod_{j=2}^k (-q^{2^j n_j}; q^{2^j})_{n_{j-1} - 2n_j} \begin{bmatrix} n_{j-1} \\ 2n_j \end{bmatrix}_{q^{2^{j-1}}} \\ = \frac{(q^{\frac{1}{2}(4^k - 2^k)}, q^{\frac{1}{2}(4^k + 2^k)}, q^{4^k}; q^{4^k})_{\infty}}{(q; q^2)_{\infty} (q^4; q^4)_{\infty}}. \end{aligned}$$

For $k = 2$ this is item (71) of Slater's list.

Proof. A k -fold application of (2.8) to (1.3) yields the polynomial identity

$$(7.10) \quad G(L; \frac{1}{2}(2^k - 1), \frac{1}{2}(2^k + 1), 2^k) \\ = \frac{1}{2}(1 + q^L) \sum_{n_1, \dots, n_{k-1} \geq 0} \prod_{j=1}^k q^{2^{j-1} n_j^2} (-q^{2^j n_j}; q^{2^j})_{n_{j-1} - 2n_j} \begin{bmatrix} n_{j-1} \\ 2n_j \end{bmatrix}_{q^{2^{j-1}}}.$$

The large L limit yields the desired theorem. \square

Theorem 7.3. *For $k \geq 2$ there holds*

$$\begin{aligned} \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{n_1^2 + 3n_2^2 + \dots + 3^{k-2}n_{k-1}^2}}{(q; q)_{2n_1}} \prod_{j=2}^k \frac{(q^{3^{j-1}}; q^{3^{j-1}})_{n_{j-1} - n_{j-1}} (1 - q^{2 \cdot 3^{j-2} n_{j-1}})}{(q^{3^{j-2}}; q^{3^{j-2}})_{n_{j-1} - 3n_j} (q^{3^{j-1}}; q^{3^{j-1}})_{2n_j}} \\ = \frac{(q^{\frac{1}{6}(9^k - 3^k)}, q^{\frac{1}{6}(9^k + 3^k)}, q^{\frac{1}{3} \cdot 9^k}; q^{\frac{1}{3} \cdot 9^k})_{\infty}}{(q; q)_{\infty}}. \end{aligned}$$

For $k = 2$ this is item (93) of Slater's list.

Proof. A k -fold application of (2.18) to (1.3) yields

$$G(L; \frac{1}{2}(3^k - 1), \frac{1}{2}(3^k + 1), 3^k) \\ = \sum_{n_1, \dots, n_{k-1} \geq 0} \prod_{j=1}^k \frac{q^{3^j n_j^2} (q^{3^j}; q^{3^j})_{n_{j-1} - n_{j-1}} (1 - q^{2 \cdot 3^{j-1} n_{j-1}})}{(q^{3^{j-1}}; q^{3^{j-1}})_{n_{j-1} - 3n_j} (q^{3^j}; q^{3^j})_{2n_j}}.$$

For $k = 1$ this is (7.4). Taking the limit and replacing $q^3 \rightarrow q$ completes the proof. \square

So far we have only presented Rogers–Ramanujan identities obtained by iterating one and the same transformation. Finally we state eight more theorems that arise when alternating (1.6) and (2.3), or (1.1) and (2.3). Eight and not two families of identities result because it not only matters with which transformation one starts, but also whether an even or odd number of iterations is carried out.

The first four theorems occur by alternating (1.6) and (2.3).

Theorem 7.4. *For k an odd integer such that $k \geq 3$ there holds*

$$(7.11) \quad \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{n_1^2}}{(q; q^2)_{2n_1}} \prod_{\substack{j=2 \\ j \equiv 0 \pmod{2}}}^{k-1} \frac{q^{2n_j^2 + n_{j+1}^2}}{(q; q)_{n_{j-1} - 2n_j} (-q; q)_{2n_j} (q^2; q^2)_{n_j - n_{j+1}} (q; q^2)_{n_{j+1}}} \\ = \frac{(q^{2^k} - 1, q^{2^k}, q^{2^{k+1} - 1}; q^{2^{k+1} - 1})_\infty}{(q; q)_\infty}.$$

Proof. Take (1.3), and in alternating fashion apply (1.6) $(k + 1)/2$ times and (2.3) $(k - 1)/2$ times, starting with (1.6). This yields an identity for

$$G(L; 2^{(k-1)/2}(1 - 2^{-k}), 2^{(k-1)/2}, 2^{(k-1)/2}; q^2)$$

which implies the theorem in the large L limit. □

To concisely state the next theorem we depart from our earlier convention that $n_0 = L$, and below the term $(q; q)_{n_{j-1} - 2n_j}$ for $j = 1$ should be interpreted as 1.

Theorem 7.5. *For k an even integer such that $k \geq 2$ there holds*

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \prod_{\substack{j=1 \\ j \equiv 1 \pmod{2}}}^{k-1} \frac{q^{2n_j^2 + n_{j+1}^2}}{(q; q)_{n_{j-1} - 2n_j} (-q; q)_{2n_j} (q^2; q^2)_{n_j - n_{j+1}} (q; q^2)_{n_{j+1}}} \\ = \frac{(q^{2^k} - 1, q^{2^k}, q^{2^{k+1} - 1}; q^{2^{k+1} - 1})_\infty}{(q^2; q^2)_\infty}.$$

For $k = 2$ this is (7.2).

Proof. Take (1.3), and in alternating fashion apply (1.6) and (2.3) each $k/2$ times starting with (1.6). This yields an identity for $G(L; 2^{k/2}(1 - 2^{-k}), 2^{k/2}, 2^{k/2})$ which implies the theorem in the large L limit. □

Theorem 7.6. *For k an odd integer such that $k \geq 3$ there holds*

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{2n_1^2}}{(-q; q)_{2n_1}} \prod_{\substack{j=2 \\ j \equiv 0 \pmod{2}}}^{k-1} \frac{q^{n_j^2 + 2n_{j+1}^2}}{(q^2; q^2)_{n_{j-1} - n_j} (q; q^2)_{n_j} (q; q)_{n_j - 2n_{j+1}} (-q; q)_{2n_{j+1}}} \\ = \frac{(q^{2^{k+1} - 2}, q^{2^{k+1}}, q^{2^{k+2} - 2}; q^{2^{k+2} - 2})_\infty}{(q^2; q^2)_\infty}.$$

We note that if we replace $q^2 \rightarrow q$, then the right-hand side equals the right-hand side of (7.11).

Proof. Take (1.3), and in an alternating fashion apply (2.3) $(k + 1)/2$ times and (1.6) $(k - 1)/2$ times, starting with (2.3). This yields an identity for

$$G(L; 2^{(k+1)/2}(1 - 2^{-k}), 2^{(k+1)/2}, 2^{(k+1)/2})$$

which implies the theorem in the large L limit. □

In the next theorem $(q^2; q^2)_{n_0 - n_1} = 1$.

Theorem 7.7. *For k an even integer such that $k \geq 2$ there holds*

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \prod_{\substack{j=1 \\ j \equiv 1 \pmod{2}}}^{k-1} \frac{q^{n_j^2 + 2n_{j+1}^2}}{(q^2; q^2)_{n_{j-1} - n_j} (q; q^2)_{n_j} (q; q)_{n_j - 2n_{j+1}} (-q; q)_{2n_{j+1}}} \\ = \frac{(q^{2^{k+1}-2}, q^{2^{k+1}}, q^{2^{k+2}-2}; q^{2^{k+2}-2})_\infty}{(q; q)_\infty}.$$

For $k = 2$ this coincides with Theorem 7.1.

Proof. Take (1.3), and in alternating fashion apply (2.3) and (1.6) each $k/2$ times starting with (2.3). This yields an identity for $G(L; 2^{k/2}(1 - 2^{-k}), 2^{k/2}, 2^{k/2}; q^2)$ which implies the theorem in the large L limit. □

The next four results are obtained by alternating (1.1) and (2.3).

Theorem 7.8. *For k an odd integer such that $k \geq 3$ there holds*

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{n_1^2}}{(q; q^2)_{n_1}} \\ \times \prod_{\substack{j=2 \\ j \equiv 0 \pmod{2}}}^{k-1} \frac{q^{2^{j/2}(n_j^2 + n_{j+1}^2)}}{(q^{2^{j/2}-1}; q^{2^{j/2}-1})_{n_{j-1} - 2n_j} (q^{2^{j/2}}; q^{2^{j/2}})_{n_j - n_{j+1}} (q^{2^{j/2}}; q^{2^{j/2+1}})_{n_{j+1}}} \\ = \frac{(q^{3 \cdot 2^{k-1} - 2^{(k+1)/2}}, q^{3 \cdot 2^{k-1} - 2^{(k-1)/2}}, q^{3 \cdot 2^k - 3 \cdot 2^{(k-1)/2}}; q^{3 \cdot 2^k - 3 \cdot 2^{(k-1)/2}})_\infty}{(q; q)_\infty}.$$

Proof. Take (1.3), and in alternating fashion apply (1.1) $(k + 1)/2$ times and (2.3) $(k - 1)/2$ times, starting with (1.1). This yields an identity for

$$G(L; 3 \cdot 2^{(k-1)/2} - 2, 3 \cdot 2^{(k-1)/2} - 1, 2^{(k-1)/2}; q)$$

which implies the theorem in the large L limit. □

Below, $(q^{1/2}; q^{1/2})_{n_0 - 2n_1} = 1$.

Theorem 7.9. *For k an even integer such that $k \geq 2$ there holds*

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \prod_{\substack{j=0 \\ j \equiv 0 \pmod{2}}}^{k-2} \frac{q^{2^{j/2}(n_{j+1}^2 + n_{j+2}^2)}}{(q^{2^{j/2}-1}; q^{2^{j/2}-1})_{n_j - 2n_{j+1}} (q^{2^{j/2}}; q^{2^{j/2}})_{n_{j+1} - n_{j+2}} (q^{2^{j/2}}; q^{2^{j/2+1}})_{n_{j+2}}} \\ = \frac{(q^{2^k - 2^{k/2}}, q^{2^k - 2^{k/2-1}}, q^{2^{k+1} - 3 \cdot 2^{k/2-1}}; q^{2^{k+1} - 3 \cdot 2^{k/2-1}})_\infty}{(q; q)_\infty}.$$

For $k = 2$ this is the first Rogers–Ramanujan identity.

Proof. Take (1.3), and in an alternating fashion apply (1.1) and (2.3) each $k/2$ times starting with (1.1). This yields an identity for $G(L; 2^{k/2+1} - 2, 2^{k/2+1} - 1, 2^{k/2})$ which implies the theorem with $q \rightarrow q^2$ in the large L limit. \square

Theorem 7.10. *For k an odd integer such that $k \geq 3$ there holds*

$$\begin{aligned} & \sum_{n_1, \dots, n_{k-1} \geq 0} q^{n_1^2} \\ & \times \prod_{\substack{j=0 \\ j \equiv 0 \pmod{2}}}^{k-3} \frac{q^{2^{j/2}(n_{j+2}^2 + 2n_{j+3}^2)}}{(q^{2^{j/2}}; q^{2^{j/2}})_{n_{j+1}-n_{j+2}} (q^{2^{j/2}}; q^{2^{j/2+1}})_{n_{j+2}} (q^{2^{j/2}}; q^{2^{j/2}})_{n_{j+2}-2n_{j+3}}} \\ & = \frac{(q^{2^{k+1}-3 \cdot 2^{(k-1)/2}}, q^{2^{k+1}-2^{(k+1)/2}}, q^{2^{k+2}-5 \cdot 2^{(k-1)/2}}; q^{2^{k+2}-5 \cdot 2^{(k-1)/2}})_{\infty}}{(q; q)_{\infty}}. \end{aligned}$$

Proof. Take (1.3), and in an alternating fashion apply (2.3) $(k + 1)/2$ times and (1.1) $(k - 1)/2$ times, starting with (2.3). This yields an identity for

$$G(L; 2^{(k+3)/2} - 3, 2^{(k+3)/2} - 2, 2^{(k+1)/2})$$

which implies the theorem with $q \rightarrow q^2$ in the large L limit. \square

Below, $(q; q)_{n_0-n_1} = 1$.

Theorem 7.11. *For k an even integer such that $k \geq 2$ there holds*

$$\begin{aligned} & \sum_{n_1, \dots, n_{k-1} \geq 0} \prod_{\substack{j=0 \\ j \equiv 0 \pmod{2}}}^{k-2} \frac{q^{2^{j/2}(n_{j+1}^2 + 2n_{j+2}^2)}}{(q^{2^{j/2}}; q^{2^{j/2}})_{n_j-n_{j+1}} (q^{2^{j/2}}; q^{2^{j/2+1}})_{n_{j+1}} (q^{2^{j/2}}; q^{2^{j/2}})_{n_{j+1}-2n_{j+2}}} \\ & = \frac{(q^{3(2^k-2^{k/2})}, q^{3 \cdot 2^k-2^{k/2+1}}, q^{3 \cdot 2^{k+1}-5 \cdot 2^{k/2}}; q^{3 \cdot 2^{k+1}-5 \cdot 2^{k/2}})_{\infty}}{(q; q)_{\infty}}. \end{aligned}$$

For $k = 2$ this is identity (61) of Slater.

Proof. Take (1.3), and in an alternating fashion apply (2.3) and (1.1) each $k/2$ times starting with (2.3). This yields an identity for $G(L; 3(2^{k/2} - 1), 3 \cdot 2^{k/2} - 2, 2^{k/2})$ which implies the theorem in the large L limit. \square

8. ROGERS-SZEGÖ POLYNOMIALS

For nonnegative integers n the Rogers-Szegö (RS) polynomials are defined as [3, Ch. 3, Examples 3–9]

$$(8.1) \quad H_n(t; q) = H_n(t) = \sum_{j=0}^n t^j \begin{bmatrix} n \\ j \end{bmatrix}.$$

By the replacements $q \rightarrow q^4$ and $j \rightarrow (n - j)/2$ in $H_n(q^{1/2}) = (-q^{1/2}; q^{1/2})_n$ [3, Ch. 3, Example 5] we find

$$(8.2) \quad \sum_{\substack{j=-n \\ j \equiv n \pmod{2}}}^n q^j \begin{bmatrix} n \\ \frac{1}{2}(n - j) \end{bmatrix}_{q^4} = q^{-n} (-q^2; q^2)_n.$$

Hence, by the quartic transformation (2.12),

$$\sum_{j=-n}^n q^j \begin{bmatrix} 2n \\ n-j \end{bmatrix} = \sum_{r=0}^n q^{n-2r} (-q^2; q^2)_r (-q^{-1}; q^2)_{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2},$$

and by its cousin (2.13),

$$\sum_{j=-n}^n \frac{q^{2j}}{1+q^{2j}} \begin{bmatrix} 2n \\ n-j \end{bmatrix} = \frac{1}{2} \sum_{r=0}^n (-1; q^2)_r (-q; q^2)_{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}.$$

By negating j it follows that the left-hand side of this last identity simplifies to

$$\frac{1}{2} \sum_{j=-n}^n \begin{bmatrix} 2n \\ n-j \end{bmatrix}.$$

If we then replace j by $n-j$ in both formulas and $r \rightarrow n-r$ in the second formula, we get

$$H_{2n}(q^{-1}) = \sum_{r=0}^n q^{-2r} (-q^2; q^2)_r (-q^{-1}; q^2)_{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}$$

and

$$H_{2n}(1) = \sum_{r=0}^n (-q; q^2)_r (-1; q^2)_{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}.$$

This suggests the following more general result.

Theorem 8.1. *The Rogers–Szegő polynomials can be expressed as*

$$(8.3) \quad H_n(t) = \sum_{r=0}^{\lfloor n/2 \rfloor} t^{2r} (-q/t; q^2)_r (-t; q^2)_{\lfloor (n+1)/2 \rfloor - r} \begin{bmatrix} \lfloor n/2 \rfloor \\ r \end{bmatrix}_{q^2}.$$

This new representation for the RS polynomials manifests the well-known facts that $H_{2n}(-1) = (q; q^2)_n$, $H_{2n+1}(-1) = 0$ and $H_n(-q) = (q; q^2)_{\lfloor (n+1)/2 \rfloor}$, which are not immediately clear from the standard definition (8.1). For recent new representations of the generating function of the Rogers–Szegő polynomials, see [34].

In the following we will give two proofs of (8.3). In the first we show that (8.3) satisfies the recurrence [3, Ch 3. Example 6]

$$(8.4) \quad H_{n+1}(t) = (1+t)H_n(t) - (1-q^n)tH_{n-1}(t)$$

which determines the RS polynomials together with the initial conditions $H_0(t) = 1$ and $H_1(t) = 1+t$. In the second more complicated and interesting proof, we establish equality between (8.1) and (8.3) using basic hypergeometric series manipulations.

First proof of (8.3). Take (8.4), replace n by $2n$ and substitute (8.3). Then use $(-t; q^2)_{n-r+1} = (-t; q^2)_{n-r}(1+tq^{2n-2r})$ on the left and

$$(1-q^{2n}) \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} = (1-q^{2n-2r}) \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}$$

on the right. All terms now pairwise cancel.

Next take (8.4), replace n by $2n-1$ and substitute (8.3). Then use

$$(8.5) \quad \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} = \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q^2} + q^{2n-2r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q^2}$$

and cancel one of the two terms on the left with one of the terms on the right. To proceed use $(-t; q^2)_{n-r} = (-t; q^2)_{n-r-1}(1 + tq^{2n-2r-2})$ on the right and replace r by $r + 1$ on the left. All resulting terms again pairwise cancel.

The final checking of the initial conditions $H_0(t) = 1$ and $H_1(t) = 1 + t$ is left to the reader. □

Second proof of (8.3). As a first step we twice use the q -binomial theorem (4.2) to expand the q -shifted factorials on the right. After this one can extract coefficients of t^j on both sides leading to the double sum

$$\sum_{r=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor - r} q^{(2r+k-j)^2+k(k-1)} \times \begin{bmatrix} r \\ 2r+k-j \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor (n+1)/2 \rfloor - r \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor n/2 \rfloor \\ r \end{bmatrix}_{q^2} = \begin{bmatrix} n \\ j \end{bmatrix}.$$

Now introduce two new summation variables r' and k' by $r' = 2r + k - j$ and $k' = j - k - r$. Eliminating k and r in favour of their primed counterparts and then dropping the primes yields

$$\sum_{k=0}^{\lfloor j/2 \rfloor} q^{(j-2k)(j-2k-1)} \begin{bmatrix} \lfloor (n+1)/2 \rfloor - k \\ j-2k \end{bmatrix}_{q^2} \begin{bmatrix} \lfloor n/2 \rfloor \\ k \end{bmatrix}_{q^2} \times {}_2\phi_2 \left[\begin{matrix} q^{-2(\lfloor n/2 \rfloor - k)}, q^{-2(j-2k)} \\ q^{-2(\lfloor (n+1)/2 \rfloor - k)}, 0 \end{matrix}; q^2, q^{3-2\sigma} \right] = \begin{bmatrix} n \\ j \end{bmatrix},$$

where $\sigma \in \{0, 1\}$ is fixed by $n + \sigma \equiv 0 \pmod{2}$. We note that the lower bound on k may be optimized to $\max(0, j - \lfloor (n+1)/2 \rfloor)$.

When n is even the ${}_2\phi_2$ series becomes a ${}_1\phi_1$ which sums to $q^{-\binom{j-2k}{2}}(-q; q)_{j-2k}$ by the $a \rightarrow \infty$ limit of (3.34) or by (8.2) with $q^2 \rightarrow 1/q$ and $j \rightarrow L - 2j$. When n is odd the ${}_2\phi_2$ sums to

$$q^{-\binom{j-2k}{2}} \frac{1 - q^{n-j+1}}{1 - q^{n-2k+1}} (-q; q)_{j-2k}$$

by the $a \rightarrow \infty$ limit of

$$(8.6) \quad {}_3\phi_2(a, bq^2, q^{-2n}; b, q^{2-2n}/a; q^2, q/a) = q^{-n} \frac{1 - bq^n}{1 - b} \frac{(-q, a; q)_n}{(a; q^2)_n}.$$

For even n we are thus left with (after replacing n by $2n$)

$$(8.7) \quad \sum_{k=0}^{\lfloor j/2 \rfloor} q^{\binom{j-2k}{2}} (-q; q)_{j-2k} \begin{bmatrix} n-k \\ j-2k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} 2n \\ j \end{bmatrix}$$

and for odd n (after replacing n by $2n + 1$) with

$$\sum_{k=0}^{\lfloor j/2 \rfloor} q^{\binom{j-2k}{2}} (-q; q)_{j-2k} \frac{1 - q^{2n-j+2}}{1 - q^{2n-2k+2}} \begin{bmatrix} n-k+1 \\ j-2k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} 2n+1 \\ j \end{bmatrix}.$$

Multiplying both sides by $(1 - q^{2n+2})/(1 - q^{2n-j+2})$ this can easily be seen to correspond to (8.7) with $n \rightarrow n + 1$. Hence we only need to consider (8.7). But this is nothing but (2.5) with $(L, j, r) \rightarrow (n, n - j, n - j + 2k)$.

It remains to prove (8.6), which for $b = a$ reduces to (3.34) and for $b = 0$ yields a companion thereof. Now by the contiguous relation [37, Eq. (3.2)]

$$\begin{aligned} {}_{r+1}\phi_r \left[\begin{matrix} a, bq, (A) \\ (B) \end{matrix}; q, z \right] &= {}_{r+1}\phi_r \left[\begin{matrix} aq, b, (A) \\ (B) \end{matrix}; q, z \right] \\ &\quad + z(b-a) \frac{\prod_{i=1}^{r-1} (1-A_i)}{\prod_{i=1}^r (1-B_i)} {}_{r+1}\phi_r \left[\begin{matrix} aq, bq, (Aq) \\ (Bq) \end{matrix}; q, z \right] \end{aligned}$$

with $(a, b, (A), (B), z, q) \rightarrow (a, b, (q^{-2n}), (b, q^{2-2n}/a), q/a, q^2)$, the ${}_3\phi_2$ series on the left-hand side of (8.6) splits into two ${}_2\phi_1$ series, both of which are summable by (3.34). After some simplifications this yields the claimed right-hand side. \square

Note added. After submission of this paper to the Mathematics ArXiv, Alain Lascoux pointed out to us that Theorem 8.1 implies that the Rogers–Szegő polynomials satisfy the recurrences

$$(8.8a) \quad H_{2n}(t) = \sum_{i=0}^n q^{\binom{i}{2}} t^i (-q; q)_i \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} H_{n-i}(t^2; q^2)$$

and

$$(8.8b) \quad H_{2n+1}(t) = (1+t) \sum_{i=0}^n q^{\binom{i+1}{2}} t^i (-q; q)_i \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} H_{n-i}(t^2; q^2).$$

Indeed, from (8.3) it follows that

$$(8.9) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{x^n H_{2n}(t)}{(q^2; q^2)_n} &= \sum_{n=0}^{\infty} x^n \sum_{r=0}^n t^{2r} \frac{(-q/t; q^2)_r}{(q^2; q^2)_r} \frac{(-t; q^2)_{n-r}}{(q^2; q^2)_{n-r}} \\ &= {}_1\phi_0(-t; -; q^2, x) {}_1\phi_0(-q/t; -; q^2, t^2x) = \frac{(-tx; q)_{\infty}}{(x, t^2x; q^2)_{\infty}}, \end{aligned}$$

where the second equality follows by an interchange of sums and the shift $n \rightarrow n+r$, and where the third equality follows from the q -binomial theorem [30, Eq. (II.3)]

$$(8.10) \quad {}_1\phi_0(a; -; q, z) = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}.$$

Mourad Ismail pointed out to us that the equality of the extremes in (8.9) has also been obtained in his recent joint paper with Dennis Stanton [36, Eq. (3.5a)] using the Askey–Wilson integral. By [3, Ch. 3, Example 3]

$$\sum_{n=0}^{\infty} \frac{x^n H_n(t)}{(q; q)_n} = \frac{1}{(x, tx; q)_{\infty}},$$

and (8.10) with $a \rightarrow -tx/z$ followed by $z \rightarrow \infty$ we thus find

$$\sum_{n=0}^{\infty} \frac{x^n H_{2n}(t)}{(q^2; q^2)_n} = \sum_{i=0}^{\infty} \frac{q^{\binom{i}{2}} (tx)^i}{(q; q)_i} \sum_{n=0}^{\infty} \frac{x^n H_n(t^2; q^2)}{(q^2; q^2)_n} = \sum_{n=0}^{\infty} x^n \sum_{i=0}^n \frac{q^{\binom{i}{2}} t^i H_{n-i}(t^2; q^2)}{(q; q)_i (q^2; q^2)_{n-i}}.$$

Comparing coefficients of x^n yields (8.8a). In much the same way one finds

$$(8.11) \quad \sum_{n=0}^{\infty} \frac{x^n H_{2n+1}(t)}{(q^2; q^2)_n} = (1+t) \frac{(-txq; q)_{\infty}}{(x, t^2x; q^2)_{\infty}} \\ = (1+t) \sum_{n=0}^{\infty} x^n \sum_{i=0}^n \frac{q^{\binom{i+1}{2}} t^i H_{n-i}(t^2; q^2)}{(q; q)_i (q^2; q^2)_{n-i}},$$

which implies (8.8b).

A more direct way of obtaining (8.8a) is by taking (2.5), multiplying both sides by t^{L-j} and then summing j over the integers from $-L$ to L . After replacing L by n and changing the order of summation on the left this gives

$$\sum_{r=0}^n q^{\binom{n-r}{2}} (-q; q)_{n-r} \left[\begin{matrix} n \\ r \end{matrix} \right]_{q^2} \sum_{\substack{j=-r \\ j \equiv r \pmod{2}}}^r t^{n-j} \left[\begin{matrix} r \\ \frac{1}{2}(r-j) \end{matrix} \right]_{q^2} = \sum_{j=-n}^n t^{n-j} \left[\begin{matrix} 2n \\ n-j \end{matrix} \right].$$

Changing $j \rightarrow n - j$ on the right and $j \rightarrow r - 2j$ followed by $r \rightarrow n - r$ on the left yields (8.8a).

Copying this latter derivation of (8.8a), but now taking (2.12) instead of (2.5) as a starting point, leads to yet another recurrence, namely

$$(8.12a) \quad H_{2n}(t) = \sum_{i=0}^n (tq)^i (-1/q; q^2)_i \left[\begin{matrix} n \\ i \end{matrix} \right]_{q^2} H_{n-i}(t^2; q^4).$$

Once this has been found it is not hard to establish the odd counterpart

$$(8.12b) \quad H_{2n+1}(t) = (1+t) \sum_{i=0}^n (tq)^i (-q; q^2)_i \left[\begin{matrix} n \\ i \end{matrix} \right]_{q^2} H_{n-i}(t^2; q^4).$$

To prove this we first multiply (8.12a) by $x^n / (q^2; q^2)_n$ and sum n over the nonnegative integers. By (8.9) and (8.10) this yields

$$(8.13) \quad \sum_{n=0}^{\infty} \frac{x^n H_n(t^2; q^4)}{(q^2; q^2)_n} = \frac{(-xt; q)_{\infty}}{(x, t^2x; q^2)_{\infty}} \frac{(xtq; q^2)_{\infty}}{(-xt; q^2)_{\infty}} = \frac{(t^2x^2q^2; q^4)_{\infty}}{(x, t^2x; q^2)_{\infty}},$$

equivalent to [36, Eq. (3.5b)]. This result together with the first line of (8.11) is equivalent to (8.12b). Specifically, taking (8.12b), multiplying both sides by $x^n / (q^2; q^2)_n$ and summing over n using (8.10), (8.11) and (8.13) yields

$$(1+t) \frac{(-txq; q)_{\infty}}{(x, t^2x; q^2)_{\infty}} = (1+t) \frac{(-txq^2; q^2)_{\infty}}{(txq; q^2)_{\infty}} \frac{(t^2x^2q^2; q)_{\infty}}{(x, t^2x; q^2)_{\infty}},$$

which is obviously true.

Finally we note that by equating right-hand sides of (8.8b) and (8.12b) and by extracting coefficients of t^k we obtain the transformation

$${}_4\phi_3 \left[\begin{matrix} a, aq^2, q^{-2n}, q^{2-2n} \\ a^2, -q^{1-2n}, -q^{3-2n} \end{matrix}; q^4, 1 \right] = q^{\binom{n}{2}} \frac{(-q; q)_n}{(-q; q^2)_n} {}_2\phi_1 \left[\begin{matrix} q^{-n}, q^{1-n} \\ -a \end{matrix}; q^2, 1 \right].$$

This can also be obtained from (3.19) and [30, Exercise 3.2 (i)]. Equating right-hand sides of (8.8a) and (8.12a) fails to produce anything new.

9. THE GENERALIZED BORWEIN CONJECTURE

9.1. **The Borwein conjecture and Bressoud’s generalization.** Some years ago Peter Borwein conjectured [8] that the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ given by

$$A_n(q^3) - qB_n(q^3) - q^2C_n(q^3) = (q, q^2; q^3)_n$$

have nonnegative coefficients. Defining

$$\begin{aligned} G(N, M; \alpha, \beta, K; q) &= G(N, M; \alpha, \beta, K) \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{Kj((\alpha+\beta)j+\alpha-\beta)/2} \begin{bmatrix} M+N \\ N-Kj \end{bmatrix}, \end{aligned}$$

it follows from (7.7) that [8]

$$\begin{aligned} A_n(q) &= G(n, n; 4/3, 5/3, 3), \\ B_n(q) &= G(n+1, n-1; 2/3, 7/3, 3), \\ C_n(q) &= G(n+1, n-1; 1/3, 8/3, 3). \end{aligned}$$

This led Bressoud [19] to a more general conjecture concerning the nonnegativity of the coefficients of $G(N, M; \alpha, \beta, K; q)$. Since we will only be concerned with the case $N = M$ we write $G(N; \alpha, \beta, K; q)$ instead of $G(N, N; \alpha, \beta, K; q)$, in accordance with (7.9). We also write $P(q) \geq 0$ for $P(q)$ a polynomial in q with nonnegative coefficients. Then the $N = M$ case of Bressoud’s generalized Borwein conjecture can be stated as follows.

Conjecture 9.1. *Let $K, L, \alpha K, \beta K$ be integers such that $0 \leq \alpha \leq K$, $0 \leq \beta \leq K$ and $1 \leq \alpha + \beta \leq 2K - 1$. Then $G(L; \alpha, \beta, K; q) \geq 0$.*

When α and β are integers the conjecture becomes [12, Thm. 1] of Andrews *et al.* For fractional values of α and or β several cases of the conjecture have been proven in [19, 35, 55, 57].

Without loss of generality one may in fact put stronger restrictions on the parameters α and β in Conjecture 9.1. By

$$G(L; \alpha, \beta, K) = G(L; \beta, \alpha, K)$$

we may assume that $0 \leq \alpha \leq \beta \leq K$. Furthermore, by

$$G(L; \alpha, \beta, K; 1/q) = q^{-L^2} G(L; K - \beta, K - \alpha, K; q)$$

we may also assume that $1 \leq \alpha + \beta \leq K$. Hence we obtain $\max(0, 1 - \beta) \leq \alpha \leq \min(\beta, K - \beta)$.

Because of the positivity preserving nature of the q -binomial transformations of Section 2 we can easily prove many cases of Conjecture 9.1. For example, iterating (1.3) using (2.8) yields the polynomial identity (7.10), which implies that

$$(9.1) \quad G(L; \frac{1}{2}(2^k - 1), \frac{1}{2}(2^k + 1), 2^k) \geq 0.$$

For $k = 1$ the polynomial identity (7.10) and the according nonnegativity of the coefficients of $G(L; 1/2, 3/2, 2)$ are due to Ismail *et al.* [35, Prop. 2 (3)].

A reformulation of the positivity preserving transformations of Section 2 in the language of the generalized Borwein conjecture reads as follows.

Lemma 9.1. *If $G(L; \alpha, \beta, K) \geq 0$, then $G(L; \alpha', \beta', K') \geq 0$ with*

- (9.2a) $\alpha' = \alpha + K, \quad \beta' = \beta + K, \quad K' = 2K,$
- (9.2b) $\alpha' = \alpha, \quad \beta' = \beta, \quad K' = 2K,$
- (9.2c) $\alpha' = \alpha + K/2, \quad \beta' = \beta + K/2, \quad K' = 2K,$
- (9.2d) $\alpha' = 2\alpha, \quad \beta' = 2\beta, \quad K' = 2K,$
- (9.2e) $\alpha' = \alpha + K, \quad \beta' = \beta + K, \quad K' = 3K.$

Proof. We will only prove (9.2a). All other cases proceed along similar lines, be it that instead of (2.3) one needs to employ (2.5), (2.8), (2.12) and (2.18). Adopting the notation of Section 3 and writing $F_{L,r}^{(2.3)}(q)$ for $q^{r^2/2} f_{L,r}(q)$, with $f_{L,r}(q)$ given by (2.4), we have by Lemma 2.1 and the assumption that $G(L; \alpha, \beta, K; q) \geq 0$,

$$\begin{aligned} 0 &\leq \sum_{r=0}^{\infty} F_{L,2r}^{(2.3)}(q) G(r; \alpha, \beta, K; q^2) \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{Kj((\alpha+\beta)j+\alpha-\beta)} \sum_{r=0}^{\infty} F_{L,2r}^{(2.3)}(q) \begin{bmatrix} 2r \\ r - Kj \end{bmatrix}_{q^2} \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{Kj((2K+\alpha+\beta)j+\alpha-\beta)} \begin{bmatrix} 2L \\ L - 2Kj \end{bmatrix} \\ &= G(L; K + \alpha, K + \beta, 2K; q). \end{aligned}$$

□

Lemma 9.1 may be complemented by two more results. First we remark that the equation following Theorem 2.5 of [57] can be recast as the following lemma.

Lemma 9.2. *For L and j integers there holds*

$$\sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{\frac{1}{2}r^2} f_{L,r}(q) \begin{bmatrix} r \\ \frac{1}{2}(r - j) \end{bmatrix} = q^{\frac{1}{2}j^2} \begin{bmatrix} 2L \\ L - j \end{bmatrix}$$

with

$$f_{L,r}(q) = \begin{bmatrix} L \\ r \end{bmatrix} \sum_{n=0}^{L-r} q^{n(n+r)} \begin{bmatrix} L - r \\ n \end{bmatrix}.$$

By the substitution $q \rightarrow 1/q$ this implies a related result.

Corollary 9.1. *For L and j integers there holds*

$$\sum_{\substack{r=0 \\ r \equiv j \pmod{2}}}^L q^{\frac{1}{4}r^2} f_{L,r}(q) \begin{bmatrix} r \\ \frac{1}{2}(r - j) \end{bmatrix} = q^{\frac{1}{4}j^2} \begin{bmatrix} 2L \\ L - j \end{bmatrix}$$

with

$$f_{L,r}(q) = \begin{bmatrix} L \\ r \end{bmatrix} \sum_{n=0}^{L-r} q^{Ln} \begin{bmatrix} L - r \\ n \end{bmatrix}.$$

Note that these results correspond to (2.1) with $k = 1$ and $\gamma = 2$ or $\gamma = 1$, but that, unlike the solutions to (2.1) presented in Section 2, $f_{L,r}(q)$ is nonfactorizable. From Lemma 9.2 and its corollary we get [57, Lemma 6.7].

Lemma 9.3. *If $G(L; \alpha, \beta, K) \geq 0$, then $G(L; \alpha', \beta', K') \geq 0$ with*

$$(9.3a) \quad \alpha' = \alpha/2 + K, \quad \beta' = \beta/2 + K, \quad K' = 2K,$$

$$(9.3b) \quad \alpha' = (\alpha + K)/2, \quad \beta' = (\beta + K)/2, \quad K' = 2K.$$

The results of Lemmas 9.1 and 9.3 can be iterated to yield a tree of conditional nonnegativity results, subject to various internal relations. For example, applying (9.2a) and then (9.2b) is equivalent to applying (9.2b) and then (9.2c) (this fact corresponds to (3.8a) with $r \rightarrow 2r$), but is also equivalent to the application of (9.2d) followed by (9.3b). Indeed in each case, starting with $G(L; \alpha, \beta, K)$ one obtains $G(L; \alpha + K, \beta + K, 4K)$. Even when ignoring these degeneracies it is very complicated to give a complete description of an arbitrary node of the tree, and for the example of the subtree generated by Lemma 9.3, which requires the theory of continued fractions, we refer to [57, Prop. 6.8]. Instead of trying to achieve maximum generality we take the easy way out and restrict ourselves to several easy to state and prove examples.

Proposition 9.1. *For K, α, β and k integers such that $\max(0, 1 - \beta) \leq \alpha \leq \min(\beta, K - \beta)$ and $k \geq 0$,*

$$G(L; \alpha + \frac{1}{2}(2^k - 1)K, \beta + \frac{1}{2}(2^k - 1)K, 2^k K) \geq 0.$$

For $\alpha = 0, \beta = 1$ and $K = 1$ this yields (9.1).

Proof. The proposition is true for $k = 0$ by the remark following Conjecture 9.1. The rest follows from (9.2c) and induction. \square

Proposition 9.2. *For K, α, β and k integers such that $\max(0, 1 - \beta) \leq \alpha \leq \min(\beta, K - \beta)$ there holds $G(L; \alpha', \beta', K') \geq 0$ with*

$$\alpha' = 2^{-k}(\alpha + \frac{5}{14}(8^k - 1)K), \quad \beta' = 2^{-k}(\beta + \frac{5}{14}(8^k - 1)K), \quad K' = 4^k K$$

for $k \geq 0$, and

$$\alpha' = 2^{1-k}(\alpha + \frac{1}{28}(3 \cdot 8^k - 10)K), \quad \beta' = 2^{1-k}(\beta + \frac{1}{28}(3 \cdot 8^k - 10)K),$$

$$K' = \frac{1}{2} \cdot 4^k K$$

for $k \geq 1$.

Proof. The first equation for $k = 0$ is true by the remark following Conjecture 9.1. Thanks to $\frac{5}{14}(8^k - 1) + 2^{3k-1} = \frac{1}{28}(3 \cdot 8^k - 10)$, applying (9.2c) to $G(L; \alpha', \beta', K')$ with (α', β', K') given by the first equation gives $G(L; \alpha', \beta', K')$ with (α', β', K') given by the second equation where $k \rightarrow k + 1$. Thanks to $\frac{1}{28}(3 \cdot 8^k - 10) + 2^{3k-2} = \frac{5}{14}(8^k - 1)$, applying (9.3b) to $G(L; \alpha', \beta', K')$ with (α', β', K') given by the second equation gives $G(L; \alpha', \beta', K')$ with (α', β', K') given by the first equation. These observations suffice to conclude the proposition by induction. \square

Reversing the order of (9.2c) and (9.3b) in the above leads to the following modification of Proposition 9.2.

Proposition 9.3. *For K, α, β and k integers such that $\max(0, 1 - \beta) \leq \alpha \leq \min(\beta, K - \beta)$ there holds $G(L; \alpha', \beta', K') \geq 0$ with*

$$\alpha' = 2^{-k}(\alpha + \frac{3}{7}(8^k - 1)K), \quad \beta' = 2^{-k}(\beta + \frac{3}{7}(8^k - 1)K), \quad K' = 4^k K$$

for $k \geq 0$, and

$$\alpha' = 2^{-k}(\alpha + \frac{1}{28}(5 \cdot 8^k - 12)K), \quad \beta' = 2^{-k}(\beta + \frac{1}{28}(5 \cdot 8^k - 12)K),$$

$$K' = \frac{1}{2} \cdot 4^k K$$

for $k \geq 1$.

Since we went through quite a bit of trouble to show that the coefficients of the polynomial in (2.19) are positive we should at least include one example that makes use of (9.2e). The next result is obtained by replacing (9.2c) in the proof of Proposition 9.2 by (9.2e).

Proposition 9.4. *For K, α, β and k integers such that $\max(0, 1 - \beta) \leq \alpha \leq \min(\beta, K - \beta)$ there holds $G(L; \alpha', \beta', K') \geq 0$ with*

$$\alpha' = 2^{-k}(\alpha + \frac{4}{11}(12^k - 1)K), \quad \beta' = 2^{-k}(\beta + \frac{4}{11}(12^k - 1)K), \quad K' = 6^k K$$

for $k \geq 0$, and

$$\alpha' = 2^{1-k}(\alpha + \frac{1}{44}(5 \cdot 12^k - 16)K), \quad \beta' = 2^{1-k}(\beta + \frac{1}{44}(5 \cdot 12^k - 16)K),$$

$$K' = \frac{1}{2} \cdot 6^k K$$

for $k \geq 1$.

We leave it to the reader to derive more examples of the above kind.

Our next examples use the result [55, Cor. 3.2].

Theorem 9.1. $G(L; b - 1/a, b, a) \geq 0$ for a, b coprime integers such that $0 < b < a$.

First we note that (9.2a) and (9.2b) nicely combine to the following statement.

Lemma 9.4. *Let k, i be integers such that $0 \leq i < 2^k$. Then $G(L; \alpha, \beta, K) \geq 0$ implies that $G(L; \alpha + iK, \beta + iK, 2^k K) \geq 0$.*

Proof. For $k = 0$ the lemma is trivially true and for $k = 1$ it corresponds to (9.2a) when $i = 1$ and to (9.2b) when $i = 0$. Induction now does the rest since application of (9.2a) to (i, k) yields (k', i') with $k' = k + 1$ and $2^k \leq i' < 2^{k+1} = 2^{k'}$ and application of (9.2b) to (i, k) yields (k', i') with $k' = k + 1$ and $0 \leq i' < 2^k$. When combined this results in (i', k') with $k' = k + 1$ and $0 \leq i' < 2^{k'}$. \square

In the same vein one can also show that (9.2b) and (9.2d) combine in a simple manner.

Lemma 9.5. *Let k, i be integers such that $0 \leq i \leq k$. Then $G(L; \alpha, \beta, K) \geq 0$ implies that $G(L; 2^i \alpha, 2^i \beta, 2^k K) \geq 0$.*

If we now invoke Theorem 9.1 we obtain the following two theorems.

Theorem 9.2. $G(L; b - 1/a, b, 2^k a) \geq 0$ for k a nonnegative integer and a, b coprime integers such that $0 < b < 2^k a$.

Proof. Obviously, by Lemma 9.4 and Theorem 9.1, $G(L; b + ia - 1/a, b + ia, 2^k a) \geq 0$ for a, b coprime integers such that $0 < b < a$ and k, i integers such that $0 \leq i < 2^k$. Replacing $b + ia$ by b this becomes $G(L; b - 1/a, b, 2^k a) \geq 0$ for a, b coprime integers such that $ia < b < (i + 1)a$ and k, i integers such that $0 \leq i < 2^k$. But since a and b are coprime the condition $ia < b < (i + 1)a$ with $0 \leq i < 2^k$ may be replaced by $0 < b < 2^k a$. \square

Theorem 9.3. $G(L; 2^i(b - 1/a), 2^i b, 2^{k+k'} a) \geq 0$ for k a nonnegative integer, a, b coprime integers such that $0 < b < 2^k a$ and k', i integers such that $0 \leq i < 2^{k'}$.

Note that this contains the previous theorem as a special case.

Proof. Simply apply Lemma 9.5 with $k \rightarrow k'$ to Theorem 9.2. □

Our final example arises by repeating the proof of Proposition 9.2 but with Theorem 9.2 as seed.

Theorem 9.4. Let k' be a nonnegative integer, and let a, b be a pair of coprime integers such that $0 < b < 2^{k'} a$. Then $G(L, \alpha, \beta, K) \geq 0$ with

$$\alpha = 2^{-k}(b - 1/a + \frac{5}{14}(8^k - 1)a), \quad \beta = 2^{-k}(b + \frac{5}{14}(8^k - 1)a), \quad K = 2^{2k+k'} a$$

for $k \geq 0$, and

$$\alpha = 2^{1-k}(b - 1/a + \frac{1}{28}(3 \cdot 8^k - 10)a), \quad \beta = 2^{1-k}(b + \frac{1}{28}(3 \cdot 8^k - 10)a),$$

$$K = 2^{2k+k'-1} K$$

for $k \geq 1$.

9.2. New representations of the Borwein polynomials. Unfortunately the positivity preserving transformations of this paper are inadequate for proving the original Borwein conjecture. Indeed, the only way to, for example, obtain $A_L(q)$ is by applying the cubic transformation (2.18) to (7.7), as was done in the proof of Proposition 7.1. But the resulting

$$A_L(q) = (1 - q^{2L}) \sum_{n=0}^{\lfloor L/3 \rfloor} \frac{q^{3n^2}(q; q)_{3n}(q^3; q^3)_{L-n-1}}{(q^3; q^3)_{2n}(q^3; q^3)_n(q; q)_{L-3n}}$$

which was first found by Andrews [8, Eq. (4.5)] is insufficient for proving that $A_L(q) \geq 0$. Another representation follows from application of (1.8) to (7.4), [50, Thm 2], but since (1.8) is not positivity preserving this again fails to prove that $A_L(q) \geq 0$. To conclude we prove alternative representations for $A_L(q)$ and $C_L(q)$ as triple sums, and use this to formulate a refinement of the Borwein conjecture.

Theorem 9.5. Let L be a nonnegative integer and $N_1 = n_1 + n_2 + n_3, N_2 = n_2 + n_3, N_3 = n_3$. Then

$$A_L(q) = \sum_{\substack{n_1, n_2, n_3 \geq 0 \\ N_1 + N_2 + N_3 \leq L}} \frac{q^{N_1^2 + N_2^2 + N_3^2}(q; q)_{L-N_1}(q; q)_{2L-N_1-N_2}}{(q; q)_{2L-2N_1}(q; q)_{L-N_1-N_2-N_3}(q; q)_{n_1}(q; q)_{n_2}(q; q)_{n_3}}$$

Proof. Define

$$(9.4) \quad \mathcal{B}(L, M, a, b) = \begin{bmatrix} L + M + a - b \\ L + a \end{bmatrix} \begin{bmatrix} L + M - a + b \\ L - a \end{bmatrix}.$$

Then, according to [14, Eq. (5.33)], the following doubly-bounded analogues of the Rogers–Ramanujan identities hold:

$$(9.5) \quad \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+2\sigma+1)/2} \mathcal{B}(L, M, j, j) = \sum_{n \geq 0} \frac{q^{n(n+\sigma)}(q; q)_{L+M}}{(q; q)_{L-n}(q; q)_{M-n}(q; q)_n},$$

where $\sigma \in \{0, 1\}$. Now for L, M, a, b integers such that not $-L + a \leq -b \leq L + a < b \leq M$ or $-L - a \leq b \leq L - a < -b \leq M$, there holds

$$(9.6) \quad \sum_{i=0}^M q^{i^2} \begin{bmatrix} 2L + M - i \\ 2L \end{bmatrix} \mathcal{B}(L - i, i, a, b) = q^{b^2} \mathcal{B}(L, M, a + b, b).$$

This result is known as the Burge transform [21, 25, 43] and can be applied to (9.5). First let us show that the conditions on the parameters (for their origin see [43]) are harmless. From $-L + a \leq -b \leq L + a < b \leq M$ one can extract the three conditions

$$(i) \ L \geq 0, \quad (ii) \ b > 0, \quad (iii) \ L + a - b < 0$$

and from $-L + a \leq -b \leq L + a < b \leq M$ it follows that

$$(iv) \ L \geq 0, \quad (v) \ b < 0, \quad (vi) \ L - a + b < 0.$$

Now in order to transform (9.5) by the Burge transform we need to take $a = b = j$. Since the inequalities (i) and (iii), and also (iv) and (vi), become mutually exclusive we can indeed utilize (9.6) to get

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} (-1)^j q^{j(7j+2\sigma+1)/2} \mathcal{B}(L, M, 2j, j) \\ &= \sum_{N_1, N_2 \geq 0} \frac{q^{N_1^2 + N_2^2 + \sigma N_2} (q; q)_L}{(q; q)_{L - N_1 - N_2} (q; q)_{N_1 - N_2} (q; q)_{N_2}} \begin{bmatrix} 2L + M - N_1 \\ 2L \end{bmatrix}. \end{aligned}$$

Here i and n have been replaced by N_1 and N_2 , respectively. We need to apply (9.6) one more time. Since now $a = 2j$ and $b = j$ the inequalities (i)–(iii) become $L \geq 0, j \geq 0$, and $L + j < 0$ which cannot occur. Similarly, the inequalities (iv)–(vi) are now $L \geq 0, j < 0$ and $L - j < 0$ which again is impossible. As a result we get

$$(9.7) \quad \begin{aligned} & \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j+2\sigma+1)/2} \mathcal{B}(L, M, 3j, j) \\ &= \sum_{\substack{n_1, n_2, n_3 \geq 0 \\ N_1 + N_2 + N_3 \leq L}} \frac{q^{N_1^2 + N_2^2 + N_3^2 + \sigma N_3} (q; q)_{L - N_1}}{(q; q)_{L - N_1 - N_2 - N_3} (q; q)_{n_2} (q; q)_{n_3}} \begin{bmatrix} 2L + M - N_1 \\ 2L \end{bmatrix} \begin{bmatrix} 2L - N_1 - N_2 \\ 2L - 2N_1 \end{bmatrix}, \end{aligned}$$

where we have replaced $(i, N_1, N_2) \rightarrow (N_1, N_2, N_3)$ and have used $n_1 = N_1 - N_2, n_2 = N_2 - N_3$ and $n_3 = N_3$. Letting M tend to infinity, the above identity simplifies to the theorem when $\sigma = 0$ and to an identity for $G(L; 1, 2, 3) = (1 + q^L)(q^3; q^3)_{L-1} / (q; q)_{L-1}$ when $\sigma = 1$. □

By modifying the above proof using an asymmetric version of the Burge transform, the following companion to Theorem 9.5 can be shown to hold.

Theorem 9.6. *Let L be a nonnegative integer and $N_1 = n_1 + n_2 + n_3, N_2 = n_2 + n_3, N_3 = n_3$. Then*

$$C_L(q) = \sum_{\substack{n_1, n_2, n_3 \geq 0 \\ N_1 + N_2 + N_3 \leq L-1}} \frac{q^{N_1^2 + N_2^2 + N_3^2 + N_1 + N_2 + N_3} (q; q)_{L - N_1 - 1} (q; q)_{2L - N_1 - N_2 - 1}}{(q; q)_{2L - 2N_1 - 1} (q; q)_{L - N_1 - N_2 - N_3 - 1} (q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}}.$$

Proof. We extend definition (9.4) to

$$\mathcal{B}_{r,s}(L, M, a, b) = \begin{bmatrix} L + M + a - b \\ L + a \end{bmatrix} \begin{bmatrix} L + M - a + b + r + s \\ L - a + r \end{bmatrix}$$

and wish to first show that

$$(9.8) \quad \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+3)/2} \mathcal{B}_{0,1}(L, M, j, j) = \sum_{n \geq 0} \frac{q^{n(n+1)}(q; q)_{L+M}}{(q; q)_{L-n}(q; q)_{M-n}(q; q)_n}.$$

To do so we subtract (9.5) with $\sigma = 1$ to get

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+3)/2} \begin{bmatrix} L + M \\ L + j \end{bmatrix} \left(\begin{bmatrix} L + M + 1 \\ L - j \end{bmatrix} - \begin{bmatrix} L + M \\ L - j \end{bmatrix} \right) = 0.$$

By (8.5) this becomes

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{5j(j+1)/2} \begin{bmatrix} L + M \\ L + j \end{bmatrix} \begin{bmatrix} L + M \\ L - j - 1 \end{bmatrix} = 0.$$

Since the expression on the left is negated after replacing the summation variable j by $-j - 1$ this obviously is true.

Now that (9.8) has been established we apply the following asymmetric version of the Burge transform [21, 43]. For L, M, a, b, r, s integers such that not $-L + a - r \leq -b \leq L + a < b + s \leq M + s$ or $-L - a \leq b \leq L - a + r < -b - s \leq M$, there holds

$$(9.9) \quad \sum_{i=\max(b, -b-s)}^M q^{i(i+b)} \begin{bmatrix} 2L + M + r - i \\ 2L + r \end{bmatrix} \mathcal{B}_{r+s,s}(L - i - s, i, a, b) = q^{b(b+s)} \mathcal{B}_{r,s}(L, M, a + b, b).$$

Before we use this transform we replace L by $L + 1$ in (9.8) and use that

$$\mathcal{B}_{0,1}(L + 1, M, j, j) = \mathcal{B}_{2,1}(L, M, j + 1, j).$$

Then we apply (9.9) with $r = s = 1, a = j + 1, b = j$. This then yields

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} (-1)^j q^{j(7j+5)/2} \mathcal{B}_{1,1}(L, M, 2j + 1, j) \\ &= \sum_{N_1, N_2 \geq 0} \frac{q^{N_1^2 + N_2^2 + N_1 + N_2}(q; q)_L}{(q; q)_{L-N_1-N_2}(q; q)_{N_1-N_2}(q; q)_{N_2}} \begin{bmatrix} 2L + M - N_1 + 1 \\ 2L + 1 \end{bmatrix}. \end{aligned}$$

We still need to check that the conditions imposed on the asymmetric Burge transform do hold. The first condition is certainly satisfied if not simultaneously

$$(i) \ L \geq -r, \quad (ii) \ 2b > s, \quad (iii) \ L + a - b - s < 0$$

and the second condition is satisfied if not simultaneously

$$(iv) \ L \geq -r, \quad (v) \ 2b < -s, \quad (vi) \ L - a + b + r + s < 0.$$

If $r = s = 1, a = j + 1, b = j$ this is easily seen to be the case.

Next we choose $r = 0$, $s = 1$, $a = 2j + 1$ and $b = j$, check that neither (i)–(iii) nor (iv)–(vi) are all satisfied, and apply (9.9) to find that

$$(9.10) \quad \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j+7)/2} \mathcal{B}_{0,1}(L, M, 3j + 1, j) \\ = \sum_{\substack{n_1, n_2, n_3 \geq 0 \\ N_1 + N_2 + N_3 \leq L-1}} \frac{q^{N_1^2 + N_2^2 + N_3^2 + N_1 + N_2 + N_3} (q; q)_{L-N_1-1}}{(q; q)_{L-N_1-N_2-N_3-1} (q; q)_{n_2} (q; q)_{n_3}} \\ \times \begin{bmatrix} 2L + M - N_1 \\ 2L \end{bmatrix} \begin{bmatrix} 2L - N_1 - N_2 - 1 \\ 2L - 2N_1 - 1 \end{bmatrix}.$$

In the large M limit this implies Theorem 9.6. □

Theorems 9.5 and 9.6 are insufficient to conclude that $A_n(q) \geq 0$ and $C_n(q) \geq 0$. It does in fact appear that the polynomials $\mathcal{A}_{L,M}(q)$ and $\mathcal{C}_{L,M}(q)$ given by (the right or left-hand side of) (9.7) with $a = 1$ and (9.10) have nonnegative coefficients.

Conjecture 9.2. *For L, M nonnegative integers*

$$\mathcal{A}_{L,M}(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j+1)/2} \mathcal{B}(L, M, 3j, j) \geq 0$$

and

$$\mathcal{C}_{L,M}(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j+7)/2} \mathcal{B}_{0,1}(L, M, 3j + 1, j) \geq 0.$$

However, since

$$A_L(q) = (q; q)_{2L} \lim_{M \rightarrow \infty} \mathcal{A}_{L,M}(q) \quad \text{and} \quad C_L(q) = (q; q)_{2L} \lim_{M \rightarrow \infty} \mathcal{C}_{L,M}(q),$$

this does not imply that $A_L(q) \geq 0$ and $C_L(q) \geq 0$. Furthermore, it is certainly not true that $(q; q)_{2L} \mathcal{A}_{L,M}(q) \geq 0$ or $(q; q)_{2L} \mathcal{C}_{L,M}(q) \geq 0$ for general L and M . (We actually believe this never to be true for positive L and finite M). Nevertheless, Theorems 9.5 and 9.6 do give rise to a two-parameter refinement of the Borwein conjecture. To describe this we need the polynomials

$$A_{L,m}(q) = \sum_{\substack{n_1, n_2, n_3 \geq 0 \\ N_1 + N_2 + N_3 = m}} \frac{q^{N_1^2 + N_2^2 + N_3^2} (q; q)_{L-N_1} (q; q)_{2L-N_1-N_2}}{(q; q)_{L-m} (q; q)_{2L-2N_1} (q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}}$$

and

$$C_{L,m}(q) = \sum_{\substack{n_1, n_2, n_3 \geq 0 \\ N_1 + N_2 + N_3 = m}} \frac{q^{N_1^2 + N_2^2 + N_3^2 + m} (q; q)_{L-N_1-1} (q; q)_{2L-N_1-N_2-1}}{(q; q)_{L-m-1} (q; q)_{2L-2N_1-1} (q; q)_{n_1} (q; q)_{n_2} (q; q)_{n_3}}.$$

Comparing these definitions with Theorems 9.5 and 9.6 shows that

$$A_L(q) = \sum_{m=0}^L A_{L,m}(q) \quad \text{and} \quad C_L(q) = \sum_{m=0}^{L-1} C_{L,m}(q).$$

Conjecture 9.3. *The polynomials $A_{L,m}(q)$ for $0 \leq m \leq L$ and $C_{L,m}(q)$ for $0 \leq m \leq L - 1$ have nonnegative coefficients.*

It is in fact not difficult to show that $A_{L,m}(q) \geq 0$ together with the initial condition $C_{L,0}(q) = 1$ implies that $C_{L,m}(q) \geq 0$. Indeed, by shifting the summation variable $n_1 \rightarrow n_1 - 1$ (so that $N_1 \rightarrow N_1 - 1$) in the expression for $C_{L,m-1}(q)$, one finds

$$C_{L+1,m}(q) = q^m A_{L,m}(q) + q^{2L+1} C_{L,m-1}(q).$$

Since Conjecture 9.3 implies that $A_L(q) \geq 0$ and $C_L(q) \geq 0$, we hope it provides a new way of tackling the Borwein conjecture. Here it should be noted that proving $A_L(q) \geq 0$ and $C_L(q) \geq 0$ is sufficient since $B_L(q) = q^{L^2-1} C_L(1/q)$.

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