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ON SOME CONSTANTS IN THE SUPERCUSPIDAL CHARACTERS OF GL_l, l A PRIME $\neq p$

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ABSTRACT. The article gives explicit values of some constants which appear in the character formula for the irreducible supercuspidal representation of $\mathrm{GL}_l(F)$ for F a local field of the residual characteristic $p \neq l$.

INTRODUCTION

Let l be a prime, let F be a non-Archimedean local field of residual characteristic $p \neq l$, let A/F be a central simple algebra of reduced degree l and let E be a field such that $F \subsetneq E \subset A$. Then either $A = D_l$, a division algebra of index l over F or $A = \mathrm{M}_l(F)$, the algebra of all $l \times l$ matrices. Moreover, every compact mod center Cartan subgroup of $A^\times$ is of the form $E^\times$ for some such E, and every irreducible supercuspidal representation of $A^\times$ corresponds to a quasi-character of some such $E^\times$ ([17], [6]).

The character formula for the irreducible supercuspidal representations of $A^\times$ has been extensively studied by many mathematicians (see especially [9] for this topic). Debacker ([8],[9]) got the formula for $\mathrm{GL}_l(F)$ under the assumption $p > l$. But it contains some undetermined constants (see Remark 3.14). The aim of this paper is to compute the constants explicitly and get complete formulas, valid on the regular elliptic set, for the supercuspidal characters of $\mathrm{GL}_l(F)$ which correspond to characters of $E^\times$ for ramified E. The reader may consult [23], in which we have discussed the case E/F unramified and we have obtained the formula containing no ambiguity (see Theorem 3.13). We do not give character values on the split torus here (see [18], [8] and [9] for results pertaining to this problem). We mention that the character values on the regular elliptic set are sufficient to uniquely determine a supercuspidal representation (or, more generally, an arbitrary discrete series representation).

Let D_n/F be a division algebra of index n. The "abstract matching theorem" of Badulescu [2], Deligne-Kazhdan-Vigneras [10] and Rogawski [22] implies the existence of a bijection between the set of irreducible representations of $D_n^\times$ and the set of essentially square-integrable representations of $\mathrm{GL}_n(F)$ which preserves the characters on the corresponding regular elliptic sets up to the sign $(-1)^{n-1}$. In the tame case (i.e. $(p,n) = 1$) Moy [17] proved the existence of a bijection between these sets of representations which respects the concrete construction of the representations by Howe [14]. In general, the relationship between these two

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bijections is unknown, but in the case n is a prime $\neq p$ it is known that the two bijections coincide ([12]). Therefore, it suffices to determine the character formula on either $\mathrm{GL}_l(F)$ or $D_l^\times$. Thus we only treat the GL case.

Let us summarize the contents of this paper, indicating its organization. Section 1 is devoted to the review of the construction of an irreducible supercuspidal representation π_θ (resp. π'_θ) of $\mathrm{GL}_l(F)$ (resp. $D_l^\times$) from a generic quasi-character θ of $E^\times$ and the known results about the representation. We note that π_θ is not always monomial, i.e., induced from a one-dimensional representation, but it can be written as a $\mathbb{Q}$-linear combination of monomial representations. In fact π_θ is written as a $\mathbb{Q}$-linear combination of the forms $\mathrm{ind}_H^{\mathrm{GL}_l(F)}\rho_\theta$, where H is a compact mod center subgroup of $\mathrm{GL}_l(F)$ and ρ_θ is a quasi-character of H.

In section 2, we compute the character of π_θ up to some root numbers. Let $G = \mathrm{GL}_l(F)$, let B be the normalizer of an Iwahori subgroup of G containing H and let $\eta_\theta = \mathrm{ind}_H^B \rho_\theta$. Since we treat only regular elliptic conjugacy classes, we consider the character χ_{π_θ} on $L^\times$, where L/F are extensions of fields of degree l. Moreover the case $L = E$ is essential. By the Frobenius formula and the result of Kutzko ([16]), we have only to calculate the sum

$$\chi_{\eta_\theta}(x) = \sum_{a \in H\backslash B} \rho_\theta(axa^{-1})$$

for $x \in E$ in order to get the character formula of π_θ. Therefore it is essential to know when $axa^{-1} \in H$, which is determined in Lemma 2.1. From this, we get the character formula of η_θ except "at the depth"(Proposition 2.2). But this formula contains the undetermined Gauss sum part $G(y, j)$, which is calculated later. From this we recover the formula of Debacker including the case $l < p$. The proof of the formula is short and simple. Moreover since we use the property "intertwining implies conjugacy"of an E/F-minimal (very cuspidal in the terminology of Carayol [5]) element as the key tool, the result may be extended to GL_n, at least when n is prime to p. The character of π_θ "at the depth"can be calculated directly by taking the explicit matrix form of $E^\times$ (Lemma 2.5). The character value at the depth has not been previously determined explicitly (see 5.6 in [8]). We can reveal that the character value is represented by some Kloosterman sum.

Section 3 is devoted to the calculation of the Gauss sum part $G(y, j)$; it appears in the character formula on $E^\times$. For this purpose, the point is that we have only to treat the character of π_θ on $U_1^* = F^\times(1+P_E) - F^\times(1+P_E^2)$. For this calculation, we use the $E^\times$-module structure of various objects. We first assume E/F is a Galois extension since the $E^\times$-module structure can be described easily for this case. This part is analogous to section 1 of [23], but everything becomes easier since we have only to treat U_1^*. When E/F is non-Galois, we use the base change lift. Let ζ be a primitive l-th root of unity and $L = F(\zeta)$. Then L is an unramified extension of F and EL/L is Galois. Therefore we can use the tools of the Galois case for $\mathrm{GL}_l(L)$. Let $\mathrm{Gal}(L/F) = \langle \tau \rangle$. By the result of Bushnell-Henniart [4], there is a base change lift η_L of η_θ to H_L^1 such that the twisted trace of η_L by τ gives the trace of η_θ (see Proposition 3.7 and Lemma 3.8). We remark that we need not assume that the characteristic of F is 0 since we do not use the Arthur-Clozel base change lift [1]. The method of calculating the twisted trace of η_L is similar to that of the Galois case. The complete character formula is stated as Theorem 3.13.

At the end of this Introduction, we compare our formula with the known results besides [8] and [9]. The same type of character formula for the division algebra case was given by Corwin, Moy and Sally [7] in the case $l \neq p$. Their formulas agree with the result given in section 2. It contains some root numbers associated with a quadratic form as in [8]. They have shown that this root number is a root of unity when $p \neq 2$. In this paper, we have determined it completely including the case $p = 2$ in section 3. Moreover we find that the Kloosterman sum appears in the character formula. These are new results of this paper. In [25], the author gave the character formula of π_θ for GL_3 by using the decomposition of π_θ as an $E^\times$-module.

Notation. Let F be a non-archimedean local field. We denote by $\mathcal{O}_F$, P_F, ϖ_F, k_F and v_F the maximal order of F, the maximal ideal of $\mathcal{O}_F$, a prime element of P_F, the residue field of F and the valuation of F normalized by $v_F(\varpi_F) = 1$. We set q to be the number of elements in k_F. Henceforth we fix an additive character ψ of F whose conductor is P_F, i.e., ψ is trivial on P_F and not trivial on $\mathcal{O}_F$. For an extension E over F, we denote by tr_E, n_E the trace and norm to F, respectively. We set $\psi_E = \psi \circ \mathrm{tr}_E$. The trace of the matrix is denoted by Tr. For an irreducible admissible representation π of $\mathrm{GL}_l(F)$, the conductoral exponent of π is defined to be the integer $f(\pi)$ such that the local constant $\varepsilon(s, \pi, \psi)$ of Godement-Jacquet [11] is the form $aq^{-s(f(\pi)-l)}$.

We call π *minimal* if

$$f(\pi) = \min_{\eta} f(\pi \otimes (\eta \circ \mathrm{Nr})),$$

where η runs through the quasi-characters of $F^\times$. Let G be a totally disconnected, locally compact group. We denote by $\widehat{G}$ the set of (equivalence classes of) irreducible admissible representations of G. For a closed subgroup H of G and a representation ρ of H, we denote by $\mathrm{Ind}_H^G \rho$ (resp. $\mathrm{ind}_H^G \rho$) the induced representation (resp. compactly induced representation) of ρ to G. For a representation π of G, we denote by $\pi|_H$ the restriction of π to H.

1. Construction of the representation

Let $l \neq p$ be an odd prime and let E be a ramified extension of F of degree l. Then E can be embedded into $\mathrm{M}_l(F)$ and, up to conjugacy, the embedding is unique. Let $G = \mathrm{GL}_l(F)$. In this section, we review the construction of supercuspidal representations of G which are parameterized by the quasi-characters of $E^\times$. Of course, this construction is well known ([5], [17]).

Definition 1.1. Let θ be a quasi-character of $E^\times$ and let $f(\theta)$ be the exponent of the conductor of θ i.e. the minimum integer such that $\mathrm{Ker}\,\theta \subset 1 + P_E^n$. Then θ is called generic if $f(\theta) \not\equiv 1 \bmod l$. For a generic character θ of $E^\times$, $\beta_\theta \in P_E^{1-f(\theta)} - P_E^{2-f(\theta)}$ is defined by

$$\theta(1+x) = \psi_E(\beta_\theta x) \quad \text{for} \quad x \in P_E^{[(f(\theta)+1)/2]}. \tag{1.1}$$

Then $F(\beta_\theta) = E$. We denote by $\widehat{E}^\times_{gen}$ the set of generic quasi-characters of $E^\times$.

We construct an irreducible supercuspidal representation of $G = \mathrm{GL}_l(F)$ from $\theta \in \widehat{E}^\times_{gen}$. For simplicity, we set $\beta = \beta_\theta$. Since E/F is tamely ramified, there exists a prime element ϖ_E of $\mathcal{O}_E$ satisfying $\varpi_E^l \in F$. Put $\varpi_F = \varpi_E^l$. We identify

$\mathrm{M}_l(F)$ with $\mathrm{End}_F\, E$ and G with $\mathrm{Aut}_F\, E$ by the F-basis $\{\varpi_E^{l-1}, \varpi_E^{l-2}, \dots, \varpi_E, 1\}$ of E, which is also an $\mathcal{O}_F$-basis of $\mathcal{O}_E$. From the lattice flag $\{P_E^i\}_{i\in\mathbb{Z}}$, we construct a maximal compact modulo center subgroup. The construction of the representation is well known. For details, see [17].

Definition 1.2. For $i \in \mathbb{Z}$, set

$$A^i = \{g \in \mathrm{M}_l(F) | g(P_E^j) \subset P_E^{j+i} \quad \text{for all} \;\; j \in \mathbb{Z}\}.$$

Put $K = (A^0)^\times, B = E^\times K$ and $K^i = 1 + A^i$ for $i \geq 1$.

Then K is an Iwahori subgroup of G and B is a normalizer of K. First we construct an irreducible representation of B from a generic quasi-character of $E^\times$.

Let θ be a generic quasi-character of $E^\times$, i.e., $f(\theta) = n \not\equiv 1 \bmod l$. There exists an element $\beta \in P_E^{1-n}$ such that $\theta(1+x) = \psi_E(\beta x)$ for $x \in P_E^m$, where $m = [(n+1)/2]$. Define ψ_β on K^m by $\psi_\beta(1+x) = \psi(\mathrm{Tr}(\beta x))$ for $x \in A^m$. Then ψ_β is a quasi-character of K^m. Put $H = E^\times K^m$ and define a quasi-character ρ_θ of H by

$$\rho_\theta(h \cdot g) = \theta(h)\psi_\beta(g) \qquad \text{for} \quad h \in E^\times, \quad g \in K^m. \tag{1.2}$$

Let J be the normalizer of ψ_β in B, i.e.,

$$J = \{a \in B \mid \psi_\beta^a = \psi_\beta\},$$

where $\psi_\beta^a(x) = \psi_\beta(a^{-1}xa)$ for $x \in K^m$. Then $J = E^\times K^{m'}$, where $m' = [n/2]$. Put $\eta_\theta = \mathrm{Ind}_H^B\, \rho_\theta$.

When n is even, i.e., $n = 2m$, then $J = H = E^\times K^m$. By the Clifford theory, η_θ is an irreducible representation of B. We put

$$\kappa_\theta = \eta_\theta. \tag{1.3}$$

If $n+1 = 2m-1$ is odd, then $J = E^\times K^{m-1}$, so we have to determine an irreducible component of $\mathrm{Ind}_H^J \rho_\theta$. For a subgroup $M \subset B$, we write $M^1 = M \cap F^\times K$. In particular, $H^1 = F^\times(1+P_E)K^m$. It is well known and not difficult to show that

$$(\mathrm{Ind}_H^J \rho_\theta)|_{J^1} = \mathrm{Ind}_{H^1}^{J^1}(\rho_\theta|_{H_1}) = q\eta. \tag{1.4}$$

This η can be extended to J in $|E^\times/F^\times(1+P_E)| = l$ ways. To determine the extension by θ, we will express ρ_θ by a linear combination of $\mathrm{Ind}_H^J\, \rho_{\theta\otimes\chi}$ ($\chi \in (E^\times/F^\times(1+P_E))$).

Lemma 1.3. *Define the virtual representation κ_θ of B by*

$$\kappa_\theta = \frac{1 - \left(\frac{q}{l}\right) q^{(l-1)/2}}{l q^{(l-1)/2}} \sum_{\chi \in (E^\times/F^\times(1+P_E))} \eta_{\theta\otimes\chi} + \left(\frac{q}{l}\right)\eta_\theta, \tag{1.5}$$

where $\left(\frac{q}{l}\right)$ is the Legendre symbol. Then κ_θ is a real representation and an irreducible component of $\mathrm{Ind}_H^B\, \rho_\theta$.

Proof. Let $\{\eta_1, \dots, \eta_l\}$ be the set of the extensions of η to J and $(E^\times/F^\times(1+P_E)) = \{\chi_1, \dots, \chi_l\}$. It follows from Lemma 3.5.35 in [17] that

$$\mathrm{Ind}_H^J\, \rho_\theta = \frac{\left(q^{(l-1)/2} - \left(\frac{q}{l}\right)\right)}{l} \sum_{i=1}^{l} \eta_i + \left(\frac{q}{l}\right)\eta_j \tag{1.6}$$

for a unique j. Let us denote ρ'_θ by this η_j. From this irreducible decomposition of $\mathrm{Ind}_H^J \rho_\theta$, we have

$$\begin{pmatrix} \mathrm{Ind}_H^J \rho_{\theta\otimes\chi_1} \\ \vdots \\ \mathrm{Ind}_H^J \rho_{\theta\otimes\chi_l} \end{pmatrix} = \left(\left(\frac{q}{l}\right) I_l + \frac{q^{(l-1)/2} - \left(\frac{q}{l}\right)}{l} T_l \right) \begin{pmatrix} \rho'_{\theta\otimes\chi_1} \\ \vdots \\ \rho'_{\theta\otimes\chi_l} \end{pmatrix},$$

where I_l is the l-th identity matrix and all coefficients of T_l are equal to 1. By the Clifford theory, $\kappa_\theta = \mathrm{Ind}_J^B \rho'_\theta$ is an irreducible representation of B. Thus we obtain the desired formula for κ_θ by calculating the inverse of the coefficient matrix. □

The following result is well known ([17], [21]).

Theorem 1.4. *Let the notation be as above and denote by $A_0(G)$ the set of equivalence classes of the supercuspidal representations of G. Put $\pi_\theta = \mathrm{ind}_B^G \kappa_\theta$. Then π_θ is an irreducible supercuspidal representation of G such that*

(1) *$\varepsilon(\pi_\theta, \psi) = \varepsilon(\theta, \psi_E)$; in particular $f(\pi_\theta) = f(\theta) + l$.*

(2) *$\bigcup_E \{\pi_\theta | \theta \in \widehat{E}^\times_{gen}\} = \{\pi \in A_0(G) | f_{\min}(\pi) \not\equiv 0 \bmod l\}$, where E runs through isomorphism classes of ramified extensions of degree l over F.*

Remark 1.5. 1. If $\pi \in A_0(G)$ and $f_{\min}(\pi) \equiv 0 \bmod l$, π can be constructed from a regular quasi-character θ of $L^\times$, where L is an unramified extension of F of degree l. The characters of such representations on elliptic conjugacy classes were completely calculated in [23].

2. The representation π_θ has depth $f(\theta)/e(E/F)$ in terms of the Moy-Prasad filtration ([8]). Since $e(E/F) = l > 1$, the depth of the representation is positive. The depth zero representation appears when E/F is unramified.

By the following proposition, we have only to calculate χ_{η_θ}.

Proposition 1.6. *The character χ_{κ_θ} of κ_θ on B is expressed by χ_{η_θ} as follows:*

$$\chi_{\kappa_\theta}(x) = \begin{cases} \left(\frac{q}{l}\right) \chi_{\eta_\theta}(x), & x \in B - F^\times K, \\ q^{-(l-1)/2} \chi_{\eta_\theta}(x), & x \in F^\times K. \end{cases} \tag{1.7}$$

Proof. Let $\chi \in E^\times / F^\times(1 + P_E)$. Since χ is trivial on $F^\times(1 + P_E)$, $\beta_{\theta\otimes\chi} = \beta_\theta$. Thus

$$\rho_{\theta\otimes\chi}(h \cdot g) = \chi(h) \rho_\theta(h \cdot g) \qquad \text{for} \quad h \in E^\times, \quad g \in K^m. \tag{1.8}$$

Due to the isomorphisms $B/F^\times K \simeq H/H^1 \simeq E^\times/F^\times(1 + P_E)$, we regard χ as a character of H and B. Then it follows from (1.8) that $\rho_{\theta\otimes\chi} = \rho_\theta \otimes \chi$. By the well-known formula, we have

$$\mathrm{Ind}_H^B(\rho_\theta \otimes \chi) = (\mathrm{Ind}_H^B \rho_\theta) \otimes \chi.$$

Therefore we get

$$\sum_{\chi \in (E^\times/F^\times(1+P_E))} \eta_{\theta\otimes\chi} = \eta_\theta \otimes (\sum_{\chi \in B/F^\times K} \chi).$$

For $x \in B$,

$$\sum_{\chi \in B/F^\times K} \chi(x) = \begin{cases} l, & x \in B - F^\times K, \\ 0, & x \in F^\times K. \end{cases}$$

Therefore we get our proposition from (1.5). □

To end this section, we quote the result of Kutzko [16] in the form that the character formula of π_θ on regular elliptic elements is essentially given by the one of κ_θ.

Theorem 1.7. *Let x be an regular elliptic element of G.*

(1) *If $F(x)/F$ is ramified and $x \notin F^\times(1+P^n_{F(x)})$,*

$$\chi_{\pi_\theta}(x) = \chi_{\kappa_\theta}(x).$$

(2) *If $F(x)/F$ is unramified and $x \notin F^\times(1+P^{[n/l]+1}_{F(x)})$,*

$$\chi_{\pi_\theta}(x) = 0.$$

Proof. These are obtained by applying Proposition 5.5 in [16] to our case. □

2. Calculation of the character

Now we begin to calculate the characters of the representations constructed in the previous section. In this section, we shall get a character formula up to some root numbers. These root numbers are calculated explicitly in the next section.

Henceforth we fix a generic character θ and put $\rho = \rho_\theta$, $\eta = \eta_\theta$ and so on. Since E/F is a totally tamely ramified extension, there exists a prime element ϖ_E of $\mathcal{O}_E$ such that $\varpi_E^l \in P_F - P_F^2$. Put $\varpi_E^l = \varpi_F$. As in the previous section, we identify $\mathrm{M}_l(F)$ with $\mathrm{End}_F(E)$ by the F-basis $\{\varpi_E^{l-1}, \varpi_E^{l-2}, \dots, \varpi_E, 1\}$, which is an $\mathcal{O}_F$-basis of $\mathcal{O}_E$. Thus we get the explicit matrix forms of various objects:

$$\varpi_E = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \varpi_F & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{2.1}$$

$$A^0 = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i \le j \\ a_{ij} \in P_F & \text{if } i > j \end{array} \right\}, \tag{2.2}$$

$$A^1 = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \cdots & \cdots & \cdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix} \middle| \begin{array}{ll} a_{ij} \in \mathcal{O}_F & \text{if } i < j \\ a_{ij} \in P_F & \text{if } i \ge j \end{array} \right\}. \tag{2.3}$$

If $q \equiv 1 \bmod l$, F has a primitive l-th root of unity ζ and E/F is a cyclic extension. Let σ be a generator of $\mathrm{Gal}(E/F)$ determined by ${}^\sigma\varpi_E = \varpi_E\zeta$. We denote the diagonal matrix $\mathrm{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \dots, \zeta)$ by ξ. Then ξ satisfies $\xi^l = 1$ and $\xi x \xi^{-1} = {}^\sigma x$ for $x \in E$.

Define a natural ring morphism R from A^0 to k_F^l by the identification of A^0/A^1 with k_F^l. We note that $R(\mathcal{O}_E^\times) = \{(\alpha, \dots, \alpha) \mid \alpha \in k_F^\times\}$ and if $R(a) = (\alpha_0, \alpha_1, \dots, \alpha_{l-1})$, $R(\varpi_E a \varpi_E^{-1}) = (\alpha_1, \alpha_2, \dots, \alpha_0)$. For convenience, we extend the subscript to $\mathbb{Z}$ by putting $\alpha_i = \alpha_{i \bmod l}$. The next lemma is the key tool for the character calculation.

Lemma 2.1. *Let $x \in P_E^i - (F + P_E^{i+1})$, let $g \in B$ and let j be a positive integer. If $gxg^{-1} \in E^\times K^j$, then*

$$g \in \begin{cases} E^\times(1+A^j) & \text{if} \quad q \not\equiv 1 \bmod l, \\ \bigcup_{k=0}^{l-1} E^\times(1+A^j)\xi^k & \text{if} \quad q \equiv 1 \bmod l. \end{cases}$$

Proof. We may assume $g \in A_0$ by replacing g by $\varpi_E^{-k} g$ if $g \in A_k$. Let $x = \varpi_E^i x_0$ for $x_0 \in \mathcal{O}_E^\times$ and $R(g) = (\alpha_0, \alpha_1, \ldots, \alpha_{l-1})$. Then

$$R(gxg^{-1}x^{-1}) = (\alpha_0\alpha_i^{-1}, \alpha_1\alpha_{i+1}^{-1}, \ldots, \alpha_{l-1}\alpha_{l-1+i}^{-1}),$$

where $\alpha_s = \alpha_{s \bmod l}$ for $s \in \mathbb{Z}$. Since $k_E = k_F$, $gxg^{-1}x^{-1} \in E^\times(1+A^1)$ implies $R(gxg^{-1}x^{-1}) \in \Delta = \{(\alpha, \ldots, \alpha) \mid \alpha \in k_F\}$. By virtue of $i \not\equiv 0 \bmod l$, we have

$$\alpha_{ki} = (\zeta')^k \alpha_0, \quad 0 \le k \le l-1,$$

for some l-th root of unity ζ'. If $q \equiv 1 \bmod l, \xi\varpi_E\xi^{-1} = \zeta\varpi_E$. Thus we get

$$g \in \begin{cases} E^\times(1+A^1) & \text{if} \quad q \not\equiv 1 \bmod l, \\ \bigcup_{k=0}^{l-1} E^\times(1+A^1)\xi^k & \text{otherwise.} \end{cases}$$

Thus we may assume $g - 1 \in A^k - (P_E^{k+1} + A^{k+1})$ for some $k \ge 1$. Put $g - 1 = \varpi_E^k g_0$ and $R(g_0) = (\gamma_0, \gamma_1, \ldots, \gamma_{l-1})$. Since

$$\begin{aligned} gxg^{-1}x^{-1} &\equiv 1 + (g-1) - x(g-1)x^{-1} \bmod A^{k+1} \\ &\equiv 1 + \varpi_E^k(g_0 - xg_0x^{-1}) \bmod A^{k+1}, \end{aligned}$$

$R(g_0 - xg_0x^{-1}) = (\gamma_0 - \gamma_i, \gamma_1 - \gamma_{1+i}, \ldots, \gamma_{l-1} - \gamma_{l-1+i})$. Therefore $gxg^{-1}x^{-1} \in E^\times K^{k+1}$ contradicts $g - 1 \in A^k - (P_E^{k+1} + A^{k+1})$. It implies that if $gxg^{-1}x^{-1} \in E^\times K^j$,

$$g \in \begin{cases} E^\times(1+A^j) & \text{if} \quad q \not\equiv 1 \bmod l, \\ \bigcup_{k=0}^{l-1} E^\times(1+A^j)\xi^k & \text{if} \quad q \equiv 1 \bmod l. \end{cases}$$

□

Put $U_{-1} = E^\times, U_0 = F^\times\mathcal{O}_E^\times, U_i = F^\times(1+P_E^i)$ for $i \ge 1$ and $U_i^* = U_i - U_{i+1}$ for $j \ge -1$. The previous lemma gives the character of η_θ on $E^\times - U_{n-1}$. We remark that $\mathrm{Aut}_F\, E = \{1\}$ if $q \not\equiv 1 \bmod l$.

Proposition 2.2. *Let $x \in U_i^*$ for $-1 \le i < n-1$. If $i > 0$, x is written in the form $x = c(1+y)$ for $c \in F$ and $y = \varpi_E^i y_0 \in \varpi_E^i \mathcal{O}_E^\times$. For $u \in k_F^\times$ and $j \in (\mathbb{Z}/l\mathbb{Z})$ such that $j \not\equiv 1/2$, we define the Gauss sum $G(u,j)$ by*

$$G(u,j) = \sum_{(\alpha_0,\ldots,\alpha_{l-1}) \in k_F^l/\Delta} \psi\left(\sum_{k=0}^{l-1} u(\alpha_{k+1} - \alpha_k)\alpha_{j+k}\right), \tag{2.4}$$

where $\Delta = \{(\alpha, \ldots, \alpha) | \alpha \in k_F\}$. Then χ_{η_θ} on U_i^* is given as follows:

$$\chi_{\eta_\theta}(x) = \begin{cases} \sum_{\sigma \in \mathrm{Aut}_F E} \theta({}^\sigma x), & i = -1, \\ q^{[(i+1)/2](l-1)} \sum_{\sigma \in \mathrm{Aut}_F E} \theta({}^\sigma x), & i > 0 \text{ and } n-i \text{ even}, \\ q^{[i/2](l-1)} \sum_{\sigma \in \mathrm{Aut}_F E} \theta({}^\sigma x) & \\ \quad G(\beta\varpi_E^{n-1} y_0 ({}^\sigma\varpi_E/\varpi_E)^i, c), & i > 0 \text{ and } n-i \text{ odd}, \end{cases}$$

where $c = i^{-1}(n+i-1)/2$ satisfies $c \not\equiv 1/2 \bmod l$.

Proof. Put ${}^a x = axa^{-1}$ for $a, x \in G$. First we treat the case $x \in U^*_{-1} = E^\times - F^\times \mathcal{O}_E^\times$. Since

$$\chi_{\eta_\theta}(x) = \sum_{a \in H \backslash B} \rho_\theta({}^a x),$$

the proof follows immediately from Lemma 2.1.

Now we treat the case $x = c(1+y)$ for $c \in F$ and $y \in P_E^i - (F + P_E^{i+1})$. We may assume $c = 1$ since $F^\times$ is the center of B. For $1 + k \in K^{[(n-i+1)/2]}$ and $a \in B$, we have

$$\begin{aligned}\chi_{\eta_\theta}(1+y) &= \sum_{a \in H \backslash B} \rho_\theta({}^a(1+y)) \\ &= C \sum_{1+k \in K^{n-i} \backslash K^{[(n-i+1)/2]}} \sum_{a \in H \backslash B} \rho_\theta({}^{a(1+k)}(1+y)),\end{aligned}$$

where $C = \frac{1}{|K^{n-i} \backslash K^{[(n-i+1)/2]}|}$. In the above expression,

$$\begin{aligned}\rho_\theta({}^{a(1+k)}(1+y)) &= \rho_\theta(1 + {}^a y + {}^a(ky - yk)) \\ &= \rho_\theta(1 + {}^a y)\rho_\theta(1 + {}^a((1+y)^{-1}(ky - yk))) \\ &= \rho_\theta(1 + {}^a y)\psi(\operatorname{Tr} \beta^a((1+y)^{-1}(ky - yk))) \\ &= \rho_\theta(1 + {}^a y)\psi(\operatorname{Tr} {}^{a^{-1}}\beta(1+y)^{-1}(ky - yk)) \\ &= \rho_\theta(1 + {}^a y)\psi(\operatorname{Tr}(y^{a^{-1}}\beta - {}^{a^{-1}}\beta y)(1+y)^{-1}k)\end{aligned}$$

since $yk^2 \in A^n$ and $a(1+y)^{-1}(ky-yk)a^{-1} \in A^m$. If $y^{a^{-1}}\beta - {}^{a^{-1}}\beta y \notin A^{1-[(n-i+1)/2]}$, the map $k \mapsto \psi(\operatorname{Tr}(y^{a^{-1}}\beta - {}^{a^{-1}}\beta y)(1+y)^{-1}k))$ is a non-trivial character of $A^{n-i} \backslash A^{[(n-i+1)/2]}$; thus

$$\sum_{k \in A^{n-i} \backslash A^{[(n-i+1)/2]}} \psi(\operatorname{Tr}(y^{a^{-1}}\beta - {}^{a^{-1}}\beta y)(1+y)^{-1}k) = 0.$$

By Lemma 3.3 in [5], $y^{a^{-1}}\beta - {}^{a^{-1}}\beta y \in A^{1-[(n-i+1)/2]}$ is equivalent to ${}^{a^{-1}}\beta \in E^\times K^{n-i-[(n-i+1)/2]}$. Thus it follows from Lemma 2.1 that

$$\chi_{\eta_\theta}(1+y) = \sum_{\sigma \in \operatorname{Aut}_F E} \sum_{1+a \in H \backslash E^\times K^{[(n-i)/2]}} \rho_\theta(1 + (1+a)^\sigma y(1+a)^{-1}).$$

By virtue of $(1+y)^{-1}(1 + {}^{(1+a)}y) \in K^m$ and $(1+y)^{-1}(1 + {}^{(1+a)}y) \equiv 1 + (1+y)^{-1}((ay - ya) + (ya - ay)a) \bmod K^n$,

$$\rho_\theta(1 + {}^{(1+a)}y) = \theta(1+y)\psi_\beta((1+y)^{-1}(ay - ya))\psi_\beta((1+y)^{-1}(ya - ay)a).$$

Since $\psi_\beta((1+y)^{-1}(ay - ya)) = \psi(\operatorname{Tr}(y\beta(1+y)^{-1} - \beta(1+y)^{-1}y)a) = 1$, $\psi_\beta((1+y)^{-1}(ya - ay)a) = \psi_\beta((ya - ay)a)$ and $|E^\times K^j / E^\times K^m| = q^{(l-1)(m-j)}$, we obtain

$$\chi_{\eta_\theta}(1+y) = \begin{cases} q^{m-(n-i)/2} \displaystyle\sum_{\sigma \in \operatorname{Aut}_F E} \theta(1 + {}^\sigma y), & n-i \text{ even}, \\ q^{m-(n-i+1)/2} \displaystyle\sum_{\sigma \in \operatorname{Aut}_F E} \theta(1 + {}^\sigma y) S(n-i, \sigma), & n-i \text{ odd}, \end{cases}$$

where

$$S(n-i,\sigma) = \sum_{a \in A^{(n-i+1)/2} + E \cap A^{(n-i-1)/2} \setminus A^{(n-i-1)/2}} \psi_\beta(({}^\sigma y a - a^\sigma y)a).$$

Now we may assume $n-i$ odd and $\sigma = 1$. Put $y = \varpi_E^i y_0$, $a = \varpi_E^{(n-i-1)/2} a_0$ and $S = S(n-i,1)$. Since $(ya - ay)a$ equals

$$\varpi_E^{n-1}(y_0 \varpi_E^{-(n-i-1)/2} a_0 \varpi_E^{(n-i-1)/2} - \varpi_E^{-(n+i-1)/2} a_0 \varpi_E^{(n+i-1)/2} y_0) a_0,$$

we have by way of the map $R : A_0/A_1 \to k_F^l$ that

$$S = \sum_{(\alpha_j) \in k_F^l/\Delta} \psi\left(\sum_{j=0}^{l-1} \beta \varpi_E^{n-1} y_0 (\alpha_{j-(n-i-1)/2} - \alpha_{j-(n+i-1)/2}) \alpha_j \right).$$

(The subscript is extended to $\mathbb{Z}$ by $\alpha_j = \alpha_{j \bmod l}$.) First replacing the subscript j by $j + (n+i-1)/2$ and then replacing α_{ij} by α_j, we get $S = G(\beta \varpi_E^{n-1} y_0, c)$. By virtue of $n \not\equiv 1 \bmod l$ and $i \not\equiv 0 \bmod l$, $c = i^{-1}(n+1-i)/2$ satisfies $c \not\equiv 1/2 \bmod l$. □

Remark 2.3. It is proved that the Gauss sum $q^{-(l-1)/2} G(u,j)$ is an eighth root of unity when $p \neq 2$ in [7] and [8]. This is the Gauss sum attached to quadratic form over k_F when $p \neq 2$. But it is not easy to calculate the determinant of the quadratic form. Moreover we need to treat the case $p = 2$ separately.

Next we calculate the character on $K^{n-1} - K^n$. We state the character formula including the case $x \notin E$. The calculation of the character on $K^{n-1} - K^n$ ("at the depth") seems difficult in [8]. Here we find it is written explicitly by the Kloosterman sum.

Definition 2.4. For $a \in k_F^\times$, we define the Kloosterman sum $\mathrm{Kl}(a)$ by

$$\mathrm{Kl}(a) = \sum_{\substack{(y_0, \ldots, y_{l-1}) \in k_F^l \\ y_0 \cdots y_{l-1} = a}} \psi(y_0 + \cdots + y_{l-1}). \tag{2.5}$$

Theorem 2.5. *Let $x = 1 + \varpi_E^{n-1} x_0$ for $x_0 = \mathrm{diag}(k_0, \ldots, k_{l-1})$ $(k_i \in \mathcal{O}_F^\times)$. Then*

$$\chi_{\eta_\theta}(x) = q^{(l-1)(m-1)} \,\mathrm{Kl}\left((\beta_\theta \varpi_E^{n-1})^l \prod_{j=0}^{l} k_j \right).$$

(Since $\beta_\theta \varpi_E^{n-1} \in \mathcal{O}_E^\times$ and $k_E^\times = k_F^\times$, we regard $\beta_\theta \varpi_E^{n-1} \bmod P_E$ as an element of $k_F^\times$.)

Proof. By the definition of η_θ, we have

$$\begin{aligned}&\chi_{\eta_\theta}(1 + \varpi_E^n \,\mathrm{diag}(k_0, \ldots, k_{l-1})) \\ =& q^{(l-1)(m-1)} \sum_{a \in E^\times K^1 \setminus B} \psi(\mathrm{Tr}\, \beta a \varpi_E^n \,\mathrm{diag}(k_0, \ldots, k_{l-1}) a^{-1}).\end{aligned}$$

It follows from (2.2) and (2.3) that the set $\{\mathrm{diag}(1, y_1, \ldots, y_{l-1}) \mid y_i \in k_F^\times\}$ makes a complete system of representatives of $E^\times K^1 \setminus B$. For convenience, put $y_0 = 1$. Since

$$\varpi_E \,\mathrm{diag}(y_0, y_1, \ldots, y_{l-1}) \varpi_E^{-1} = \mathrm{diag}(y_0, \ldots, y_{l-1}, 1),$$

we have

$$\operatorname{Tr}\beta\,\operatorname{diag}(y_0,y_1,\dots,y_{l-1})\varpi_E^{n-1}\operatorname{diag}(k_0,\dots,k_{l-1})\operatorname{diag}(y_0,y_1,\dots,y_{l-1})^{-1} \equiv \beta\varpi_E^{n-1}\sum_{i=0}^{l-1}k_i y_{i-n+1}/y_i \bmod P_F.$$

By replacing y_i by $k_i y_{i-n+1}/y_i$, we get our lemma. □

On K^n, the character of $\pi=\pi_\theta$ becomes a constant function on regular elliptic conjugacy classes.

Lemma 2.6. *Let x be an regular elliptic element in K^n. Then*

$$\chi_\pi(x)=q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}.$$

Proof. We use the Deligne-Kazhdan correspondence ([10], [22]). There exists an irreducible representation π' of $D_l^\times$ such that

$$\chi_\pi=\chi_{\pi'}\quad\text{on the regular elliptic conjugacy classes.}$$

Since the correspondence preserves the conductoral exponents, π' is trivial on $1+P_D^{f(\theta)}$. Thus χ_π is also constant on K^n. This constant is expressed by the local character expansion and equals $q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}$ (see for example [5], 7.4). □

The character formula on regular elliptic conjugacy classes outside $E^\times$ can be easily obtained.

Lemma 2.7. *Let x be an regular elliptic element of B. If x satisfies the condition that $F(x)\not\simeq E$ and x is not conjugate to an element of $F^\times K^n$, then $\chi_\pi(x)=0$.*

Proof. See Lemma 3.3 in [16]. □

3. Calculation of Gauss sums

In this section, we determine the Gauss sum part $G(y,n-i)$ explicitly. Since $G(y,n-i)$ depends only on $n-i \bmod l$ and $y \bmod P_E$, we have only to treat the character of η_θ on U_1^* by making n big enough. We have only to treat the case that n is even by replacing n by $n+l$ if necessary.

Lemma 3.1. *Assume $n=2m$. Then for $x\in U_1^*$,*

$$\chi_{\eta_\theta}(x)=\sum_{\sigma\in\operatorname{Aut}_F E}\ \sum_{a\in H\backslash E^\times K^{m-1}}\rho_\theta(a^\sigma x a^{-1}). \tag{3.1}$$

Proof. It follows from Lemma 2.1 that $axa^{-1}\in H$ implies $a\in E^\times K^{m-1}$. Hence our lemma. □

For the calculation of the sum in the above lemma, we use the $E^\times$-module structure of various objects. When E/F is a Galois extension, it is easy to treat. Thus we first assume E/F is Galois, i.e., $q\equiv 1 \bmod l$. We recall that ξ is the diagonal matrix $\operatorname{diag}(1,\zeta^{l-1},\zeta^{l-2},\dots,\zeta)$, where ζ is an l-th root of unity in F and

ξ satisfies $\xi^l = 1$ and $\xi x \xi^{-1} = {}^\sigma x$ for $x \in E$, where σ is the generator of $\mathrm{Gal}(E/F)$ determined by ${}^\sigma\varpi_E = \varpi_E \zeta$. By the explicit matrix form of E and A_i, we obtain

$$\begin{array}{llllllll} \mathrm{M}_l(F) & = E & \oplus & E\xi & \oplus & \cdots & E\xi^{l-1} \\ A^0 & = \mathcal{O}_E & \oplus & \mathcal{O}_E\xi & \oplus & \cdots & \mathcal{O}_E\xi^{l-1} \\ A^1 & = P_E & \oplus & P_E\xi & \oplus & \cdots & P_E\xi^{l-1} \\ \cdots & & & & & & \\ A^{l-1} & = P_E^{l-1} & \oplus & P_E^{l-1}\xi & \oplus & \cdots & P_E^{l-1}\xi^{l-1}. \end{array} \tag{3.2}$$

Lemma 3.2. *A complete system of representatives of $H\backslash E^\times K^{m-1}$ is given by*

$$\{1 + \varpi_E^{m-1}\alpha_1\xi + \cdots + \varpi_E^{m-1}\alpha_{l-1}\xi^{l-1} \mid \alpha_i \in k_F\}.$$

Proof. This is obvious from (3.2). □

For $a = 1 + \alpha_1\xi + \cdots + \alpha_{l-1}\xi^{l-1} \in A^{m-1}$, $\rho(axa^{-1})$ for $x \in U_1^*$ can be expressed explicitly in terms of $\alpha_1, \ldots, \alpha_{l-1}$. First, we determine the coefficients of a^{-1} with respect to the E-basis $\{1, \xi, \ldots, \xi^{l-1}\}$.

Lemma 3.3. *For $a = \sum_{j=0}^{l-1} \alpha_j \xi^j$ $(\alpha_j \in E)$, define $\Lambda(a) \in \mathrm{M}_l(E)$ by*

$$\Lambda(a) = ({}^{\sigma^j}\alpha_{i-j \bmod l})_{0 \le i,j \le l-1}$$

and let $\Lambda_k(a)$ be the $(1, k+1)$-cofactor of $\Lambda(a)$. Then

$$a^{-1} = \sum_{j=0}^{l-1} \frac{\Lambda_j(a)}{\det \Lambda(a)} \xi^j.$$

Proof. Our lemma follows from Cramer's formula. □

Lemma 3.4. *Assume $n = 2m$ and $3(m-1) \ge 2m$. Let $c \in F^\times, y \in P_E^{m-1}$ and $a = 1 + \sum_{j=1}^{l-1} \alpha_j \xi^j \in K^{m-1}$. Then*

$$\rho_\theta(ac(1+y)a^{-1}) = \theta(c(1+y))\psi_E\left(\sum_{j=1}^{l-1} (\beta\alpha_j {}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\beta\alpha_{l-j} {}^{\sigma^{-j}}\alpha_j)y\right).$$

Proof. It is obvious that we may assume $c = 1$. Since

$$\begin{aligned} g^{-1}aga^{-1} &= 1 + (g^{-1}(a-1)g - (a-1))a^{-1} \\ &= 1 + \left(\sum_{j=1}^{l-1} ({}^{\sigma^j}gg^{-1} - 1)\alpha_j\xi^j\right) a^{-1}, \end{aligned}$$

$\sum_{j=1}^{l-1} ({}^{\sigma^j}gg^{-1} - 1)\alpha_j\xi^j \in A^m$ and $\mathrm{Tr}(\beta x \xi^j) = 0$ for all $x \in E$, we have

$$\begin{aligned} \rho_\theta(g^{-1}aga^{-1}) &= \psi_\beta\left((\sum_{j=1}^{l-1} ({}^{\sigma^j}gg^{-1} - 1)\alpha_j\xi^j)a^{-1}\right) \\ &= \psi_\beta\left(\sum_{j=1}^{l-1} ({}^{\sigma^j}gg^{-1} - 1)\alpha_j {}^{\sigma^j}(f_{l-j}(a))\right), \end{aligned}$$

where $f_j(a) \in E$ is defined by $a^{-1} = \sum_{j=0}^{l-1} f_j(a)\xi^j$. Put $g = 1 + y$. In the last equation, $\beta \in P_E^{1-n}$, $f_{l-j} \in P_E^{m-1}$ and ${}^{\sigma^j}gg^{-1} - 1 \equiv {}^{\sigma^j}y - y \bmod P_E^{2m-2}$. Thus we get

$$\rho_\theta(g^{-1}aga^{-1}) = \psi_E\left(\sum_{j=1}^{l-1}({}^{\sigma^{-j}}\beta f_{l-j}(a)^{\sigma^{-j}}\alpha_j - \beta^{\sigma^j}(f_{l-j}(a))\alpha_j)y\right)$$

by virtue of $\mathrm{tr}_E\, u^{\sigma^j}v = \mathrm{tr}_E\, {}^{\sigma^{-j}}uv$ for any $u, v \in E$. It follows from Lemma 3.3 that

$$f_{l-j}(a) = \frac{\Lambda_{l-j}(a)}{\det \Lambda(a)} \equiv \alpha_{l-j} \bmod P_E^{2m-2}.$$

By the assumption $3m - 3 \geq 2m$, we obtain the desired formula. □

Proposition 3.5. *Assume $q \equiv 1 \bmod l$.*

(1) *If $n = 2m$ and $m \geq 3$,*

$$\chi_{\eta_\theta}(x) = q^{(l-1)/2}\sum_{j=0}^{l-1}\theta({}^{\sigma^j}x) \quad \textit{for} \quad x \in U_1^*.$$

(2) *For any integer $j \not\equiv 1/2 \bmod l$ and $y \in \mathcal{O}_F^\times$, $G(y, j) = q^{(l-1)/2}$.*

Proof. By Lemmas 3.1, 3.2 and 3.4, we have for $c \in F^\times$ and $y \in 1 + P_E$

$$\chi_{\eta_\theta}(c(1+y)) = \sum_{i=0}^{l-1}\theta(c(1 + {}^{\sigma^i}y)) \sum_{(\alpha_1,\ldots,\alpha_{l-1}) \in (P_E^{m-1}/P_E^m)^{l-1}} f(\alpha_1, \ldots, \alpha_{l-1}; {}^{\sigma^i}y),$$

where

$$f(\alpha_1, \ldots, \alpha_{l-1}; y) = \psi_E\left(\sum_{j=1}^{l-1}(\beta\alpha_j{}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\beta\alpha_{l-j}{}^{\sigma^{-j}}\alpha_j)y\right).$$

Put $S_j = \{(\alpha_1, \ldots, \alpha_{l-1}) \in (P_E^{m-1}/P_E^m)^{l-1} \mid \alpha_k = 0 \text{ for } k < j, \alpha_j \neq 0\}$ and $I_j(y) = \sum_{(\alpha_1,\ldots,\alpha_{l-1}) \in S_j} f(\alpha_1, \ldots, \alpha_{l-1}; y)$. Then

$$\chi_{\eta_\theta}(c(1+y)) = \sum_{i=0}^{l-1}\theta(c(1 + {}^{\sigma^i}y))\sum_{j=1}^{l-1} I_j({}^{\sigma^i}y).$$

If $\alpha_1 = \cdots = \alpha_{(l-1)/2} = 0$, $f(\alpha_1, \ldots, \alpha_{l-1}; y) = 0$. Thus we have

$$\sum_{j=(l+1)/2}^{l-1} I_j(y) = q^{(l-1)/2}.$$

For $1 \leq j \leq (l-1)/2$, $I_j(y)$ is proportional to

$$\sum_{\alpha_{l-j} \in P_E^m/P_E^{m+1}} \psi_E((\beta\alpha_j{}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\beta\alpha_{l-j}{}^{\sigma^{-j}}\alpha_j)y).$$

Since $\alpha_j \neq 0$, the map $\alpha_{l-j} \mapsto \beta\alpha_j{}^{\sigma^j}\alpha_{l-j} - {}^{\sigma^{-j}}\beta\alpha_{l-j}{}^{\sigma^{-j}}\alpha_j$ is a bijection from P_E^{m-1}/P_E^m to k_F. Therefore $I_j(y) = 0$. Consequently we get the first part of our lemma. $G(y, n-1) = q^{(l-1)/2}$ follows from Proposition 2.2 and the first part. Since $G(y, j)$ depends only on $j \bmod l$ an $y \in k_F^\times$, $G(y, j) = q^{(l-1)/2}$ for any j. □

Next we assume $q-1 \not\equiv 0 \bmod l$ and $n = 2m$. In this situation, it is rather difficult to describe $E^\times$-module structure of various objects since F has no l-th primitive root of unity and E/F is not Galois. In order to apply the result of the Galois case, we use the base change lift of simple characters by Bushnell-Henniart [4]. Let ζ be a primitive l-th root of unity and $L = F(\zeta)$. Then L/F is an unramified extension of degree d, where d is the smallest integer satisfying $q^d \equiv 1 \bmod l$. The generator τ of $\mathrm{Gal}(L/F)$ is determined by ${}^\tau\zeta = \zeta^k$, where $k = r^{(l-1)/d}$and r is a generator of $(\mathbb{Z}/l\mathbb{Z})^\times$. We add the subscript L to the base changed objects. Then $\mathrm{M}_l(L) = \mathrm{M}_l(F) \otimes_F L$ and $E_L = E \otimes_F L \simeq EL$. E_L is a ramified Galois extension over L of degree l, an unramified extension over E of degree d with $\mathrm{Gal}(E_L/E) = \mathrm{Gal}(L/F) = \langle\tau\rangle$ and a non-Abelian Galois extension over F of degree ld. (We embed E into E_L by the map $x \mapsto x \otimes 1$.)

As in the previous section, we identify $\mathrm{M}_l(L)$ with $\mathrm{End}_L E_L$ and $G_L = \mathrm{GL}_l(L)$ with $\mathrm{Aut}_L E_L$ by the L-basis $\{\varpi_E^{l-1}, \ldots, \varpi_E, 1\}$ of E_L, which is also an $\mathcal{O}_L$-basis of $\mathcal{O}_{E_L}$. By the lattice flag $\{P_{E_L}^i\}_{i\in\mathbb{Z}}$, we define

$$A_L^i = \{f \in \mathrm{M}_l(L) | f(P_{E_L}^j) \subset P_{E_L}^{j+i} \quad \text{for all } \ j \in \mathbb{Z}\}.$$

Put $K_L = (A_L^0)^\times, B_L = E_L^\times K_L$, $K_L^i = 1 + A_L^i$ for $i \geq 1$ and $H_L = L^\times(1+P_{E_L})K_L^m$.

Definition 3.6. Let θ be a generic character of $E^\times$ with $f(\theta) = n$ and $\theta(1+x) = \psi(\mathrm{tr}_E(\beta x))$ for $x \in P_E^m$. We define a base change lift θ_L of θ to $L^\times$ by $\theta_L = \theta \circ n_{E_L/E}$. Then $\theta_L(1+x) = \psi_L(\mathrm{tr}_{E_L/L}\, \beta x)$ for $x \in P_{E_L}^m$. (Recall $m = [(n+1)/2]$.) Let $H^1 = F^\times(1+P_E)K^m \subset H$. The base change lift ρ_L of $\rho|_{H^1}$ to $H_L^1 = L^\times(1+P_{E_L})K_L^m$ is defined by

$$\rho_L(h \cdot g) = \theta_L(h)\psi_L(\mathrm{Tr}\, \beta(g-1)) \qquad \text{for} \quad h \in L^\times(1+P_{E_L}), \quad g \in K_L^m.$$

Now we apply the result of Bushnell-Henniart [4] to our case and get the character relation between ρ_L and ρ. Put $U_{E_L,i} = L^\times(1+P_{E_L}^i)$ for $i > 0$ and $U_{E_L,i}^* = U_{E_L,i} - U_{E_L,i+1}$. By (12.19) Corollary in [4] and the fact $\langle\tau\rangle$-fixed space ${}^{\langle\tau\rangle}(L^\times K_L^i)$ is equal to $F^\times K^i$, we get the following result.

Proposition 3.7. *Let $x \in U_{E_L,1}$. Between the set*

$$\{g \in H^1 \backslash (E^\times K^{m-1})^1 \mid g n_{E_L/E}(x) g^{-1} \in H^1\}$$

and the set

$$\{h \in H_L^1 \backslash (E_L^\times K_L^{m-1})^1 \mid h x^\tau h^{-1} \in H_L^1\},$$

there is a bijection ψ with the property

$$\rho_L(\psi(g) x^\tau (\psi(g))^{-1}) = \rho(g n_{E_L/E}(x) g^{-1}).$$

Combining this with Lemma 3.1, we have:

Lemma 3.8. *For $x \in U_{E_L,1}$,*

$$\chi_{\eta_\theta}(n_{E_L/E}(x)) = \sum_{\substack{a \in H_L^1 \backslash (E_L^\times K_L^{m-1})^1 \\ a x^\tau a^{-1} \in H_L}} \rho_L(a x^\tau a^{-1}). \tag{3.3}$$

Since $n_{E_L/E}(L^\times(1+P_{E_L}^i)) = F^\times(1+P_E^i)$, it suffices to calculate the right-hand side of (3.3) for $x \in U_{E_L,1}^*$.

As in the Galois case, set $\xi = \mathrm{diag}(1, \zeta^{l-1}, \zeta^{l-2}, \ldots, \zeta) \in \mathrm{M}_l(L)$. Then ξ satisfies $\xi^l = 1$, ${}^\tau\xi = \xi^k$ and

$$\xi x \xi^{-1} = {}^\sigma x \quad \text{for any} \quad x \in E_L,$$

where σ is the generator of $\mathrm{Gal}(E_L/L)$ determined by ${}^\sigma\varpi_E = \varpi_E\zeta$. Moreover we have $\tau\sigma\tau^{-1} = \sigma^k$ and

$$\begin{array}{lllllllll} \mathrm{M}_l(L) & = E_L & \oplus & E_L\xi & \oplus & \cdots & E_L\xi^{l-1} \\ A_L^0 & = \mathcal{O}_{E_L} & \oplus & \mathcal{O}_{E_L}\xi & \oplus & \cdots & \mathcal{O}_{E_L}\xi^{l-1} \\ A_L^1 & = P_{E_L} & \oplus & P_{E_L}\xi & \oplus & \cdots & P_{E_L}\xi^{l-1} \\ \cdots & & & & & & \\ A_L^{l-1} & = P_{E_L}^{l-1} & \oplus & P_{E_L}^{l-1}\xi & \oplus & \cdots & P_{E_L}^{l-1}\xi^{l-1}. \end{array} \tag{3.4}$$

We note that any element of K_L^1 can be written in the form $(1 + \alpha_1\xi + \alpha_2\xi^2 + \cdots + \alpha_{l-1}\xi^{l-1})$ for $\alpha_i \in P_{E_L}$.

Lemma 3.9. *Let $i < m$ and $a = 1 + \alpha_1\xi + \alpha_2\xi^2 + \cdots + \alpha_{l-1}\xi^{l-1}$ for $\alpha_j \in \mathcal{O}_E$ and $x \in U^*_{E_L,i}$. Then $ax^\tau a^{-1} \in H_L$ is equivalent to $\alpha_j \in P_{E_L}^{m-i}$ and $\alpha_{hk^j} = {}^{\tau^j}\alpha_h$ for $j = 0, 1, \ldots, d-1$ and $h = 1, r, \ldots, r^{(l-1)/d-1}$. (The subscript of α_j is extended to $\mathbb{Z}$ by $\alpha_j = \alpha_{j \bmod l}$.)*

Proof. It follows from Lemma 3.2 that if $a^{-1}x^\tau a \in H_L$, there exist $\gamma_0 \in \mathcal{O}_E^\times$ and $\gamma_j \in P_{E_L}^m$ for $1 \le j \le l-1$ such that

$$\begin{aligned}(1 + \alpha_1\xi + \cdots + \alpha_{l-1}\xi^{l-1})x = \gamma_0(1 &+ \gamma_1\xi + \gamma_2\xi^2 + \cdots + \gamma_{l-1}\xi^{l-1}) \\ &(1 + {}^\tau\alpha_1\xi^k + {}^\tau\alpha_2\xi^{2k} + \cdots + {}^\tau\alpha_{l-1}\xi^{(l-1)k}).\end{aligned}$$

This implies

$$\begin{aligned} x &= \gamma_0(1 + \gamma_{l-k}{}^{\sigma^{l-k}\tau}\alpha_1 + \gamma_{l-2k}{}^{\sigma^{l-2k}\tau}\alpha_2 + \cdots + \gamma_k{}^{\sigma^k\tau}\alpha_{l-1}) \\ \alpha_k{}^{\sigma^k}x &= \gamma_0(\gamma_k + {}^\tau\alpha_1 + \gamma_{l-k}{}^{\sigma^{l-k}\tau}\alpha_2 + \cdots + \gamma_{2k}{}^{\sigma^{2k}\tau}\alpha_{l-1}) \\ &\cdots\cdots \\ \alpha_{l-k}{}^{\sigma^{l-k}}x &= \gamma_0(\gamma_{l-k} + \gamma_{l-2k}{}^{\sigma^{l-2k}\tau}\alpha_1 + \cdots + {}^\tau\alpha_{l-1}). \end{aligned}$$

Thus we have

$$\alpha_{hk}{}^{\sigma^{hk}}x = x^\tau\alpha_h \bmod P_{E_L}^m \quad (h \in (\mathbb{Z}/l\mathbb{Z})^\times).$$

By eliminating $\alpha_{hk}, \alpha_{hk^2}, \ldots, \alpha_{hk^{d-1}}$, we get

$$\alpha_h = n_{E_L/E}(x)^{\sigma^k} n_{E_L/E}(x)^{-1}\alpha_h \bmod P_{E_L}^m.$$

Since $n_{E_L}(x)^{\sigma^k} n_{E_L/E}(x)^{-1} \in 1 + P_E^i - P_E^{i+1}$, $\alpha_k \in P_{E_L/E}^{m-i}$. By $x^{\sigma^k}x^{-1} \in 1 + P_{E_L}^i$, we obtain $\alpha_h \in P_{E_L}^{m-i}$ and $\alpha_{hk^j} = {}^{\tau^j}\alpha_h \bmod P_{E_L}^m$ for $j = 0, 1, \ldots, d-1$ and $h = 1, r, \ldots, r^{(l-1)/d-1}$. $\square$

Lemma 3.10. *Assume $n = 2m$ and $m \ge 3$. Let $x \in 1 + P_{E_L} - P_{E_L}^2$ and $a = 1 + \sum_{i=1}^{(l-1)/d}\sum_{j=1}^{d} {}^{\tau^j}\alpha_{r^i}\xi^{r^ik^j}$ for $\alpha_{r^i} \in P_{E_L}^{m-1}$. Then*

$$\rho_L(ax^\tau a^{-1}x^{-1}) = \psi_E\left(\sum_{i=1}^{(l-1)/d} \mathrm{tr}_{E_L/E}(u_i - {}^{\sigma^{-r^i}}u_i)\,\mathrm{tr}_{E_L/E}(x-1)\right), \tag{3.5}$$

where $u_i = \gamma\alpha_{r^i}{}^{\sigma^{r^i}}\alpha_{-r^i}$.

Proof. It follows from Lemma 3.9 that ${}^{\tau}aa^{-1} \in H_L$. Since $\rho_L({}^{\tau}aa^{-1}) = 1$, we have

$$\rho_L(ax^{\tau}a^{-1}g^{-1}) = \rho_L(axa^{-1}g^{-1}).$$

By the same way as Lemma 3.4, we have

$$\rho_L(ax^{\tau}a^{-1}x^{-1}) = \psi_{E_L}\left(\sum_{i=1}^{(l-1)/d}\sum_{j=1}^{d}(v_{i,j} - {}^{\sigma^{-r^i k^j}}v_{i,j})(x-1)\right),$$

where $v_{i,j} = \beta^{\tau^j}\alpha_{r^i}{}^{\tau^j\sigma^{r^i}}\alpha_{-r^i}$. Since $\sigma^{-r^i k^j}\tau^j = \tau^j\sigma^{-r^i}$ and ${}^{\tau}\beta = \beta$, we have

$$\begin{aligned}\sum_{j=1}^{d}(v_{i,j} - {}^{\sigma^{-r^i k^j}}v_{i,j}) &= \sum_{j=1}^{d}(\beta^{\tau^j}\alpha_{r^i}{}^{\tau^j\sigma^{r^i}}\alpha_{-r^i} - {}^{\tau^j\sigma^{-r^i}}\beta^{\tau^j\sigma^{-r^i}}\alpha_{r^i}{}^{\tau^j}\alpha_{-r^i})\\ &= \mathrm{tr}_{E_L/E}(\beta\alpha_{r^i}{}^{\sigma^{r^i}}\alpha_{-r^i} - {}^{\sigma^{-r^i}}\beta^{\sigma^{-r^i}}\alpha_{r^i}\alpha_{-r^i}).\end{aligned}$$

This implies (3.5). □

It is time to get the character value of χ_η on U_1^*.

Proposition 3.11. *Assume $q \not\equiv 1 \bmod l$.*

(1) *Let $x \in 1 + P_{E_L} - P_{E_L}^2$ and $n = 2m > 6$. Then*

$$\chi_\eta(n_{E_L/E}(x)) = \left(\frac{q}{l}\right) q^{(l-1)/2}\theta(n_{E_L/E}(x)).$$

(2) *For any integer $j \not\equiv 1/2 \bmod l$ and $y \in \mathcal{O}_F^\times$, $G(y,j) = \left(\frac{q}{l}\right) q^{(l-1)/2}$.*

Proof. By Proposition 3.7, Lemmas 3.8, 3.9, 3.10 and 3.11, we have

(3.6)

$$\chi_\eta(n_{E_L/E}(x)) = \theta_L(x)\sum_{(\alpha_{r^i})}\psi_E\left(\sum_{i=1}^{(l-1)/d}\mathrm{tr}_{E_L/E}(u_i - {}^{\sigma^{-r^i}}u_i)\,\mathrm{tr}_{E_L/E}(x-1)\right),$$

where $u_i = \beta\alpha_{r^i}{}^{\sigma^{r^i}}\alpha_{-r^i}$ and $(\alpha_{r^i})_{1\le i\le(l-1)/d} \in (P_{E_L}^{m-1}/P_{E_L}^{m})^{(l-1)/d}$. First we assume $(l-1)/d$ is odd. Then $\left(\frac{q}{l}\right) = -1$, d is even and ${}^{\tau^{d/2}}\alpha_{r^i} = \alpha_{-r^i}$. Let E_i be the $\langle\sigma^{r^i}\tau^{d/2}\rangle$-fixed field. Then E_L/E_i is a quadratic unramified extension, $\alpha_{r^i}{}^{\sigma^{r^i}\tau^{d/2}}\alpha_{r^i} = n_{E_L/E_i}(\alpha_{r^i})$, n_{E_L/E_i} induces a surjection from $\varpi_E^{m-1}\mathcal{O}_{E_L}/1 + P_{E_L}$ to $\varpi_E^{2m-2}\mathcal{O}_{E_i}/1 + P_{E_i}$ and each fiber of the induced map has $q^{d/2} + 1$ elements. Moreover the map $x \mapsto \mathrm{tr}_{E_L/E_i}(x - {}^{\sigma^{-r^i}}x)$ induces a surjective k_F-linear map from $P_{E_i}^{2m-2}/P_{E_i}^{2m-1}$ to $P_E^{-1}/\mathcal{O}_E$. Thus we have

$$\sum_{\alpha_{r^i}\in P_{E_L}^{m-1}/P_{E_L}^{m}}\psi_E\left(\sum_{i=1}^{(l-1)/d}\mathrm{tr}_{E_L/E}(u_i - {}^{\sigma^{-r^i}}u_i)\,\mathrm{tr}_{E_L/E}(x-1)\right) = 1 - (q^{d/2}+1).$$

Putting this into (3.6), we get

$$\begin{aligned}\chi_\eta(n_{E_L/E}(x)) &= \theta_L(x)(1-(q^{d/2}+1))^{(l-1)/d}\\ &= -q^{(l-1)/2}\theta_L(x),\end{aligned}$$

and it follows from Proposition 2.2 that $G(y,j) = -q^{(l-1)/2}$ for all $y \in k_F$ and j odd. Now we assume $(l-1)/d$ is even. Then $\left(\frac{q}{l}\right) = 1$ and it follows from the same argument as in the proof of Proposition 3.5 that

$$\begin{aligned}\chi_\eta(n_{E_L/E}(x)) &= \theta_L(x)|k_{E_L}|^{(l-1)/2d} \\ &= q^{(l-1)/2}\theta_L(x).\end{aligned}$$

By Proposition 2.2 and the fact $G(y,j)$ depends only on j mod l, we have $G(y,j) = q^{(l-1)/2}$ for all $y \in k_F$ and j. □

Summing up the above results, we get the following formula for $G(y,j)$.

Proposition 3.12. *For $y \in k_F^\times$ and $j \in (\mathbb{Z}/l\mathbb{Z})$ such that $j \not\equiv 1/2$,*

$$G(y,j) = \left(\frac{q}{l}\right) q^{(l-1)/2}.$$

From Theorems 1.7, 2.5, Lemmas 2.6, 2.7, and Propositions 1.6, 2.2 and 3.12, we get the complete character table of π_θ.

Theorem 3.13. *Let E be a ramified extension of F with degree $l \neq p$, θ a generic quasi-character of $E^\times$ with $f(\theta) = n$ and $\pi = \pi_\theta$ the irreducible supercuspidal representation of $\mathrm{GL}_l(F)$ defined in section 1. Put $U_0 = F^\times \mathcal{O}_E^\times$, $U_j = F^\times(1 + P_E^j)$ and $U_j^* = U_j - U_{j+1}$ for $j \geq 1$. Let x be an regular elliptic element of $\mathrm{GL}_l(F)$ and let $\mathrm{Aut}_F E$ be the group of automorphism of E over F.*

1. *If $F(x)|F$ is unramified, then*

x	$\chi_\pi(x)$
$x \notin F^\times(1 + P_{F(x)}^n)$	0
$c(1+y)(c \in F^\times, y \in P_{F(x)}^n)$	$q^{(n-2)(l-1)/2}\frac{q^l-1}{q-1}\theta(c).$

2. *If $F(x)/F$ is ramified and $F(x) \not\simeq E$, then*

x	$\chi_\pi(x)$
$x \notin F^\times(1 + P_{F(x)}^n)$	0
$c(1 + \varpi_E^{n-1}\,\mathrm{diag}(k_0, \dots, k_{l-1}) + z)$ $(c \in F^\times, k_i \in k_F^\times, z \in P_{F(x)}^n)$	$q^{(n-2)(l-1)/2}\theta(c)\,\mathrm{Kl}((\beta\varpi_E^{n-1})^l \prod_{j=0}^{l-1} k_j)$
$c(1+y) \quad (c \in F^\times, y \in P_{F(x)}^n)$	$q^{(n-2)(l-1)/2}\frac{(q^l-1)}{q-1}\theta(c).$

3. *When $x \in E$, then*

x	$\chi_{\pi_\theta}(x)$
$x \in U_j^* \quad (0 \leq j \leq n-1)$	$\left(\frac{q}{l}\right)^{n-j} q^{j(l-1)/2} \sum_{\sigma \in \mathrm{Aut}_F E} \theta({}^\sigma x)$
$c(1 + \varpi_E^{n-1}x_0) \quad (c \in F^\times, x_0 \in \mathcal{O}_E^\times)$	$q^{(n-2)(l-1)/2}\frac{q^l-1}{q-1}\theta(c)\,\mathrm{Kl}(\beta\varpi_E^{n-1}x_0)$
$c(1+y) \quad (c \in F^\times, y \in P_E^n)$	$q^{(n-2)(l-1)/2}\frac{q^l-1}{q-1}.$

(See (2.5) *for the definition of the Kloosterman sum* $\mathrm{Kl}(a)$*.)*

Remark 3.14. 1. Combining with the formula for the unramified case [23] (see Remark 1.5), we get the complete character table for all supercuspidal representations of $\mathrm{GL}_l(F)$ on regular elliptic conjugacy classes when $p \neq l$.

2. For the case $p > l$, Debacker got the following character table for the supercuspidal representations of $\mathrm{GL}_l(F)$ ([9], Lemma 15):

$$\Theta_\pi(\gamma) = \begin{cases} C_0(t)\lambda(\sigma) \sum_{\omega\in\mathcal{W}} \phi({}^w t) & \text{if } n(\gamma)=0 \text{ and } \gamma \sim t \text{ with } t \in G', \\ C_0(t)\lambda(\sigma) \sum_{w\in\mathcal{W}} \phi({}^w t)\gamma_{(X_\pi, {}^\omega Y)} & \text{if } 0 < n(\gamma) < r \text{ and } \gamma \sim t, \text{ where} \\ & \quad t = z(1+Y) \text{ with } z \in Z \text{ and} \\ & \quad Y \in \mathfrak{g}'_{n(\gamma)} \setminus (\mathfrak{z}_{n(\gamma)} + \mathfrak{g}'_{n(\gamma)^+}), \\ C(t) \sum_{k\in G_x/M_0 G_{x,0+}} \phi({}^k t) & \text{if } n(\gamma) = r \text{ and } \gamma \sim t, \text{ where} \\ & \quad t = z(1+T) \text{ with } z \in Z \text{ and} \\ & \quad T \in \mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r^+} \text{ is } x\text{-generic} \\ \text{the local character} & n(\gamma) > r, \\ \text{expansion } (\mathrm{mod}\, Z) & \\ 0 & \text{otherwise.} \end{cases}$$

(For the notations, see Appendix B.6.6 in [9].) In this paper, we determined $\gamma_{(X_\pi, {}^\omega Y)}$ and

$$\sum_{k\in G_x/M_0 G_{x,0+}} \phi({}^k t)$$

explicitly.

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