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REPRESENTATION DIMENSION: AN INVARIANT UNDER STABLE EQUIVALENCE

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Dedicated To Daidai Cha

Abstract. In this paper, we prove that the representation dimension is an invariant under stable equivalence.

1. Introduction

Throughout this paper, all algebras are basic artin algebras, all modules are finitely generated left modules, and compositions are from right to left. For an artin algebra Λ, $mod(\Lambda)$ denotes the category of finitely generated Λ modules. Given two Λ modules X, Y, we write ${}_{\Lambda}(X, Y)$, and sometimes (X, Y) if there is no confusion, for $Hom_{\Lambda}(X, Y)$, and we denote by $add(X)$ the full subcategory of $mod(\Lambda)$ consisting of all (finite) sums of summands of X. Call X a generator-cogenerator of Λ, if $add(X)$ contains all projective and all injective Λ modules.

Auslander introduced the concept of representation dimension of an artin algebra to measure how far it is from being of representation finite type. Most methods on it were developed by Auslander [1]. It is important to investigate the properties of representation dimension under various equivalence relations such as stable equivalence, derived equivalence, and so on. Auslander [6] proved that an algebra which is stable equivalent to a hereditary algebra has representation dimension no larger than 3. In particular, representation dimension is left invariant under the stable equivalence in this case. In this paper, we proved that the representation dimension is an invariant under any stable equivalence.

Theorem 1.1. *Let Λ and $\Lambda^{'}$ be two stably equivalent artin algebras. Then*

$$rep.dim(\Lambda) = rep.dim(\Lambda^{'}).$$

2. Definitions and preliminaries

In this section, we give a lemma on the calculation of the representation dimension and some results on stable equivalence. First we recall the definition of coherent functors. Assume $\mathcal{A}$ is an additive category, a functor from $\mathcal{A}^{op}$ to the

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category of abelian groups is called coherent if there is a morphism $f : A \to B$, for $A, B \in \mathcal{A}$, which induces an exact sequence of functors:

$$_{\mathcal{A}}(-, A) \to_{\mathcal{A}} (-, B) \to F \to 0,$$

that is,

$$_{\mathcal{A}}(X, A) \to_{\mathcal{A}} (X, B) \to F(X) \to 0$$

is exact for any X in $\mathcal{A}$.

Now suppose that Λ is an artin algebra. For an arbitrary Λ module M, $\widehat{add(M)}$ denotes the category of all coherent functors from $(add(M))^{op}$ to the category of abelian groups. It is well known that $\widehat{add(M)}$ is an abelian category. All functors of the form $_{\Lambda}(-, C), C \in mod(\Lambda)$, are in $\widehat{add(M)}$ and the projective objects of $\widehat{add(M)}$ are precisely those functors $_{\Lambda}(-, A), A \in add(M)$ (see [1]).

Then we can describe the representation dimension of an artin algebra Λ as follows:

$$rep.dim(\Lambda) = min\{gl.dim(\widehat{add(M)})|M \text{ is a generator-cogenerator of } \Lambda\}.$$

For an arbitrary Λ module M, we define $sd(M)$ as follows:

$$sd(M) = sup\{pd(-, C) \text{ in } \widehat{add(M)}|C \in mod(\Lambda)\}.$$

Lemma 2.1. *Let Λ be an artin algebra which is not semi-simple. If M is a generator-cogenerator of Λ, we have $gl.dim(\widehat{add(M)}) = sd(M) + 2$. In particular,*

$$rep.dim(\Lambda) = inf\{sd(M)|M \textit{ is a generator-cogenerator of } \Lambda\} + 2.$$

Proof. Suppose M is a generator-cogenerator of Λ.

If $add(M) = mod(\Lambda)$, then $sd(M) = 0$ and $gl.dim(\widehat{add(M)}) = 2$, for Λ is of representation finite type in this case (see [1]).

Now we suppose $add(M) \neq mod(\Lambda)$.

For an arbitrary $F \in \widehat{add(M)}$, we have a Λ morphism $f : M_1 \to M_0$, which induced the following exact sequence in $\widehat{add(M)}$:

$$0 \to (-, ker(f)) \to (-, M_1) \to (-, M_0) \to F \to 0.$$

Then we have $pd(F) \leq pd(-, ker(f)) + 2$. Then $gl.dim(\widehat{add(M)}) \leq sd(M) + 2$.

Conversely suppose C is a Λ module not in $add(M)$, we have the minimal injective copresentation of C in $mod(\Lambda)$:

$$0 \to C \to I_0 \to I_1,$$

which induced an exact sequence in $\widehat{add(M)}$:

$$0 \to (-, C) \to (-, I_0) \to (-, I_1) \to G \to 0.$$

Then $pd((-, C)) + 2 = pd(G)$. Thus we have $sd(M) + 2 \leq gl.dim(\widehat{add(M)})$.

Then $gl.dim(\widehat{add(M)}) = sd(M)+2$, and the second conclusion follows easily. $\square$

Now we give some notation and results about the stable equivalence of artin algebras (see [3], [4], [5]). Let Λ be an artin algebra, then we denote dy $\underline{mod}(\Lambda)$ the stable module category of Λ. The objects of $\underline{mod}(\Lambda)$ are the same as those of $mod(\Lambda)$, which we write as $\underline{C}$, for $C \in mod(\Lambda)$; morphism sets in $\underline{mod}(\Lambda)$ is

${}_{\Lambda}\underline{(X,Y)} ={}_{\Lambda} (X,Y)/P(X,Y)$, where $P(X,Y)$ is the subgroup of ${}_{\Lambda}(X,Y)$ consisting of morphisms which factor through projectives. We denote by $mod_P(\Lambda)$ the full subcategory of $mod(\Lambda)$ consisting of all modules without projective summands, and by $mod_I(\Lambda)$ the full subcategory of $mod(\Lambda)$ consisting of all modules without injective summands. $\underline{mod}(mod(\Lambda))$ is the category of all (finitely presented) coherent functors from $mod(\Lambda)^{op}$ to the abelian group category which vanish on projective modules. The projective objects of $\underline{mod}(mod(\Lambda))$ are precisely those functors $(-,\underline{C}), C \in mod_P(\Lambda)$, where $(-,\underline{C})(X) = \underline{(X,C)}$. The injective objects are precisely those functors $Ext^1(-,C), C \in mod_I(\Lambda)$. The functor $mod_P(\Lambda) \to \underline{mod}(mod(\Lambda))$ defined by $C \mapsto (-,\underline{C})$ is a natural embedding. If $\alpha : \underline{mod}(\Lambda) \to \underline{mod}(\Lambda^{'})$ is an equivalence, then there is an induced equivalence $\underline{mod}(mod(\Lambda)) \to \underline{mod}(mod(\Lambda^{'}))$, and an induced correspondence between $mod_P(\Lambda)$ and $mod_P(\Lambda^{'})$, which we both denote by α (see [3], [4]).

We now give some lemmas of Auslander which we will use later.

Lemma 2.2 ([4]). *Let Λ be an artin algebra, and let*

$$0 \to A \to B \to C \to 0$$

be an exact sequence without split summands. Then we have the following

(1) *the induced exact sequence of functors*

$$(-,\underline{B}) \to (-,\underline{C}) \to F \to 0$$

is the minimal projective presentation of F in $\underline{mod}(mod(\Lambda))$, where F is the cokernel of the induced morphism of functors $(-,\underline{B}) \to (-,\underline{C})$;

(2) *the induced exact sequence of functors*

$$0 \to F \to Ext^1(-,A) \to Ext^1(-,B)$$

is the minimal injective copresentation of F in $\underline{mod}(mod(\Lambda))$. □

Lemma 2.3 ([5]). *Let Λ be an artin algebra, and $A, B \in mod_I(\Lambda)$. Then $A \simeq B$ if and only if $Ext^1(-,A) \simeq Ext^1(-,B)$.* □

We recall that for an artin algebra Λ, $e(\Lambda)$ is the set of indecomposable non-injective Λ modules A such that if $0 \to A \to B \to C \to 0$ is an almost split sequence, then either A or B is projective.

Lemma 2.4 ([4]). *Let $\alpha : \underline{mod}(\Lambda) \to \underline{mod}(\Lambda^{'})$ be an equivalence, which induced the equivalence $\alpha : \underline{mod}(mod(\Lambda)) \to \underline{mod}(mod(\Lambda^{'}))$. If A is a Λ module not in $e(\Lambda)$, then $\alpha(Ext^1_{\Lambda}(-,A)) \simeq Ext^1_{\Lambda^{'}}(-,\alpha(A))$.* □

3. Proof of the main theorem

In this section we fix two artin algebras Λ and $\Lambda^{'}$, which are not semi-simple, and an equivalence $\alpha : \underline{mod}(\Lambda) \to \underline{mod}(\Lambda^{'})$ with its inverse β. We want to prove the two algebras have the same representation dimension. Before we give the proof of the main theorem we give some notation and lemmas.

Define $n(\Lambda)$ to be the set of indecomposable non-projective non-injective Λ modules A such that $\alpha(Ext^1_{\Lambda}(-,A)) \not\simeq Ext^1_{\Lambda^{'}}(-,\alpha(A))$. Define $n(\Lambda^{'})$ to be the set of indecomposable non-projective non-injective $\Lambda^{'}$ modules $A^{'}$ such that $\beta(Ext^1_{\Lambda^{'}}(-,A^{'})) \not\simeq Ext^1_{\Lambda}(-,\beta(A^{'}))$. Obviously $n(\Lambda) \subseteq e(\Lambda)$ and $n(\Lambda^{'}) \subseteq e(\Lambda^{'})$.

Lemma 3.1. (1) *If P is an indecomposable projective non-injective Λ module, then*

$$\alpha(Ext^1_\Lambda(-,P)) \simeq Ext^1_{\Lambda'}(-,Q)$$

for some indecomposable $\Lambda^{'}$ module Q which is either projective or in $n(\Lambda^{'})$.

(2) *If Q is an indecomposable projective non-injective $\Lambda^{'}$ module, then*

$$\beta(Ext^1_{\Lambda'}(-,Q)) \simeq Ext^1_\Lambda(-,P)$$

for some indecomposable Λ module P which is either projective or in $n(\Lambda)$.

(3) *If I is an indecomposable injective non-projective Λ module, then $\alpha(I)$ is either injective or in $n(\Lambda^{'})$.*

(4) *If J is an indecomposable injective non-projective $\Lambda^{'}$ module, then $\beta(J)$ is either injective or in $n(\Lambda)$.*

(5) *If A is a module in $n(\Lambda)$, then $\alpha(A)$ is either injective non-projective or in $n(\Lambda^{'})$.*

(6) *If $A^{'}$ is a module in $n(\Lambda^{'})$, then $\beta(A')$ is either injective non-projective or in $n(\Lambda)$.*

Proof. It suffices to prove (1), (3) and (5).

(1) Suppose that P is an indecomposable projective non-injective Λ module, then $\alpha(Ext^1_\Lambda(-,P)) \simeq Ext^1_{\Lambda'}(-,Q)$ for some indecomposable non-injective $\Lambda^{'}$ module Q. If Q is non-projective and not in $n(\Lambda^{'})$, then we have

$$Ext^1_\Lambda(-,P) \simeq \beta\alpha(Ext^1_\Lambda(-,P)) \simeq \beta(Ext^1_{\Lambda'}(-,Q)) \simeq Ext^1_\Lambda(-,\beta(Q)),$$

which forces by Lemma 2.3, $P \simeq \beta(Q)$, a contradiction. So Q is either projective or in $n(\Lambda^{'})$.

(3) Suppose that I is an indecomposable injective non-projective Λ module. If $\alpha(I)$ is neither injective nor in $n(\Lambda^{'})$. Then we have $Ext^1_{\Lambda'}(-,\alpha(I)) \neq 0$ and

$$0 \neq \beta(Ext^1_{\Lambda'}(-,\alpha(I))) \simeq Ext^1_\Lambda(-,\beta(\alpha(I))) \simeq Ext^1_\Lambda(-,I) = 0,$$

a contradiction. This proves that $\alpha(I)$ is either injective or in $n(\Lambda^{'})$.

(5) Suppose that A is in $n(\Lambda)$, then $\alpha(A)$ is non-projective for A is non-projective. If $\alpha(A)$ is non-injective, then

$$Ext^1_{\Lambda'}(-,\alpha(A)) \ncong \alpha(Ext^1_\Lambda(-,A)) \cong \alpha(Ext^1_\Lambda(-,\beta(\alpha(A))))$$

and

$$\beta(Ext^1_{\Lambda'}(-,\alpha(A))) \ncong Ext^1_\Lambda(-,\beta(\alpha(A))).$$

We have that $\alpha(A)$ is in $n(\Lambda^{'})$, by the definition of $n(\Lambda^{'})$. □

Lemma 3.2. *Suppose that $M \in mod_P(\Lambda)$ and that $add(M)$ contains all modules in $n(\Lambda)$ and all injective non-projective Λ modules. Then $add(\alpha(M))$ contains all modules in $n(\Lambda^{'})$ and all injective non-projective $\Lambda^{'}$ modules. In particular, if $M \oplus \Lambda$ is a generator-cogenerator of Λ with $add(M \oplus \Lambda)$ containing all modules in $n(\Lambda)$, then $\alpha(M) \oplus \Lambda^{'}$ is a generator-cogenerator of $\Lambda^{'}$ with $add(\alpha(M) \oplus \Lambda^{'})$ containing all modules in $n(\Lambda^{'})$.*

Proof. Suppose $A^{'}$ is a $\Lambda^{'}$ module which is either injective non-projective or in $n(\Lambda^{'})$. Then by Lemma 3.1 $\beta(A^{'})$ is either injective non-projective or in $n(\Lambda)$. We have that $\beta(A^{'})$ is in $add(M)$ and hence that $A^{'}$ is in $add(\alpha(M))$. □

The next conclusion is straightforward.

Lemma 3.3. *Assume C is an indecomposable non-projective Λ module. Let*

$$0 \longrightarrow A \oplus S \oplus P_1 \longrightarrow B \oplus P_0 \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence in $mod(\Lambda)$ which has no split summands, such that P_1, P_0 projective, $A, B \in mod_P(\Lambda)$, A has no summand in $n(\Lambda)$ and S is a sum of modules in $n(\Lambda)$. Then there is an exact sequence without split summands in $mod(\Lambda^{'})$:

$$0 \longrightarrow A' \oplus S' \oplus Q_1 \longrightarrow \alpha(B) \oplus Q_0 \xrightarrow{g'} \alpha(C) \longrightarrow 0$$

such that $\alpha(\overline{g}) = g^{'}$, Q_1, Q_0 is projective, $A^{'} \in mod_P(\Lambda^{'})$, $A^{'}$ has no summands in $n(\Lambda^{'})$, and $S^{'}$ is a sum of modules in $n(\Lambda^{'})$. In this case, we have $\alpha(A)$ is a summand of $A^{'} \oplus S^{'}$ and that $\beta(A^{'})$ is a summand of $A \oplus S$.

Moreover, if $M \in mod_P(\Lambda)$, and

$$0 \to_\Lambda (-, A \oplus S \oplus P_1) \to_\Lambda (-, B \oplus P_0) \to_\Lambda (-, C) \to 0,$$

is exact in $add\widehat{(M \oplus} \Lambda)$, then

$$0 \to_{\Lambda'} (-, A^{'} \oplus S^{'} \oplus Q_1) \to_{\Lambda'} (-, \alpha(B) \oplus Q_0) \to_{\Lambda'} (-, \alpha(C)) \to 0,$$

is exact in $add(\widehat{\alpha(M)} \oplus \Lambda')$.

Proof.

$$0 \longrightarrow A \oplus S \oplus P_1 \longrightarrow B \oplus P_0 \xrightarrow{g} C \longrightarrow 0$$

is exact without split summands. Then by Lemma 2.2 we have

$${}_\Lambda(-, \underline{B}) \xrightarrow{{}_\Lambda(-,\underline{g})} {}_\Lambda(-, \underline{C}) \longrightarrow F \longrightarrow 0$$

is the minimal projective presentation of $F = cok({}_\Lambda(-, \underline{g}))$ in $\underline{mod}(mod(\Lambda))$;

$$0 \to F \to Ext^1_\Lambda(-, A \oplus S \oplus P_1) \to Ext^1_\Lambda(-, B \oplus P_0)$$

is the the minimal injective copresentation of F in $\underline{mod}(mod(\Lambda))$.

Apply the functor α, and we get

$${}_{\Lambda'}(-, \alpha(\underline{B})) \xrightarrow{{}_{\Lambda'}(-,\alpha(\underline{g}))} {}_{\Lambda'}(-, \alpha(\underline{C})) \longrightarrow \alpha(F) \longrightarrow 0$$

is the minimal projective presentation of $\alpha(F)$ in $\underline{mod}(mod(\Lambda^{'}))$;

$$0 \to \alpha(F) \to \alpha(Ext^1_\Lambda(-, A \oplus S \oplus P_1)) \to \alpha(Ext^1_\Lambda(-, B \oplus P_0))$$

is the the minimal injective copresentation of $\alpha(F)$ in $\underline{mod}(mod(\Lambda^{'}))$.

Then there exists $f : \alpha(B) \to \alpha(C)$ such that $\underline{f} = \alpha(\underline{g})$. Then there is some projective $\Lambda^{'}$ module Q_0 and $h : Q_0 \to \alpha(C)$ such that the composition of h and the natural epimorphism $\alpha(C) \to cok(f)$ is a projective cover. Define $g^{'} : \alpha(B) \oplus Q_0 \to \alpha(C)$ by $g^{'}(b, q) = f(b) + h(q)$, for all $b \in \alpha(B), q \in Q_0$. Clear $\underline{g^{'}} = \alpha(\underline{g})$.

We get an exact sequence of $\Lambda^{'}$ modules:

$$0 \longrightarrow A' \oplus S' \oplus Q_1 \longrightarrow \alpha(B) \oplus Q_0 \xrightarrow{g'} \alpha(C) \longrightarrow 0$$

where Q_1 is projective, $A^{'} \in mod_P(\Lambda^{'})$, $A^{'}$ has no summand in $n(\Lambda^{'})$, and $S^{'}$ is a sum of modules in $n(\Lambda^{'})$. It is easy to check that this exact sequence has no split summands. Then by Lemma 2.2, we have the minimal injective copresentation of $\alpha(F)$ in $\underline{mod}(mod(\Lambda^{'}))$,

$$0 \to \alpha(F) \to Ext^1_{\Lambda^{'}}(-, A^{'} \oplus S^{'} \oplus Q_1) \to Ext^1_{\Lambda^{'}}(-, \alpha(B) \oplus Q_0).$$

By the uniqueness of injective copresentation of $\alpha(F)$, we have $Ext^1_{\Lambda^{'}}(-, A^{'} \oplus S^{'} \oplus Q_1) \simeq \alpha(Ext^1_{\Lambda}(-, A \oplus S \oplus P_1))$. Then we have $\alpha(Ext^1_{\Lambda}(-, A)) \simeq Ext^1_{\Lambda^{'}}(-, \alpha(A))$ is a summand of $Ext^1_{\Lambda^{'}}(-, A^{'} \oplus S^{'} \oplus Q_1)$, and by Lemma 2.3, $\alpha(A)$ is a summand of $A^{'} \oplus S^{'} \oplus Q_1$. But $\alpha(A) \in mod_P(\Lambda^{'})$ and Q_1 is projective, then we have $\alpha(A)$ is a summand of $A^{'} \oplus S^{'}$. Similarly we have $\beta(A^{'})$ is a summand of $A \oplus S$.

Now we go on to prove the second part of the lemma. Suppose $M \in mod_P(\Lambda)$, and

$$0 \to_{\Lambda} (-, A \oplus S \oplus P_1) \to_{\Lambda} (-, B \oplus P_0) \to_{\Lambda} (-, C) \to 0$$

is exact in $add\widehat{(M \oplus \Lambda)}$.

We show that

$$0 \to_{\Lambda^{'}} (-, A^{'} \oplus S^{'} \oplus Q_1) \to_{\Lambda^{'}} (-, \alpha(B) \oplus Q_0) \to_{\Lambda^{'}} (-, \alpha(C)) \to 0$$

is exact in $add(\widehat{\alpha(M)} \oplus \Lambda')$.

It suffices to show that for any indecomposable $Y \in add(\alpha(M) \oplus \Lambda^{'})$,

$$0 \to_{\Lambda^{'}} (Y, A^{'} \oplus S^{'} \oplus Q_1) \to_{\Lambda^{'}} (Y, \alpha(B) \oplus Q_0) \to_{\Lambda^{'}} (Y, \alpha(C)) \to 0$$

is exact.

It suffices to show that any $\Lambda^{'}$ morphism $f : Y \to \alpha(C)$ factors through $g^{'}$.

It suffices to prove for the case $Y \in add(\alpha(M))$. We have $\underline{f} : \underline{Y} \to \alpha(\underline{C})$ and $\beta(\underline{f}) : \beta(\underline{Y}) \to \underline{C}$. Then there exists $h : \beta(Y) \to C$ such that $\underline{h} = \beta(\underline{f})$.

Since $\beta(Y) \in add(M)$ and ${}_{\Lambda}(-, B \oplus P_0) \to_{\Lambda} (-, C) \to 0$ is exact in $add\widehat{(M \oplus \Lambda)}$, then ${}_{\Lambda}(\beta(Y), B \oplus P_0) \to_{\Lambda} (\beta(Y), C) \to 0$ is exact. Then there exists $h_1 : \beta(Y) \to B \oplus P_0$ such that $h = gh_1$,

$$\begin{tikzcd}
& \beta(Y) \arrow[dl, "h_1"'] \arrow[d, "h"] & \\
B \oplus P_0 \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[dl, "f_1"'] \arrow[d, "f"] & \\
\alpha(B) \oplus Q_0 \arrow[r, "g^{'}"] & \alpha(C) \arrow[r] & 0
\end{tikzcd}$$

Thus $\underline{f} = \alpha\beta(\underline{f}) = \alpha(\underline{h}) = \alpha(\underline{g})\alpha(\underline{h_1}) = \underline{g}^{'}\alpha(\underline{h_1})$. There is some $f_1 : Y \to \alpha(B) \oplus Q_0$ such that $\underline{f_1} = \alpha(\underline{h_1})$. Then $\underline{f} = \underline{g}^{'}\ \underline{f_1} = \underline{g^{'} f_1}$. Then $f = g^{'} f_1 + \varphi$, for some $\varphi : Y \to \alpha(B) \oplus Q_0$ which factors through projective modules (hence also factors through $g^{'}$). Thus f factors through $g^{'}$. This finishes the proof. $\square$

Lemma 3.4. *Let $M \in mod_P(\Lambda)$, and S a module in $n(\Lambda)$. Then we have $sd(M \oplus S) \leq sd(M)$. Moreover, we have $rep.dim(\Lambda) = inf\{sd(M) | M$ is a generator-cogenerator of Λ such that $add(M)$ contains all modules in $n(\Lambda)\} + 2$.*

Proof. For any indecomposable Λ module C, we write $pd(-, C)$ for $pd_{\widehat{add(M)}}(-, C)$ and $pd_s(-, C)$ for $pd_{\widehat{add(M \oplus S)}}(-, C)$. We show $pd_s(-, C) \leq pd(-, C)$.

We prove by induction on $pd(-, C)$.

If $pd(-, C) = 0$, then $C \in add(M)$, $C \in add(M \oplus S)$, and we are $pd_s(-, C) = 0$. The conclusion is true.

Suppose the conclusion is true for $pd(-,C) \leq m-1, m \geq 1$. Now we prove it is true for $pd(-,C) = m$.

When $C \simeq S$, $pd_s(-,C) = 0$, and we are done.

When $C \not\cong S$, we have an exact sequence in $mod(\Lambda)$:

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow 0$$

which induces exact sequence in $\widehat{add(M)}$:

$$0 \to (-,A) \to (-,B) \to (-,C) \to 0,$$

with $B \in add(M)$ (that is, $(-,B)$ is projective in $\widehat{add(M)}$). Then we have $pd(-,C)$ $= pd(-,A) + 1$.

We now prove that the sequence is also exact in $\widehat{add(M \oplus S)}$. It suffices to show that for any Λ morphism $f : S \to C$, f factors through π.

$S \in n(\Lambda)$, then $S \in e(\Lambda)$, and there is an almost split sequence

$$0 \longrightarrow S \xrightarrow{i} P \longrightarrow TrDS \longrightarrow 0$$

with P projective.

For $C \not\cong S$, we have that f is not a split monomorphism. Then there exists $g : P \to C$ such that $f = gi$. But P is a projective Λ module, then there is $h : P \to B$ such that $g = \pi h$. Thus $f = \pi hi$, that is, f factors through π,

$$\begin{tikzcd}
 & & 0 \arrow[r] & S \arrow[r, "i"] \arrow[d, "f"'] & P \arrow[r] \arrow[dl, dotted, "g"] \arrow[dll, dotted, "h"'] & TrDS \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "\pi"'] & C \arrow[r] & 0 & &
\end{tikzcd}$$

We get that

$$0 \to (-,A) \to (-,B) \to (-,C) \to 0$$

is exact in $\widehat{add(M \oplus S)}$. Then we have $pd_s(-,C) = pd_s(-,A) + 1$. By induction, $pd_s(-,A) \leq pd(-,A)$, then $pd_s(-,C) \leq pd(-,C)$.

We have proved that for any indecomposable Λ module C, $pd_s(-,C) \leq pd(-,C)$, which implies $sd(M \oplus \Lambda \oplus S) \leq sd(M \oplus \Lambda)$. This finishes the proof. $\square$

Lemma 3.5. *Suppose $M \in mod_P(\Lambda)$ and $M \oplus \Lambda$ is a generator-cogenerator of Λ with $add(M)$ containing all modules in $n(\Lambda)$. Let $N = \alpha(M)$. Then we have $sd(M \oplus \Lambda) = sd(N \oplus \Lambda^{'})$, and $gl.dim(\widehat{add(M \oplus \Lambda)}) = gl.dim(\widehat{add(N \oplus \Lambda')})$.*

Proof. We first prove $sd(M \oplus \Lambda) = sd(N \oplus \Lambda^{'})$. It suffices to prove that for any indecomposable non-projective Λ module C, $pd_{\widehat{add(M\oplus\Lambda)}}(-,C) = pd_{\widehat{add(N\oplus\Lambda')}}(-,\alpha(C))$. We write $pd(-,C)$ for $pd_{\widehat{add(M\oplus\Lambda)}}(-,C)$ and $pd(-,\alpha(C))$ for $pd_{\widehat{add(N\oplus\Lambda')}}(-,\alpha(C))$.

We proceed by induction on $pd(-,C)$. When $pd(-,C) = 0$, we have $C \in add(M \oplus \Lambda)$, then $\alpha(C) \in add(N \oplus \Lambda^{'})$, thus $pd(-,\alpha(C)) = 0$.

Suppose the conclusion is true for $pd(-,C) \leq m-1, (m \geq 1)$. Now suppose $pd(-,C) = m$. There exists an exact sequence in $mod(\Lambda)$,

$$0 \longrightarrow A \oplus S \oplus P_1 \longrightarrow B \oplus P_0 \xrightarrow{g} C \longrightarrow 0$$

where P_1, P_0 is projective, $A, B \in mod_P(\Lambda)$, A has no summands in $n(\Lambda)$, S is a sum of modules in $n(\Lambda)$, and $B \in add(M)$, such that the induced sequence

$$0 \rightarrow_{\Lambda} (-, A \oplus S \oplus P_1) \rightarrow_{\Lambda} (-, B \oplus P_0) \rightarrow_{\Lambda} (-, C) \rightarrow 0$$

is exact in $add\widehat{(M \oplus \Lambda)}$. Then $(-, B \oplus P_0)$ is projective in $add\widehat{(M \oplus \Lambda)}$, and we have that $pd(-,C) = pd(-, A \oplus S) + 1$. But S is a sum of modules in $n(\Lambda)$, and $add(M)$ contains all modules in $n(\Lambda)$, then we have that $pd(-,C) = pd(-,A) + 1$, and, by induction, that $pd(-,A) = pd(-, \alpha(A))$.

Then by Lemma 3.3, we have an exact sequence in $mod(\Lambda^{'})$,

$$0 \longrightarrow A^{'} \oplus S^{'} \oplus Q_1 \longrightarrow \alpha(B) \oplus Q_0 \xrightarrow{g^{'}} \alpha(C) \longrightarrow 0$$

such that $\alpha(\underline{g}) = \underline{g^{'}}$, Q_1, Q_0 is projective, $A^{'} \in mod_P(\Lambda^{'})$, $A^{'}$ has no summands in $n(\Lambda^{'})$, $S^{'}$ is a sum of modules in $n(\Lambda^{'})$, $\alpha(A)$ is a summand of $A^{'} \oplus S^{'}$, and that $\beta(A^{'})$ is a summand of $A \oplus S$, which induced an exact sequence in $add\widehat{(N \oplus \Lambda')}$

$$0 \rightarrow_{\Lambda'} (-, A^{'} \oplus S^{'} \oplus Q_1) \rightarrow_{\Lambda'} (-, \alpha(B) \oplus Q_0) \rightarrow_{\Lambda'} (-, \alpha(C)) \rightarrow 0,$$

with $\alpha(B) \in add(N)$, and hence ${}_{\Lambda'}(-, \alpha(B) \oplus Q_0)$ is projective in $add\widehat{(N \oplus \Lambda')}$.

Then we get $pd(-, \alpha(C)) = pd(-, A^{'} \oplus S^{'}) + 1 = pd(-, A^{'}) + 1$, since $add(N) = add(\alpha(M))$ contains all modules in $n(\Lambda^{'})$ by Lemma 3.2.

Thus we have that $pd(-,C) = pd(-,A)+1 = pd(-,\alpha(A))+1 \leq pd(-, A^{'} \oplus S^{'}) + 1 = pd(-, A^{'}) + 1 = pd(-, \alpha(C))$, since $\alpha(A)$ is a summand of $A^{'} \oplus S^{'}$. Conversely $pd(-, \alpha(C)) = pd(-, A^{'}) + 1 \leq pd(-, \alpha(A) \oplus \alpha(S)) + 1 = pd(-, \alpha(A)) + 1 = pd(-, A) + 1 = pd(-, C)$, since $\beta(A^{'})$ is a summand of $A \oplus S$ and hence $A^{'}$ is a summand of $\alpha(A) \oplus \alpha(S)$ with $\alpha(S)$ either injective non-projective or in $n(\Lambda^{'})$, which is contained in $add(N)$.

Now we have proved that for any indecomposable non-projective Λ module C, $pd(-,C) = pd(-, \alpha(C))$. It follows easily that $sd(M \oplus \Lambda) = sd(N \oplus \Lambda^{'})$.

The next conclusion follows by Lemma 2.1. □

Now we can easily prove our main theorem.

Proof of Theorem 1.1. We only prove the case that Λ with $\Lambda^{'}$ are both basic and neither semi-simple.

By Lemma 2.1, Lemma 3.4 and Lemma 3.5, we have $rep.dim(\Lambda) = inf\{sd(M) | M$ is a generator-cogenerator of Λ such that $add(M)$ contains all modules in $n(\Lambda)\} + 2 = inf\{sd(N) | N$ is a generator-cogenerator of $\Lambda^{'}$ such that $add(N)$ contains all modules in $n(\Lambda^{'})\} + 2 = rep.dim(\Lambda^{'})$. □

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