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SPIKE-LAYERED SOLUTIONS FOR AN ELLIPTIC SYSTEM WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We prove the existence of nonconstant positive solutions for a system of the form $-\varepsilon^2\Delta u+u=g(v)$, $-\varepsilon^2\Delta v+v=f(u)$ in Ω, with Neumann boundary conditions on $\partial\Omega$, where Ω is a smooth bounded domain and f, g are power-type nonlinearities having superlinear and subcritical growth at infinity. For small values of ε, the corresponding solutions u_ε and v_ε admit a unique maximum point which is located at the boundary of Ω.

Introduction

In this paper we are concerned with the following system of elliptic equations with Neumann boundary conditions:

$$(\mathrm{P})\qquad -\varepsilon^2\Delta u+u=g(v),\quad -\varepsilon^2\Delta v+v=f(u)\ \text{ in } \Omega,\qquad \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0\ \text{ on } \partial\Omega,$$

where $\varepsilon>0$ is a small parameter and Ω is a C^2 bounded domain of $\mathbb{R}^N$, with $N\geqslant 3$. As a model problem, we consider the case where

$$(0.1)\qquad f(s)s=|s|^\alpha+|s|^p,\quad g(s)s=|s|^\beta+|s|^q,\quad 2<\alpha\leqslant p,\quad 2<\beta\leqslant q,$$

and

$$(0.2)\qquad \frac{1}{p}+\frac{1}{q}>\frac{N-2}{N}\cdot$$

Our main result goes as follows (see Section 4 for a more general theorem).

Theorem 0.1. *Under assumptions* (0.1) *and* (0.2)*, there exists* $\varepsilon_0>0$ *such that for any* $0<\varepsilon<\varepsilon_0$ *problem* (P) *has nonconstant positive solutions* u_ε, $v_\varepsilon\in C^2(\overline{\Omega})$. *Moreover, both functions* u_ε *and* v_ε *attain their maximum value at some unique and common point* $x_\varepsilon\in\partial\Omega$.

We point out that, in contrast with Theorem 0.1, only *constant* positive solutions are expected to exist for *large* values of ε (see [4, 14]).

Results of this type for semilinear Schrödinger *equations* with Neumann boundary conditions, where this "spike-layer pattern" appears as $\varepsilon\to 0$ for ground-state solutions, were obtained e.g. in [10], [12], [14], [15], [16], among others. We refer the reader to the nice introduction in [10] for a survey and a discussion of the problem.

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To the best of our knowledge, the recent work [4] is the first attempt at proving that a similar phenomenon can be observed in Hamiltonian systems such as (P). We may summarize as follows the three main difficulties that one has to face in dealing with such a problem. On the one hand, it can be observed that in the study of problem (P) an important role is played by the two "limit problems"

$$(0.3) \qquad -\Delta u = g(v), \qquad -\Delta v = f(u), \qquad u, v \in C^2(\mathbb{R}^N), \quad u, v > 0,$$

and

$$(0.4) \qquad -\Delta u + u = g(v), \qquad -\Delta v + v = f(u), \qquad u, v \in C^2(\mathbb{R}^N), \quad u, v > 0.$$

It turns out that under conditions (0.1), (0.2) a nonexistence result for (0.3) is not known (except when $p, q < 2N/(N-2)$, cf. [8]) and a uniqueness result for (0.4) is also not known. This is in contrast with the scalar equations $-\Delta u = u^{p-1}$ and $-\Delta u + u = u^{p-1}$ in $\mathbb{R}^N$, with $2 < p < 2N/(N-2)$. At this point we should recall that our subcritical condition at infinity (0.2) is a quite natural one; it states that $(p, q) \in \mathbb{R}^2$ lies below the so-called "critical hyperbola", a notion first introduced in [7], [9], [11], [17].

On the other hand, in the scalar case (single equation) the crucial estimates are obtained by comparing energy levels of the energy functional associated to the problem; this is in turn achieved thanks to the mountain-pass variational characterization of the ground-state level. In our case, the quadratic part of the energy functional I associated to the problem (see Section 1 for the definition of I) has no definite sign and the underlying minimax theorem (an infinite-dimensional linking) is of more complex nature; for example, it does not seem to follow readily from minimax arguments that ground-state solutions are indeed nonconstant solutions of problem (P), for small values of ε.

In [4], the authors propose an indirect approach based on a dual variational formulation of the problem (a method which cannot be applied e.g. in the presence of nonlocal terms in the equations), they restrict themselves to the special case where $f(s)s = |s|^p$, $g(s)s = |s|^q$ and $2 < p, q < 2N/(N-2)$ and they deduce similar (though less precise, as far as the information on the maximum points x_ε is concerned) conclusions as in Theorem 0.1. Ours, instead, is a direct approach. We consider general convex nonlinearities f and g (see assumption (H) in Section 4) and work with the energy functional I defined over the natural Hilbert space $H^1(\Omega) \times H^1(\Omega)$. Our starting point is the observation that ground-state critical points of I indeed correspond to nonconstant solutions of the problem since constant positive solutions of (P) are expected to have an increasingly large Morse index as ε decreases to 0 (see Remark 4.2).

Another key ingredient in our argument is an estimate from above on the ground-state energy level of the functional I. It is based on a new variational characterization (of Nehari type) of the ground-state level for strongly indefinite functionals; cf. Theorem 3.1.

The proof of our main result is postponed to Section 4 (cf. Theorem 4.1). Before that, we make a preliminary discussion on the properties of solutions of general nonlinear problems such as (P), concerning L^∞ a priori bounds (Section 1), the shape of its positive solutions (Section 2) and estimates on the ground-state energy level (Section 3). We prefer to state these preliminary results in separate sections in order to emphasize several aspects of the problem which can be of independent interest and which envolve different assumptions on the nonlinearities f and g.

1. Solutions with bounded Morse index

In this section we discuss a priori bounds for solutions of the system

$$\text{(P)}\qquad -\varepsilon^2\Delta u+u=g(v),\quad -\varepsilon^2\Delta v+v=f(u)\ \text{ in }\Omega,\qquad \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0\ \text{ on }\partial\Omega.$$

We shall assume the following conditions on f and g.

(H1) $f,g\in C^1(\mathbb{R})$, $f(0)=0=f'(0)$, $g(0)=0=g'(0)$ and there exist real numbers ℓ_1, $\ell_2>0$ and $p,q>2$ such that $\frac{1}{p}+\frac{1}{q}>\frac{N-2}{N}$ and

$$\lim_{|s|\to\infty}\frac{f'(s)}{|s|^{p-2}}=\ell_1,\qquad \lim_{|s|\to\infty}\frac{g'(s)}{|s|^{q-2}}=\ell_2. \tag{1.1}$$

(H2) $f'(s)\geqslant 0$ and $g'(s)\geqslant 0$ for every $s\in\mathbb{R}$.

We define the energy functional

$$I(u,v)=\int_\Omega\left(\varepsilon^2\langle\nabla u,\nabla v\rangle+uv-F(u)-G(v)\right), \tag{1.2}$$

where $F(s):=\int_0^s f(\xi)\,d\xi$ and $G(s):=\int_0^s g(\xi)\,d\xi$. Under our assumptions, it can happen that, say, $q>2N/(N-2)>p$ and so I may not be well defined over the Hilbert space $H^1(\Omega)\times H^1(\Omega)$. One can handle this situation by working with fractional Sobolev spaces, as in e.g. [19]. Since, anyway, it is our aim to prove a priori bounds for solutions of (P), we find it more convenient to consider a modified problem, by combining ideas from [2], [3], [18].

To this purpose, assume without loss of generality that $q\geqslant p>2$ and $p<2N/(N-2)$ and, for any given sequence $(a_j)\subset\mathbb{R}$, $a_j\to+\infty$, let $g_j(s)=A_j|s|^{p-2}s+B_j$ for $s\geqslant a_j$, $g_j(s)=g(s)$ for $|s|\leqslant a_j$ and $g_j(s)=\tilde A_j|s|^{p-2}s+\tilde B_j$ for $s\leqslant -a_j$, where the coefficients are chosen in such a way that g_j is C^1. Thus, in view of (1.1), we see that $A_j=\left(\frac{\ell_2}{p-1}+\mathrm{o}(1)\right)a_j^{q-p}=\tilde A_j$ and $B_j=\left(\frac{\ell_2(p-q)}{(p-1)\,(q-1)}+\mathrm{o}(1)\right)a_j^{q-1}=-\tilde B_j$.

The corresponding energy functional is given by (1.2) with $G_j:=\int_0^s g_j(\xi)\,d\xi$ in place of G. We still denote by I the energy functional. It is a C^2 functional defined over the Hilbert space

$$E:=H^1(\Omega)\times H^1(\Omega),\qquad ||(u,v)||^2:=||u||^2+||v||^2,$$

where we denote $||u||^2=||u||^2_{H^1(\Omega)}=\int_\Omega\left(|\nabla u|^2+u^2\right)$. We also denote $||u||_r:=||u||_{L^r(\Omega)}$, for any $r\in[1,\infty]$. The critical points of I correspond to $C^2(\overline{\Omega})$ solutions of the modified problem

$$\text{(P)}_j\qquad -\varepsilon^2\Delta u+u=g_j(v),\quad -\varepsilon^2\Delta v+v=f(u)\ \text{ in }\Omega,\qquad \frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0\ \text{ on }\partial\Omega.$$

The second derivative $I''(u,v)$ evaluated at a critical point (u,v) is given by

$$I''(u,v)(\phi,\psi)(\phi,\psi)=\int_\Omega\left(2\varepsilon^2\langle\nabla\phi,\nabla\psi\rangle+2\phi\psi-f'(u)\phi^2-g_j'(v)\psi^2\right). \tag{1.3}$$

It can be observed that the following orthogonal splitting holds:

$$E=E^-\oplus E^+,\qquad \text{where }\ E^\pm:=\{(\phi,\pm\phi),\ \phi\in H^1(\Omega)\},$$

so that, denoting by Q the quadratic term of the energy functional, namely

$$Q(u,v)=\int_\Omega\left(\varepsilon^2\langle\nabla u,\nabla v\rangle+uv\right),$$

we have that Q is positive definite (resp. negative definite) in E^+ (resp. in E^-). If (u, v) is a solution of $(\mathrm{P})_j$, we denote by $m(u, v)$ its relative Morse index, as defined in [1, 2] (the definition of $m(u, v)$ is recalled in the proof of Lemma 1.2 below). Now we can state the main result of this section.

Theorem 1.1. *Assume* (H1) *and* (H2). *For any given sequence $\varepsilon_j \to 0$ and $a_j \to +\infty$, let u_j, v_j be solutions of problem* $(\mathrm{P})_j$ *(with $\varepsilon = \varepsilon_j$). If there exists $k \in \mathbb{N}$ such that $m(u_j, v_j) \leqslant k$ for every j, then there exists $K > 0$ such that*

$$||u_j||_\infty + ||v_j||_\infty \leqslant K, \qquad \forall j.$$

In particular, u_j and v_j are solutions of problem (P), *for large values of j.*

We anticipate that in proving Theorem 0.1 we shall find solutions having relative Morse index $\leqslant 1$; thus Theorem 1.1 above states that, for small ε, not only these solutions are bounded in $L^\infty(\Omega)$ but also that in the process of actually finding these solutions we can assume that the conditions in assumption (H1) hold with $2 < q = p < 2N/(N-2)$. Further conclusions on u_j and v_j will be derived in Proposition 1.6 at the end of this section.

The proof of Theorem 1.1 is based on the following simple fact.

Lemma 1.2. *Assume* (H1) *and* (H2) *and let u, v be any solutions of problem* $(\mathrm{P})_j$. *If there exist $\lambda > 0$ and $k+1$ functions $\varphi_1, \dots, \varphi_{k+1} \in H^1(\Omega)$ having disjoint supports, such that*

$$I''(u,v)(\varphi_i, \lambda\varphi_i)(\varphi_i, \lambda\varphi_i) < 0, \qquad \forall i = 1, \dots, k+1,$$

then $m(u, v) \geqslant k + 1$.

Proof. Denote by V the negative eigenspace of $I''(u, v)$. By definition (see [1, 2]), $m(u, v) = m_{(E^+, E^-)}(u, v) = \dim_{E^-} V$, where the relative dimension is defined as follows:

$$\dim_{E^-} V = \dim\left(V \cap (E^-)^\perp\right) - \dim\left(V^\perp \cap E^-\right). \tag{1.4}$$

We denote by W the subspace $W = \{(\lambda\phi, -\phi) : \phi \in H^1(\Omega)\}$ and apply a general identity in [1, page 3], [2, page 401]:

$$\dim_{E^-} V = \dim_W V + \dim_{E^-} W.$$

Using the definition (1.4) with V in place of W, it follows easily that $\dim_{E^-} W = 0$ (recall that $(E^-)^\perp = E^+$). On the other hand, since $f' \geqslant 0$ and $g_j' \geqslant 0$, we see from the expression in (1.3) that $W \subset V$. In conclusion,

$$m(u, v) = \dim_W V = \dim(V \cap W^\perp) - \dim(V^\perp \cap W) = \dim(V \cap W^\perp).$$

Now, the space $X := \mathrm{span}\{(\varphi_i, \lambda\varphi_i) : i = 1, \dots, k+1\}$ has dimension $k+1$ and is contained in $V \cap W^\perp$, and this completes the proof. □

We will also rely on a Liouville-type theorem for solutions having finite index, as defined in [5], [18].

Definition 1.3. Let f_∞, $g_\infty \in C^1(\mathbb{R})$ and u, v be C^2 functions satisfying $-\Delta u = g_\infty(v)$, $-\Delta v = f_\infty(u)$ in ω, $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ on $\partial\omega$, where $\omega := \{x : \langle x, y_0\rangle < d_0\}$ for some $y_0 \in \mathbb{R}^N$, $y_0 \neq 0$, and $d_0 \in]-\infty, +\infty]$. We say that (u, v) has *finite index* if there exists $R_0 > 0$ with the property that for every $\varphi \in H^1(\omega)$ such that $\varphi = 0$ in $B_{R_0}(0) \cap \omega$ it holds that

$$2\int_\omega |\nabla\varphi|^2 - \int_\omega f'_\infty(u)\varphi^2 - \int_\omega g'_\infty(v)\varphi^2 \geqslant 0. \tag{1.5}$$

We are mainly concerned with the case where f_∞ and g_∞ are power-type nonlinearities. Our next result is stated in a form suitable to our purposes. We consider a C^1 function g_∞ satisfying, for some $c_1,\ c_2 > 0$, $2 < p < 2N/(N-2)$, $p \leqslant q$, $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$ and any $s \in \mathbb{R}$, the following three conditions:

(*a*) $c_1|s|^q \leqslant g_\infty(s)s \leqslant c_2|s|^q$,

(*b*) $g_\infty(s)s \leqslant qG_\infty(s)$,

(*c*) $(p-1)g_\infty(s)s \leqslant g'_\infty(s)s^2$.

Proposition 1.4. *Let $g_\infty \in C^1(\mathbb{R})$ and $f_\infty(s) = c\,|s|^{p-2}s$, with $c > 0$ and $2 < p < 2N/(N-2)$ and suppose (u, v) has finite index, in the sense of Definition* 1.3.

(*i*) *If $g_\infty = 0$, then $u = 0$.*

(*ii*) *If g_∞ satisfies conditions $(a), (b), (c)$ above, then $u = 0 = v$.*

Proof. Since the problem is invariant by translations, we may assume that $\omega = \mathbb{R}^N$ or else $\omega = \{x : \langle x, y_0\rangle < 0\}$. We may also assume that $c = 1$, i.e. $f_\infty(s) = |s|^{p-2}s$ with $2 < p < 2N/(N-2)$. For any large $R > 0$, we denote by φ a cut-off function $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \leqslant \varphi \leqslant 1$ in $\mathbb{R}^N$, $\varphi = 1$ in $B_R(0)$, $\varphi = 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$ and $||\nabla\varphi||_\infty \leqslant C/R$. We also denote by C some generic constant whose value may change from place to place but is independent of R.

(*i*) Suppose $g_\infty = 0$, so that $-\Delta u = 0$ and $-\Delta v = |u|^{p-2}u$ in ω. We argue similarly to [18, Proposition 10]. Fix any smooth function Ψ such that $\Psi = 0$ in $B_{R_0}(0)$. Then assumption (1.5) reads as

$$(p-1)\int_\omega |u|^p\Psi^2\varphi^2 \leqslant 2\int_\omega |\nabla(u\Psi\varphi)|^2. \tag{1.6}$$

If we expand the right-hand member of (1.6) and use the equation $-\Delta u = 0$ we easily see that

$$\int_\omega |u|^p\varphi^2 \leqslant C\left(1 + \int_\omega u^2|\nabla\varphi|^2\right), \tag{1.7}$$

for some constant $C = C(R_0)$. If we replace φ by φ^m with m large and apply Hölder's inequality in (1.7), we conclude that $\int_{B_R(0)\cap\omega} |u|^p$ is bounded independently of R. Thus, $u \in L^p(\omega)$. In particular, $\int_{B_R(0)\cap\omega} u^2 = \mathrm{o}(R^2)$ as $R \to \infty$. Using this fact in the equation $-\Delta u = 0$ (multiply both members by $u\varphi^2$ and use Schwarz's inequality) we deduce that $\int_\omega |\nabla u|^2 = 0$. Since also $u \in L^p(\omega)$, we conclude that $u = 0$.

(*ii*) Suppose $-\Delta u = g_\infty(v)$ and $-\Delta v = f_\infty(u) = |u|^{p-2}u$ with g_∞ satisfying conditions (*a*), (*b*), (*c*) above. Arguing as before, the inequality in (1.7) now reads as

$$(p-1)\int_\omega f_\infty(u)u\varphi^2 + \int_\omega g'_\infty(v)u^2\varphi^2 \leqslant C\,(1 + \int_\omega u^2|\nabla\varphi|^2) + 2\int_\omega g_\infty(v)u\varphi^2.$$

Using condition (*c*) and the inequality $2uv \leqslant u^2 + v^2$ in the last integral above, we deduce that

$$(p-1)\int_\omega f_\infty(u)u\varphi^2 \leqslant C\,(1 + \int_\omega u^2|\nabla\varphi|^2) + \int_\omega g_\infty(v)v\varphi^2. \tag{1.8}$$

At this point we use [3, Theorem 5A], according to which, for any $\varepsilon > 0$,

$$\int_\omega g_\infty(v)v\varphi^m \leqslant \frac{1+\varepsilon}{1-\varepsilon} \int_\omega f_\infty(u)u\varphi^m + \mathrm{o}(1), \qquad \text{as } R \to \infty, \tag{1.9}$$

provided m is taken sufficiently large (in [3] it is assumed that $g_\infty(v)v = |v|^q$ but an inspection of its proof shows that in our case (1.9) holds as well, thanks to assumption (a)). Combining (1.8) with φ replaced by $\varphi^{m/2}$ and (1.9) yields that $u \in L^p(\omega)$ (compare with (1.7)). Thanks to (1.9), $v \in L^q(\omega)$ also. Then, according to [3, Corollary 5B and Lemma 6B], we conclude that

$$\int_\omega \langle \nabla u, \nabla v\rangle = \int_\omega f_\infty(u)u = \int_\omega g_\infty(v)v < \infty. \tag{1.10}$$

Once (1.10) is settled, we may use a Pohožaev-Rellich type identity, according to which

$$\int_\omega \langle \nabla u, \nabla v\rangle = \frac{N}{N-2} \int_\omega (F_\infty(u) + G_\infty(v)). \tag{1.11}$$

This follows as in [3, Lemma 6D]; one can also deduce (1.11) as in [13, 20], by computing $\mathrm{div}(\varphi W)$, where W is the vector field

$$W(x) = \langle \nabla v, x\rangle \nabla u + \langle \nabla u, x\rangle \nabla v - \langle \nabla u, \nabla v\rangle x - F_\infty(u)x - G_\infty(v)x,$$

and by passing to the limit as $R \to \infty$ (observe that indeed $\int_{\partial\omega} \mathrm{div}(\varphi W) = 0$, thanks to the Neumann boundary conditions and to our definition of ω).

We conclude by combining (1.10), (1.11) and assumption (b), which together lead to

$$\frac{N-2}{N}\int_\omega |u|^p = \frac{1}{p}\int_\omega |u|^p + \int_\omega G_\infty(v) \geqslant \frac{1}{p}\int_\omega |u|^p + \frac{1}{q}\int_\omega g_\infty(v)v = (\frac{1}{p}+\frac{1}{q})\int_\omega |u|^p.$$

Since $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$, this implies that $u = v = 0$ and concludes the proof of the proposition. □

Proof of Theorem 1.1 *completed.* We first give a sketch of the proof. We will prove that if $||u_j||_\infty + ||v_j||_\infty \to \infty$ (along a subsequence), then it is possible to find $x_j \in \overline{\Omega}$, $\alpha_j > 0$, $\beta_j > 0$, $\lambda_j \to 0^+$ such that both functions

$$\widetilde{u}_j(x) = \frac{1}{\alpha_j}\, u(\lambda_j\varepsilon_j x + x_j), \qquad \widetilde{v}_j(x) = \frac{1}{\beta_j}\, v(\lambda_j\varepsilon_j x + x_j), \tag{1.12}$$

which satisfy

$$-\Delta\widetilde{u}_j + \lambda_j^2\widetilde{u}_j = \frac{\lambda_j^2}{\alpha_j}\, g_j(\beta_j\widetilde{v}_j), \quad -\Delta\widetilde{v}_j + \lambda_j^2\widetilde{v}_j = \frac{\lambda_j^2}{\beta_j}\, f(\alpha_j\widetilde{u}_j)$$

in $\Omega_j := \frac{1}{\lambda_j\varepsilon_j}(\Omega - x_j)$, are uniformly bounded and, up to subsequences, they converge in C^2_{loc} to functions u and v, respectively, satisfying some limit problem

$$-\Delta u = g_\infty(v), \quad -\Delta v = f_\infty(u) \ \text{ in } \omega, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \partial\omega, \tag{1.13}$$

for some C^1 functions f_∞ and g_∞; here, of course, ω is either the whole space $\mathbb{R}^N$ or else $\omega = \{x : \langle x, n(x_0)\rangle < d_0\}$, where $x_0 \in \partial\Omega$ is a limit point of (x_j), $n(x_0)$ is the unit outward normal at the point x_0 and $d_0 = \lim \frac{\mathrm{dist}(x_j,\partial\Omega)}{\lambda_j\varepsilon_j}$.

Moreover, we will have that

$$\frac{\alpha_j}{\beta_j}\lambda_j^2 f'(\alpha_j\widetilde{u}_j) \to f'_\infty(u) \qquad \text{and} \qquad \frac{\beta_j}{\alpha_j}\lambda_j^2 g'_j(\beta_j\widetilde{v}_j) \to g'_\infty(v), \tag{1.14}$$

uniformly in compact sets. In particular, the limit function (u, v) has finite index. Otherwise, according to Definition 1.3, we could find an arbitrarily large number of functions $\varphi \in H^1(\omega)$ with compact support, such that

$$E(\varphi) := 2\int_\omega |\nabla\varphi|^2 - \int_\omega f'_\infty(u)\varphi^2 - \int_\omega g'_\infty(v)\varphi^2 < 0.$$

Then, by letting $\varphi_j(x) := \varphi((x-x_j)/(\lambda_j\varepsilon_j)) \in H^1(\Omega)$, we see from (1.3) and (1.14) that

$$\frac{1}{\lambda_j^{N-2}\varepsilon^N}\, I''(u_j, v_j)(\varphi_j, \frac{\beta_j}{\alpha_j}\varphi_j)(\varphi_j, \frac{\beta_j}{\alpha_j}\varphi_j) \to E(\varphi) < 0.$$

For large values of j, this would yield the existence of $k+1$ functions having disjoint supports, $\varphi_{1,j}, \dots, \varphi_{k+1,j}$, such that

$$I''(u_j, v_j)(\varphi_{i,j}, \frac{\beta_j}{\alpha_j}\varphi_{i,j})(\varphi_{i,j}, \frac{\beta_j}{\alpha_j}\varphi_{i,j}) < 0, \qquad \forall i = 1, \dots, k+1.$$

It would then follow from Lemma 1.2 that $m(u_j, v_j) \geqslant k+1$, contradicting our assumption that $m(u_j, v_j) \leqslant k$ for every j.

So, we deduce that (u, v) has indeed finite index and we are in a position to apply Proposition 1.4. Then our specific choice for $x_j, \lambda_j, \alpha_j, \beta_j$ will lead to a contradiction, and the proof of Theorem 1.1 will be complete.

Thus, to conclude the argument it remains to show that x_j, λ_j, α_j and β_j can be properly chosen. This, in turn, will depend on the asymptotic value of $a_j/||v_j||_\infty$ and so, for the sake of clarity, we consider several (exhaustive) cases and try to conclude that a contradiction arises in each of it. We observe that this is in contrast with the direct arguments that one uses when no truncation of the nonlinearity is involved; those correspond to Case 1 below, but since in our case the estimates are not "symmetric" with respect to p and q, we must also discuss further situations in the blow-up procedure (Cases 2, 3 and 4 below; as mentioned before, the good point of this is that once Theorem 1.1 is proved, then we will be able to work directly in the natural functional space $H^1(\Omega) \times H^1(\Omega)$ associated to problem (P)). As in [3], we denote

$$M_j := \sup_{x\in\overline{\Omega}} \{\max\{|u_j(x)|^{1/q}, |v_j(x)|^{1/p}\}\}.$$

By assumption, $M_j \to \infty$. Up to a subsequence, let

$$\ell := \lim \frac{a_j}{M_j^p} \in [0, \infty].$$

Case 1. We first consider the easiest case when $\ell > 0$ (we may have that $\ell = \infty$). Let λ_j be given by $\lambda_j^2 M_j^{pq-p-q} = 1$, $y_j \in \overline{\Omega}$ be such that

$$M_j = \max\{|u_j(y_j)|^{1/q}, |v_j(y_j)|^{1/p}\}$$

and set $\alpha_j = M_j^q$, $\beta_j = M_j^p$. Finally, let $\widetilde{u}_j$, $\widetilde{v}_j$ be as in (1.12), with $x_j = y_j$. Then $||\widetilde{u}_j||_\infty \leqslant 1$, $||\widetilde{v}_j||_\infty \leqslant 1$ and one can easily check that, up to subsequences, $\widetilde{u}_j \to u$, $\widetilde{v}_j \to v$ in C^2_{loc} and u, v satisfy a limit system (1.13) with $f_\infty(s) = \frac{\ell_1}{p-1}|s|^{p-2}s$ and g_∞ given by $g_\infty(s) = \frac{\ell_2}{q-1}|s|^{q-2}s$ for $|s| \leqslant \ell$ and $g_\infty(s) = \frac{\ell_2}{p-1}\ell^{q-p}|s|^{p-2}s +$

$\frac{\ell_2(p-q)}{(p-1)\,(q-1)}\ell^{q-1}$ for $s > \ell$ (and g_∞ is odd, i.e. $g_\infty(-s) = -g_\infty(s)$). It is only a matter of patience to check that g_∞ satisfies the conditions (a), (b), (c) which are mentioned in Proposition 1.4 and that (1.14) holds as well (in fact, the conditions hold for $|s| \leqslant 1$, but this is sufficient for our purposes since $||u||_\infty \leqslant 1$ and $||v||_\infty \leqslant 1$; analogous computations can be found in [18]). Now, according to our previous considerations and to Proposition 1.4 (ii), it follows that $u = v = 0$. This is impossible since, by definition, either $|\widetilde{u}_j(0)| = 1$ or $|\widetilde{v}_j(0)| = 1$ for every j.

Case 2. Suppose now that $\ell = 0$. Proceeding exactly as in Case 1 we arrive at the limit system $-\Delta u = 0$, $-\Delta v = \frac{\ell_1}{p-1}|u|^{p-2}u$ so that, according to Proposition 1.4 (i), we must have $u = 0$. This means that

$$M_j^p = ||v_j||_\infty = |v_j(y_j)| \qquad \text{and} \qquad \frac{a_j}{||v_j||_\infty} \to 0.$$

In particular, $||v_j||_\infty \to \infty$. Suppose that, for a subsequence, $||u_j||_\infty \leqslant ||v_j||_\infty$. Then we let

$$\widetilde{u}_j(x) = \frac{u_j(\lambda_j\varepsilon_j x + y_j)}{||v_j||_\infty}, \qquad \widetilde{v}_j(x) = \frac{v_j(\lambda_j\varepsilon_j x + y_j)}{||v_j||_\infty}, \qquad \lambda_j^2\, A_j\, ||v_j||_\infty^{p-2} = 1,$$

where A_j was introduced at the beginning of Section 1. Thus $\lambda_j \to 0$, $||\widetilde{u}_j|| \leqslant 1$, $||\widetilde{v}_j||_\infty = 1$ and it is easy to check that we arrive at the limit system $-\Delta u = |v|^{p-2}v$, $-\Delta v = 0$. It follows from Proposition 1.4 (i) that $v = 0$, and this contradicts the fact that $|\widetilde{v}_j(0)| = 1$ for every j.

Case 3. At this point we are reduced to the study of the case when

$$\frac{a_j}{||v_j||_\infty} \to 0 \qquad \text{and} \qquad ||v_j||_\infty \leqslant ||u_j||_\infty. \tag{1.15}$$

Suppose first that

$$\frac{||v_j||_\infty}{||u_j||_\infty}\, A_j^{1/p} \to \infty. \tag{1.16}$$

Then we let

$$\widetilde{u}_j(x) = \frac{u_j(\lambda_j\varepsilon_j x + y_j)}{||u_j||_\infty}, \qquad \widetilde{v}_j(x) = \frac{v_j(\lambda_j\varepsilon_j x + y_j)}{||v_j||_\infty}, \qquad \lambda_j^2 = \frac{||u_j||_\infty}{A_j\, ||v_j||_\infty^{p-1}}.$$

Notice that, according to (1.16), $\lambda_j^{-2} \geqslant ||v_j||_\infty^{p-2}\, A_j^{(p-1)/p}$ so that indeed $\lambda_j \to 0$. Then, as in Case 2, we arrive at the limit system $-\Delta u = |v|^{p-2}v$, $-\Delta v = 0$. Thus $v = 0$ and again this contradicts the fact that $|\widetilde{v}_j(0)| = 1$ for every j.

Case 4. It remains to consider the case when (1.15) holds but (1.16) does not, i.e. $||v_j||_\infty\, A_j^{1/p} \leqslant C\, ||u_j||_\infty$ for some C and every j. In this case we let

$$\widetilde{u}_j(x) = \frac{u_j(\lambda_j\varepsilon_j x + x_j)}{||u_j||_\infty}, \qquad \widetilde{v}_j(x) = \frac{v_j(\lambda_j\varepsilon_j x + x_j)}{||v_j||_\infty}, \qquad \lambda_j^2 = \frac{||v_j||_\infty}{||u_j||_\infty^{p-1}},$$

where x_j is such that $|u_j(x_j)| = ||u_j||_\infty$. Observe that $\lambda_j^2 \leqslant \frac{1}{||u_j||_\infty^{p-2}} \to 0$. Up to a subsequence, let

$$\overline{\lambda} := \lim \frac{\lambda_j^2\, A_j\, ||v_j||_\infty^{p-1}}{||u_j||_\infty}.$$

Notice that $0 \leqslant \overline{\lambda} < \infty$. Then one gets a limit system $-\Delta u = \overline{\lambda}|v|^{p-2}v$, $-\Delta v = \frac{\ell_1}{p-1}|u|^{p-2}u$. By Proposition 1.4 we conclude that $u = 0$ (not that $v = 0$, since it may happen that $\overline{\lambda} = 0$) and this contradicts the fact that $|\widetilde{u}_j(0)| = 1$ for every j. The proof of Theorem 1.1 is complete. □

Remark 1.5. In the proof of Theorem 1.1 we have not used the assumption that $f(0) = 0 = f'(0)$ and $g(0) = 0 = g'(0)$, and the monotonicity of f and g was used only in Lemma 1.2; our key assumption was in fact the asymptotics (1.1). However, in Section 4 we will restrict ourselves to special nonlinearities satisfying all the assumptions in this section.

We end this section with a remark that will be useful in Section 2. Let us summarize some facts proved in Theorem 1.1. Besides (H1) and (H2), we will assume the following:

(H3) There exists $\delta > 0$ such that $f'(s)s^2 \geqslant (1+\delta)f(s)s$ for every $s \in \mathbb{R}$, and similarly for g.

It is trivial to check that this property is stable under truncation, i.e. that the modified function g_j also satisfies a similar inequality (with a smaller δ).

Now, suppose u_j and v_j are solutions of problem (P) for some sequence $\varepsilon_j \to 0$ and that $m(u_j, v_j)$ is bounded. Then we know that $||u_j||_\infty$ and $||v_j||_\infty$ are also bounded sequences. For given $x_j \in \overline{\Omega}$, let

$$\widetilde{u}_j(x) = u_j(\varepsilon_j x + x_j), \quad \widetilde{v}_j(x) = v_j(\varepsilon_j x + x_j), \quad x \in \Omega_j := \frac{1}{\varepsilon_j}(\Omega - x_j). \tag{1.17}$$

Up to subsequences, $\widetilde{u}_j \to u$ and $\widetilde{v}_j \to v$ in C^2_{loc} and u, v satisfy

$$-\Delta u + u = g(v), \quad -\Delta v + v = f(u) \ \text{ in } \omega, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \partial\omega, \tag{1.18}$$

where either $\omega = \mathbb{R}^N$ or else, in case $x_j \to x_0 \in \partial\Omega$,

$$\omega = \{x : \langle x, n(x_0)\rangle < d_0\}, \qquad \text{with } \ d_0 := \lim \frac{\mathrm{dist}\,(x_j, \partial\Omega)}{\varepsilon_j}, \tag{1.19}$$

where $n(x_0)$ is the unit outward normal at the point x_0. Moreover, (u, v) has finite index in the sense of Definition 1.3 and, thanks to assumption (H3), the latter property implies that $u, v \in H^1(\omega)$. (This is contained in the proof of Proposition 1.4 (*ii*) although a simpler, less technical proof could be provided, since we may now assume that (H1) holds with $p = q < 2N/(N-2)$.) In particular, (u, v) has finite energy, i.e.

$$I_\infty(u, v) := \int_\omega (\langle \nabla u, \nabla v\rangle + uv - F(u) - G(v)) < \infty.$$

In our next proposition we provide sufficient conditions to ensure that

$$I_j(\widetilde{u}_j, \widetilde{v}_j) := \int_{\Omega_j} (\langle \nabla \widetilde{u}_j, \nabla \widetilde{v}_j\rangle + \widetilde{u}_j\widetilde{v}_j - F(\widetilde{u}_j) - G(\widetilde{v}_j)) \to I_\infty(u, v) \tag{1.20}$$

as $j \to \infty$.

Proposition 1.6. *Assume* (H1), (H2), (H3) *and, using the previous notations, suppose that* u_j, v_j *are such that* $m(u_j, v_j) \leqslant 1$ *and* $(u, v) \neq (0, 0)$*. Then strong convergence holds, i.e. for every* $\delta > 0$ *there exist* $R > 0$ *and* $j_0 \in \mathbb{N}$ *such that*

$$\int_{\Omega_j \cap \{x:\, |x| \geqslant R\}} (f(\widetilde{u}_j)\widetilde{u}_j + g(\widetilde{v}_j)\widetilde{v}_j) \leqslant \delta, \qquad \forall j \geqslant j_0. \tag{1.21}$$

Proof. We cannot have both inequalities $I''_\infty(u,v)(u\varphi,u\varphi)(u\varphi,u\varphi) \geqslant 0$ and $I''_\infty(u,v)(v\varphi,v\varphi)(v\varphi,v\varphi) \geqslant 0$ for *every* test function φ with compact support. Otherwise, we would deduce, as in the proof of Proposition 1.4, that for some constant C independent of φ,

$$\int (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2 + f(u)u + g(v)v)\varphi^2 \leqslant C \int (u^2+v^2)|\nabla\varphi|^2, \tag{1.22}$$

implying that $u = v = 0$. Thus there exists some test function φ_1 with support in some ball of radius R_1 such that, say, $I''_\infty(u,v)(u\varphi_1,u\varphi_1)(u\varphi_1,u\varphi_1) < 0$. Since $\widetilde{u}_j \to u$ and $\widetilde{v}_j \to v$ in C^1_{loc}, also $I''_j(\widetilde{u}_j,\widetilde{v}_j)(u\varphi_1,u\varphi_1)(u\varphi_1,u\varphi_1) < 0$ for large values of j, and so, according to Lemma 1.2, since $m(u_j,v_j) \leqslant 1$ we must have $I''_j(\widetilde{u}_j,\widetilde{v}_j)(\Psi,\Psi)(\Psi,\Psi) \geqslant 0$ for any $\Psi \in H^1(\Omega_j)$ which vanishes identically in $B_{R_1}(0)$.

Now, given $\delta > 0$, choose R so large that $\int_{\omega\cap(B_{2R}(0)\setminus B_R(0))}(u^2+v^2) < \delta$. If j is large, $\int_{\Omega_j\cap(B_{2R}(0)\setminus B_R(0))}(\widetilde{u}_j^2+\widetilde{v}_j^2) < \delta$ also. Fix a cut-off function φ such that $\varphi = 0$ in $B_R(0)$ and $\varphi = 1$ in $B^c_{2R}(0)$. Since $I''_j(\widetilde{u}_j,\widetilde{v}_j)(\Psi,\Psi)(\Psi,\Psi) \geqslant 0$ for $\Psi = \varphi\widetilde{u}_j$ and for $\Psi = \varphi\widetilde{v}_j$, we can argue as in (1.22) to arrive at

$$\int_{\Omega_j\cap B^c_{2R}(0)} (|\nabla\widetilde{u}_j|^2 + |\nabla\widetilde{v}_j|^2 + \widetilde{u}_j^2 + \widetilde{v}_j^2 + f(\widetilde{u}_j)\widetilde{u}_j + g(\widetilde{v}_j)\widetilde{v}_j) \leqslant C\delta.$$

Since δ is arbitrary, this proves (1.21) (and also (1.20), since we have strong convergence in compact subsets of $\overline{\omega}$). □

Remark 1.7. In applying Proposition 1.6, one must choose x_j in such a way that $\widetilde{u}_j$ and $\widetilde{v}_j$ converge to *nonzero* functions u and v. In connection with this, it should be noted that *if* $u_j > 0$ *and* $v_j > 0$ *in* Ω *then*

$$\liminf_{j\to\infty} \min\{||u_j||_\infty, ||v_j||_\infty\} > 0. \tag{1.23}$$

This follows from standard arguments, observing that $w_j := u_j + v_j$ satisfies

$$-\varepsilon_j^2\,\Delta w_j = u_j(\frac{f(u_j)}{u_j} - 1) + v_j(\frac{g(v_j)}{v_j} - 1). \tag{1.24}$$

Indeed, it cannot happen that $-\Delta w_j < 0$ in a neighborhood of a maximum point of w_j (even at the boundary, thanks to Hopf's lemma); since the right-hand member of (1.24) is negative for small values of $u_j + v_j$ (as follows from (H1)), we conclude that $\liminf ||w_j||_\infty > 0$, say $\liminf ||u_j||_\infty > 0$. Then, since $-\varepsilon_j^2\,\Delta u_j = g(v_j) - u_j$, by the very same reason we must have $g(v_j(x_j)) \geqslant u_j(x_j) := ||u_j||_\infty$, so that $\liminf ||v_j||_\infty > 0$ also.

2. Spike-layer patterns

In this section we assume that the conditions (H1)-(H3) of Section 1 hold and combine the precedent conclusions with standard symmetry arguments in order to study the shape of *positive* solutions of our system

$$\text{(P)} \quad -\varepsilon^2\Delta u + u = g(v), \quad -\varepsilon^2\Delta v + v = f(u) \ \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \partial\Omega.$$

Theorem 2.1. *Assume* (H1), (H2), (H3) *hold and let* u_ε, v_ε *be positive solutions of* (P) *such that* $m(u_\varepsilon, v_\varepsilon) \leqslant 1$. *Suppose that there exist* ε_1, $C > 0$ *such that, for every* $0 < \varepsilon \leqslant \varepsilon_1$, u_ε *has a maximum point satisfying*

$$\operatorname{dist}(x_\varepsilon, \partial\Omega) \leqslant C\,\varepsilon. \tag{2.1}$$

Then, for any sufficiently small value of ε, x_ε *is the unique maximum point of* u_ε *and also of* v_ε, *and* $x_\varepsilon \in \partial\Omega$.

Proof. Using an argument by contradiction we see that it is enough to prove that any sequence of solutions of (P) with $\varepsilon = \varepsilon_j \to 0$ has some subsequence satisfying the final conclusions in the statement of Theorem 2.1. In the sequel we will still denote by u_j, v_j, x_j any subsequences that we may have to extract from the original sequence u_{ε_j}, v_{ε_j}, x_{ε_j}. We recall from Theorem 1.1 that we may assume that (H1) holds with $2 < p = q < 2N/(N-2)$. We split the argument into several steps.

1. It follows from (2.1) that $x_j \to P \in \partial\Omega$. For simplicity of notation, we assume that the unit outward normal at the point P is $n(P) = e_N = (0, \dots, 0, 1)$ and denote $x_N = \langle x, e_N\rangle$ for any $x \in \mathbb{R}^N$. We claim that

$$\lim_{j\to\infty} \frac{\operatorname{dist}(x_j, \partial\Omega)}{\varepsilon_j} = 0. \tag{2.2}$$

Indeed, denote by d_0 the limit above and let $\widetilde{u}_j \to u$, $\widetilde{v}_j \to v$ be the blow-up scheme mentioned in (1.17)–(1.19). Here, $\omega = \{x : x_N < d_0\}$. Since $u(0) \neq 0$ (cf. (1.23)), both u and v are positive. Reflect u and v with respect to the hyperplane $x_N = d_0$, so that the system in (1.18) will be satisfied in the whole space $\mathbb{R}^N$. Since $u, v \in H^1(\mathbb{R}^N)$, it follows from standard elliptic regularity theory that $\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0$. Since, moreover, $f(0) = 0 = f'(0)$ and $f' \geqslant 0$ (and similarly for g), it follows from [6, Theorem 2] that u is radially symmetric (and also radially strictly decreasing) with respect to its (unique) maximum point $x_0 \in \mathbb{R}^N$; the same holds for v, with respect to the same point x_0. Since $u(0) = \max_{\mathbb{R}^N} u$ by construction, we must have $x_0 = 0$. Thus, also $d_0 = 0$, since u is radially symmetric with respect to the origin and since it has been reflected with respect to $\partial\omega$.

2. Let y_j be any maximum point of v_j in $\overline{\Omega}$ and let us prove that there exists $c > 0$ such that, for every large j,

$$|x_j - y_j| \leqslant C\,\varepsilon_j. \tag{2.3}$$

To that purpose, let $\overline{u}_j \to \overline{u}$, $\overline{v}_j \to \overline{v}$ be the blow-up scheme mentioned in (1.17)–(1.19), with y_j in place of x_j. Since $\overline{v}(0) \neq 0$, it follows from Proposition 1.6 that for any $\delta > 0$ there exists $R > 0$ such that, for every large j,

$$\int_{\bar{\Omega}_j \setminus B_R(0)} (f(\overline{u}_j)\overline{u}_j + g(\overline{v}_j)\overline{v}_j) \leqslant \delta,$$

where $\bar{\Omega}_j := \frac{1}{\varepsilon_j}(\Omega - y_j)$. Denoting $z_j := \frac{y_j - x_j}{\varepsilon_j}$, the above inequality reads as

$$\int_{\Omega_j \setminus B_R(z_j)} (f(\widetilde{u}_j)\widetilde{u}_j + g(\widetilde{v}_j)\widetilde{v}_j) \leqslant \delta. \tag{2.4}$$

So, if $|z_j| \to \infty$ we conclude from (1.21) and (2.4) that for every $\delta > 0$ there exists $j_0 \in \mathbb{N}$ such that

$$\int_{\Omega_j} (f(\widetilde{u}_j)\widetilde{u}_j + g(\widetilde{v}_j)\widetilde{v}_j) \leqslant 2\delta, \qquad \forall j \geqslant j_0.$$

This implies $\int_\omega (f(u)u + g(v)v) = 0$, whence $u = v = 0$. This contradiction completes the proof of (2.3).

3. The proof now uses more or less standard arguments. At first we improve the conclusion in (2.3) by showing that if y_j is any maximum point of v_j in $\overline{\Omega}$, then

$$\lim_{j\to\infty} \frac{|x_j - y_j|}{\varepsilon_j} = 0. \tag{2.5}$$

Indeed, according to (2.3) we have that, similar to x_j, (y_j) converges to the same limit point $P \in \partial\Omega$ and, similar to (2.2), $\text{dist}(y_j, \partial\Omega)/\varepsilon_j$ is bounded. Thus, as in step 1 above, we have that $\text{dist}(y_j, \partial\Omega)/\varepsilon_j \to 0$ as $j \to \infty$. As a consequence, the blow-up sequences $\overline{u}_j$, $\overline{v}_j$ defined in step 2 have limit functions $\overline{u}$ and $\overline{v}$ defined in the same hyperplane as u and v (namely, $\omega = \{x : x_N < 0\}$) and $\overline{u}$, $\overline{v}$ attain their (unique) maximum point at the origin.

Now, according to (2.3), let $z_0 \in \mathbb{R}^N$ be such that $\frac{y_j - x_j}{\varepsilon_j} \to z_0$. Since

$$\overline{u}_j(x) = \widetilde{u}_j(x + \frac{y_j - x_j}{\varepsilon_j}) \to u(x + z_0) \quad \text{pointwise,}$$

we conclude that $\overline{u}(x) = u(x + z_0)$ (and similarly for $\overline{v}$). In particular, u has 0 and z_0 as maximum points, so that $z_0 = 0$ and this establishes (2.5).

This also shows that if y_j is any maximum point of v_j, then both sequences $\widetilde{u}_j(x) = u_j(\varepsilon_j x + x_j)$ and $\overline{u}_j(x) = u_j(\varepsilon_j x + y_j)$ have the same limit function u (and similarly for $\widetilde{v}_j$ and $\overline{v}_j$).

4. Next we prove that, for large j,

$$x_j \in \partial\Omega \qquad \text{and} \qquad y_j = x_j.$$

To prove this, we first observe that, as functions of $r = |x|$, u and v satisfy $-(r^{N-1}u')' = r^{N-1}(g(v) - u)$, $-(r^{N-1}v')' = r^{N-1}(f(u) - v)$, and $-Nu''(0) = g(v(0)) - u(0)$, $-Nv''(0) = f(u(0)) - v(0)$ (see e.g. [19, page 1452]). Since they are nonconstant functions, the local uniqueness theorem for ordinary differential equations implies that either $u''(0) \neq 0$ or $v''(0) \neq 0$. Assume the former, i.e. that u has a nondegenerate maximum point at $x = 0$. It then follows from the convergence $\widetilde{u}_j \to u$ in $C^2_{\text{loc}}(\overline{\omega})$ that u_j has a nondegenerate maximum point at x_j, over some ball $B_{\delta\varepsilon_j}(x_j)$ with a small $\delta > 0$, i.e. $\langle \nabla u_j(x), y\rangle \neq 0$ for every $x \in B_{\delta\varepsilon_j}(x_j)$, $x \neq x_j$, and for every vector $y \in \mathbb{R}^N$ with $|y| = 1$. Since $\frac{\partial u_j}{\partial n} = 0$ on $\partial\Omega$, it follows from (2.2) that $x_j \in \partial\Omega$.

Similarly, as we already proved that $\overline{u}_j \to u$, u_j has a strict *local* nondegenerate maximum point at y_j, over some small ball $B_{\delta'\varepsilon_j}(y_j)$. Since $x_j \in B_{\delta'\varepsilon_j}(y_j)$ for large j (cf. (2.5)), we must have that $y_j = x_j$.

The same conclusion can be derived in the case $v''(0) \neq 0$.

5. We have proved that any maximum point y_j of v_j must coincide with the given maximum point x_j of u_j, if j is sufficiently large. Now we can interchange the roles of u_j and v_j to conclude that our original sequence u_j, v_j, admits a subsequence satisfying all the requirements in the statement of Theorem 2.1. □

It is known that the property (2.1) is related to a "ground-state" feature of the solutions u_ε, v_ε. We prefer to discuss this matter in our next section, since it is mainly related with convexity properties of f and g rather than the positivity of the solutions.

3. An energy estimate of mountain-pass type

In this section we establish an estimate on the energy functional associated to our problem

$$\text{(P)} \quad -\varepsilon^2\Delta u + u = g(v), \quad -\varepsilon^2\Delta v + v = f(u) \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega.$$

We will assume that condition (H2) of Section 1 holds in a stronger form, namely

$$\text{(H2)}' \qquad f^2(s) \leqslant 2f'(s)F(s) \text{ for every } s \in \mathbb{R} \text{ and similarly for } g.$$

In our proof of Theorem 3.1 below, condition (H2)$'$ will play the role of the convexity-type condition that is usually assumed in the scalar case (see e.g. [10, page 884]), namely $sf'(s) \geqslant f(s)$ for every $s \in \mathbb{R}$. We observe that both conditions are satisfied in case f is convex, i.e. $f'' \geqslant 0$.

Now, let I be the energy functional associated to (P) (cf. (1.2)) and recall our notations $E = H^1(\Omega) \times H^1(\Omega) = E^- \oplus E^+$, where $E^- = \{(\phi, -\phi),\ \phi \in H^1(\Omega)\}$.

Theorem 3.1. *Assume that* (H1) *with* $2 < p = q < 2N/(N-2)$, (H2)$'$ *and* (H3) *hold and let* u, v *be solutions of problem* (P) *such that* $v \neq -u$. *Then*

$$\sup_{E^- \oplus \mathbb{R}^+(u,v)} I \;=\; I(u,v).$$

Remark 3.2. We have put some restrictions in (H1) just in order to ensure that I is indeed well defined in $H^1(\Omega) \times H^1(\Omega)$. It will be clear from the proofs that Theorems 3.1 and 3.5 hold true (with a more technical proof) under the general assumptions of (H1), provided one works with fractional Sobolev spaces. In view of Theorem 1.1, in the context of the present paper this question is irrelevant.

Proof of Theorem 3.1. For simplicity of notation, we let $\varepsilon = 1$. We must prove that, for any $\phi \in H^1(\Omega)$ and any $t \geqslant 0$, $I((\phi, -\phi) + t(u,v)) \leqslant I(u,v)$. By continuity, we may assume that $t \neq 1$, and so this is equivalent to

$$\alpha(t) := I(t(u,v) + (1-t)(\phi, -\phi)) \leqslant I(u,v) = \alpha(1). \tag{3.1}$$

It can be checked that $\alpha(t)$ has indeed a maximum point, because $\alpha(t) \to -\infty$ as $|t| \to \infty$, since $v \neq -u$ (we shall prove a more general fact in Lemma 3.4 below). We claim that

$$\alpha(0) \leqslant 0, \qquad \alpha'(1) = 0 \qquad \text{and} \qquad \alpha''(1) < 0. \tag{3.2}$$

Indeed, the second statement is obvious, the first statement follows from the fact that $F \geqslant 0$ and $G \geqslant 0$, while a direct computation leads to

$$\alpha''(1) = -2||\phi||^2 + i(f) + i(g), \tag{3.3}$$

where

$$i(f) := \int_\Omega (f(u)u - 2f(u)\phi - f'(u)(u-\phi)^2).$$

Using assumption (H3), we easily see that $i(f) \leqslant -\delta \int_\Omega f(u)u$ for some small $\delta > 0$.

Next we denote by $\beta(t)$ the function $\alpha(t)$ evaluated at points where $\alpha'(t) = 0$ (an explicit expression for $\beta(t)$ is provided below). A straightforward computation shows that

$$\beta(0) \leqslant 0, \qquad \beta(1) = \alpha(1) > 0 \qquad \text{and} \qquad \beta'(1) > 0. \tag{3.4}$$

In fact, $\beta(0) = \alpha(0) \leqslant 0$, $\beta(1) = \alpha(1)$ by definition (since $\alpha'(1) = 0$) and $\beta'(1)$ can be seen to be equal to $-\alpha''(1)/2 > 0$. We observe that indeed $\alpha(1) > 0$, since (H3) implies that $f(s)s \geqslant (2+\delta)F(s)$ (and similarly for g), so that, using (P),

$$\begin{aligned}\alpha(1) = I(u,v) &= \int_\Omega ((\frac{1}{2}f(u)u - F(u)) + (\frac{1}{2}g(v)v - G(v))) \\ &\geqslant (\frac{1}{2} - \frac{1}{2+\delta}) \int_\Omega (f(u)u + g(v)v) > 0.\end{aligned}$$

Now, suppose by contradiction that α has a maximum point $t_0 \geqslant 0$, $t_0 \neq 1$. Then either $t_0 \in]0,1[$ or else $t_0 \in]1,\infty[$. In the first case we have that $\beta(0) \leqslant 0$ and $\beta(t_0) > \beta(1)$, and so

$$\exists\, \overline{t} > 0 : \qquad \beta'(\overline{t}) = 0 \quad \text{and} \quad \beta(\overline{t}) > 0. \tag{3.5}$$

On the other hand, if $\alpha(t_0) > \alpha(1)$ and $t_0 > 1$, since $\alpha'(1) = 0$ and $\alpha''(1) < 0$, there exists $t_1 \in]1, t_0[$ such that $\alpha(t_1) < \alpha(1)$ and $\alpha'(t_1) = 0$; thus, $\beta(t_1) < \beta(1)$ and the fact that $\beta'(1) > 0$ implies that again (3.5) is satisfied (for some $\overline{t} \in]1, t_1[$).

Our final property of β contradicts (3.5) and ends the proof of Theorem 3.1: we claim that for any $t \geqslant 0$,

$$\beta'(t) = 0 \;\Rightarrow\; \beta(t) \leqslant 0. \tag{3.6}$$

We prove (3.6) by a direct computation. Namely, if t is such that $\alpha'(t) = 0$, then the quadratic part of $\alpha(t)$ has the value

$$\frac{1}{2}\int_\Omega (f(\overline{u})t(u-\phi) + g(\overline{v})t(v-\psi)) + \frac{1}{2}Q(\overline{u},\psi) + \frac{1}{2}Q(\phi,\overline{v}),$$

where Q was defined just before Theorem 1.1, $\psi := -\phi$, $\overline{u} := \phi + t(u-\phi)$ and $\overline{v} := \psi + t(v-\psi)$. Thus an explicit expression for $\beta(t)$ is

$$2\beta(t) = Q(\overline{u},\psi) + Q(\phi,\overline{v}) - 2\int_\Omega (F(\overline{u}) + G(\overline{v})) + \int_\Omega (f(\overline{u})t(u-\phi) + g(\overline{v})t(v-\psi)).$$

Similarly, if $\beta'(t) = 0$, then the quadratic terms in the expression of $\beta(t)$ can be expressed using the nonlinearities f and g, leading to

$$-2\beta(t) = 2||\phi||^2 + \gamma(f) + \gamma(g),$$

where

$$\gamma(f) = \int_\Omega (f'(\overline{u})t^2(u-\phi)^2 - 2f(\overline{u})t(u-\phi) + 2F(\overline{u})).$$

Of course, (H2)$'$ implies that $\gamma(f) \geqslant 0$. □

In Section 4 we will use a slightly more general result that we describe now. In the following we denote by $u, v \in H^1(\mathbb{R}^N_+) \cap C^2(\mathbb{R}^N_+)$ a given solution of the problem

$$(\mathrm{P})_\infty \quad -\Delta u + u = g(v), \quad -\Delta v + v = f(u) \;\text{ in } \mathbb{R}^N_+, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \;\text{ on } \partial\mathbb{R}^N_+,$$

where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$ and, following a familiar procedure, we "concentrate" u and v at a point of the boundary of Ω. Without loss of generality, we assume that $0 \in \partial\Omega$. If $\alpha : \mathbb{R}^{N-1} \to \partial\Omega$ is a C^2 diffeomorphism onto a neighborhood of $0 \in \partial\Omega$ such that $\alpha(0) = 0$ and $D\alpha(0) = Id$, let $\Phi(y,t) := \alpha(y) - t\, n(\alpha(y))$, where $n(x)$ stands for the unit outward normal of Ω at the point $x \in \partial\Omega$. Since $\frac{\partial\Phi}{\partial t}(0) = -n(0)$ and $\frac{\partial\Phi}{\partial y}(0) = Id$, Φ admits a C^2 local inverse $\Psi = \Phi^{-1} : B_r(0) \subset \mathbb{R}^N \to \mathbb{R}^N$ for some $r > 0$, such that $\Psi(0) = 0$. By construction, $\langle D\Psi(x), n(x)\rangle = (0,1) \in \mathbb{R}^{N-1} \times \mathbb{R}$

for every $x \in \partial\Omega$ lying near 0. Thus, the function $u \circ \Psi$ satisfies $\frac{\partial(u\circ\Psi)}{\partial n} = 0$ on $\partial\Omega$ (near the point zero).

Now, given a sequence $\varepsilon_j \to 0^+$, let $\Omega_j := \frac{1}{\varepsilon_j}\Omega$ and

$$u_j(x) = u\left(\frac{\Psi(\varepsilon_j x)}{\varepsilon_j}\right)\, \chi(\Psi(\varepsilon_j x)) \in H^1(\Omega_j), \tag{3.7}$$

where $\chi \in C^\infty(\mathbb{R}^N)$ is radially symmetric (in particular, $\frac{\partial\chi}{\partial n} = 0$ on $\partial\mathbb{R}^N_+$), $\chi = 1$ in $B_{r/2}(0)$ and $\chi = 0$ in $\mathbb{R}^N \setminus B_r(0)$. By construction, $\frac{\partial u_j}{\partial n}(x) = 0$ for every $x \in \partial\Omega_j$. We define v_j in a similar way. We list some relevant properties of u_j and v_j. The proofs are elementary and so we merely sketch them.

Lemma 3.3. *Assume* (H1) *holds with* $2 < p = q < 2N/(N-2)$, *let* $u, v \in H^1(\mathbb{R}^N_+) \cap C^2(\mathbb{R}^N_+)$, $v \neq -u$, *be a solution of problem* $(\mathrm{P})_\infty$ *and consider* u_j, v_j *as in* (3.7). *Then, as* $j \to \infty$, *and uniformly for* $\phi_j \in H^1(\Omega_j)$, $||\phi_j|| \leqslant 1$,

(*i*) $\int_{\Omega_j} |u - u_j|^p = \mathrm{o}(1)$.

(*ii*) $\liminf_{j\to\infty} \frac{1}{|s_j|^p} \int_{\Omega_j} F(t_j\phi_j + s_j u_j) > 0$, *if* $t_j/s_j \to 0$.

(*iii*) $\liminf_{j\to\infty} \frac{1}{|t_j|^p} \int_{\Omega_j} (F(t_j\phi_j + s_j u_j) + G(-t_j\phi_j + s_j v_j)) > 0$ *if* $|t_j| \to \infty$ *and* $s_j/t_j \to \ell \in \mathbb{R}$, $\ell \neq 0$.

(*iv*) $\int_{\Omega_j} f(u_j)\phi_j = \int_{\Omega_j} \frac{f(\overline{u}_j)}{\overline{u}_j} u_j\phi_j + \mathrm{o}(1)$ *if* $\overline{u}_j := \phi_j + t_j(u_j - \phi_j)$, $t_j \to 1$.

Proof. (*i*) Using Lebesgue's dominated convergence theorem it follows easily that $\int_{\Omega_j \cap B_R(0)} |u - u_j|^p = \mathrm{o}(1)$ for any $R > 0$. It is also easy to see that given $\varepsilon > 0$ there exists $R > 0$ such that $\int_{\Omega_j \setminus B_R(0)} |u_j|^p \leqslant \varepsilon$ for every large j.

(*ii*) Denote $\mathcal{A}_j := \{x \in \Omega_j : |t_j\phi_j(x) + s_j u_j(x)| \geqslant 1\}$. Using (H1) we see that

$$\begin{aligned}
\frac{1}{|s_j|^p}\int_{\Omega_j} F(t_j\phi_j + s_j u_j) &= \frac{1}{|s_j|^p}\int_{\mathcal{A}_j} F(t_j\phi_j + s_j u_j) + \mathrm{o}(1) \\
&\geqslant \frac{c}{|s_j|^p}\int_{\mathcal{A}_j} |t_j\phi_j + s_j u_j|^p + \mathrm{o}(1) \\
&= \frac{c}{|s_j|^p}\int_{\Omega_j} |t_j\phi_j + s_j u_j|^p + \mathrm{o}(1) \\
&= c\int_{\Omega_j} |u_j|^p + \mathrm{o}(1) = c\int_{\mathbb{R}^N_+} |u|^p > 0.
\end{aligned}$$

(*iii*) If the claim was not true then, as in (*ii*), we would conclude that

$$\int_{\Omega_j} |\phi_j + \frac{s_j}{t_j} u_j|^p + \int_{\Omega_j} |-\phi_j + \frac{s_j}{t_j} v_j|^p = \mathrm{o}(1).$$

Then $\frac{s_j}{t_j}\int_{\Omega_j} |u_j + v_j|^p = \mathrm{o}(1)$, whence $\int_{\mathbb{R}^N_+} |u+v|^p = 0$, contradicting our assumption that $v \neq -u$.

(*iv*) We have $u_j - \overline{u}_j = \lambda_j\,(\overline{u}_j - \phi_j)$, with $\lambda_j := (1 - t_j)/t_j \to 0$, and

$$\int_{\Omega_j} (f(u_j)\phi_j - \frac{f(\overline{u}_j)}{\overline{u}_j} u_j\phi_j) = \int_{\Omega_j} (f(u_j) - f(\overline{u}_j))\,\phi_j - \int_{\Omega_j} \frac{f(\overline{u}_j)}{\overline{u}_j}\lambda_j\,(\overline{u}_j - \phi_j)\,\phi_j.$$

Both integrals above can be bounded by

$$C\,\lambda_j \int_{\Omega_j} (1 + |u_j|^{p-2} + |\phi_j|^{p-2})\,(|u_j| + |\phi_j|)\,\phi_j.$$

Since (ϕ_j) is bounded in $L^p(\Omega_j)$, we conclude using Hölder's inequality. □

We denote by I_j (resp. I_∞) the energy functional obtained from (1.2) by setting $\varepsilon = 1$ and by replacing Ω by Ω_j (resp. by replacing Ω by $\mathbb{R}^N_+$).

Lemma 3.4. *Under the assumptions of Lemma* 3.3*, if* $\phi_j \in H^1(\Omega_j)$ *and* $||\phi_j|| = 1$*, then*

$$I_j(t_j\,(\phi_j, -\phi_j) + s_j\,(u_j, v_j)) \to -\infty \quad \text{as } |t_j| + |s_j| \to \infty.$$

Proof. This follows readily from (ii) and (iii) in Lemma 3.3. □

Theorem 3.5. *Assume that* (H1) *with* $2 < p = q < 2N/(N-2)$, (H2)$'$ *and* (H3) *hold, let* $u, v \in H^1(\mathbb{R}^N_+) \cap C^2(\mathbb{R}^N_+)$*,* $v \neq -u$*, be a solution of problem* $(\mathrm{P})_\infty$ *and consider* u_j*,* v_j *as in* (3.7)*. Then, as* $j \to \infty$*,*

$$\sup_{E^- \oplus \mathbb{R}^+(u_j, v_j)} I_j \;=\; I_j(u_j, v_j) + \mathrm{o}(1) \;=\; I_\infty(u, v) + \mathrm{o}(1).$$

Proof. 1. Let $t_j\,(\phi_j, -\phi_j) + s_j\,(u_j, v_j)$ be a maximizing sequence for the supremum above, with $t_j \in \mathbb{R}$, $s_j \geqslant 0$ and $||\phi_j|| = 1$. According to Lemma 3.4, (t_j) and (s_j) are bounded. Suppose first that $s_j \to 1$ and consider the real function

$$\theta_j(t) = I_j(t\,(\phi_j, -\phi_j) + s_j\,(u_j, v_j)), \qquad t \in \mathbb{R}.$$

Let T_j be a maximum point for θ_j. A similar (and easier) argument as in the proof of (iv) in Lemma 3.3 shows that $\theta_j'(0) \to 0$, since $s_j \to 1$. Since moreover $\theta_j'(T_j) = 0$ and $\theta_j''(t) \leqslant -2$ for every j and every t, it follows that $T_j \to 0$. Then, clearly, $\theta_j(T_j) = I_j(u_j, v_j) + \mathrm{o}(1)$ and so $\theta_j(t_j) = I_j(u_j, v_j) + \mathrm{o}(1)$ as well.

2. Assume now that $\liminf_{j\to\infty} |s_j - 1| > 0$. Observing that $t_j\phi_j + s_j u_j = \psi_j + s_j(u_j - \psi_j)$, where $\psi_j = \frac{t_j}{1-s_j}\,\phi_j$ is bounded, to complete the proof of Theorem 3.5 it remains to show that, for any $C > 0$ and any $\phi_j \in H^1(\Omega_j)$ with $||\phi_j|| \leqslant C$,

$$I_j(t\,(u_j, v_j) + (1-t)\,(\phi_j, -\phi_j)) \leqslant I_j(u_j, v_j) + \mathrm{o}(1), \quad \forall t \geqslant 0.$$

That is, denoting by $\alpha_j(t)$ the left-hand member above and by selecting a maximum point t_j of α_j, we must prove that

$$\alpha_j(t_j) \leqslant \alpha_j(1) + \mathrm{o}(1). \tag{3.8}$$

The argument is very similar to the one in the proof of Theorem 3.1 and so we only stress the differences. First, it is clear that (3.8) holds in case $t_j \to 1$. So, in view of a contradiction, suppose that

$$\liminf_{j\to\infty} |t_j - 1| > 0 \qquad \text{and} \qquad \liminf_{j\to\infty} |\alpha_j(t_j) - \alpha_j(1)| > 0. \tag{3.9}$$

Next, we observe that, similar to (3.2),

$$\alpha_j(0) \leqslant 0, \qquad \alpha_j'(1) = \mathrm{o}(1) \qquad \text{and} \qquad \sup_{|t-1|<\varepsilon} \alpha_j''(t) < 0, \tag{3.10}$$

for some small $\varepsilon > 0$. Indeed, it can be observed that u_j and v_j satisfy, over Ω_j,

$$-\Delta u_j + u_j = g(v_j) + \mu_j(x), \qquad -\Delta v_j + v_j = f(u_j) + \nu_j(x), \tag{3.11}$$

where $\mu_j(x)$ and $\nu_j(x)$ are o(1), in the sense that $\int_{\Omega_j} |\mu_j(x)\phi_j(x)|\,dx = \mathrm{o}(1)$ as $j \to \infty$ and similar to $\nu_j(x)$, provided $||\phi_j||$ is bounded; this follows by straightforward computations, using the definitions of u_j and v_j and also (i) in Lemma 3.3. Then, for example in computing $\alpha_j''(t)$, using (3.11) we arrive at a similar expression as in (3.3), with $i(f)$ replaced by

$$i(f)_j = \int_{\Omega_j} (f(u_j)u_j - 2f(u_j)\phi_j - f'(\overline{u}_j)(u_j - \phi_j)^2) + \mathrm{o}(1),$$

where $\overline{u}_j = \phi_j + t(u_j - \phi_j)$. But, thanks to assumption (H3) and to (iv) in Lemma 3.3, if t is close to 1,

$$\begin{aligned} i(f)_j &= \int_{\Omega_j} \left(\frac{f(\overline{u}_j)}{\overline{u}_j}\, u_j^2 - 2\frac{f(\overline{u}_j)}{\overline{u}_j}\, u_j\, \phi_j - f'(\overline{u}_j)\,(u_j - \phi_j)^2 \right) + \mathrm{o}(1) \\ &\leqslant -\delta \int_{\Omega_j} \frac{f(\overline{u}_j)}{\overline{u}_j}\, u_j^2 + \mathrm{o}(1) \\ &= -\delta \int_{\Omega_j} f(u_j)\, u_j + \mathrm{o}(1) = -\delta \int_{\mathbb{R}^N_+} f(u)\, u + \mathrm{o}(1), \end{aligned}$$

and this leads to the last inequality in (3.10). We observe that (3.9) and (3.10) imply that

$$\exists \widetilde{t}_j \text{ (lying between } t_j \text{ and } 1) \ : \alpha_j'(\widetilde{t}_j) = 0 \ \text{ and } \ \alpha_j(\widetilde{t}_j) < \alpha_j(1) - \rho, \tag{3.12}$$

for every large j and some positive constant ρ independent of j.

Now, denote by β_j the function α_j evaluated at points where $\alpha_j' = 0$. Using (3.11), it follows as in (3.4) that

$$\beta_j(0) \leqslant 0,\ \beta_j(1) = \alpha_j(1) + \mathrm{o}(1) > 0 \ \text{ and } \ 2\beta_j'(1) = -\alpha_j''(1) + \mathrm{o}(1) > 0. \tag{3.13}$$

Then, using (3.12) and (3.13) we conclude as in (3.5) that β_j' vanishes at some point $\overline{t}_j$ such that $\beta_j(\overline{t}_j) > 0$. But the property in (3.6) was derived with no integration by parts (i.e. (3.11) was not used), and so it holds for β_j as well. Since this property is in contradiction with the existence of the point $\overline{t}_j$, this establishes (3.8) and completes the proof of Theorem 3.5. □

4. Proof of the main result

Consider the problem

$$\text{(P)} \quad -\varepsilon^2 \Delta u + u = g(v), \quad -\varepsilon^2 \Delta v + v = f(u) \ \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \ \text{ on } \partial\Omega.$$

We assume:

(H) $f, g \in C^1(\mathbb{R})$, $f(0) = 0 = f'(0)$, $g(0) = 0 = g'(0)$ and there exist real numbers ℓ_1, $\ell_2 > 0$ and $p, q > 2$ such that $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$ and

$$\lim_{|s|\to\infty} \frac{f'(s)}{|s|^{p-2}} = \ell_1, \qquad \lim_{|s|\to\infty} \frac{g'(s)}{|s|^{q-2}} = \ell_2. \tag{4.1}$$

Moreover, for some $\delta > 0$ and every $s \in \mathbb{R}$, $s \neq 0$,

$$f(s)s \geqslant (2+\delta)\, F(s) > 0 \quad \text{ and } \quad f^2(s) \leqslant 2f'(s)F(s), \tag{4.2}$$

and similarly for g.

Clearly, the conditions in (H) are equivalent to conditions (H1), (H2)$'$ and (H3) in the preceding sections. Our main result includes Theorem 0.1 as a special case (see also Remark 4.2 for a related, more simple, result).

Theorem 4.1. *Under assumptions* (H)*, there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *problem* (P) *has nonconstant positive solutions* u_ε*,* $v_\varepsilon \in C^2(\overline{\Omega})$*. Moreover, both functions* u_ε *and* v_ε *attain their maximum value at some unique and common point* $x_\varepsilon \in \partial\Omega$*.*

Proof. Set $f(s) = g(s) = 0$ for $s \leqslant 0$. We use the framework and notations introduced in Section 1. In view of Theorem 1.1, we assume that (4.1) holds with $2 < p = q < 2N/(N-2)$. Using our assumptions, it is well known (and easy to ckeck) that the energy functional I associated to problem (P) (cf. (1.2)) satisfies the conditions of Benci-Rabinowitz's saddle point theorem, namely: the Palais-Smale condition is satisfied and

$$I \leqslant 0 \quad \text{in } E^-, \qquad I \geqslant \rho > 0 \quad \text{in } E^+ \cap \partial B_r(0),$$

for some small $r > 0$, $\rho > 0$ (which may depend on ε); moreover, if $R = R(\varepsilon) > 0$ is sufficiently large and $e = (e_1, e_2) \in E$, $e_1 > 0$, $e_2 > 0$, then

$$\sup_{(E^- \oplus \mathbb{R}^+ e) \cap \partial B_R(0)} I \leqslant 0.$$

Then, according to [1, Theorem 1.1], I has a critical point having relative Morse index $\leqslant 1$ whose energy value lies between ρ and $\sup_{E^- \oplus \mathbb{R}^+ e} I$.

Let $\mathcal{N} := \{z \in E : I'(z) = 0,\ I(z) \geqslant \rho\}$. Then $\mathcal{N}$ is nonempty and, since I satisfies the Palais-Smale condition,

$$c := \inf_{\mathcal{N}} I$$

is attained at some critical point $(u_\varepsilon, v_\varepsilon) \in \mathcal{N}$. Since $f = g = 0$ over $\mathbb{R}^-$, $u_\varepsilon > 0$ and $v_\varepsilon > 0$. By Theorem 3.1, $c = I(u_\varepsilon, v_\varepsilon) = \sup_{E^- \oplus \mathbb{R}^+(u_\varepsilon, v_\varepsilon)} I$ and so, again by [1, Theorem 1.1] we may assume that $m(u_\varepsilon, v_\varepsilon) \leqslant 1$.

Let x_ε be any maximum point of u_ε in $\overline{\Omega}$. Suppose that

$$\frac{\operatorname{dist}(x_j, \partial\Omega)}{\varepsilon_j} \to \infty \tag{4.3}$$

for some sequence $x_j = x_{\varepsilon_j}$, $\varepsilon_j \to 0$. Let $\widetilde{u}_j(x) = u_{\varepsilon_j}(\varepsilon_j x + x_j) \to u$, $\widetilde{v}_j(x) = v_{\varepsilon_j}(\varepsilon_j x + x_j) \to v$ be the blow-up scheme described in (1.17)–(1.19). In our case, $\omega = \mathbb{R}^N$ and, by [6, Theorem 2], we may assume that both u and v are radially symmetric with respect to the origin. According to (1.20),

$$\begin{aligned}
I_j(\widetilde{u}_j, \widetilde{v}_j) + \mathrm{o}(1) &= \int_{\mathbb{R}^N} (\langle \nabla u, \nabla v \rangle + uv - F(u) - G(v)) \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{2} f(u)u - F(u)\right) + \int_{\mathbb{R}^N} \left(\frac{1}{2} g(v)v - G(v)\right) \\
&= 2 \int_{\mathbb{R}^N_+} \left(\frac{1}{2} f(u)u - F(u)\right) + 2 \int_{\mathbb{R}^N} \left(\frac{1}{2} g(v)v - G(v)\right) \\
&> \int_{\mathbb{R}^N_+} \left(\frac{1}{2} f(u)u - F(u)\right) + \int_{\mathbb{R}^N} \left(\frac{1}{2} g(v)v - G(v)\right) \\
&= \int_{\mathbb{R}^N_+} (\langle \nabla u, \nabla v \rangle + uv - F(u) - G(v)) .
\end{aligned}$$

We have used the first inequality in (4.2). On the other hand, by our construction of the critical level c and by Theorem 3.5,

$$I_j(\widetilde{u}_j, \widetilde{v}_j) \leqslant \int_{\mathbb{R}^N_+} (\langle \nabla u, \nabla v \rangle + uv - F(u) - G(v)) + \mathrm{o}(1).$$

This contradicts the strict inequality above and shows that (4.3) cannot hold. The conclusion that the solutions are nonconstant, as well as the remaining conclusions in Theorem 4.1 then follow from Theorem 2.1. □

Remark 4.2. We observe that the second condition in (4.2) is not needed if one merely seeks for nonconstant positive solutions of problem (P), without concern about the location of the maximum points of the solutions. For example, assume that (H1), (H2) and (H3) hold (cf. Section 1). Then, if u, v are positive constant solutions of (P) we have that

$$f'(u) + g'(v) \geqslant (1+\delta)\left(\frac{f(u)}{u} + \frac{g(v)}{v}\right) = (1+\delta)\left(\frac{v}{u} + \frac{u}{v}\right) \geqslant 2\,(1+\delta)$$

and so, using (1.3) with $\psi = \phi$, we see that $m(u, v) \geqslant 2$ if ε is sufficiently small, while (P) has nonzero positive solutions having relative Morse index $\leqslant 1$, as we saw in the proof of Theorem 4.1.

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