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ENTIRE SOLUTIONS OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS AND FACTORIZATION OF PARTIAL DERIVATIVES

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Abstract. We show that the problem of characterizing entire solutions of certain partial differential equations and the problem of characterizing common right factors of partial derivatives of meromorphic functions in $\mathbf{C}^2$ are closely related, and characterizations will be given using their relations.

1. Introduction

In this paper, we will consider the following two problems: (a) characterize entire solutions of certain well-known partial differential equations; and (b) characterize common right factors (in the sense of composition) of partial derivatives of meromorphic functions in $\mathbf{C}^2$. We will show that while these two problems are of independent interests in partial differential equations and complex analysis, they are closely related and characterizations will be given using their relations.

We begin with the Fermat type partial differential equations

$$(u_{z_1})^m + (u_{z_2})^n = 1, \tag{1.1}$$

defined using the Fermat type varieties

$$z_1^m + z_2^n = 1 \tag{1.2}$$

in $\mathbf{C}^2$ by analogy with the well-known equation in the Fermat's last theorem, where $u_{z_j} = \frac{\partial u}{\partial z_j}$. The partial differential equations (1.1) in the real variable case arise in wave propagation theory and in the study of characteristic surfaces; and when $m = n = 2$ it is one of the main equations of geometric optics (see e.g. [3] and [5]). The equations (1.1) are clearly related to the functional equations $f^m + g^n = 1$. The study of these equations goes back to Montel ([13], [8]) and Cartan ([2]), who showed that entire solutions f and g of the equations $f^m + g^n = 1$ must be both constant for the cases $m = n \geq 3$ and for the more general cases $\frac{1}{m} + \frac{1}{n} < 1$, respectively. As a matter of fact, when $\frac{1}{m} + \frac{1}{n} < 1$, one can show that the variety $z_1^m + z_2^n = 1$ is a Kobayashi hyperbolic manifold in $\mathbf{C}^2$, which implies that there are no non-constant entire holomorphic mappings (f, g) to this manifold ([16]). As a consequence, the partial differential equation $(u_{z_1})^m + (u_{z_2})^n = 1$ does not have any non-linear entire solutions when $\frac{1}{m} + \frac{1}{n} < 1$. In the "critical" case that

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$m = n = 2$, the relation between entire solutions of (1.1) and of (1.2) is more subtle and suggestive (cf. Theorem 2.1 and Corollary 2.2 below). On one hand, the functional equation $f^2 + g^2 = 1$ defined by (1.2) has obviously non-constant entire solutions $f = \cos h$ and $g = \sin h$ for any non-constant entire function h in $\mathbf{C}^2$. [Moreover, any entire solutions f and g of the equation $f^2 + g^2 = 1$ can be written as $f = \cos h$ and $g = \sin h$ for some entire function h. The proof of this fact is immediate (cf. [12], [15]): Since $(f + ig)(f - ig) = 1$, $f + ig = e^{ih}$ and then $f - ig = e^{-ih}$ for an entire function h, which clearly implies the result by solving f and g from the last two equations.] On the other hand, any entire solutions of the partial differential equation (1.1) are necessarily linear when $m = n = 2$ ([10]).

The motivation for writing this paper was provided by our different proof of the above result when $m = n = 2$, which follows almost immediately from an application of Nevanlinna theory (see Corollary 2.2). The proof is based on the observation that in this case the above-mentioned solutions $f = \cos h$ and $g = \sin h$ of $f^2 + g^2 = 1$, and thus the partial derivatives u_{z_1} and u_{z_2} of an entire solution u of (1.1), have a common right factor h in the sense of composition. This observation leads us to relate the characterization of the entire solutions of the partial differential equations with the characterization of the entire functions in $\mathbf{C}^2$ whose partial derivatives have a common right factor. The latter is a factorization problem for entire functions. We refer the reader to [6] and [11] and extensive references therein for previous work on factorization of entire and meromorphic functions. The problem of characterizing entire functions whose partial derivatives have a common right factor will, in turn, lead us to consider characterizations of entire solutions of another interesting partial differential equation, namely, $u_{z_1} = Cu^m u_{z_2}$, which is the well-known inviscid Burgers equation when $m = 1$, arising in the study of the motion of fluids of small viscosity in real variables (see e.g. [14]).

We proceed as follows. We first establish a theorem (Theorem 2.1) on the linearity of a meromorphic function in $\mathbf{C}^2$ when its partial derivatives have a common right factor. This theorem will immediately yield the linearity of entire solutions of (1.1) when $m = n = 2$ (Corollary 2.2). Then we characterize, in Theorem 2.3, all entire solutions of the partial differential equation $u_{z_1} = Cu^m u_{z_2}$. Using this characterization together with Theorem 2.1 we will then be able to characterize common right factors of partial derivatives of an arbitrary meromorphic function in $\mathbf{C}^2$ (Theorem 2.4); and we then use Theorem 2.4 to characterize all meromorphic functions in $\mathbf{C}^2$ whose partial derivatives have a common right factor (Theorem 2.5). As a consequence, Theorem 2.5 describes the meromorphic solutions u in $\mathbf{C}^2$ of the system of the partial differential equations of the form $\frac{\partial u}{\partial z_j} = f_j(g), j = 1, 2$ (Corollary 2.6). Also, as a corollary we characterize entire solutions of the partial differential equation $(u_{z_1})^m + (u_{z_1})^n = 1$ for arbitrary positive integers m and n (Corollary 2.7). All the results obtained in this paper either appear as both necessary and sufficient characterizations or are shown to be complete by examples. The detailed results will be stated in the next section, and the proofs will be given in the last section.

As the paper deals with partial differential equations, as well as entire and meromorphic functions, some results and tools from partial differential equations and complex variables will be needed. Besides that, we will also employ Nevanlinna theory in the proofs. We will assume that the reader is familiar with basic elements of these topics (see e.g. [1], [9], [16], [17]).

2. Results

We start with the following Theorem 2.1 concerning factorization of partial derivatives of meromorphic functions. Recall that an entire or a meromorphic function $u = f(g)$ in $\mathbf{C}^n$ is said to have f and g as left and right factors, respectively, if f is a non-linear meromorphic function in the complex plane $\mathbf{C}$ and g is an entire function in $\mathbf{C}^n$. The left factor f is required to be non-linear to avoid trivial factorizations like $u = f(g)$ with $f(w) = w$ in $\mathbf{C}$ and $g(z) = u(z)$. Note, however, that a factorization of u with a linear right factor in $\mathbf{C}^n$ is not a trivial factorization and is not always possible.

Theorem 2.1. *Let u be a meromorphic function in $\mathbf{C}^2$. If $u_{z_1} = f_1(g)$ and $u_{z_2} = f_2(g)$ have a common right factor g, and $\frac{f_1'}{f_2'}$ is transcendental, then g must be a constant and thus u must be a linear function in $\mathbf{C}^2$.*

As mentioned in the Introduction, Theorem 2.1 immediately yields the linearity of entire solutions of (1.1) with $m = n = 2$:

Corollary 2.2. *Any entire solution of the partial differential equation $(u_{z_1})^2 + (u_{z_2})^2 = 1$ in $\mathbf{C}^2$ is linear.*

Proof. Let u be an entire solution of the equation. Then $u_{z_1} = \cos h$ and $u_{z_2} = \sin h$, where h is an entire function in $\mathbf{C}^2$. We see that u satisfies the conditions in Theorem 2.1. Thus, u is linear. □

Remark. It is easy to see that the condition that $\frac{f_1'}{f_2'}$ is transcendental cannot be dropped in Theorem 2.1. For example, for the entire function $u = \sin e^{z_1+z_2}$, we have that $u_{z_1} = e^{z_1+z_2} \cos e^{z_1+z_2} = u_{z_2}$. Thus, the partial derivatives have a common right factor $e^{z_1+z_2}$. However, this right factor is not constant and u is not linear. Thus, it is a natural goal to characterize common right factors of partial derivatives of an arbitrary meromorphic function and, furthermore, characterize all meromorphic functions whose partial derivatives have a common right factor. We are able to attain this goal by utilizing the following characterization of entire solutions of the inviscid Burgers equations $u_{z_1} = Cu^m u_{z_2}$, which is of independent interest by itself and will be also used in Corollary 2.7 to characterize entire solutions of $(u_{z_1})^m + (u_{z_1})^n = 1$.

Theorem 2.3. *A function u is an entire solution of the partial differential equation $u_{z_1} = Cu^m u_{z_2}$ in $\mathbf{C}^2$, where $C \neq 0$ is a constant and $m \geq 0$ is an integer, if and only if u is a constant when $m > 0$; and $u = f(z_2 + Cz_1)$ when $m = 0$, where f is an entire function in the complex plane.*

Theorem 2.3 together with Theorem 2.1 enables us to characterize common right factors of partial derivatives of an arbitrary meromorphic function in $\mathbf{C}^2$.

Theorem 2.4. *Let u be a meromorphic function in $\mathbf{C}^2$. If $u_{z_1} = f_1(g)$ and $u_{z_2} = f_2(g)$ have a common right factor g, then either g is a constant, or $g = f(z_2 + cz_1)$ and $f_1 = cf_2 + d$, where f is a non-constant entire function in the complex plane, and $c \neq 0$ and d are two constants.*

Remark. The case that $g = f(z_2 + cz_1)$ and $f_1 = cf_2 + d$ in Theorem 2.4 can indeed occur. For example, consider $u = (z_1 + z_2)^5 + z_1$. Then

$$u_{z_1} = 5(z_1 + z_2)^4 + 1 = (5w^2 + 1) \circ (z_1 + z_2)^2 = f_1(g)$$

and

$$u_{z_2} = 5(z_1 + z_2)^4 = (5w^2) \circ (z_1 + z_2)^2 = f_2(g),$$

where

$$g = (z_1 + z_2)^2, f_1(w) = 5w^2 + 1, f_2(w) = 5w^2.$$

Thus, u_{z_1} and u_{z_2} have a non-constant common right factor g. Clearly,

$$g = f(z_2 + cz_1), \quad f_1 = cf_2 + d$$

with $f = w^2, c = 1, d = 1$, as stated in Theorem 2.4.

Theorem 2.4 then enables us to characterize all meromorphic functions in $\mathbf{C}^2$ whose partial derivatives have a common right factor.

Theorem 2.5. *Let u be a meromorphic function in $\mathbf{C}^2$. Then its partial derivatives u_{z_1} and u_{z_2} have a common right factor if and only if u is a linear function, or $u = c_1 z_1 + f(z_2 + c_2 z_1)$, where f is a meromorphic function in the complex plane such that f' is non-linear, and c_1 and $c_2 \neq 0$ are two constants.*

Theorem 2.5 immediately yields the following consequence, which describes meromorphic solutions u to the systems of partial differential equations of the form $\frac{\partial u}{\partial z_j} = f_j(g), j = 1, 2$.

Corollary 2.6. *Let u be a meromorphic solution of the system of $\frac{\partial u}{\partial z_j} = f_j(g), j = 1, 2$, in $\mathbf{C}^2$, where f_js are non-linear meromorphic functions in $\mathbf{C}$ and*

$$g = g(z, u, u_{z_1}, u_{z_2}, u_{z_1 z_1}, u_{z_1 z_2}, u_{z_2 z_2}, \cdots)$$

is an entire function of the variables $z, u, u_{z_1}, u_{z_2}, u_{z_1 z_1}, u_{z_1 z_2}, u_{z_2 z_2}, \cdots$. Then u is either a linear function, or is of the form $u = c_1 z_1 + f(z_2 + c_2 z_1)$, where f is a meromorphic function in $\mathbf{C}$ such that f' is non-linear, and c_1 and $c_2 \neq 0$ are two constants.

Proof. Let u be a meromorphic solution of the system. Then its partial derivatives u_{z_1} and u_{z_2} have a common right factor g. Thus, the conclusion follows from Theorem 2.5. □

Remark. The form $u = c_1 z_1 + f(z_2 + c_2 z_1)$ in Corollary 2.6 can indeed occur. We use the same example in the Remark for Theorem 2.4. The entire function $u = (z_1 + z_2)^5 + z_1$ is a solution of the system $\frac{\partial u(z)}{\partial z_j} = f_j(g), j = 1, 2$, with $f_1(w) = 5w^2 + 1$, $f_2(w) = 5w^2$, and $g = (z_1 + z_2)^2$. This solution u is of the form $u = c_1 z_1 + f(z_2 + c_2 z_1)$ with $f = w^5, c_1 = 1, c_2 = 1$.

Combining the known results we have the following

Corollary 2.7. *Let m and n be arbitrary positive integers. A function u is an entire solution of the partial differential equation $u_{z_1}^m + u_{z_2}^n = 1$ in $\mathbf{C}^2$ if and only if $u = c_1 z_1 + c_2 z_2 + c_3$ is a linear function when $(m, n) \neq (1, 1)$; and $u = z_1 + f(z_2 - z_1)$ when $(m, n) = (1, 1)$, where c_j's are constants satisfying that $c_1^m + c_2^n = 1$, and f is an entire function in the complex plane.*

Corollary 2.7 follows easily from the known results mentioned above and Theorem 2.3 (see the proof). The case $(m, n) \neq (1, 1)$ can also follow from a result in [7] with a different way, which gave a generalization of the result in [10] using the Hamilton-Jacobi equations.

3. Proofs of the results

Proof of Theorem 2.1. We will prove that g is a constant, which then implies that u_{z_1} and u_{z_2} are constant and thus that u is a linear function. Suppose to the contrary that g is not a constant. It is clear that $u_{z_1 z_2} = f_1'(g) g_{z_2}$ and $u_{z_2 z_1} = f_2'(g) g_{z_1}$. Thus, $f_1'(g) g_{z_2} = f_2'(g) g_{z_1}$, or

$$\frac{f_1'}{f_2'}(g) = \frac{g_{z_1}}{g_{z_2}}, \tag{2.1}$$

from which it follows that g must be transcendental since otherwise the left-hand side of (2.1) would be transcendental while the right-hand side of (2.1) would be a rational function. We then use the following theorem in our paper [4]: If F is a transcendental meromorphic function in $\mathbf{C}$ and G is a transcendental entire function in $\mathbf{C}^2$, then

$$\lim_{r\to\infty} \frac{T(r, F(G))}{T(r, G)} = +\infty.$$

Here $T(r, F)$ denotes the Nevanlinna characteristic function. Applying this result to the left-hand side of (2.1), we obtain that

$$T(r, \frac{g_{z_1}}{g_{z_2}})/T(r, g) = T(r, \frac{f_1'}{f_2'}(g))/T(r, g) \to +\infty$$

as $r \to +\infty$. However,

$$T(r, \frac{g_{z_1}}{g_{z_2}}) \leq T(r, g_{z_1}) + T(r, g_{z_2}) + O(1) = O\{T(r, g)\}$$

as $r \to \infty$ outside a set of finite Lebesgue measure by the logarithmic derivative lemma (see e.g. [18]), a contradiction. This completes the proof. $\square$

Proof of Theorem 2.3. The sufficiency is obvious. For the necessity, let $w = u(z_1, z_2)$ be an entire solution of the partial differential equation

$$u_{z_1} = C u^m u_{z_2}. \tag{2.2}$$

Note that the characteristic equations for a quasi-linear partial differential equation $a(x, y, w) w_x + b(x, y, w) w_y = c(x, y, w)$ are

$$\frac{dx}{dt} = a(x, y, w), \quad \frac{dy}{dt} = b(x, y, w), \quad \frac{dw}{dt} = c(x, y, w)$$

and the integral surface $w = u(x, y)$ is the union of characteristic curves (see e.g. [9, p. 11]). Thus, the characteristic equations for (2.2) is

$$\frac{dz_1}{dt} = 1, \frac{dz_2}{dt} = -C w^m, \frac{dw}{dt} = 0.$$

The solution u of (2.2) is a solution that obviously satisfies the following initial conditions:

$$z_1 = 0, \quad z_2 = s, \quad \text{and} \quad w = u(0, s) := f(s),$$

where s is a parameter. Clearly f is an entire function in the complex plane. We then obtain, in view of the initial conditions, the parametric representation

$$z_1 = t, \quad z_2 = -C w^m t + s, \quad w = f(s)$$

for the solution $w = u(z_1, z_2)$, from which it follows that

$$u(z_1, z_2) = f(z_2 + C u^m z_1). \tag{2.3}$$

If $m = 0$, then $u = f(z_2 + C z_1)$, which is the desired result.

Next consider the case $m > 0$. We claim that f is a polynomial. Suppose that f is transcendental. Then u will be also transcendental, since otherwise the left-hand side of (2.3) is a polynomial, while the right-hand side of (2.3) is transcendental, which is absurd. But then, in view of (2.3), we arrive at the following contradiction:

$$1 = \frac{T(r,u)}{T(r,u)} = \frac{T(r, f(z_2 + Cu^m z_1))}{T(r, z_2 + Cu^m z_1)} \frac{T(r, z_2 + Cu^m z_1)}{T(r,u)} \to +\infty,$$

using the fact that

$$\frac{T(r, f(z_2 + Cu^m z_1))}{T(r, z_2 + Cu^m z_1)} \to +\infty$$

by the theorem of [4] mentioned in the proof of Theorem 2.1, and the fact that

$$\frac{T(r, z_2 + Cu^m z_1)}{T(r,u)} \geq c_0$$

for a positive constant c_0 and large r by the properties of characteristic functions. This shows that f must be a polynomial. We can then write (2.3) to

$$u^n = a_{n-1}u^{n-1} + a_{n-2}u^{n-2} + \cdots + a_1 u + a_0, \tag{2.4}$$

where a_j's are rational functions and $n \geq 1$. We assert that u must also be a polynomial. Otherwise, we will have that $T(r, a_j) = o\{T(r,u)\}, j = 0, 1, 2, \cdots, n-1$. Note that when $u \geq 1$,

$$\begin{aligned} |u| &= |\frac{u^n}{u^{n-1}}| = |\frac{a_{n-1}u^{n-1} + a_{n-2}u^{n-2} + \cdots + a_0}{u^{n-1}}| \\ &\leq |a_{n-1}| + |a_{n-2}| + \cdots + |a_0|. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} T(r,u) &= \int_{|z|=r} \log^+ |u(z)| d\sigma + O(1) \\ &= \int_{|z|=r, |u(z)| \geq 1} \log^+ |u(z)| d\sigma + O(1) \\ &\leq \sum_{j=0}^{n-1} T(r, a_j) + O(1) = o\{T(r,u)\}, \end{aligned}$$

which implies that u is a constant, a contradiction. (Here, $d\sigma$ is the usual normalized volume form on the sphere $\{z \in \mathbf{C}^2 : |z| = r\}$.) We thus have showed that both f and u are polynomials. Clearly the degrees of the two sides of (2.3), as polynomials in z_1 and z_2, are different and thus (2.3) cannot hold unless f is a constant, which implies that u is a constant. This completes the proof. □

Proof of Theorem 2.4. If g is a constant, we have nothing to prove. Thus, in the following we assume that g is not a constant.

Since $u_{z_1 z_2} = f_1'(g) g_{z_2}$ and $u_{z_2 z_1} = f_2'(g) g_{z_1}$, we have that $f_1'(g) g_{z_2} = f_2'(g) g_{z_1}$, or $\frac{f_1'}{f_2'}(g) = \frac{g_{z_1}}{g_{z_2}}$. (Note that the left factors f_1 and f_2 are non-linear by definition, and thus f_1' and f_2' are not identically zero.) If $\frac{f_1'}{f_2'}$ is transcendental, then by Theorem 2.1, g is a constant, a contradiction. Thus, $\frac{f_1'}{f_2'}$ is a rational function. That is,

$$\frac{f_1'}{f_2'}(w) = C \frac{(w-a_1)^{n_1}(w-a_2)^{n_2} \cdots (w-a_t)^{n_t}}{(w-b_1)^{m_1}(w-b_2)^{m_2} \cdots (w-b_s)^{m_s}}, \tag{2.5}$$

where $w \in \mathbf{C}$, a_j and b_j are distinct complex numbers, C is a non-zero complex number, and m_j, n_j are non-negative integers. We then have that

$$C(g-a_1)^{n_1}(g-a_2)^{n_2}\cdots(g-a_t)^{n_t}g_{z_2} = (g-b_1)^{m_1}(g-b_2)^{m_2}\cdots(g-b_s)^{m_s}g_{z_1}. \tag{2.6}$$

It is easy to see that there is a constant α such that $g(z_1,\alpha)$ is not a constant function in z_1, since otherwise we would have that $g_{z_1}(z_1,\alpha) \equiv 0$ for all $z_1 \in \mathbf{C}$ and all $\alpha \in \mathbf{C}$. Thus $g_{z_1} \equiv 0$ in $\mathbf{C}^2$, and then $g_{z_2} \equiv 0$ in $\mathbf{C}^2$ by (2.6), which implies that g is a constant, a contradiction. Consider the one variable function $g(z_1,\alpha)$ in (2.6). It is easy to check that the two sides of (2.6), as entire functions in z_1, will have different multiplicities at a possible zero of $g(z_1,\alpha) - a_j$, which is of course absurd, and thus implies that each a_j must be a Picard value of $g(z_1,\alpha)$. However, $g(z_1,\alpha)$ has at most one Picard value. Thus, we know that the value of the index t in (2.5) is exactly 1. (Note here that we cannot argue this way directly for $g(z_1,z_2)$ in $\mathbf{C}^2$, since the multiplicity or divisor of a partial derivative of g at a zero of g might be higher than the one of g at the zero in $\mathbf{C}^2$, which however never happens in one complex variable.) Using symmetry or the same argument as above, we must also have that $s = 1$. Hence, (2.6) reduces to

$$C(g-a_1)^{n_1}g_{z_2} = (g-b_1)^{m_1}g_{z_1}. \tag{2.7}$$

We claim that at least one of n_1 and m_1 must be zero. Suppose to the contrary that $n_1 > 0$ and $m_1 > 0$. We will prove that a_1 and b_1 are both Picard values of g, which is of course impossible since g has at most one Picard value. If a_1 is not a Picard value of g, then $g(z_0) - a_1 = 0$ for some $z_0 \in \mathbf{C}^2$, while $g(z_0) \neq b_1$ since $a \neq b$. Thus, the function $G(z) := \frac{g_{z_2}(z)}{(g(z)-b_1)^{m_1}}$ is analytic in a neighborhood $N_{z_0} \subset \mathbf{C}^2$ of z_0. By (2.7), we have that

$$\frac{g_{z_1}(z)}{(g(z)-a_1)^{n_1}} = CG(z).$$

Integrating this equality, we obtain that for $z \in N_{z_0}$,

$$g(z) - a_1 = e^{Q_1(z)} \quad \text{when} \quad n_1 = 1;$$

and

$$\frac{1}{(1-n_1)(g(z)-a_1)^{n_1-1}} = Q_2(z) \quad \text{when} \quad n_1 \neq 1,$$

where Q_1 and Q_2 are analytic in N_{z_0}. We see that in the both cases, $g(z) \neq a_1$ in N_{z_0}, and, in particular, $g(z_0) \neq a_1$, a contradiction. This shows that a is a Picard value of g. The same argument will show that b is a Picard value of g. This proved our claim that at least one of n_1 and m_1 must be zero. Hence, (2.7) reduces to

$$g_{z_1} = C(g-b)^m g_{z_2},$$

where m is an integer, which might be negative, and b is a complex number. We assert that actually $m = 0$. Suppose that $m > 0$. (If $m < 0$ we proceed in the same way as below for the identity $g_{z_2} = C^{-1}(g-b)^{-m}g_{z_1}$.) Let $G = g - b$. Then G is an entire solution of the partial differential equation $u_{z_1} = Cu^m u_{z_2}$. By Theorem 2.3, when $m > 0$, G is a constant and thus g is a constant, a contradiction. Hence $m = 0$. We thus have that $g_{z_1} = Cg_{z_2}$. By Theorem 2.3 again, $g = f(z_2 + Cz_1)$, where f is an entire function, and f is non-constant. (Otherwise g is a constant.) Since $m = 0$, we also have that $\frac{f_1'}{f_2'} = C$ by (2.5), which implies that $f_1 = Cf_2 + d$, where d is a constant. The proof is thus complete. □

Proof of Theorem 2.5. The sufficiency is quite obvious. In fact, if u is linear, then u_{z_1} and u_{z_2} are constants and thus have a constant common right factor. In the second case, $u = c_1z_1 + f(z_2 + c_2z_1)$. Thus, $u_{z_1} = c_1 + c_2f'(z_2 + c_2z_1)$ and $u_{z_2} = f'(z_2 + c_2z_1)$ have a common right factor $z_2 + c_2z_1$ in view of the assumption that f' is non-linear and $c_2 \neq 0$. (Note that a left factor in a factorization is non-linear by the definition.)

Next we prove the necessity. If $u_{z_1} = f_1(g)$ and $u_{z_2} = f_2(g)$ have a common right factor g, where g is an entire function in $\mathbf{C}^2$, and f_1 and f_2 are non-linear meromorphic functions in the complex plane, then by Theorem 2.4, either g is a constant, or $g(z_1, z_2) = f_0(z_2 + cz_1)$ and $f_1 = cf_2 + d$, where f_0 is an entire function in the complex plane, and $c \neq 0$ and d are two constants.

In the first case where g is a constant, u_{z_1} and u_{z_2} are constants and thus u is a linear function, which is the desired conclusion.

In the second case where $g = f_0(z_2 + cz_1)$ and $f_1 = cf_2 + d$, we may assume that u is non-linear. (Otherwise the theorem is already true.) Then f_0 is non-constant, since otherwise g is a constant, which implies that u is linear, a contradiction. We see that

$$u_{z_1} = f_1(g) = (cf_2 + d) \circ g = cf_2(g) + d = cu_{z_2} + d$$

or

$$u_{z_1} - cu_{z_2} = d. \tag{2.8}$$

Note that the characteristic equations for the partial differential equation (2.8) is

$$\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = -c, \quad \frac{dw}{dt} = d,$$

and the solution $u(z_1, z_2)$ of (2.8) is a solution satisfying the initial conditions

$$z_1 = 0, \quad z_2 = s, \quad \text{and} \quad w = u(0, s) := f(s),$$

where s is a parameter. We claim that f is a meromorphic function in the one variable $s \in \mathbf{C}$. To see this, we only need to verify that $f(s) \not\equiv \infty$. In fact,

$$u_{z_2}(0, z_2) = f_2(g(0, z_2)) = f_2(f_0(z_2)).$$

Since $f_0(z_2)$ is a non-constant entire function in z_2 and $f_2 \not\equiv \infty$, $u_{z_2}(0, z_2)$ is a meromorphic function in z_2, which implies that $u(0, z_2)$, or $f(z_2)$, is a meromorphic function in z_2, which proves the claim. In view of the initial conditions, we obtain the parametric representation

$$z_1 = t, z_2 = -ct + s, w = td + f(s)$$

for the solution $w = u(z_1, z_2)$, from which it follows that

$$u(z_1, z_2) = dz_1 + f(z_2 + cz_1) := c_1z_1 + f(z_2 + c_2z_1), \tag{2.9}$$

where $c_1 = d$ and $c_2 = c \neq 0$. Furthermore,

$$u_{z_1} = c_1 + c_2f'(z_2 + c_2z_1), u_{z_2} = f'(z_2 + c_2z_1).$$

We see that the function f' cannot be linear, since otherwise it is easy to see that u_{z_1} and u_{z_2} cannot have a common right factor. (Note that a left factor in a factorization must be non-linear.) This completes the proof. □

Proof of Corollary 2.7. The sufficiency is obvious. We prove the necessity. The case when $m \geq 2, n \geq 2$ was mentioned in the Introduction. Thus, we may assume that $m = 1$ by symmetry. Then $u_{z_1} + u_{z_2}^n = 1$, and thus $u_{z_2 z_1} + n u_{z_2}^{n-1} u_{z_2 z_2} = 0$. Hence, u_{z_2} is an entire solution of the equation in Theorem 2.3 with $m = n-1$. By Theorem 2.3, when $n > 1$, u_{z_2} (and thus u_{z_1} from the given equation) is a constant, which implies that u is linear; and when $n = 1$, $u_{z_2} = f_1(z_2 - z_1)$ for an entire function f_1 in $\mathbf{C}$, which implies from the given equation that $u_{z_1} = 1 - f_1(z_2 - z_1)$ and then that $u = z_1 + f(z_2 - z_1)$, where f is an antiderivative of f_1. This completes the proof. □

References

1. C.A. Berenstein and R. Gay, *Complex Variables*, Springer-Verlag, New York, 1991. MR1107514 (92f:30001)
2. H. Cartan, *Sur les zeros des combinaisons linéaires de p functions holomorphes données*, Mathematica (Cluf) **7** (1933), 5-31.
3. R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol II, partial differential equations*, Interscience, New York, 1962. MR1013360 (90k:35001)
4. D.C. Chang, B.Q. Li, and C.C. Yang, *On composition of meromorphic functions in several complex variables*, Forum Math **7** (1995), 77-94. MR1307956 (95i:32004)
5. P.R. Garabedian, *Partial Differential Equations*, Wiley, New York, 1964. MR0162045 (28:5247)
6. F. Gross, *Factorization of Meromorphic Functions*, U.S. Government Printing Office, Washington, D.C., 1972. MR0407251 (53:11030)
7. J. Hemmati, *Entire solutions of first-order nonlinear partial differential equations*, Proc. Amer. Math. Soc. **125** (1997), 1483-1485. MR1396979 (97g:35022)
8. A.V. Jategaonkar, *Elementary proof of a theorem of P. Motel on entire functions*, J. London Math. Soc. **40** (1965), 166-170. MR0170007 (30:248)
9. F. John, *Partial Differential Equations*, Springer-Verlag, New York, 1982. MR0831655 (87g:35002)
10. D. Khavinson, *A note on entire solutions of the eiconal equation*, Amer. Math. Mon. **102** (1995), 159-161. MR1315596 (95j:35132)
11. B.Q. Li and C.C. Yang, *Factorization of meromorphic functions in several complex variables*, Contemporary Mathematics **142** (1993), 61-74. MR1208784 (94a:32006)
12. J. Markushevich, *Entire Functions*, Amer. Elsevier Pub. Co., New York, 1966. MR0199395 (33:7541b)
13. P. Montel, *Lecons sur les families normales de functions analytique et leurs applications*, Gauthier-Villars, Paris, 1927.
14. J. Rauch, *Partial Differential Equations*, Springer-Verlag, New York, 1991. MR1223093 (94e:35002)
15. E.G. Saleeby, *Entire and meromorphic solutions of Fermat type partial differential equations*, Analysis **19** (1999), 369-376. MR1743529 (2001c:35050)
16. B.V. Shabat, *Introduction to Complex Analysis, part II, Functions of several variables*, Translation Mathematical Monographs, Vol. 110, American Mathematical Society, Providence, RI, 1992. MR1192135 (93g:32001)
17. W. Stoll, *Introduction to the Value Distribution Theory of Meromorphic Functions*, Springer-Verlag, New York, 1982. MR0672787 (84a:32041)
18. A. Vitter, *The lemma of the logarithmic derivative in several complex variables*, Duke Math. J. **44** (1977), 89-104. MR0432924 (55:5903)

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