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AN APPROXIMATE UNIVERSAL COEFFICIENT THEOREM

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ABSTRACT. An approximate Universal Coefficient Theorem (AUCT) for certain C^*-algebras is established. We present a proof that Kirchberg-Phillips's classification theorem for separable nuclear purely infinite simple C^*-algebras is valid for C^*-algebras satisfying the AUCT instead of the UCT. It is proved that two versions of AUCT are in fact the same. We also show that C^*-algebras that are locally approximated by C^*-algebras satisfying the AUCT satisfy the AUCT. As an application, we prove that certain simple C^*-algebras which are locally type I are in fact isomorphic to simple AH-algebras. As another application, we show that a sequence of residually finite-dimensional C^*-algebras which are asymptotically nuclear and which asymptotically satisfies the AUCT can be embedded into the same simple AF-algebra.

1. INTRODUCTION

The Universal Coefficient Theorem for C^*-algebras was first introduced by L. G. Brown in the connection with the study of extensions of C^*-algebras ([Br]). A C^*-algebra, A satisfies the Universal Coefficient Theorem (UCT) if for any σ-unital C^*-algebra, B, one has the following short exact sequence:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A,B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0,$$

where γ has degree 0 and δ has degree 1. It was shown by Rosenberg and Schochet [RS] that every C^*-algebra, in the so-called "bootstrap" class satisfies the Universal Coefficient Theorem (UCT). It is a very important tool in KK-theory. It was pointed out by M. Rørdam (see [Rr2]) that a quotient of KK, namely KL, is more relevant in the theory of classification of nuclear C^*-algebras and perhaps in many other applications. Therefore it might be possible that a version of the UCT for KL would suffice for many important purposes. However, we need to define KL without using the UCT. Moreover, without the UCT, one needs to establish a number of facts before one can effectively use the functor KL.

In this paper, we will present an approximate version of the Universal Coefficient Theorem. We will show that such an approximate version of UCT for KL is indeed sufficient for many purposes, in particular, for the classification of nuclear simple C^*-algebras.

There are two possible approximate versions of UCT for KL. When A satisfies the UCT, stably approximately trivial extensions of A by SB are represented by $Pext(K_*(A), K_{*-1}(B))$, the pure extensions of the abelian groups. If

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$E_0(K_*(A), K_{*-1}(B))$ denotes the quotient

$$\mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_{*-1}(B))/Pext(K_*(A), K_{*-1}(B)),$$

one possible version of the approximate UCT is

$$0 \to E_0(K_*(A), K_{*-1}(B)) \to KL(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0.$$

A more often used UCT in classification theory is the Dadarlat-Loring multi-coefficient UCT:

$$0 \to Pext(K_*(A), K_{*-1}(B)) \to KK(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to 0.$$

The Dadarlat-Loring's UCT plays an increasingly important role in the study of classification of nuclear C^*-algebras. Correspondingly another approximate UCT for KL is

$$KL(A, B) = \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)).$$

In the absence of the UCT, we will show these two versions of approximate UCT for KL are in fact equivalent at least for separable nuclear C^*-algebras. This enables us to define the approximate UCT (AUCT) for KL. Of course C^*-algebras satisfying the UCT also satisfy the AUCT. At this moment, it is not clear that the AUCT and UCT are the same in general. Nevertheless, our results do show that, for example, a separable nuclear purely infinite simple C^*-algebra, that satisfies the AUCT satisfies the UCT. In section 3, in the absence of UCT, we strengthen our earlier uniqueness theorem for homomorphisms using KL. It is clear that $KL(A, B)$ should be the right tool for stably approximately unitarily equivalent homomorphisms (from A to B). However, a general version of this uniqueness theorem was not obtained earlier. With this uniqueness theorem, we show that for classification of separable nuclear C^*-algebras, only KL is needed. We then present a proof that the Kirchberg-Phillips classification theorem for separable nuclear purely infinite simple C^*-algebras works under the assumption that these C^*-algebras satisfy the AUCT (instead of UCT). This also applies to the case that C^*-algebras have tracial topological rank zero.

This may lead one to speculate that perhaps more nuclear C^*-algebras satisfy the approximate UCT for KL, or at least the approximate UCT for KL may be somewhat easier to establish.

It was shown by Schochet ([S1]), using L. G. Brown's mapping telescope construction, that if $A = \lim_n(A_n, h_n)$, where A_n are separable C^*-algebras satisfying the UCT and h_n are monomorphisms, then A satisfies the UCT. Suppose that A is a C^*-algebra, which satisfies the following local approximation property: for any $\varepsilon > 0$ and any finite subset $\mathcal{F}$, there is a C^*-subalgebra B of A which satisfies the AUCT such that

$$\mathrm{dist}(x, B) < \varepsilon \ \text{ for all } \ x \in \mathcal{F}.$$

C^*-algebras satisfying this kind of local approximation property have appeared more often recently. It was shown in [DE1] that there are C^*-algebras which can be locally approximated by homogeneous C^*-algebras but are not expressible as inductive limits of homogeneous C^*-algebras. The question is whether a C^*-algebra that has the above-mentioned local approximation property satisfies the approximate UCT for KL. Another purpose of this paper is to answer this question (see 6.16 and 6.17).

Let A be a unital separable simple C^*-algebra, with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and with a unique tracial state. It is shown in

[Ln9] that, if, in addition, A is locally approximable by type I C^*-algebras, then the tracial topological rank of A is zero. However, in order to apply the classification theorem in [Ln8], one needs another assumption: A satisfies the UCT. It is easy to see that A is nuclear. By 6.16, A automatically satisfies the AUCT. We will also show that the classification theorem in [Ln8] holds for C^*-algebras satisfying the AUCT (instead of the UCT) (see 5.10). Thus the classification theorem (5.10) can also be applied to the above mentioned C^*-algebra A. This is one of our original motivations for establishing the AUCT. As another application, we show that a sequence of residually finite-dimensional C^*-algebras which is asymptotically nuclear and asymptotically satisfies the (A)UCT can be embedded into one AF-algebra (see 7.8).

2. The functor $KL(-,-)$

We use the following notation:

(i) Let B be a C^*-algebra. In what follows, we denote by SB the suspension of B, i.e., $SB = C_0((-1,1), B)$.

(ii) Let $\mathcal{K}$ be the C^*-algebra of compact operators on l^2. Let B be a C^*-algebra. We denote by $M(B)$ the multiplier algebra of B. We denote by $Q(B)$ the stable quotient $M(B\otimes\mathcal{K})/B\otimes\mathcal{K}$.

(iii) Denote by $\mathcal{N}$ the so called "bootstrap" class of C^*-algebras (see [RS]).

(iv) For each n, set

$$\mathbb{I}_n = \{f \in C([0,1], M_n) : f(0) = 0 \quad \text{and} \quad f(1) \in \mathbb{C}\cdot 1\}.$$

It follows from [Lo2] that $\mathbb{I}_n$ is a semiprojective C^*-algebra.

(v) Let $\mathbb{Q}$ be the field of rational numbers. Denote by Q the UHF-algebra with $K_0(Q) = \mathbb{Q}$ and $[1_Q] = 1$.

Definition 2.1. Let A be a nuclear C^*-algebra, and let B be a σ-unital C^*-algebra. Let

$$0 \to B \to E \to A \to 0$$

be an extension and let $\tau : A \to M(B)/B$ be the Busby invariant. Recall that the above extension is said to be trivial if the extension splits. Let $\pi : M(B) \to M(B)/B$ be the quotient map. Then if E is trivial, there is homomorphism $\sigma : A \to M(B)$ such that $\tau = \pi\circ\sigma$.

An extension $\tau : A \to M(B)/B$ of A by B determined by

$$0 \to B \to E \to A \to 0$$

is said to be *approximately trivial* if there are trivial extensions $\tau_n : A \to M(B)/B$ such that

$$\|\tau_n(a) - \tau(a)\| \to 0 \quad \text{as} \quad n \to \infty$$

for every $a \in A$.

Recall that an extension $\tau : A \to Q(B)$ determined by

$$0 \to B\otimes\mathcal{K} \to E \to A \to 0$$

is stably trivial if $\tau\oplus\tau_0$ is trivial for some trivial extension $\tau_0 : A \to Q(B)$. We say τ is *stably approximately trivial* if there is a trivial extension τ_0 such that $\tau\oplus\tau_0$ is approximately trivial.

Two extensions τ_1 and τ_2 are stably unitarily equivalent if there is a trivial extension τ_0 and there is a unitary $u \in Q(B)$ such that $u^*(\tau_1\oplus\tau_0)u = \tau_2\oplus\tau_0$.

An extension τ is *absorbing* if $\tau \oplus \tau_0$ is unitarily equivalent to τ for all trivial extensions τ_0. Denote by $Ext(A, B)$ the set of stable unitary equivalence classes of extensions of A by $B \otimes \mathcal{K}$. It becomes a semigroup with zero element represented by stably trivial extensions. Denote by $Ext^{-1}(A, B)$ the set of invertible elements in $Ext(A, B)$. It is known (see [A] and [CE]) that $Ext(A, B)$ is a group, provided A is nuclear. Denote by $\mathcal{T}(A, B)$ the set of equivalence classes of stably approximately trivial extensions.

As in Schochet ([S4]), there is a metric on $Ext(A, B)$, and $\mathcal{T}(A, B)$ is precisely the closure of the zero (stably trivial extensions) element in that topology.

Lemma 2.2. *Let A and B be σ-unital C^*-algebras. The subset*

$$\mathcal{T}(A, B) \cap Ext^{-1}(A, B)$$

is a group.

Proof. It is clear that the sum of two stably approximately trivial extensions is again a stably approximately trivial extension. Thus $\mathcal{T}(A, B)$ is closed under the addition. Let $y \in Ext^{-1}(A, B) = KK^1(A, B)$. Suppose that $x \in KK(A, A)$ is the inverse of $[\mathrm{id}_A]$ in $KK(A, A)$. Then $x \times y = -y$. It follows from a result of Schochet (see [S3]) that the Kasparov product is continuous with respect to the Salinas topology. Therefore, if $y \in \mathcal{T}(A, B)$ which is in the closure of the zero, then $x \times y \in \mathcal{T}(A, B)$. Therefore $-y \in \mathcal{T}(A, B)$. $\square$

Definition 2.3. Let B be a σ-unital C^*-algebra. We identify $KK(A, B)$ with $KK^1(SA, B) = Ext^{-1}(SA, B)$. We follow the notation first introduced by Rørdam ([Rr2]). Let B be a σ-unital C^*-algebra. Define

$$KL(A, B) = Ext^{-1}(SA, B)/Ext^{-1}(SA, B) \cap \mathcal{T}(SA, B).$$

Let $\Pi : KK(A, B) \to KL(A, B)$ be the quotient map.

It should be noted that previously $KL(A, B)$ was only defined for C^*-algebras satisfying the UCT since its definition depended on the UCT (a slightly more general definition using pure extensions also appeared). Here we define KL without assuming that A satisfies the UCT nor do we assume that A is nuclear (even though we are mostly interested in the case that A is nuclear). Therefore, to effectively use KL, we need to establish certain facts without the UCT.

Let $f : A \to B$ be a homomorphism. Let $f_1 : SA \to SB$ be the induced homomorphism. For each $x \in Ext^{-1}(SB, D)$, define $f^*(x) = [\tau \circ f_1 \oplus t]$, where $[\tau] = x$ and $t : A \to Q(D)$ is a trivial extension. We see that $f^*(\mathcal{T}(SB, D)) \subset \mathcal{T}(SA, D)$. Define $f^*(\Pi(x)) = \Pi(f^*(x))$. This is well defined. From this we see that $KL(-, D)$ is a contravariant functor from σ-unital C^*-algebras to abelian groups.

Let $f : B \to D$ be a homomorphism, where both B and D are assumed to be σ-unital. If B is unital, then f extends uniquely a homomorphism $\tilde{f} : M(B \otimes \mathcal{K}) \to M(D \otimes \mathcal{K})$. Define $\bar{f} : Q(B) \to Q(D)$ to be the induced map. For $x \in Ext^{-1}(SA, B)$, we define $f_*(x) = [\bar{f} \circ \tau \oplus t]$, where $[\tau] = x$ and t is a trivial extension. If B is not unital, let $\tilde{f} : \tilde{B} \to \tilde{D}$ be the extension. We still use $\tilde{f}$ for the extension from $\tilde{B} \otimes \mathcal{K}$ to $\tilde{D} \otimes \mathcal{K}$. Note that both $\tilde{B} \otimes \mathcal{K}$ and $\tilde{D} \otimes \mathcal{K}$ are σ-unital. $\tilde{f}$ extends to a homomorphism from $M(\tilde{B} \otimes \mathcal{K})$ to $M(\tilde{D} \otimes \mathcal{K})$. We identify $M(\tilde{B} \otimes \mathcal{K})$ and $M(\tilde{D} \otimes \mathcal{K})$ with the C^*-subalgebra of $M(B \otimes \mathcal{K})$ and $M(D \otimes \mathcal{K})$, respectively. Thus $\tilde{f}$ induces a homomorphism $\bar{f} : M(\tilde{B} \otimes \mathcal{K})/B \otimes \mathcal{K} \to M(\widetilde{D} \otimes \mathcal{K})/D \otimes \mathcal{K}$. It follows from 1.4

in [H] that there is an isometry $v \in M(B \otimes \mathcal{K})$ such that

$$vM(B \otimes \mathcal{K})v^* \subset M(\tilde{B} \otimes \mathcal{K}).$$

Thus, for any $[\tau] \in Ext(SA, B)$, $v\tau(a)v^* \in M(\tilde{B} \otimes \mathcal{K})$. We define $f_*([\tau]) = [\bar{f} \circ v\tau v^*]$. From the definition of f_*, it is clear that $f_*(\mathcal{T}(SA, B)) \subset \mathcal{T}(SA, D)$. Define $f_*(\Pi(x)) = \Pi(f_*(x))$. Then, $KL(A, -) = Ext^{-1}(SA, -)$ becomes a covariant functor from σ-unital C^*-algebras to abelian groups. If we use the fact that $f_*(x) = x \times [f]$, then it also easily follows from the result of Schochet (see [S3]) that the Kasparov product is continuous.

Since f_* and f^* both map the closure of the zero to the closure of zero, from the fact that $KK(-, A)$ and $KK(A, -)$ are homotopy invariant, stable and split exact, we obtain the following:

Proposition 2.4. (1) *If A is a separable C^*-algebra, then $KL(A, -)$ is a homotopy invariant, stable and split exact covariant functor from σ-unital C^*-algebras to abelian groups.*

(2) *If A is a σ-unital C^*-algebra, then $KL(-, A)$ is a homotopy invariant, stable and split exact covariant functor from separable C^*-algebras to abelian groups.*

Definition 2.5. Let A be a σ-unital nuclear C^*-algebra and let B be a σ-unital C^*-algebra. We define $Ext_{ap}(A, B)$ to be the quotient $Ext(A, B)/\mathcal{T}(A, B)$.

Remark 2.6. The set $\mathcal{T}(A, B)$ is the same as the closure of zero in $Ext(A, B)$ in [S3]. Schochet (Theorem 3.1 in [S3]) showed that the Kasparov product is continuous. Thus, one can define the Kasparov product on KL. His proof works for the non-nuclear case, too.

3. Some uniqueness theorems

Definition 3.1. For any σ-unital C^*-algebra, A, there is an embedding $A \to A \otimes \mathcal{K}$ defined by $a \mapsto a \otimes e_{11}$, where $\{e_{ij}\}$ is a fixed system of matrix units for $\mathcal{K}$. One can then identify $M(A)$ with $(1 \otimes e_{11})M(A \otimes \mathcal{K})(1 \otimes e_{11})$ which gives an embedding $M(A) \to M(A \otimes \mathcal{K})$. Let A be a C^*-algebra, and let B be a σ-unital stable C^*-algebra. We set $C = \prod_{n=1}^\infty B$ and $C_0 = \bigoplus_{n=1}^\infty B$, the product and direct sum of a sequence of C^*-algebras each of which is B, respectively. We obtain a map $\phi_1 : M(C) \to M(C \otimes \mathcal{K})$. We will identify $M(\prod_{n=1}^\infty B)$ with $\prod_{n=1}^\infty M(B)$. Let $\phi_2 : C \otimes \mathcal{K} \to (C/C_0) \otimes \mathcal{K}$ be the quotient map. Then (by [Ped1]) there is a (surjective) homomorphism, again denoted by ϕ_2, from $M(C \otimes \mathcal{K})$ to $M((C/C_0) \otimes \mathcal{K})$ which extends ϕ_2. Put $\phi = \phi_2 \circ \phi_1$. We will use π for both quotient maps from $M(C)$ to $M(C/C_0)$ and from $M((C/C_0) \otimes \mathcal{K})$ to $M((C/C_0) \otimes \mathcal{K})/((C/C_0) \otimes \mathcal{K})$.

We have the following lemma:

Lemma 3.2.

$$\bigoplus_{n=1}^{\infty} M(B) \subset \ker \phi$$

and

$$\bigoplus_{n=1}^{\infty} M(B) + \prod_{n=1}^{\infty} B \subset \ker \pi \circ \phi.$$

Proof. Let $b \in \bigoplus_{n=1}^{\infty} M(B)$. We need to show that $\phi_1(b)c, c\phi_1(b) \in (\bigoplus_{n=1}^{\infty} B) \otimes \mathcal{K}$ for any $c \in (\prod_{n=1}^{\infty} B) \otimes \mathcal{K}$. Since we may assume that b is self-adjoint, we only need to show that $\phi_1(b)c \in \bigoplus_{n=1}^{\infty} B$. The element $\phi(b)$ may be identified with an infinite matrix whose only non-zero entry is b at the $(1,1)$ position. Write $c = (c_{ij})$, where $c_{ij} \in \prod_{n=1}^{\infty} B$. Then $\phi_1(b)c = (d_{ij})$, where $d_{ij} = 0$ if $i \neq 1$ and $d_{1j} = bc_{1j}$ for all j. Since $b \in \bigoplus_{n=1}^{\infty} M(B)$, each $d_{1j} \in \bigoplus_{n=1}^{\infty} B$. Since $c \in (\prod_{n=1}^{\infty} B) \otimes \mathcal{K}$, one obtains that $\| \sum_{j=n}^{\infty} c_{1j}^* c_{1j} \| \to 0$ as $n \to \infty$. This implies that

$$\| \sum_{j=n}^{\infty} d_{1j}^* d_{1j} \| \leq \|b\|^2 \| \sum_{j=n}^{\infty} c_{1j}^* c_{1j} \| \to 0 \text{ as } n \to \infty.$$

Therefore $(d_{ij}) \in (\bigoplus_{n=1}^{\infty} B) \otimes \mathcal{K}$.

The second inclusion holds because ϕ_1 maps $\prod_{n=1}^{\infty} B$ to $(\prod_{n=1}^{\infty} B) \otimes \mathcal{K}$ and ϕ_2 maps $(\prod_{n=1}^{\infty} B) \otimes \mathcal{K}$ to $(\prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B) \otimes \mathcal{K}$. □

Definition 3.3. Let B be a σ-unital stable C^*-algebra, and let A be a nuclear C^*-algebra. Let $\tau : A \to M(B)/B$ be an extension of A by B. Denote by $\pi_0 : M(B) \to M(B)/B$ the quotient map. Define $j : B \to \prod_{n=1}^{\infty} B$ by $j(b) = (b, b, ...,)$ (the constant sequence). Again, we let $C = \prod_{n=1}^{\infty} B$ and $C_0 = \bigoplus_{n=1}^{\infty} B$. Let $\pi_1 : C \to C/C_0$ be the quotient map. The map j extends to a map from $M(B)$ to $\prod_{n=1}^{\infty} M(B) = M(\prod_{n=1}^{\infty} B)$. Put $\Psi = \pi_1 \circ j$. Then Ψ is a monomorphism from B to C/C_0. The map π_1 can be extended to a (surjective) map from $M(\prod_{n=1}^{\infty} B)$ to $M(C/C_0)$. Thus we obtain an extension of Ψ which maps $\tilde{\Psi} : M(B) \to M(C/C_0)$. Moreover, as in 3.1, Ψ can also be extended to $\tilde{\Psi} : M(B) \to M((C/C_0) \otimes \mathcal{K})$. Denote by $\bar{\Psi}$ the induced map from $M(B)/B$ to $M(C/C_0)/(C/C_0)$ (and the induced map from $M(B)/B$ to $M((C/C_0) \otimes \mathcal{K})/((C/C_0) \otimes \mathcal{K})$). With the above notation, $\Psi_*([\tau]) = [\tilde{\Psi} \circ \tau]$. Suppose that $\phi' : A \to M(B)$ is a contractive completely positive linear map such that $\pi_0 \circ \phi = \tau$. With notation in 3.1, we also have $[\bar{\Psi} \circ \tau] = [\pi \circ \phi_2 \circ \phi_1 \circ j \circ \varphi]$, where $\pi : M((C/C_0) \otimes \mathcal{K}) \to M((C/C_0) \otimes \mathcal{K})/((C/C_0) \otimes \mathcal{K})$ is the quotient map. In other words, $\Psi_*([\tau]) = [\tilde{\Psi} \circ \tau] = [\pi \circ \phi_2 \circ \phi_1 \circ j \circ \varphi]$.

Consider $\Psi_* : Ext(A, B) \to Ext(A, \prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B)$. Using the Kasparov product, we have $\Psi_*([\tau]) = [\tau] \times [\Psi]$ (see, for example, 18.7.2 in [B]), where $[\Psi]$ is viewed as an element in $KK(B, \prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B)$.

Theorem 3.4. *Let A be a separable nuclear C^*-algebra and let B be a σ-unital stable C^*-algebra. Then*

$$\mathcal{T}(A, B) \subset \ker \Psi_*.$$

Proof. Suppose that $\tau : A \to M(B)/B$ is an extension in $\mathcal{T}(A, B)$. Let $\phi : A \to M(B)$ be a contractive completely positive linear map such that $\pi \circ \phi = \tau$, where $\pi : M(B) \to M(B)/B$. By adding an absorbing trivial extension, if necessary, we may assume that there is a sequence of trivial extensions $\tau_n : A \to M(B)/B$ such that

$$\lim_{n \to \infty} \tau_n(a) = \tau(a) \text{ for all } a \in A.$$

Let $h_n : A \to M(B)$ be a monomorphism such that $\pi \circ h_n = \tau_n$, where $\pi : M(B) \to M(B)/B$ is the quotient map. Let $H : A \to \prod_{n=1}^{\infty} M(B)$ be defined by $H(a) = (h_1(a), ..., h_n(a), ...)$. Then, with notation as in 3.3,

$$H(a) - j \circ \varphi(a) \in \bigoplus_{n=1}^{\infty} M(B) + \prod_{n=1}^{\infty} B_n$$

for all $a \in A$. Then it follows from 3.2 that

$$\pi \circ \phi_1 \circ \phi_2(H(a) - j \circ \varphi(a)) = 0 \text{ for all } a \in A.$$

This implies that

$$\Psi_*([\tau]) = [\pi \circ \phi_1 \circ \phi_2 \circ H].$$

Since $\phi_1 \circ \phi_2 \circ H$ is a homomorphism from A to $M((\prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B) \otimes \mathcal{K})$, we have that $\pi \circ \phi_1 \circ \phi_2 \circ H$ is trivial. Therefore

$$\Psi_*([\tau]) = 0.$$

□

Corollary 3.5. *Let A be a separable nuclear C^*-algebra, and let B be a σ-unital C^*-algebra. Suppose that $h_i : A \to B$ are two homomorphisms ($i = 1, 2$) such that $[h_1] = [h_2]$ in $KL(A, B)$. Let $H_i : A \to \prod_{n=1}^{\infty} B$ be the homomorphism defined by $H_i(a) = (h_i(a), h_i(a), ..., h_i(a), ...)$, $i = 1, 2$, and let $\pi_1 : \prod_{n=1}^{\infty} B \to \prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B$ be the quotient map. Then*

$$[\pi_1 \circ H_1] = [\pi_1 \circ H_2] \text{ in } KK(A, \prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B).$$

Proof. We identify $KL(A, B)$ with $Ext_{ap}(SA, B)$. Let $[h_i]$ be the image of h_i in $Ext_{ap}(SA, B)$. Note that $\pi_1 \circ H_i = \pi_1 \circ j \circ h_i = \Psi \circ h_i$ for $i = 1, 2$. Then $[\pi_1 \circ H_i] = \Psi_*([h_i]) = [h_i] \times [\Psi]$, where Ψ is defined in 3.3. Let $\tau \in KK^1(SA, B)$ be represented by $[h_1] - [h_2]$; then $[\tau] \times [\Psi] = 0$, by 3.4. Therefore $([h_1] - [h_2]) \times [\Psi] = 0$, or $\Psi_*([\tau]) = 0$. Hence $\Psi_*([h_1]) = \Psi_*([h_2])$, i.e.,

$$[\pi_1 \circ H_1] = [\pi_1 \circ H_2] \text{ in } KK(A, \prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B).$$

□

Definition 3.6. Let A and B be C^*-algebras and let $\phi : A \to B$ be a linear map. We say that ϕ is *full* if the closed ideal generated by $\phi(a)$ is B for each non-zero $a \in A$.

The following was first proved in [Ln2]. We will give an improvement in 3.9. See also the Remark 3.8 below.

Theorem 3.7. *Let A be a separable unital nuclear C^*-algebra, and let B be a unital C^*-algebra. Suppose that $h_1, h_2 : A \to B$ are two unital homomorphisms such that*

$$[h_1] = [h_2] \text{ in } KK(A, B).$$

Suppose that $h_0 : A \to B$ is a full unital monomorphism. Then, for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there is an integer n and a unitary $w \in U(M_{n+1}(B))$ such that

$$\|w^* \mathrm{diag}(h_1(a), h_0(a), \dots, h_0(a))w - \mathrm{diag}(h_2(a), h_0(a), \dots, h_0(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Remark 3.8. The version of Theorem 3.7 in [Ln2] assumed that A is a unital simple C^*-algebra, and h_0 is a unital. What one needs is that $(h_0, h_0, ..., h_0, ...)$ gives an absorbing extension. From the proof in [Ln2], it is easy to see that one only needs to assume that $\phi_0(a)$ is full. This was observed in [DE1]. In fact, in [DE1], the nuclearity of A can be replaced by the condition that homomorphisms are nuclear. A simplified and more elementary proof of 3.7 can be found in 5.6.4 in [Ln7].

A version of the following was obtained in [Ln2] (see also 5.9.9 in [Ln7]). The following theorem is much more general.

Theorem 3.9. *Let A be a separable unital nuclear C^*-algebra, and let B be a unital C^*-algebra. Suppose that $h_1, h_2 : A \to B$ are two unital homomorphisms such that*

$$[h_1] = [h_2] \text{ in } KL(A, B).$$

Suppose that $h_0 : A \to B$ is a full unital monomorphism. Then, for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there is an integer n and a unitary $W \in U(M_{n+1}(B))$ such that

$$\|W^* \mathrm{diag}(h_1(a), h_0(a), \dots, h_0(a))W - \mathrm{diag}(h_2(a), h_0(a), \dots, h_0(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof. Let $C = \prod_{n=1}^{\infty} B$, let $C_0 = \bigoplus_{n=1}^{\infty} B$ and let $\pi : C \to C/C_0$ be the quotient map. Let H_1 and H_2 be as in 3.5. Then by 3.5, $[\pi \circ H_1] = [\pi \circ H_2]$ in $KK(A, \prod_{n=1}^{\infty} B / \bigoplus_{n=1}^{\infty} B)$. Let $H_0 : A \to \prod_{n=1}^{\infty} B$ be defined by $H_0(a) = \{h_0(a)\}$ for all $a \in A$. Since h_0 is full, it follows that, for any $a \neq 0$, there are $x_1, \dots, x_n$ and $y_1, \dots, y_n \in B$ such that $\sum_{i=1}^{n} x_i h_0(a) y_i = 1_B$. Let $X_i = (x_i, x_i, ..., x_i, ...)$ and $Y_i = (y_i, y_i, ..., y_i, ...)$ in $\prod_{n=1}^{\infty} B$. Then

$$\sum_{i=1}^{n} X_i H_0(a) Y_i = 1_C.$$

This implies that H_0 is full (in $\prod_{n=1}^{\infty} B$). Thus $\pi \circ H_0$ is also full. It follows from 3.7, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is an integer n and a unitary $w \in U(M_{n+1}(C/C_0))$ such that

$$\begin{aligned} \|w^* \mathrm{diag}(\pi \circ H_1(a), \pi \circ H_0(a), ..., \pi \circ H_0(a))w \\ - \mathrm{diag}(\pi \circ H_2(a), \pi \circ H_0(a), ..., \pi \circ H_0(a))\| < \varepsilon/2 \end{aligned}$$

for all $a \in \mathcal{F}$. It follows that there is a unitary $z = (W_1, W_2, ..., W_N, ...) \in M_{n+1}(C)$ such that $\pi(z) = w$. Therefore, for sufficiently large N,

$$\|W_N^* \mathrm{diag}(h_1(a), h_0(a), \dots, h_0(a))W_N - \mathrm{diag}(h_2(a), h_0(a), \dots, h_0(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$. □

Definition 3.10. Denote by $\mathcal{C}_1$ the class of σ-unital C^*-algebras A satisfying the following:

(1) The canonical homomorphism $U(pAp)/U(pAp)_0 \to K_1(pAp)$ is an isomorphism, for all full projections $p \in A$.

(2) There is $L > 0$ such that $cel(pAp) \le L$ for all full projections $p \in A$, i.e., for every $u \in U(pAp)_0$, there are self-adjoint elements $h_1, \cdots, h_n \in pAp$ such that $\sum_{i=1}^{n} \|h_i\| \le L$ and $u = exp(ih_1) \cdot exp(ih_2) \cdots exp(ih_n)$.

(3) If p and q are two full projections in $A \otimes \mathcal{K}$ with $[p] = [q]$ in $K_0(A)$, then there is $w \in A \otimes \mathcal{K}$ such that $w^* w = p$ and $ww^* = q$.

Every purely infinite simple C^*-algebra, is in $\mathcal{C}_1$ ([Cu] and [P1]).

The following is an important result of Rørdam (Theorem 5.1 in [Rr1]):

Lemma 3.11. *Let A be in $\mathcal{C}_1$. Suppose that $h_1, h_2 : \mathcal{O}_2 \to A$ are two homomorphisms with $h_1(1) = p$ and $h_2(1) = q$. Suppose that both p and q are full. Then, for* *any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is $u \in A$ with $u^*u = p$ and $uu^* = q$ such that*

$$\|u^*h_1(a)u - h_2(a)\| < \varepsilon \quad \textit{for all } a \in \mathcal{F}.$$

Definition 3.12. Let A and B be two C^*-algebras. A homomorphism $h : A \to B$ is said to factor through $\mathcal{O}_2$ if there are homomorphisms $\phi_1 : A \to \mathcal{O}_2$ and $\phi_2 : \mathcal{O}_2 \to B$ such that $h = \phi_2 \circ \phi_1$.

By the above lemma of Rørdam, for a unital C^*-algebra, A, if $B \in \mathcal{C}_1$ and if $h_1, h_2 : A \to B$ are full homomorphisms and both factor through $\mathcal{O}_2$, then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is $u \in B$ with $u^*u = h_1(1)$ and $uu^* = h_2(1)$ such that

$$\|u^*h_1(a)u - h_2(a)\| < \varepsilon \text{ for } a \in \mathcal{F}.$$

The following is a result of Kirchberg ([K2]):

Lemma 3.13. *Let A be a unital separable nuclear purely infinite simple C^*-algebra, and let B be a unital C^*-algebra, in $\mathcal{C}_1$. Suppose that $h : A \to B$ is a unital homomorphism. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is $u \in M_2(B)$ with $u^*u = 1_B$ and $uu^* = 1 \oplus p$ such that*

$$\|u^*\mathrm{diag}(h(a), h_0(a))u - h(a)\| < \varepsilon \quad \textit{for all } a \in \mathcal{F},$$

where $h_0 : A \to pBp$ is a unital homomorphism which factors through $\mathcal{O}_2$ and $p \in B$ is a full projection. Moreover, h_0 can be chosen independent of ε and $\mathcal{F}$.

The following was proved in [P2] (Theorem 4.1.1) under the condition that $[h_1] = [h_2]$ in $KK(A, B)$ (the conclusion there is slightly stronger). This is perhaps precisely the KL analog of the asymptotic unitary equivalence result in [P2]. Moreover, it is also known that under the assumption that A satisfies the UCT, the following theorem holds (see for example, [Ln2]). However, Theorem 3.14 does not assume that A satisfies the UCT nor does it assume that $[h_1] = [h_2]$ in $KK(A, B)$.

Theorem 3.14. *Let A be a separable unital nuclear purely infinite simple C^*-algebra, and let $h_i : A \to B$ be two full homomorphisms, $i = 1, 2$, where B is a unital C^*-algebra, in $\mathcal{C}_1$. Suppose that*

$$[h_1] = [h_2] \quad \textit{in } KL(A, B).$$

Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is $u \in B$ with $u^*u = h_2(1)$ and $uu^* = h_1(1)$ such that

$$\|u^*h_1(a)u - h_2(a)\| < \varepsilon \quad \textit{for all } a \in \mathcal{F}.$$

Proof. To simplify notation, without loss of generality, we may assume that $h_1(1) = h_2(1)$. By replacing h_1 by $u^*(\mathrm{diag}(h_1 \oplus h_0)(a))u$, as in 3.13, we obtain a full homomorphism $h_0' : A \to B$ which factors through $\mathcal{O}_2$ (h_0' can be taken to be uh_0u^*). By applying 3.9, we obtain an integer n and a unitary $U \in M_{n+1}(B)$ such that

$$\|U^*\mathrm{diag}(h_1(a), h_0'(a), ..., h_0'(a))U - \mathrm{diag}(h_2(a), h_0'(a), ..., h_0'(a))\| < \varepsilon/3$$

for all $a \in \mathcal{F}$. Since all full homomorphisms that factor through $\mathcal{O}_2$ are approximately unitarily equivalent, by applying 3.13 we obtain $V \in M_{n+1}(B)$ with $V^*V = h_1(1)$ and $VV^* = h_1(1) \oplus h_0'(1) \oplus \cdots \oplus h_0'(1)$ such that

$$\|V^*\mathrm{diag}(h_1(a), h_0'(a), ..., h_0'(a))V - h_1(a)\| < \varepsilon/3$$

for all $a \in \mathcal{F}$. Similarly, there exists $W \in M_{n+1}(B)$ with $W^*W = h_2(1)$ and $WW^* = h_2(1) \oplus h_0'(1) \oplus \cdots \oplus h_0'(1)$ such that

$$\|W^*\mathrm{diag}(h_2(a), h_0'(a), ..., h_0'(a))W - h_2(a)\| < \varepsilon/3$$

for all $a \in \mathcal{F}$. Put $u = V^*UW$. Then $u^*u = h_2(1)$ and $uu^* = h_1(1)$ such that

$$\|u^*h_1(a)u - h_2(a)\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for all $a \in \mathcal{F}$. □

Remark 3.15. One could add an additional condition that there is a full embedding from $\mathcal{O}_2$ into C^*-algebras in the class $\mathcal{C}_1$. However, in Theorem 3.14, by assuming that there are full homomorphisms from purely infinite simple C^*-algebras to B, we actually assumed that there is a full embedding of $\mathcal{O}_2$ to B.

4. The definition of AUCT

In this section, we will discuss the definition of the AUCT.

Definition 4.1. Let

$$0 \to G_0 \to G \xrightarrow{p} G_1 \to 0$$

be a short exact sequence of abelian groups. So G is an extension of G_1 by G_0. Recall that the extension is said to be *pure* if for any finitely generated subgroup G_1' of G_1, there is a homomorphism $j : G_1' \to G$ such that $p \circ j = \mathrm{id}_{G_1'}$. The set of pure extensions is denoted by $Pext(G_1, G_0)$. It is a subgroup of $Ext_{\mathbb{Z}}(G_1, G_0)$. We denote by $E_0(G_1, G_0)$ the quotient $Ext_{\mathbb{Z}}(G_1, G_0)/Pext(G_1, G_0)$. The quotient map will be denoted by $\tilde{\Pi}$.

Definition 4.2. Let A be a nuclear C^*-algebra, and let B be a σ-unital C^*-algebra. Denote by γ_A (or γ if A is understood) the natural map from $Ext(A, B)$ to $\mathrm{Hom}(K_*(A), K_*(B))$. It was first observed by Larry Brown that there is a homomorphism $\kappa : \ker\gamma \to \mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(B))$. To be more specific, in the case of extensions, if we have a short exact sequence

$$0 \to B \to E \to A \to 0,$$

then the corresponding six-term exact sequence in K-theory is

$$\begin{array}{ccccc} K_0(B) & \to & K_0(E) & \to & K_0(A) \\ \uparrow_{\gamma} & & & & \downarrow_{\gamma} \\ K_1(A) & \leftarrow & K_1(E) & \leftarrow & K_1(B)\,. \end{array}$$

Thus we have $\gamma : Ext(A, B) \to Hom(K_i(A), K_{i-1}(B))$. If $\gamma = 0$, then the six-term exact sequence breaks into two short exact sequences:

$$0 \to K_0(B) \to K_0(E) \to K_0(A) \to 0 \quad \text{and} \quad 0 \to K_1(B) \to K_1(E) \to K_1(A) \to 0.$$

This gives an element in $\mathrm{Ext}_{\mathbb{Z}}(K_i(A), K_i(B))$.

If $x \in \mathcal{T}(A, B)$, then it is straightforward to prove that $\gamma_A(x) = 0$. So $\mathcal{T}(A, B) \subset \ker\gamma$.

The following is certainly known in the case that A satisfies the Universal Coefficient Theorem. But we do not assume that A satisfies the UCT below.

Lemma 4.3. *Let A be a nuclear C^*-algebra, and let B be a σ-unital C^*-algebra. Then*

$$\kappa(\mathcal{T}(A, B)) \subset Pext(K_*(A), K_*(B)).$$

Proof. We now recall 3.1 and 3.3 and use the notation there. We may assume that B is stable. Let $\tau : A \to M(B)/B$ be an absorbing approximately trivial extension. By 4.2, $\kappa(\tau)$ gives an element in $\mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(B))$:

$$0 \to K_i(B) \to G_i \to K_i(A) \to 0.$$

Let $F_i = \prod_{n=1}^{\infty} G_i$ and let $j : G_i \to F_i$ be defined by $j(g) = (g, g, ..., g, ...)$ (the constant sequence) for $i = 0, 1$. Let $N_i^{(0)} = \bigoplus_{n=1}^{\infty} K_i(B)$ and let $\eta_i : F_i \to F_i/N_i^{(0)}$ be the quotient map, $i = 0, 1$. Note that $K_i(C_0) = N_i^{(0)}$. It follows from 3.4 that $\Psi_*([\tau]) = 0$. By 2.9 in [GL1], since B is stable, $K_i(C) = \prod_{n=1}^{\infty} K_i(B)$ and $K_i(C/C_0) = \prod_{n=1}^{\infty} K_i(B)/N_i^{(0)}$. Therefore

$$0 \to \prod_{n=1}^{\infty} K_i(B)/N_i^{(0)} \to \eta_i \circ j(G_i) \to K_i(A) \to 0$$

splits for $i = 0, 1$. Let $g \in K_i(A)$ such that $k \cdot g = 0$ for some positive integer k. Suppose that $f \in j(G_i)$ such that $\eta_i(f) = g$. Then $k \cdot f \in N_i^{(0)}$. Since f is a constant sequence, this implies that $k \cdot f = 0$. Therefore the extension

$$0 \to K_i(B) \to G_i \to K_i(A) \to 0$$

is pure. □

Definition 4.4. Let A be a nuclear C^*-algebra, and let B be a σ-unital C^*-algebra. Since $\mathcal{T}(A, SB) \subset \ker \gamma_A$, we obtain a homomorphism

$$\bar{\gamma}_A : KL(A, B) \to \mathrm{Hom}(K_*(A), K_*(B)).$$

It follows from above that κ induces a homomorphism

$$\bar{\kappa} : \ker \bar{\gamma}_A \to E_0(K_*(A), K_*(B)).$$

Definition 4.5. Let C_n be a commutative C^*-algebra, with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and let $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Then $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$ (see [S2]). One has the following six-term exact sequence (see [S2]):

$$\begin{array}{ccccc} K_0(A) & \to & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \to & K_1(A) \\ \uparrow_{\mathbf{k}} & & & & \downarrow_{\mathbf{k}} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A)\,. \end{array}$$

In [DL2], $K_i(A, \mathbb{Z}/n\mathbb{Z})$ is identified with $KK^i(\mathbb{I}_n, A)$ for $i = 0, 1$.

As in [DL2], we use the notation

$$\underline{K}(A) = \bigoplus_{i=0,1,n\in\mathbb{Z}_+} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect the direct sum decomposition and the so-called Bockstein operations (see [DL2]). It follows from the definition in [DL2], that if $x \in KK(A, B)$, then the Kasparov product $KK^i(\mathbb{I}_n, A) \times x$ gives an element in $KK^i(\mathbb{I}_n, B)$ which we identify with $\mathrm{Hom}(K_i(A, \mathbb{Z}/n\mathbb{Z}), K_i(B, \mathbb{Z}/n\mathbb{Z}))$. Thus one obtains a map $\Gamma : KK(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$. It is shown by Dadarlat and Loring ([DL2]) that, if A is in $\mathcal{N}$, then, for any σ-unital C^*-algebra, B, the map Γ is surjective and $\ker \Gamma = Pext(K_*(A), K_*(B))$.

Without assuming A satisfies the UCT, we have the following:

Lemma 4.6. *Let A and B be a σ-unital C^*-algebras. Then the map $\Gamma : KK(A,B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ maps $\mathcal{T}(SA,B)$ to zero.*

Proof. Without loss of generality, we may assume that A is stable. Let $x \in KK(\mathbb{I}_n, A)$. It follows from [DL1], [Lo1] and [Lo2] that $\mathbb{I}_n$ is semiprojective and there is a homomorphism $h : \mathbb{I}_n \to A$ such that $[h] = x$ in $KK(\mathbb{I}_n, A)$. Then, in $KK(\mathbb{I}_n, A) \times KK(A,B)$, $x \times y = h^*(y)$ for all $y \in KK(A,B)$. If $y \in \mathcal{T}(SA,B)$, then, by 2.3, $h^*(y) \in \mathcal{T}(S\mathbb{I}_n, B)$. Since $\mathbb{I}_n$ is separable and nuclear, we may write $KK^1(S\mathbb{I}_n, B) = Ext(S\mathbb{I}_n, B)$. However, $S\mathbb{I}_n$ satisfies the UCT. Therefore, since $K_i(\mathbb{I}_n)$ is finitely generated, by a result of Schochet [S4] (see also 5.9.11 in [Ln7]), we have $KK^1(S\mathbb{I}_n, B) = KL(\mathbb{I}_n, B)$—this can be proved directly without using the UCT. In other words, $\mathcal{T}(S\mathbb{I}_n, B) = 0$. Therefore $[x \times \tau] = 0$. This implies that $KK(\mathbb{I}_n, A) \times y = 0$ for any $y \subset \mathcal{T}(SA,B) \subset KK(A,B)$. By the definition of Γ (see [DL2]), we conclude that $\Gamma|_{\mathcal{T}(SA,B)} = 0$. $\square$

Corollary 4.7. *Let A and B be σ-unital C^*-algebras. Then the map Γ gives a map from $KL(A,B)$ to $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.*

Definition 4.8. Let A be a σ-unital C^*-algebra. We say A satisfies the Approximate Universal Coefficient Theorem I (AUCT1) if for any σ-unital C^*-algebra, B, the map Γ from $KK(A,B)$ to $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ is surjective and its kernel is $\mathcal{T}(A,B)$. In other words, we have

$$KL^0(A,B) = \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \quad \text{and} \quad KL^1(A,B) = \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(SB)).$$

Now let A be a separable nuclear C^*-algebra. We say A satisfies the Approximate Universal Coefficient Theorem II (AUCT2) if for any σ-unital C^*-algebra, B, the map $\bar{\gamma} : KL(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$ is surjective and map $\bar{\kappa} : \ker\bar{\gamma} \to E_0(K_*(A), K_*(B))$ is an isomorphism, i.e., there is an exact sequence

$$0 \to E_0(K_*(A), K_*(B)) \to KL(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

which is natural in each variable.

We will show that (AUCT1) is equivalent to (AUCT2) if A is assumed to be separable and nuclear.

It follows from the definition in [DL1] that, if $x \in KK(B,A)$, then the Kasparov product $KK(\mathbb{I}_n, B) \times x$ gives a homomorphism $\Gamma(x) : \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(D)) \to \mathrm{Hom}_\Lambda(\underline{K}(B), \underline{K}(D))$ for any C^*-algebra, D.

Lemma 4.9. *Let A and B be two σ-unital C^*-algebras. Suppose that A satisfies the AUCT1 and there is $\bar{x} \in KL(B,A)$ such that $\bar{\gamma}(\bar{x}) \in \mathrm{Hom}(K_*(B), K_*(A))$ is an isomorphism. Then $\Gamma(\bar{x}) : \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(D)) \to \mathrm{Hom}_\Lambda(\underline{K}(B), \underline{K}(D))$ is an isomorphism.*

Proof. Since the map from $KK(B,A)$ to $KL(B,A)$ is surjective, we obtain an element $x \in KK(B,A)$ such that its image in $KL(B,A)$ is $\bar{x}$. The Kasparov product by x gives a map

$$\Gamma(x) : \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(D)) \to \mathrm{Hom}_\Lambda(\underline{K}(B), \underline{K}(D))$$

which is defined, for each n, i, $(i = 0, 1)$ $\phi_{n,i} \in \mathrm{Hom}(K_i(B, \mathbb{Z}/n\mathbb{Z}), K_i(D, \mathbb{Z}/n\mathbb{Z}))$, by

$$\Gamma(x)(\phi_{n,i})(y) = \phi_{n,i}(y \times x),$$

where $y \in K_i(B, \mathbb{Z}/n\mathbb{Z}) = KK(\mathbb{I}_n, B)$. It follows that $\Gamma(z)$ gives (for each k) the following commutative diagram:

$$\begin{tikzcd}
K_0(B) \arrow[rr] \arrow[dr, "\gamma(x)"] & & K_0(B, \mathbf{Z}/k\mathbf{Z}) \arrow[rr] \arrow[d, "\times x"'] & & K_1(B) \arrow[dl, "\gamma(x)"'] \arrow[ddd] \\
& K_0(A) \arrow[r] & K_0(A, \mathbf{Z}/k\mathbf{Z}) \arrow[r] & K_1(A) \arrow[d] & \\
& K_0(A) \arrow[u] & K_1(A, \mathbf{Z}/k\mathbf{Z}) \arrow[l] & K_1(A) \arrow[l] & \\
K_0(B) \arrow[uuu] \arrow[ur, "\gamma(x)"] & & K_1(B, \mathbf{Z}/k\mathbf{Z}) \arrow[ll] \arrow[u, "\times x"] & & K_1(B) \arrow[ll] \arrow[ul, "\gamma(x)"']
\end{tikzcd}$$

Since $\gamma(x)$ is an isomorphism, by the five lemma, we see that the homomorphism $(-) \times x$ is in fact an isomorphism. By the naturality of the Kasparov product, this implies that $\Gamma(x)$ is an isomorphism.

Lemma 4.10. *Let A and B be two nuclear C^*-algebras and let D be a σ-unital C^*-algebra. Suppose that there is an element $x \in KK(A, B)$ such that $\gamma(x)$ is an isomorphism in* $\mathrm{Hom}(K_*(A), K_*(B))$. *Denote again*

$$\gamma_A : KK(A, D) \to \mathrm{H}om(K_*(A), K_*(D))$$

and

$$\gamma_B : KK(B, D) \to \mathrm{H}om(K_*(B), K_*(D)).$$

Then the Kasparov product gives the following (with one arrow missing) commutative diagram:

$$\begin{tikzcd}
\ker\gamma_B \arrow[rrr, "\Pi"] \arrow[dr, "\times x"] \arrow[ddd, "\kappa_B"'] & & & \ker\bar{\gamma}_B \arrow[ddd, "\bar{\kappa}_B"'] \\
& \ker\gamma_A \arrow[r] \arrow[d, "\kappa_A"'] & \ker\bar{\gamma}_A \arrow[d, "\bar{\kappa}_A"'] & \\
& \mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(D)) \arrow[r] & E_0(K_*(A), K_*(D)) & \\
\mathrm{Ext}_{\mathbb{Z}}(K_*(B), K_*(D)) \arrow[rrr, "\tilde{\Pi}"] \arrow[ur, "\theta"] & & & E_0(K_*(B).K_*(D)) \arrow[ul, "\bar{\theta}"']
\end{tikzcd}$$

Moreover, both θ and $\bar{\theta}$ are isomorphisms.

Proof. Fix a geometric injective resolution

$$0 \to F \xrightarrow{g} C \to SD \to 0.$$

The associated K-theory sequence degenerates to

$$0 \to K_i(D) \to K_i(F) \xrightarrow{g_*} K_i(C) \to 0 \ \ (i = 0, 1)$$

which is an injective resolution of $K_i(D)$ $(i = 0, 1)$. The six-term exact sequence of KK-theory (for any nuclear C^*-algebra, A) is

$$\cdots \to KK^i(A, F) \xrightarrow{\omega} KK^i(A, C) \to KK^i(A, D) \to KK^{i-1}(A, F) \to \cdots$$

which unsplices to two exact sequences:

$$0 \to \operatorname{coker} \omega \to KK^i(A, D) \to \ker \omega \to 0.$$

The Kasparov product gives the following commutative diagram:

$$\begin{tikzcd}
KK^i(B,F) \arrow[rrr, "\omega"] \arrow[dr] \arrow[ddd] & & & KK^i(B,C) \arrow[dl] \arrow[ddd] \\
& \mathrm{Hom}(K_*(B),K_*(F)) \arrow[r] \arrow[d] & \mathrm{Hom}(K_*(B),K_*(C)) \arrow[d] & \\
& \mathrm{Hom}(K_*(A),K_*(F)) \arrow[r] & \mathrm{Hom}(K_*(A),K_*(C)) & \\
KK^i(A,F) \arrow[rrr, "\omega"] \arrow[ur] & & & KK^i(A,C) \arrow[ul]
\end{tikzcd}$$

where, the unnamed horizontal maps are $\mathrm{Hom}(1, g_*)$. We identify (see the proof of Theorem 4.1 in [RS])

$$\ker(\mathrm{Hom}(1, g_*)) = \mathrm{Hom}(K_*(A), K_*(D))$$

$(\ker(\mathrm{Hom}(1, g_*)) = \mathrm{Hom}(K_*(B), K_*(D)))$ and

$$\mathrm{coker}(\mathrm{Hom}(1, g_*)) = \mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(D))$$

$(\mathrm{coker}(\mathrm{Hom}(1, g_*)) = \mathrm{Ext}_{\mathbb{Z}}(K_*(B), K_*(D)))$. Since in the above diagram, the two shorter vertical maps are isomorphism, we obtain an isomorphism

$$\theta : \mathrm{Ext}_{\mathbb{Z}}(K_*(B), K_*(D)) \to \mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(D)).$$

Since $\gamma(x)$ is an isomorphism, θ maps pure extensions to pure extensions so we also obtain another map

$$\bar{\theta} : E_0(K_*(B), K_*(D)) \to E_0(K_*(A), K_*(D)).$$

Note both θ and $\bar{\theta}$ are isomorphisms. □

Theorem 4.11. *Let A be a separable nuclear C^*-algebra. Then (AUCT1) and (AUCT2) are equivalent.*

Proof. (1) AUCT1 implies AUCT2: Suppose that A is a separable nuclear C^*-algebra, which satisfies the (AUCT1). Let $B \in \mathcal{N}$ such that $K_i(A) \cong K_i(B)$. Since A satisfies the AUCT1, there is an element $x \in KL^i(A, B)$ such that $\bar{\gamma}(x) \in \mathrm{Hom}(K_*(A), K_*(B))$ is an isomorphism. Thus by 4.9 the map z gives an isomorphism $\Pi(z) : KL^i(A, D) \to KL^i(B, D)$. Let

$$\bar{\gamma}_A : KL(A, D) \to \mathrm{Hom}(K_*(A), K_*(D))$$

and

$$\bar{\gamma}_B : KL(B, D) \to \mathrm{Hom}(K_*(B), K_*(D))$$

be maps as defined in 4.4. Then the induced maps $\eta : \mathrm{Hom}(K_*(A), K_*(D)) \to \mathrm{Hom}(K_*(B), K_*(D))$ and $\theta : \ker\gamma_A \to \ker\gamma_B$ are isomorphisms. On the other hand, by the AUCT1, $\ker\bar{\gamma}_A = E_0(K_*(A), K_*(D))$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to & E_0(K_*(B), K_*(D)) \to & KL(B, D) \to & \mathrm{Hom}(K_*(B), K_*(D)) & \to 0 \\
& \downarrow_{\theta} & \downarrow_{\Pi(x)} & \downarrow_{\eta} & \\
0 \to & \ker\bar{\gamma}_A \to & KL(A, D) \to & \mathrm{Hom}(K_*(A), K_*(D)) & \to 0.
\end{array}$$

By applying 4.10 we obtain a (complete) commutative diagram:

$$\begin{tikzcd}
\ker\gamma_B \arrow[rrr, "\Pi"] \arrow[dr, "\times x"] \arrow[ddd, "\kappa_B"'] & & & \ker\bar{\gamma}_B \arrow[dl, "\tilde{\Pi}"'] \arrow[ddd, "\bar{\kappa}_B"] \\
& \ker\gamma_A \arrow[r] \arrow[d, "\kappa_A"'] & \ker\bar{\gamma}_A \arrow[d, "\bar{\kappa}_A"'] & \\
& \mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(D)) \arrow[r] & E_0(K_*(A), K_*(D)) & \\
\mathrm{Ext}_{\mathbb{Z}}(K_*(B), K_*(D)) \arrow[ur, "\theta"] \arrow[rrr, "\Pi"] & & & E_0(K_*(B).K_*(D)) \arrow[ul, "\bar{\theta}"']
\end{tikzcd}$$

with θ and $\bar{\theta}$ being isomorphisms. Since B satisfies the UCT, $\bar{\kappa}_B$ is an isomorphism. We have also shown that $\Pi(x)$ is an isomorphism. Therefore $\bar{\kappa}_A$ is also an isomorphism. This proves that A satisfies (AUCT2).

(2) AUCT2 implies AUCT1: Now we assume that A satisfies the (AUCT2). Again let $B \in \mathcal{N}$ such that $K_i(A) = K_i(B)$. Since A satisfies the AUCT2, there is an element $x \in KK(A, B)$ such that $\gamma(x)$ gives the isomorphisms. Since B satisfies the UCT, the Kasparov product gives, for any σ-unital C^*-algebra, D, the following commutative diagram:

$$\begin{array}{ccccc}
0 \to & \mathrm{Ext}_{\mathbb{Z}}(K_*(B), K_{*-1}(D)) \to & KK(B, D) \to & \mathrm{Hom}(K_*(B), K_*(D)) & \to 0 \\
& \downarrow_{\theta'} & \downarrow_{\Pi(x\times(-))} & \downarrow_{\gamma(x)} & \\
0 \to & E_0(K_*(A), K_*(D)) \to & KL(A, D) \to & \mathrm{Hom}(K_*(A), K_*(D)) & \to 0.
\end{array}$$

Applying the commutative diagram of 4.10, we see that $\theta' = \tilde{\Pi} \circ \theta$ (here $\tilde{\Pi}$ is the map from $\mathrm{Ext}_{\mathbb{Z}}(K_*(A), K_*(B))$ to $E_0(K_*(A), K_*(D))$). Therefore

$$\ker\Pi((-) \times x) = Pext(K_*(B), K_*(D)).$$

But $Pext(K_*(B), K_*(D)) = Pext(K_*(B), K_{*-1}(SD))$ corresponds to $\mathcal{T}(B, SD)$ (see [S4] and 5.9.11 in [Ln7]). Therefore $\ker\Pi((-) \times x) = \mathcal{T}(B, SD)$. So the Kasparov product gives an isomorphism from $KL(B, D)$ to $KL(A, D)$.

On the other hand, $\Gamma(x)$ gives the following commutative diagram (by 4.6):

$$\begin{array}{ccc}
KL(B, D) & \xrightarrow{\Gamma} & \mathrm{Hom}_\Lambda(\underline{K}(B), \underline{K}(D)) \\
\downarrow_{\Pi(x\times(-))} & & \downarrow \Gamma(x) \\
KL(A, D) & \xrightarrow{\Gamma} & \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(D)).
\end{array}$$

From what we have shown the left vertical map is an isomorphism. That the right vertical map is an isomorphism follows from 4.9. Since B satisfies the UCT, it follows from [DL2] the upper horizontal map is also an isomorphism. Thus the above commutative diagram shows that the lower horizontal map has to be surjective and injective. Therefore A satisfies the (AUCT1). □

Definition 4.12. Let A be a separable nuclear C^*-algebra. We say A satisfies the Approximate Universal Coefficient Theorem (AUCT), if either A satisfies the AUCT1 or AUCT2.

Corollary 4.13. *Suppose that A and B are two σ-unital nuclear C^*-algebras which satisfy the AUCT. Suppose that $K_i(A) \cong K_i(B)$ $(i = 0, 1)$. Then there are elements $z \in KL(A, B)$ and $z^{-1} \in KL(B, A)$ such that $\Gamma(z^{-1})$ is an isomorphism from $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(D))$ onto $\mathrm{Hom}_\Lambda(\underline{K}(B), \underline{K}(D))$ for any σ-unital C^*-algebra, D with inverse given by $\Gamma(z)$.*

Theorem 4.14. *Let A be a separable nuclear C^*-algebra, which satisfies the AUCT and let D be a σ-unital C^*-algebra. If $\tau \in Ext(A, D)$ so that τ gives two pure extensions*

$$0 \to K_i(D) \to K_i(E) \to K_i(A) \to 0,$$

then τ is stably approximately trivial.

Proof. Suppose that $[\tau] \in Ext(A, D) = KK(A, D)$ so that τ gives two pure extensions as described in the theorem. By the AUCT (AUCT2), $\Pi([\tau]) = 0$. So τ is stably approximately trivial. $\square$

5. Applications to classification of simple nuclear C^*-algebras

Definition 5.1. Let $A_1, A_2, ..., A_n, ...$ and $B_1, B_2, ..., B_n, ...$ be two sequences of C^*-algebras. Let ε_n, δ_n and η_n be decreasing sequences of positive numbers such that

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty, \ \sum_{n=1}^{\infty} \delta_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \eta_n < \infty.$$

Let $\phi_n : A_n \to A_{n+1}$, $\psi_n : B_n \to B_{n+1}$ and $L_n : A_n \to B_n$ be sequences of contractive completely positive linear maps. Suppose that $\mathcal{F}_1^{(n)} \subset \mathcal{F}_2^{(n)} \subset \cdots \subset \mathcal{F}_k^{(n)} \subset \cdots$ and $\mathcal{G}_1^{(n)} \subset \mathcal{G}_2^{(n)} \subset \cdots \subset \mathcal{G}_k^{(n)} \subset \cdots$ be finite subsets of A_n and B_n such that $\bigcup_k \mathcal{F}_k^{(n)}$ and $\bigcup_k \mathcal{G}_k^{(n)}$ are dense in A_n and B_n, respectively ($n = 1, 2, ...$). Suppose also that $\phi_n \circ \cdots \circ \phi_k(\mathcal{F}_k^{(n)}) \subset \mathcal{F}_1^{(n+1)}$ and $\psi_n \circ \cdots \circ \psi_k(\mathcal{G}_{k+1}^{(n)}) \subset \mathcal{G}_1^{(n+1)}$. Moreover

$$\|\phi_n(a'a) - \phi_n(a')\phi_n(a)\| < \varepsilon_n, \qquad \|L_n(a'a) - L_n(a')L_n(a)\| < \delta_n$$

for all $a, a' \in \mathcal{F}_1^{(n)}$, and

$$\|\psi_n(bb') - \psi_n(b)\psi_n(b')\| < \eta_n$$

for all $b, b' \in \mathcal{G}_1^{(n)}$. Set $A = \lim_n(A_n, \phi_n)$ and $B = \lim_n(B_n, \psi_n)$ (generalized inductive limits in the sense in [BK1]). Suppose further that

$$\|\psi_{n+1} \circ L_n(a) - L_{n+1} \circ \phi_n(a)\| < \varepsilon_n$$

for all $a \in \mathcal{F}_1^{(n)}$. Then we say the following diagram is (one-sided) approximately intertwining:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & \cdots A \\
\downarrow_{L_1} & & \downarrow_{L_2} & & \downarrow_{L_3} & & \\
B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_3} & B_3 & \xrightarrow{\psi_3} & \cdots B
\end{array}$$

It follows from an argument of Elliott that there is a homomorphism $h : A \to B$ which completes the above approximated intertwining diagram (see [Ell1] and for a more elaborated late version, see 1.10.14 and 1.10.15 [Ln7]).

If furthermore, there are contractive completely positive linear map $H_n : B_n \to A_{n+1}$ such that

$$\|H_n(bb') - H_n(b)H_n(b')\| < \delta_n, \ \|H_n \circ L_n(a) - \phi_n(a)\| < \eta_n$$

and

$$\|L_{n+1} \circ H_n(b) - \psi_n(b)\| < \eta_n$$

for all $b, b' \in \mathcal{G}_1^{(n)}$ and $a \in \mathcal{F}_1^{(n)}$, then there are isomorphisms $h : A \to B$ and $h^{-1} : B \to A$.

Definition 5.2. Let $\phi_i : A \to B$ be two maps ($i = 1, 2$) and $\mathcal{F} \subset A$. For $\varepsilon > 0$, we will write

$$\phi_1 \approx_\varepsilon \phi_2\,, \quad \text{on } \mathcal{F}, \text{ if}$$
$$\|\phi_1(a) - \phi_2(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

The following result strengthens (marginally) the Kirchberg and Phillips's ([P2] and [K2]) classification theorem. With 3.14, we can replace the assumption that A and B satisfy the UCT by the condition that they satisfy the AUCT. The possibility of this approach first appeared in [Ln2].

Theorem 5.3. *Let A and B be two separable nuclear purely infinite simple C^*-algebras satisfying the AUCT.*

(i) *If both A and B are unital, then $A \cong B$ if and only if*

$$(K_0(A), [1_A]) \cong (K_0(B), [1_B]) \quad \textit{and} \quad K_1(A) \cong K_1(B).$$

(ii) *If both A and B are stable, then $A \cong B$ if and only if*

$$K_i(A) \cong K_i(B), \quad i = 0, 1.$$

Proof. We will prove (i) only. The "only if" part follows easily. So we will prove the "if" part. Let $\alpha \in \mathrm{Hom}(K_*(A), K_*(B))$ be an isomorphism as given. By 4.13, since both A and B satisfy the AUCT, there are elements $x \in KK(A, B)$ and $y \in KK(B, A)$ such that $\Pi(x \times y) = [\mathrm{id}_A]$ and $\Pi(y \times x) = [\mathrm{id}_B]$ in $KL(A, A)$ and $KL(B, B)$, respectively. It follows by 4.1.1 in [P2] that there are unital homomorphisms $h_1 : A \to B$ and $\phi_1' : B \to A$ such that $[h_1] = [x]$ and $[\phi_1'] = [y]$. Moreover $[\phi_1' \circ h_1] = [\mathrm{id}_A]$ in $KL(A, A)$ and $[h_1 \circ \phi_1'] = [\mathrm{id}_B]$ in $KL(B, B)$.

Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ be finite subsets such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A. Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$ be finite subsets such that $\bigcup_{n=1}^\infty \mathcal{G}_n$ is dense in B.

It follows from 3.14 that there is a unitary $u_1 \in A$ such that

$$\mathrm{ad}u \circ \phi_1' \circ h_1 \approx_{1/2} \mathrm{id}_A \quad \text{on } \mathcal{F}_1.$$

Set $\phi_1 = \mathrm{ad}u \circ \phi_1'$. Let $S_1 = \mathcal{F}_1$ and $S_1' = \mathcal{G}_1 \cup h_1(S_1)$. It follows again from 3.14 that there is a unitary $v_1 \in B$ such that

$$\mathrm{ad}v \circ h_1 \circ \phi_1 \approx_{1/4} \mathrm{id}_B.$$

Define $h_2 : A \to B$ by $\mathrm{ad}v_1 \circ h_1$. If we continue this process, we obtain the following approximately intertwining diagram:

$$\begin{array}{ccccccccc}
A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & A & \xrightarrow{\mathrm{id}_A} & \cdots \longrightarrow & A \\
\downarrow_{h_1} & \nearrow_{\phi_1} & \downarrow_{h_2} & \nearrow_{\phi_2} & \downarrow_{h_3} & & & \\
B & \xrightarrow{\mathrm{id}_B} & B & \xrightarrow{\mathrm{id}_B} & B & \xrightarrow{\mathrm{id}_B} & \cdots \longrightarrow & B.
\end{array}$$

Therefore $A \cong B$. □

In the absence of the AUCT, we have the following version of the Kirchberg-Phillips theorem:

Theorem 5.4. *Let A and B be two separable nuclear purely infinite simple C^*-algebras.*

(i) *If both A and B are non-unital and there is $x \in KK(A, B)$ and $y \in KK(B, A)$ such that $\Pi(x \times y) = [\mathrm{id}_A]$ and $\Pi(y \times x) = [\mathrm{id}_B]$, then $A \cong B$.*

(ii) *If both A and B are unital, and there is $x \in KK(A, B)$ and $y \in KK(B, A)$ such that $\Pi(x \times y) = [\mathrm{id}_A]$, $\Pi(y \times x) = [\mathrm{id}_B]$ and $\gamma(x)([1_A]) = [1_B]$, then $A \cong B$.*

Corollary 5.5. *Let A be a separable nuclear purely infinite simple C^*-algebra. If A satisfies the AUCT, then $A \in \mathcal{N}$.*

Proof. The proof of this follows from the fact that separable nuclear purely infinite simple C^*-algebras in $\mathcal{N}$ exhaust all possible K_i-groups ([Rr2]) and the above theorem. □

Definition 5.6. Let A be a C^*-algebra, and let C_n be as in Definition 4.5. Let $\mathbf{P}(A)$ be the set of all projections in $M_\infty(A)$, $M_\infty(C(S^1)\otimes A)$, $M_\infty(A\otimes C_m)$ and $M_\infty(C(S^1)\otimes A\otimes C_m))$. Let B be another C^*-algebra, and let $L : A \to B$ be a completely positive linear map. Then L induces maps from $A\otimes C_m \to B\otimes C_m$, from $C(S^1)\otimes A\otimes C_m$ to $C(S^1)\otimes B\otimes C_m$, namely, $L\otimes \mathrm{id}$. For convenience, we will also denote the induced map by L.

Let A and B be C^*-algebras, let $L : A \to B$ be a contractive completely positive linear map, let $\varepsilon > 0$ and let $\mathcal{F} \subset A$ be a subset. L is said to be $\mathcal{F}$-ε-multiplicative if

$$\|L(xy) - L(x)L(y)\| < \varepsilon$$

for all $x, y \in \mathcal{F}$.

Given a projection $p \in \mathbf{P}(A)$, if L is $\mathcal{G}$-ε-multiplicative with sufficiently large $\mathcal{G}$ and sufficiently small ε, $L(p)$ is close to a projection. Let $L(p)'$ be that projection. Fix finite subsets of $\mathcal{P}_1 \subset \mathbf{P}(A)$. It is easy to see that $L(p)'$ and $L(q)'$ are in the same equivalence class of projections of $\mathbf{P}(A)$ if p and q are in $\mathcal{P}_1$ and are in the same equivalence class of projections of $\mathbf{P}(A)$, provided that $\mathcal{F}$ is sufficiently large and ε is sufficiently small. We use $[L](p)$ for the class of the projections containing $L(p)'$.

In what follows, whenever we write $[L](p)$, we assume that $\mathcal{F}$ is sufficiently large and ε is sufficiently small so that $[L](p)$ are well defined on $\mathcal{P}_1$.

Suppose that q is in $\mathbf{P}(A)$ with $[q] = k[p]$ for some integer k, by adding sufficiently many elements (partial isometries) in $\mathcal{F}$, we can assume that $[L](q) = k[L](p)$.

Suppose that G is a finitely generated group generated by $\mathcal{P}$ and $G = \mathbb{Z}^n \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \mathbb{Z}/k_m\mathbb{Z}$. Let $g_1, g_2, ..., g_n$ be free generators of $\mathbb{Z}^n$ and let $t_i \in \mathbb{Z}/k_i\mathbb{Z}$ be the generator with order k_i, $i = 1, 2, ..., m$. Since every element in $K_0(C)$ (for any unital C^*-algebra, C) may be written as $[p_1] - [p_2]$ for projections $p_1, p_2 \in A\otimes M_l$, for some $l > 0$, with sufficiently large $\mathcal{F}$ and sufficiently small ε, one can define $[L](g_j)$ and $[L](t_i)$. Moreover (with sufficiently large $\mathcal{F}$ and sufficiently small ε), the order of $[L](t_i)$ divides k_i. Then we can define a map $[L]|_G$ by defining $[L](\sum_i^n n_i g_i + \sum_j^m m_j t_j) = \sum_i^k n_i [L](g_i) + \sum_j^m m_j [L](t_j)$. Note, in general, that $[L]|_{\mathcal{P}}$ may *not* coincide with $[L]|_G$ on $\mathcal{P}$. However, if $\mathcal{F}$ is large enough and ε is small enough, they coincide. In what follows, we say $[L]|_G$ is well defined and write $[L]|_G$ if

(1) $[L]$ is well defined on $\{g_1, g_2, ..., g_n, t_1, ..., t_m\}$ with the order of $[L](t_i)$ dividing k_i,

(2) $[L]|_{\mathcal{P}} = [L]|_G$ on $\mathcal{P}$.

(See 1.6 and 1.8 in [Ln4].)

Definition 5.7 (4.6 in [Ln2], see also [GL2])**.** Fix $l \ge 1$, $b \ge \pi$ and $M \ge 1$. We say a unital C^*-algebra $A \in \mathbf{C}_{(l,b,M)}$ if

(a) for any projections p, $q \in M_K(A)$ with $[p] = [q]$ in $K_0(A)$, $p \oplus 1_{M_{Kl}(A)}$ is Murry-von Neumann equivalent to $q \oplus 1_{M_{Kl}(A)}$ for all K,

(b) the canonical map $U(M_l(A))/U_0(M_l(A)) \to K_1(A)$ is surjective,

(c) the exponential length of $M_m(A)$, $cel(M_m(A)) \le b$ for all m,

(d) if $k > 0$ and $-l[1_A] \le kx \le l[1_A]$, then $-lMk[1_A] \le x \le lMk[1_A]$ for all $x \in K_0(A)$.

If A is a separable C^*-algebra, with real rank zero, stable rank one and with weakly unperforated $K_0(A)$, then $A \in \mathbf{C}_{1,\pi,1}$ (see [GL2]). In particular, if A is a simple C^*-algebra, with $TR(A) = 0$, then $A \in \mathbf{C}_{1,\pi,1}$ ([Ln6]).

Definition 5.8. A contractive completely positive linear map $L : A \to B$, where B is unital, is said to be $(N(a), K(a))_{a\in\mathcal{F}}$ -full, if for each $a \in \mathcal{F} \subset A_+ \setminus \{0\}$ there are $x_1(a), ..., x_{N(a)}(a) \in B$ such that $\|x_i(a)\| \le K(a)$ and

$$\sum_{i=1}^{N(a)} x_i(a)^* L(a) x_i(a) = 1_B.$$

Also if $\mathcal{F} \subset A$, we denote by $\mathcal{F}^+$ the set $\{(\frac{(a^*+a)}{2})_+, (\frac{(a-a^*)}{2i})_+ : a \in \mathcal{F}\} \setminus \{0\}$.

With 3.9, exactly as in the proof of Theorem 5.3 in [Ln2] (see also 6.3.1 in [Ln7]), we obtain the following:

Theorem 5.9. *Let A be a nuclear separable C^*-algebra, satisfying the AUCT and $l \ge 1$, $b \ge \pi$ and $M \ge 1$. Then, for any finite subset $\mathcal{F} \subset A$, $\varepsilon > 0$ and $T : A \to (\mathbb{N}_+, \mathbb{R}_+)$, $a \mapsto (N(a), K(a))$, there exist a finite subset $\mathcal{G} \subset A$, a positive number $\delta > 0$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ (they do not depend on T but depends on A, ε and $\mathcal{F}$) such that, for any unital C^*-algebra, $B \in \mathbf{C}_{(l,b,M)}$ and any $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps, $\phi, \psi : A \to B$ and any $\{(N(a), K(a)) : a \in \mathcal{G}^+\}$-full unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\sigma : A \to B$, if*

$$[\phi_1]|_{\mathcal{P}} = [\phi_2]|_{\mathcal{P}},$$

then there exists an integer $k > 0$ and a unitary $U \in M_{k+1}(B)$ such that

$$\|U^*(diag(\phi(a), \sigma(a), ..., \sigma(a)))U - diag(\psi(a), \sigma(a), ..., \sigma(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof. The proof is merely a minor modification of that of 5.3 in [Ln2] (see also 6.3.1 in [Ln7]). We will refer to that proof. The only thing that we are required to add to the proof of 5.3 in [Ln2] (6.3.1 in [Ln7]) is to show that $\bar{\Sigma} : A \to \prod_n B_n / \bigoplus_n B_n$ is a full monomorphism. Given $a \in A_+ \setminus \{0\}$, there are $x_i^{(n)}(a) \in B_n$, $i = 1, ..., N(a)$, such that $\|x_i^{(n)}(a)\| \le K(a)$ and

$$\sum_{i=1}^{N(a)} x_i^{(n)}(a)^* \sigma_n(a) x_i^{(n)}(a) = 1_{B_n}$$

for all large n. Let $x_i(a) = \{x_i^{(n)}\}$ (with any first few $x_i^{(n)}(a)$). Then $x_i(a) \in \prod_n B_n$ such that

$$\sum_{i=1}^{N(a)} \pi(x_i(a))^* \bar{\Sigma}(a) \pi(x_i(a)) = 1_{C/C_0},$$

where $C = \prod_n B_n$, $C_0 = \bigoplus_n B_n$ and $\pi : C \to C/C_0$ as in the proof of 5.3 in [Ln2]. This shows that $\bar{\Sigma}$ is full as well as injective (since $\bar{\Sigma}(a) \ne 0$ for any $a \in A$). The rest of the proofs are exactly the same. $\square$

The following theorem was proved in [Ln8] with the condition that both A and B satisfy the UCT. With 5.9, we can strengthen it as follows.

Theorem 5.10. *Let A and B be two separable unital nuclear simple C^*-algebras with tracial topological rank zero and satisfying the $AUCT$. Then $A \cong B$ if and only if*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Since there are separable unital simple C^*-algebras A in $\mathcal{N}$ with $TR(A) = 0$ with any given weakly unperforated $K_0(A)$ with the Reisz property and countable abelian group $K_1(A)$, we obtain the following

Corollary 5.11. *Let A be a unital separable nuclear simple C^*-algebra, with tracial topological rank zero and satisfying the $AUCT$. Then A is in $\mathcal{N}$.*

6. An approximate Universal Coefficient Theorem

In this section, we establish a theorem which says a certain class of C^*-algebras satisfies the approximate Universal Coefficient Theorem for KL. By [RS], we note that all C^*-algebras in $\mathcal{N}$ satisfy the UCT. We will show that C^*-algebras that are "locally in" $\mathcal{N}$ satisfy the AUCT. Moreover, we show that C^*-algebras which "locally" satisfy the AUCT satisfy the AUCT. We start with the following lemma.

Lemma 6.1. *Suppose that B is a σ-unital stable C^*-algebra. Then $Q(B) \in \mathbf{C}_{(1,3\pi,1)}$.*

Proof. Since B is stable, for any $m > 0$, there is a partial isometry $v \in M_m(Q(B))$ such that

$$v^*v = 1_{Q(B)} \quad \text{and} \quad vv^* = 1_{M_m(Q(B))}.$$

Thus, every projection in $M_m(Q(B))$ is equivalent to a projection in $Q(B)$. This implies that $Q(B)$ satisfies condition (d) in 5.7 for $M = 1$. Furthermore, suppose that $p, q \in M_l(Q(B))$ and $[p] = [q]$ in $K_0(Q(B))$. We may assume that $p \oplus 1_{M_m(Q(B))}$ is equivalent to $q \oplus 1_{M_m(Q(B))}$. This implies (a) in 5.7.

Let $u \in U(M_m(Q(B))$. Then

$$v^*uv \in U(Q(B)).$$

It is standard that $\mathrm{diag}(v^*uv, 1_{M_{2m}(Q(B))})$ and $\mathrm{diag}(u, 1_{M_m(Q(B))})$ are in the same path-connected component of $U(M_{2m+1}(Q(B))$. This proves that $Q(B)$ satisfies (b) for $l = 1$ in 5.7.

Let $u \in U_0(M_m(Q(B))$. There is a unitary $w \in M_m(M(B)) = M(M_m(B))$. It follows from [Zh2] (see also [M]) that

$$cel(M(M_m(B))) \leq 3\pi.$$

This implies that $cel(M_m(Q(B))) \leq 3\pi$ (for all m). □

Definition 6.2. Let A be a nuclear and separable C^*-algebra, and let $h : A \to B(l^2)$ be an injective homomorphism such that $\pi \circ h : A \to B(l^2)/\mathcal{K}$ is also injective, where $\pi : B(l^2) \to B(l^2)/\mathcal{K}$ is the quotient map. Suppose that B is σ-unital. There is an obvious embedding $\imath : B(l^2) \to M(B \otimes \mathcal{K})$. The composition $\imath \circ h : A \to M(B \otimes \mathcal{K})$ gives an absorbing trivial extension $\tau_0 = \pi \circ \imath \circ h : A \to Q(B)$ (see [Ka]).

We have the following:

Proposition 6.3. *Let $\tau : A \to Q(B)$ be an injective homomorphism. If τ gives an absorbing extension, then τ is full.*

Proof. Let $\tau_0 = \pi \circ \imath \circ h$ be the trivial absorbing extension defined 6.2. We will show that τ_0 is full. Let $a \in A$ be a non-zero element. The embedding $\imath : B(l^2) \to M(B \otimes \mathcal{K})$ gives an embedding $\bar{\imath} : B(l^2)/\mathcal{K} \to Q(B)$. Let $D = \bar{\imath}(B(l^2)/\mathcal{K})$. Then D is a unital simple C^*-algebra. Thus the (closed) ideal generated by $\tau_0(a)$ contains the identity of D. Since the identity of D is the identity of $Q(B)$, we see that τ_0 is full. Therefore $\tau_0 \oplus \tau$ is also full. Since $\tau_0 \oplus \tau$ is unitarily equivalent to τ, τ is full. □

Lemma 6.4. *Let A be a C^*-algebra, with real rank zero and let $a \in A_+ \setminus \{0\}$. Then for any $\varepsilon > 0$, there is a non-zero projection $p \in A$ such that*

$$a \geq (\|a\| - \varepsilon)p.$$

Proof. Let $f \in C_0((0, \|a\|])_+$ such that $f(t) = 0$ if $t \in [0, \|a\| - \varepsilon]$ and $f(t) = 1$ if $t \in [\|a\| - \varepsilon/2, \|a\|]$. Set $b = f(a)$. Then $b \neq 0$. Let $B = Her(b) = \overline{bAb}$. Since $RR(A) = 0$, there exists a non-zero projection $p \in B$. It is clear from the construction that

$$a \geq (\|a\| - \varepsilon)p.$$

□

Lemma 6.5. *Let A be any separable C^*-algebra, and let B be a unital purely infinite simple C^*-algebra. Let $\mathcal{G}$ be a finite subset of A_+. Suppose that $\phi : A \to B$ is a contractive completely positive linear map such that $\|\phi(a)\| \geq 1/2\|a\|$ for all $a \in \mathcal{G}$. Suppose that $(N(a), K(a)) = (1, 3)$ for all $a \in \mathcal{G}$. Then ϕ is $(1, 3))_{a \in \mathcal{G}}$-full.*

Proof. It follows from [Zh1] that $RR(B) = 0$. If follows from Lemma 6.4 that

$$\phi(a) \geq 1/3p \quad \text{for some nonzero projection in } B.$$

Since B is a purely infinite simple C^*-algebra,, there is a projection $e \leq p$ and a partial isometry $v \in B$ such that $v^*v = 1$ and $vv^* = e$. Hence $v^*pv = 1$. Thus we obtain an element $c \in B$ with $\|c\| \leq 3$ such that

$$c^*\phi(a)c = 1.$$

□

Theorem 6.6. *Let A be a nuclear separable C^*-algebra, satisfying the AUCT. Then, for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$ there exist a finite subset $\mathcal{G} \subset A$, a positive number $\delta > 0$ and a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ such that, for any σ-unital stable C^*-algebra, B and any $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\phi, \psi : A \to Q(B)$ and any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $\sigma : A \to Q(B)$ which satisfies $\|\sigma(a)\| \geq 1/2\|a\|$ for all $a \in \mathcal{G}^+$ and ϕ factors through $Q(\mathcal{K})$, if*

$$[\phi_1]|_{\mathcal{P}} = [\phi_2]|_{\mathcal{P}},$$

then there exists an integer $k > 0$ and a unitary $U \in M_{k+1}(Q(B))$ such that

$$\|U^* diag(\phi(a), \sigma(a), ..., \sigma(a))U - diag(\psi(a), \sigma(a), ..., \sigma(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof. We have shown that $Q(B) \in \mathbf{C}_{1,3\pi,1}$ in 6.1. Since $Q(\mathcal{K})$ is purely infinite and simple, there are $c(a) \in Q(\mathcal{K})$ with $\|c(a)\| \le 3$ such that

$$c(a)^*\sigma(a)c(a) = 1_{Q(\mathcal{K})}$$

for $a \in \mathcal{G}^+$. Since $Q(\mathcal{K})$ is embedded unitally into $Q(B)$, this implies that σ are $(1,3)_{\mathcal{G}^+}$-full. Therefore the theorem follows from 5.9. □

Definition 6.7. Fix a C^*-algebra, A satisfying the AUCT and $l \ge 1$, $b \ge \pi$ and $M \ge 1$. Let $\mathcal{F} \subset A$ be a finite subset and let $\varepsilon > 0$. Let $\mathcal{G} \subset A$ and $\mathcal{P} \subset \mathbf{P}(A)$ be finite subsets and let $\delta > 0$. We say that $(\mathcal{F}, \varepsilon, \mathcal{G}, \mathcal{P}, \delta)$ is a 5-tuple satisfying requirements in 5.9 if the conclusion of 5.9 holds for any $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps $\phi, \psi, \sigma : A \to B$ (with $B \in \mathbf{C}_{l,b,M}$) that also satisfy other conditions of 5.9. We say $(\mathcal{F}, \varepsilon, \mathcal{G}, \mathcal{P}, \delta)$ is a 5-tuple satisfying the requirements for the maps in collection $\mathcal{M}$, if the conclusion of 5.9 holds for all $\mathcal{G}$-δ-multiplicative maps ϕ, ψ and σ (to a unital C^*-algebra, $B \in \mathbf{C}_{l,b,M}$) in $\mathcal{M}$. This usage is only for the convenience in this paper.

Fix $\mathcal{F}$ and ε as above. There is a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ such that for any $\mathcal{G}$ and δ, $(\mathcal{F}, \varepsilon, \mathcal{G}, \mathcal{P}, \delta)$ is a 5-tuple satisfying the requirements of 5.9 for all homomorphisms (since homomorphisms are multiplicative).

Let $B = A_1 \oplus A_2 \oplus \cdots \oplus A_m$, where each A_i satisfies the AUCT. Fix a finite subset $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_m$, where $\mathcal{F}_i \subset A_i$. Suppose that $\mathcal{G}_i \subset A_i$ and $\mathcal{P}_i \subset \mathbf{P}(A_i)$ are finite subsets and $\delta_i > 0$ so that $(\mathcal{F}_i, \varepsilon, \mathcal{G}_i, \mathcal{P}_i, \delta_i)$ is a 5-tuple satisfying the requirements of 5.9. Let $\delta = \min\{\delta_1, ..., \delta_m\}$, $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \mathcal{G}_m$ and $\mathcal{P} = \mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_m$. Then it is easy to see that $(\mathcal{F}, \varepsilon, \mathcal{G}, \mathcal{P}, \delta)$ is a 5-tuple satisfying the requirements of 5.9 (for B).

Suppose that $h : A \to A$ is an automorphism. Then $(h(\mathcal{F}), \varepsilon, h(\mathcal{G}), [h](\mathcal{P}), \delta)$ is a 5-tuple satisfying the requirements of 5.9 if $(\mathcal{F}, \varepsilon, \mathcal{G}, \mathcal{P}, \delta)$ is.

Definition 6.8. Let A be a C^*-algebra, satisfying the AUCT and let $l \ge 1$, $b \ge \pi$ and $M \ge 1$ be fixed. Let B be a σ-unital C^*-algebra. Let $(\mathcal{F}, \varepsilon, \mathcal{G}, \mathcal{P}, \delta)$ be a 5-tuple satisfying the requirements in 5.9 (for $\mathbf{C}_{l.b,M}$) for all $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps ϕ from A for which $[\phi]|_{\mathcal{P}}$ is also well defined. We say a contractive completely positive linear map $\phi : A \to B$ is a $(\mathcal{F}, \varepsilon, \mathcal{G}, \mathcal{P}, \delta)$-map if it is $\mathcal{G}$-δ-multiplicative. Note that $\mathcal{G}$ and δ are used to described the degree of multiplicativity of the map and that it does not require that B belongs to $\mathbf{C}_{l,b,M}$.

Definition 6.9. Let A be a separable C^*-algebra, which is a generalized inductive limit of C^*-algebras A_n (and write $A = \lim_n(A_n, \phi_n)$, in the sense of Blackadar and Kirchberg ([BK1]). Suppose that each A_n satisfies the AUCT. We use $\phi_{n,\infty} : A_n \to A$ for the contractive completely positive linear map induced by the (generalized) inductive limit. Let $\mathcal{S}_1, \mathcal{S}_2, ...$ be a sequence of finite subsets such that the union is dense in A. We say that A satisfies the property (P) if there are finite subsets $\mathcal{F}_n \subset A_{k(n)}$ such that $\phi_{k(n),\infty}(\mathcal{F}_n) \supset \mathcal{S}_n$, $\phi_{k(n),k(n)+1}(\mathcal{F}_n) \subset \mathcal{F}_{n+1}$, and $\phi_{k(n),k(n+1)}$ and $\phi_{k(n),\infty}$ are $(\mathcal{F}_n, \varepsilon_n, \mathcal{G}_n, \mathcal{P}_n, \delta_n)$-maps with

$$\phi_{k(n),k(n)+1}](\mathcal{P}_n) \subset \mathcal{P}_{n+1}, \|\phi_{k(n),,k(n)+1}(a)\| \ge 1/2\|a\|$$

for all $a \in \mathcal{G}^+$ and there is $\alpha_n \in \mathrm{Hom}_\Lambda(\underline{K}(A_n), \underline{K}(A))$ such that

$$[\phi_{n,\infty}]|_{\mathcal{P}_n} = (\alpha_n)|_{\mathcal{P}_n},$$

where $\sum_{n=k}^\infty \varepsilon_n < \infty$ and $\sum_{n=1}^\infty \delta_n < \infty$.

Note that the property (P) depends on the number l, b and M.

Definition 6.10. Let A be a separable C^*-algebra, and let $\mathcal{C}$ be a class of nuclear C^*-algebras. Suppose that there is a sequence of C^*-subalgebras $\{A_n\}$ of A in the class $\mathcal{C}$ such that, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is A_n with

$$dist(x, A_n) < \varepsilon \quad \text{for all} \quad x \in \mathcal{F}.$$

Then we say A is *locally in* $\mathcal{C}$.

We say that A is locally AUCT, if $\mathcal{C}$ is the class of nuclear C^*-algebras which satisfy the AUCT.

The class of C^*-algebras which are locally in $\mathcal{N}$ will be denoted by $\mathcal{N}_l$. It is easy to see that A is locally in $\mathcal{N}_l$ must be in $\mathcal{N}_l$. Moreover, if A is in $\mathcal{N}_l$, then A is locally AUCT.

Proposition 6.11. *Every C^*-algebra, which is locally AUCT satisfies the property* (P) *(for any $l \geq 1$, $b \geq \pi$ and $M \geq 1$).*

Proof. Fix $l \geq 1$, $b \geq \pi$ and $M \geq 1$. We use $j_n : A_n \to A$ for the embedding. Let $\{x_1, x_2, ...,\}$ be a dense sequence of A. Without loss of generality, we may assume that $x_n \in A_n$. Let $0 < \varepsilon_n < \min\{1/2^n, \sum_{i=1}^{n-1} \varepsilon_n\}$. We may assume that $\mathcal{S}_1 = \{x_1\} = \mathcal{F}_1$. Let $\delta_1 > 0$ and let finite sets $\mathcal{G}_1 \subset A_1$ and $\mathcal{P}_1 \subset \mathbf{P}(A_1)$ be corresponding to $\varepsilon_1/2$ and $\mathcal{F}_1$ as required in 5.9. We may assume that $x_1 \in \mathcal{G}_1$. There is $k(2) > 1$ and $y_2 \in A_{k(2)}$ such that

$$\|x_1 - y_2\| < \varepsilon_2/2 \quad \text{and} \quad \mathrm{dist}(x, A_{k(2)}) < \varepsilon_2/2$$

for all $x \in \mathcal{G}_1$.

Let $\mathcal{S}_2 = \mathcal{G}_1 \cup \{x_2\}$. Since $A_{k(2)}$ is nuclear, there is a contractive completely positive linear map $L_1 : A \to A_{k(2)}$ such that

$$\|L_1(a) - a\| < \min\{\varepsilon_1/2, \delta_1/2\}$$

for $a \in \mathcal{G}_1$ (see for example 2.3.12 in [Ln7]).

Let $\phi_1 = (L_1)|_{A_1}$. Without loss of generality, we may assume that

$$[\phi_1]|_{\mathcal{P}_1} = [j_1]|_{\mathcal{P}_1}.$$

Set $\mathcal{F}_2 = L_1(\mathcal{G}_1) \cup \{x_2\}$. Let $\delta_2 > 0$, let finite sets $\mathcal{G}_2 \subset A_{k(2)}$ and let $\mathcal{P}_2 \subset \mathbf{P}(A_{k(2)})$ be corresponding to $\varepsilon_2/2$ and $\mathcal{F}_2$ as required in 5.9. There is $k(3) > k(2)$ and $y_2, y_3 \in A_{k(3)}$ such that

$$\|x_i - y_i\| < \varepsilon_2/2 \quad \text{and} \quad \mathrm{dist}(x, A_{k(3)}) < \varepsilon_2/2$$

for all $x \in \mathcal{G}_2$.

Let $\mathcal{S}_3 = \mathcal{G}_2 \cup \{x_3\}$. Since $A_{k(3)}$ is nuclear, there is a contractive completely positive linear map $L_2 : A \to A_{k(3)}$ such that

$$\|L_2(a) - a\| < \min\{\varepsilon_2/2, \delta_2/2\}$$

for $a \in \mathcal{G}_2$.

Let $\phi_2 = (L_2)|_{A_{k(2)}}$. Without loss of generality, we may assume that

$$[\phi_2]|_{\mathcal{P}_2} = [j_{k(2)}]|_{\mathcal{P}_2}.$$

Continuing this way, we obtain a sequence of maps $\phi_n : A_{k(n)} \to A_{k(n+1)}$. From the construction, we see that ϕ_n is a $(\mathcal{F}_n, \varepsilon_n, \mathcal{G}_n, \mathcal{P}_n, \delta_n)$-map. Set $B = \lim_n(A_{k(n)}, \phi_n)$. This is a generalized inductive limit in the sense of Blackadar and Kirchberg ([BK1]). Since j_n is a monomorphism, $[j_n] \in KK(A_n, A)$. From the above construction, B satisfies the property (P).

We also have the following approximate intertwining:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\phi_1} & A_{k(2)} & \xrightarrow{\phi_2} & A_{k(3)} & \xrightarrow{\phi_3} & \cdots\longrightarrow & B \\ \downarrow_{j_1} & \nearrow_{L_1} & \downarrow_{j_2} & \nearrow_{L_2} & \downarrow_{j_3} & & & \\ A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & \cdots\longrightarrow & A. \end{array}$$

Therefore $A \cong B$. So A satisfies the property (P). $\square$

Definition 6.12. Let $h : A \to B$ be a homomorphism. Then h induces homomorphisms from $K_*(A, \mathbb{Z}/k\mathbb{Z})$ to $K_*(B, \mathbb{Z}/k\mathbb{Z})$. By the naturality, we obtain a homomorphism $\Gamma(h) \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$. Let $\tau : A \to Q(B)$ be a trivial extension. There is a monomorphism $\sigma : A \to M(B \otimes \mathcal{K})$ such that $\pi \circ \sigma = \tau$, where $\pi : M(A \otimes \mathcal{K}) \to Q(B)$ is the quotient map. Since $K_*(M(A \otimes \mathcal{K})) = \{0\}$, we see that $\Gamma(\tau) = 0$. Thus, each element in $[\tau] \in Ext(A, B)$ gives an element $\Gamma(\tau) \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q(B)))$. Therefore we obtain a homomorphism $\Gamma : Ext(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q(B)))$. Since there is a surjective map from $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q(B)))$ to $\mathrm{Hom}(K_*(A), K_{*-1}(B))$, we also obtain a homomorphism $\gamma : Ext(A, B) \to \mathrm{Hom}(K_*(A), K_*(B))$.

Lemma 6.13. *Let A be a nuclear C^*-algebra, satisfying the property* (P) *(for $\mathbf{C}_{1,3\pi,1}$). Then for any σ-unital C^*-algebra, the map*

$$\Gamma : Ext(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q(B)))$$

is surjective.

Proof. We may also assume that B is stable. Let $\beta \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q(B)))$.

Let $\mathcal{S}_n$, $\mathcal{F}_n$, $\mathcal{G}_n$, ε_n, δ_n and $\alpha_n \in \mathrm{Hom}_\Lambda(\underline{K}(A_n), \underline{K}(A))$ be as in Definition 6.8.

Let $\tau_0 : A \to Q(B)$ be an absorbing trivial extension (of A by B) which factors through $B(l^2) \to Q(\mathcal{K}) \to Q(B)$. So τ_0 is a full monomorphism.

To save notation, without loss of generality, we may assume that $k(n) = n$ in the definition of 6.8.

Let $\beta_n \in \mathrm{Hom}_\Lambda(\underline{K}(A_n), \underline{K}(Q(B)))$ be such that $\beta \circ \alpha_n = \beta_n$. Since A_1 satisfies the AUCT, there is a an extension $\tau_1 : A_1 \to Q(B)$ such that τ_1 induces β_1. Since $[e] \oplus [1_{Q(B)}] = [e]$, for any projection $e \in Q(B)$, we may set $L_1 = (\sigma_1 \oplus \tau_0) \circ \phi_{1,\infty} : A_1 \to Q(B)$. Note that $\Gamma(\tau_0) = 0$ and $\tau_0 \circ \phi_{1,\infty}$ factors through $Q(\mathcal{K})$.

Since A_2 satisfies the AUCT, there is an extension $\tau_2 : A_2 \to Q(B)$ such that τ_2 induces β_2. As above, we set $L_2' = \tau_2 \oplus (\tau_0 \circ \phi_{2,\infty}) : A_2 \to Q(B)$. Note that for any integer m, there is an isometry $V_m \in M_m(Q(\mathcal{K})) \subset M_m(Q(B))$ such that $V^*\tau_0 V = \tau_0 \oplus \cdots \oplus \tau_0$. So

$$V^*\tau_0 \circ \phi_{1,\infty} V = \tau_0 \circ \phi_{1,\infty} \oplus \tau_0 \circ \phi_{1,\infty} \oplus \cdots \oplus \tau_0 \circ \phi_{1,\infty}.$$

Note also that

$$\tau_0 \circ \phi_{2,\infty} \circ \phi_1 = \tau_0 \circ \phi_{1,\infty} \quad \text{and} \quad L_2' \circ \phi_1 = \tau_2 \circ \phi_1 \oplus \tau_0 \circ \phi_{1,\infty}.$$

Furthermore

$$[\tau_2 \circ \phi_1]|_{\mathcal{P}_1} = [\beta \circ \alpha_2] \circ [\phi_1]|_{\mathcal{P}_1} = [\beta] \circ [\phi_{2,\infty}] \circ [\phi_1]|_{\mathcal{P}_1}.$$

Thus, by applying 6.6, there is a unitary $U_1 \in Q(B)$ such that

$$\mathrm{ad}\, U_1 \circ L_2' \circ \phi_1 \approx_{1/2} L_1 \ \text{ on } \ \mathcal{F}_1.$$

Set $L_2 = \mathrm{ad}U_1 \circ L_2'$. So

$$\begin{array}{ccc} A_1 & \xrightarrow{\phi_1} & A_2 \\ \downarrow_{L_1} & & \downarrow_{L_2} \\ Q(B) & \xrightarrow{\mathrm{id}} & Q(B) \end{array}$$

is approximately commutative on $\mathcal{F}_1$ within ε_1.

We note that we have assumed that $\phi_{n,n+1}(\mathcal{F}_n) \subset \mathcal{F}_{n+1}$.

Continuing this way, since A_{n+1} satisfies the AUCT, there is an extension $\tau_{n+1} : A_{n+1} \to Q(B)$ such that τ_{n+1} induces β_{n+1}. We set $L'_{n+1} = \tau_n \oplus (\tau_0 \circ \phi_{n,\infty}) : A_{n+1} \to Q(B)$. By the construction ($\tau_0$ is unitarily equivalent to any finite copies of it), applying 6.6 again, there exists a unitary $U_n \in Q(B)$ such that

$$\mathrm{ad}\, U_n \circ L'_{n+1} \circ \phi_n \approx_{1/2^n} L_n \quad \text{on } S_n.$$

Set $L_{n+1} = \mathrm{ad}U_n \circ L'_{n+1}$. Hence we obtain an approximate intertwining:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & \cdots \longrightarrow & & A \\ \downarrow_{L_1} & & \downarrow_{L_2} & & \downarrow_{L_3} & & & & \\ Q(B) & \xrightarrow{\mathrm{id}} & Q(B) & \xrightarrow{\mathrm{id}} & Q(B) & \xrightarrow{\mathrm{id}} & \cdots \longrightarrow & & Q(B). \end{array}$$

Therefore we obtain the following approximate intertwining:

$$\begin{array}{ccccccccc} A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & A & \xrightarrow{\mathrm{id}} & \cdots \longrightarrow & & A \\ \downarrow_{\Psi_1} & & \downarrow_{\Psi_2} & & \downarrow_{\Psi_3} & & & & \\ Q(B) & \xrightarrow{\mathrm{id}} & Q(B) & \xrightarrow{\mathrm{id}} & Q(B) & \xrightarrow{\mathrm{id}} & \cdots \longrightarrow & & Q(B). \end{array}$$

Consequently we obtain, by Elliott's argument, a homomorphism $L : A \to Q(B)$ induced by the above approximate intertwining. To make sure that L is a monomorphism, set $\tau = L \oplus \tau_0$, where $\tau_0 : A \to Q(B)$ is an absorbing extension. Since

$$[L_n]|_{\mathcal{P}_n} = \beta|_{\mathcal{P}_n},$$

it is evident from the construction that τ induces β. $\square$

Theorem 6.14. *If A is a nuclear C^*-algebra, satisfying the property* (P) *(for $\mathbf{C}_{1,3,1}$), then the map Γ induces an isomorphism*

$$\overline{\Gamma} : KL^1(A,B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q(B))).$$

Proof. It follows from 6.13 that it suffices to show that $\mathrm{ker}i\,\Gamma = \mathcal{T}(A,B)$. Let $\tau : A \to Q(B)$ be an extension. By 4.6, we will show that if $[\tau] \in \ker \Gamma$, then $[\tau] \in \mathcal{T}(A,B)$. Let $\tau_0 : A \to Q(B)$ be an absorbing trivial extension. Note again that τ_0 is unitarily equivalent to $\tau_0 \oplus \tau_0 \oplus \cdots \oplus \tau_0$ (for any finitely many τ_0). It then suffices to show that, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a unitary $U \in Q(B)$ such that

$$\mathrm{ad}\, U \circ \tau_0 \approx_\varepsilon \tau \oplus \tau_0 \quad \text{on } \mathcal{F}.$$

Let $\tau_n = \tau|_{A_n}$. Since $\Gamma([\tau]) = 0$, one has $\Gamma([\tau_n]) = \Gamma([\tau \circ j_n]) = 0$. Since $\sigma_n = (\tau_0)|_{A_n}$ are absorbing trivial extensions of A_n by B, it follows from 6.6 that for any $\varepsilon > 0$ and any finite subset $\mathcal{S}_n \subset A_n$, there is a unitary $W_n \in Q(B)$ such that

$$\mathrm{ad}\, W_n \circ \sigma_n \approx_\varepsilon \tau_n \oplus \sigma_n \quad \text{on } \mathcal{S}_n.$$

Since $\bigcup_n A_n$ is dense in A, we conclude (from the above) that $\tau \in \mathcal{T}$. $\square$

Proposition 6.15. *Let B be a σ-unital C^*-algebra. Then $\underline{K}(Q(B)) = \underline{K}(SB)$.*

Proof. We have the following six-term exact commutative diagram:

$$\begin{array}{ccccc} K_0(B\otimes\mathcal{K}\otimes C_k) & \to & K_0(M(B\otimes\mathcal{K})\otimes C_k) & \to & K_0(Q(B)\otimes C_k) \\ \uparrow & & & & \downarrow \\ K_1(Q(B)\otimes\mathcal{K}\otimes C_k) & \leftarrow & K_1(M(B\otimes\mathcal{K})\otimes C_k) & \leftarrow & K_1(B\otimes\mathcal{K})\otimes C_k) \end{array}$$

By [M], $K_i(M(B\otimes\mathcal{K})\otimes C_k)=0$ $(i=0,1)$. Therefore we have

$$K_i(Q(B),\mathbb{Z}/k\mathbb{Z})=K_{i-1}(B,\mathbb{Z}/k\mathbb{Z})\quad(i=0,1).$$

Since $SB\otimes C_k=S(B\otimes C_k)$, we have $K_i(SB,\mathbb{Z}/k\mathbb{Z})=K_{i-1}(B,\mathbb{Z}/k\mathbb{Z})$, $i=0,1$. □

Theorem 6.16. *Let A be a nuclear C^*-algebra satisfying the property* (P). *Then A satisfies the AUCT.*

Proof. It follows from 6.14 and 6.15 that

$$KL^1(A,B)=\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(SB))$$

and

$$KL^0(A,B)=KL^1(A,SB)=\mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B)).$$

□

The main result in this section is:

Corollary 6.17. (i) *Every C^*-algebra which is local AUCT satisfies the AUCT.*
(ii) *Every C^*-algebra in $\mathcal{N}_l$ satisfies the AUCT.*

Theorem 6.18. *Let A and B be two unital separable simple C^*-algebras with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and with unique tracial states. Suppose also that both A and B are locally type I. Then $A\cong B$ if and only if*

$$(K_0(A),K_0(A)_+,[1_A],K_1(A))\cong(K_0(B),K_0(B)_+,[1_B],K_1(B)).$$

Proof. It follows from [Ln9] that $TR(A)=TR(B)=0$. Since both A and B are locally type I, both A and B are nuclear. Moreover A, $B\in\mathcal{N}_l$. Therefore both A and B satisfy the AUCT. Thus 5.10 applies. □

Remark 6.19. It should be noted that the class of C^*-algebras in $\mathcal{N}$ is closed under inductive limits. However, Dadarlat and Eilers in [DE1] gave an example of a C^*-algebra which is locally homogeneous but it cannot be written as an inductive limit of homogeneous C^*-algebras. There is no reason to believe, at this moment, that C^*-algebras locally in $\mathcal{N}_l$ are in $\mathcal{N}$. However, here is the question: are there any separable nuclear C^*-algebras which satisfy the AUCT but not UCT?

7. AF-embedding

This section concerns embeddability of a separable RFD-algebra into a simple AF-algebra. Recall that a C^*-algebra, A is said to be residually finite dimensional (RFD), if there exists a separating family of finite-dimensional irreducible representations. It was shown in [D] that a nuclear separable RFD algebra which is homotopically dominated by an AF-algebra embeds in an AF-algebra and in [Ln5] that a nuclear separable RFD algebra satisfying the UCT can be embedded into a (simple) AF-algebra. More recently, among other things, Dadarlat shows ([D3]) that a unital separable nuclear embeddable RFD C^*-algebra, satisfying the UCT can be embedded into a simple AF-algebra. In this section we consider a sequence of RFD C^*-algebras.

Definition 7.1. Let $\{A_n\}$ be a sequence of unital RFD C^*-algebras. We say that $\{A_n\}$ is *asymptotically nuclear and asymptotically satisfies the AUCT*, if there are unital injective homomorphisms $h_n : A_n \to A_{n+1}$ such that the unital C^*-algebra $A = \lim_n(A_n, h_n)$ is nuclear and satisfies the AUCT.

As an application of results of section 2 and the last section, we show that, if $\{A_n\}$ is asymptotically nuclear and asymptotically satisfies the AUCT, then every A_n can be embedded into a same unital simple AF-algebra. It is important to note that we do not assume that A_n satisfies the UCT nor do we assume that A_n is nuclear.

7.2. Let $A = \lim_{n\to\infty}(A_n, \phi_n)$ be a unital nuclear C^*-algebra, where each A_n is a residually finite-dimensional (RFD) C^*-algebra, and each map ϕ_n is injective. We also assume that each A_n is separable. It is proved in [BK1] that every strong NF C^*-algebra, is such a C^*-algebra. We may write $A = \overline{\bigcup_{n=1}^\infty A_n}$, where $A_n \subset A_{n+1}$ and each A_n is a separable RFD C^*-algebra. We may further assume that each A_n has a unit and that the unit is the same as the unit of A. Denote by $\imath_n : A_n \to A$ the embedding.

Let $\{a_{i,j}\}$ be dense sequences of A_j. Let $\pi_{n,1}, ..., \pi_{n,s(n)}$ be finite-dimensional irreducible representations of A_n such that

$$max\{\|\pi_{n,k}(a_{i,j})\| : 1 \le k \le s(n)\} > \|a_{i,j}\| - 1/n\|a_{i,j}\|$$

for $1 \le i, j \le n$. Without loss of generality, we may assume that $s(n) \ge n$.

Suppose that $\pi_{i,j}$ has rank $r(i,j)$. Let $r(2) = \sum_{i=1}^{s(1)} r(1,i) + 1, r(n+1) = n(r(n)(\sum_i^{s(n)} r(n,i))) + r(n)$, $n = 1, 2, ...$ Let $C'_1 = A_1$, $C'_2 = A_2 \otimes M_{r(2)}$,..., $C'_n = A_n \otimes M_{r(n)}$, ... Define $j'_{n,n+1} : C'_n \to C'_{n+1}$ by

$$f \mapsto \mathrm{diag}(f, \pi_{n,1} \otimes \mathrm{id}_{r(n)}(f), \pi_{n,1} \otimes \mathrm{id}_{r(n)}(f), ..., \pi_{n,s(n)} \otimes \mathrm{id}_{r(n)}(f)),$$

where $f \in C'_n$ and $\pi_{n,i} \otimes \mathrm{id}_{r(n)}(f)$ repeats n times. Also, the image of $\pi_{n,i} \otimes \mathrm{id}_{r(n)}$ is in $(\mathbb{C} \cdot \mathrm{id}_{A_n}) \otimes M_{r(n)}$.

Let $C_1 = A_1 \otimes Q, ..., C_n = A_n \otimes Q, ...$ There is an isomorphism $R'_n : Q \otimes M_{r(n)} \to Q$ which gives $(R'_n)_* : \mathbb{Q} \to \mathbb{Q}$ by $(R'_n)_*(r) = \frac{r}{r(n)}$. Set $R_n : A_n \otimes (M_{r(n)} \otimes Q) \to C_n$ by $R_n(a \otimes b) = a \otimes R'_n(b)$. Also set $j_{n,n+1} = R_{n+1} \circ (j'_{n,n+1} \otimes \mathrm{id}_Q)$.

Let $H'_n : A_n \to F_n$ be defined by

$$f \mapsto \mathrm{diag}(\pi_{n,1} \otimes \mathrm{id}_{r(n)}(f), \pi_{n,1} \otimes \mathrm{id}_{r(n)}(f), ..., \pi_{n,s(n)} \otimes \mathrm{id}_{r(n)}(f)),$$

where F_n is a finite-dimensional C^*-subalgebra of $M_{r(n+1)}$. Let $H_n : C_n \to Q$ be defined by $H_n = R_{n+1} \circ (H'_n \otimes \mathrm{id}_Q)$.

Since $s(n) \ge n$, we compute that

$$t(j_{n,n+1}(1) - H_n(1)) \le 1/n^2$$

for all tracial states t on C_{n+1}.

Define $B = \lim_n(C_n, j_{n,n+1})$. In what follows, the map $C_n \to B$ will be denoted by $j_{n,\infty}$.

Similar constructions of this type previously appeared in [Go] and [D2].

Lemma 7.3. *B is a unital separable simple C^*-algebra, with $TR(B) = 0$ with divisible $K_0(B)$, and with a unique tracial state.*

See the proof of Lemma 6 in [Ln5] and 4.3 (and 4.4) in [Ln3].

Lemma 7.4. *Let F be a nuclear C^*-algebra, and A be the closure of $\bigcup_n A_n$, where $A_n \subset A_{n+1}$, $n = 1, 2, ...$ Suppose that there is a contractive completely positive linear map $\phi : F \to A$. Then, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset F$, there is an integer $n > 0$ and a contractive completely positive linear map $\psi : F \to A_n$ such that*

$$\|\phi(a) - \psi(a)\| < \varepsilon \quad \textit{for all } a \in \mathcal{F}.$$

Proof. There are (cf. 2.3.13 in [Ln7]) maps $L_n : F \to A_n$ for all large n such that $\|L_n(a)\| \le \|a\|$,

$$\|L_n(a) - \phi(a)\| \to 0, \quad \text{as } n \to \infty.$$

Define a map $L : F \to \prod_n A_n$ by $L(a) = \{L_n(a)\}$ for all $a \in A$. Let $\pi : \prod_n A_n \to \prod_n A_n/\oplus_n A_n$ be the quotient map. Then $\pi \circ L : F \to \prod_n A_n/\oplus_n A_n$ is a contractive completely positive linear map. Since F is nuclear, by [CE], there is a contractive completely positive linear map $\Psi : F \to \prod_n A_n$ such that $\pi \circ \Psi = \pi \circ L$. Write $\Phi(a) = \{\psi_n(a)\}$, where each $\psi_n : F \to A_n$ is a contractive completely positive linear map. We have

$$\|\psi_n(a) - L_n(a)\| \to 0 \quad \text{as } n \to \infty.$$

Thus lemma follows. □

Corollary 7.5. *Let A be a unital separable nuclear C^*-algebra, which is the closure of the union of increasing sequence $\{A_n\}$. Then for any finite subset $\mathcal{F} \subset A_n$ and $\varepsilon > 0$, there is an integer $k(n) \ge n$ and a contractive completely positive linear map $L : A \to A_{k(n)}$ such that*

$$\|L(a) - a\| < \varepsilon \quad \textit{for all } a \in A_n.$$

Proof. This follows from 7.4 immediately. □

Lemma 7.6. *B is nuclear and satisfies the AUCT.*

Proof. Let $D = A \otimes Q$. Since $D = \lim_{n\to\infty} A \otimes M_{n!}$, D satisfies the AUCT. Let $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$ be a dense sequence of finite subsets of B. Without loss of generality, we may assume that $\mathcal{S}_n \subset j_{n,\infty}(C_n)$. We may also assume that $\mathcal{S}_n \subset \mathcal{S}_{n+1}$. Let $\mathcal{F}_n \in C_n$ so that $j_{n,\infty}(\mathcal{F}_n) = \mathcal{S}_n$. Let $\varepsilon_n > 0$ so that $\varepsilon_n < 1/2^{n+1}$. Note that D satisfies the AUCT. Let $\mathcal{G}_n$ be a finite subset of D and let $\mathcal{P}_n$ be a finite subset of $\mathbf{P}(D)$ and $\delta_n > 0$ be as in 5.9. We may assume, without loss of generality, that $\delta_n < \varepsilon_n$ and $\sum_{k=n+1}^{\infty} \delta_k < \delta_n$. By passing to a subsequence if necessary, and with an error no more than $\delta_n/2$, we may assume that $\mathcal{G}_n \subset C_{n+1}$.

Let $\eta_n > 0$. By applying 7.5 and by passing to a subsequence, we may assume that there is a contractive completely positive linear map $L_n' : A \to A_n$ such that

$$\|L_n'(a_{i,j}) - a_{i,j}\| < \eta_n,$$

$i, j = 1, 2, ..., n$. Set $L_n = L_n' \otimes \mathrm{id}_Q$. By choosing sufficiently small η_n, we may assume that

$$\|L_n(c) - c\| < \delta_n/2$$

for $c \in \mathcal{F}_n$.

Set $\Psi_n = \imath_{n+1} \circ j_{n,n+1} \circ L_n$. Let $D' = \lim_n(D, \Psi_n)$ (a generalized inductive limit in the sense of Blackadar and Kirchberg ([BK1])). From the definition, since each D is nuclear, it is easy to see that D' is nuclear.

Then the following diagram is approximately intertwining:

$$\begin{array}{ccccccccc} D & \xrightarrow{\Psi_1} & D & \xrightarrow{\Psi_2} & D & \xrightarrow{\Psi_3} & \cdots\longrightarrow & D' \\ \downarrow_{L_1} & \nearrow_{\imath_2\circ j_{1,2}} & \downarrow_{L_2} & \nearrow_{\imath_3\circ j_{2,3}} & \downarrow_{L_3} & & & \\ C_1 & \xrightarrow{j_{1,2}} & C_2 & \xrightarrow{j_{2,3}} & C_3 & \xrightarrow{j_{3,4}} & \cdots\longrightarrow & B. \end{array}$$

So, $D' \cong B$. We may identify these two C^*-algebras.

Let $\mathcal{F}'_n = j_{n,n+1}(\mathcal{F}_n)$. Then, we may write

$$\mathcal{F}'_n = R_{n+1}(\{diag(f, H'_n(f)) : f \in \mathcal{F}_n\}),$$

where $H'_n : C_n \to M_{\bar{r}(n)} \otimes Q$. Set

$$\mathcal{P}'_n = [R_{n+1}](\mathcal{P}_n) \oplus [H_n](\mathcal{P}_n).$$

We note that $j_{n,\infty}|_{H_n(\mathcal{F}_n)}$ is a homomorphism, $M_{\bar{r}(1)} \otimes Q \cong Q$ and R_{n+1} is an automorphism. By 6.7, $(\mathcal{F}'_n, \varepsilon_n, \mathcal{G}'_n, \mathcal{P}'_n, \delta_n)$ is a 5-tuple (for D) that satisfies the conditions in 6.7. Let $G(\mathcal{P}'_n)$ be the group generated by $\mathcal{P}'_n$. Let $G_0 = G(\mathcal{P}'_n) \cap K_0(D)$ and $G_1 = G(\mathcal{P}'_n) \cap K_1(D)$. Then $[j_{n,\infty}]$ gives two homomorphisms from G_0 to $K_0(B)$ and G_1 to $K_1(B)$. Since both $K_0(B)$ and $K_1(B)$ are divisible, there is $\alpha_n^i : K_i(D) \to K_i(B)$ which extends these two homomorphisms. Since D satisfies the AUCT and both $K_i(D)$ and $K_i(B)$ are divisible, by the AUCT, $KL(D, B) = \mathrm{Hom}(K_*(D), K_*(B))$. Therefore there is $\alpha_n \in KK(D, B)$ such that

$$[j_{n,\infty}]|_{\mathcal{P}'_n} = (\alpha_n)|_{\mathcal{P}'_n}.$$

This shows that $B \cong D' = \lim_n(D, \Psi_n)$ satisfies the property (P). Hence by 6.16 B satisfies AUCT. □

Proposition 7.7. *B can be embedded into C.*

Proof. Since B is a unital separable simple C^*-algebra, with $TR(B) = 0$ and satisfies the AUCT, by 5.11, B has the UCT. There is $\alpha_i : K_i(B) \to K_i(C)$ which is positive and $h([1_B]) = [1_C]$. It follows that B is an AH-algebra. It follows from [EG] that there is an (injective) homomorphism $h : B \to C$ such that $h_{*i} = \alpha$. □

Since each A_n was embedded into B, we obtain the following:

Theorem 7.8. *Let $\{A_n\}$ be a sequence of RFD C^*-algebras which is asymptotically nuclear and asymptotically satisfies the UCT. Then every A_n can be embedded into a same unital simple AF-algebra.*

We note again that we do not assume that A_n satisfies the UCT (or AUCT) or that A_n is nuclear.

Corollary 7.9. *Let A be a unital separable nuclear RFD C^*-algebra, which satisfies the AUCT. Then A can be embedded into a simple AF-algebra.*

Proof. Write $A = \lim_{n\to\infty}(A, \mathrm{id}_A)$. Then this corollary follows from 7.8. □

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