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THE SMOOTHING PROPERTY FOR A CLASS OF DOUBLY NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. We consider a class of doubly nonlinear parabolic equations used in modeling free boundaries with a finite speed of propagation. We prove that nonnegative weak solutions satisfy a smoothing property; this is a well-known feature in some particular cases such as the porous medium equation or the parabolic p-Laplace equation. The result is obtained via regularization and a comparison theorem.

1. INTRODUCTION

This paper deals with a class of partial differential equations—doubly nonlinear parabolic equations—that have recently attracted a lot of attention. They arise in many different physical contexts such as, for instance, the description of turbulent filtration in porous media, or the flow of a gas through a porous medium in a turbulent regime; in general, doubly nonlinear parabolic equations are used to model processes obeying a nonlinear Darcy law (see [8], [14], and the references given therein).

Typical examples of such parabolic PDEs are equations of the form

$$u_t = \Delta_p(|u|^{m-1}u), \qquad m(p-1) > 1 \ ,$$

where Δ_p is the p-Laplacian, which are used in modeling phenomena involving a free boundary with a finite speed of propagation. These degenerate equations exhibiting a doubly nonlinearity generalize the porous medium equation ($p = 2$) and the parabolic p-Laplace equation ($m = 1$).

The aim of the paper is to show that nonnegative solutions of a class of doubly nonlinear parabolic equations satisfy the *smoothing property*, i.e., the estimate

$$u_t \geq -\frac{c}{t}\, u \ , \tag{1.1}$$

where c is a constant depending only on the data. The smoothing property (1.1) implies the *regularizing property*

$$\|u_t\|_{L^1(\mathbb{R}^d)} \leq \frac{2c}{t}\, \|u_0\|_{L^1(\mathbb{R}^d)} \ ,$$

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as will be shown below, and plays a crucial role in the study of the finite speed of propagation of the free boundary (see, e.g., [5, 12], where the porous medium equation is treated) or the proof of regularity results for the solutions (cf. [13]).

The smoothing property is known for some particular degenerate equations although a systematic approach to the matter is still lacking. For instance, let b be nondecreasing, $b(0) = 0$, and

$$c_1 \le \frac{b(s)\, b''(s)}{(b'(s))^2} \le c_2$$

for certain constants $c_1, c_2 > 0$. Then there is a constant c, depending only on c_1 and c_2, such that solutions of

$$u_t - \Delta b(u) = 0$$

satisfy estimate (1.1) (cf. [6, 12]). A first proof of the smoothing property was given in [1] for the case of the porous medium equation

$$u_t = \Delta u^m \qquad (m > 1).$$

In [17] the smoothing property was obtained for the one-dimensional porous medium equation with weak absorption

$$u_t = \partial_x \partial_x u^m - c u^k \qquad (c \ge 0,\ k \ge m).$$

The n-dimensional case is treated in [26]. Further proofs for other classes of porous medium type equations can be found in [6], [7], and [23]. The parabolic p–Laplace equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

was considered in [15], with $p > 2 - \frac{2}{d+1}$. A proof of the smoothing property for solutions of the doubly nonlinear parabolic equation

$$u_t = \partial_x(|\partial_x b(u)|^{p-2} \partial_x b(u))$$

in the one-dimensional case is given in [14].

The paper is organized as follows. The problem and the main result are stated in the next section. Then, in section 3, we introduce a regularized problem and study its most important properties. Section 4 contains the proof of the smoothing property as a consequence of a series of intermediate lemmas; essential use is made of the regularization and a comparison argument.

2. The main result

Let $x \in \mathbb{R}^d$, $d \ge 2$, and $t \in (0, T]$, with $T < \infty$. We consider the Cauchy problem

$$\text{(2.1)} \quad \begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \displaystyle\sum_{i=1}^{d} \partial_i a_i(\nabla b(u(x,t))) - f(u(x,t)) & \text{in} \quad \mathbb{R}^d \times (0, T], \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}^d, \end{cases}$$

where

$$\begin{aligned} b(z) &= |z|^{m-1} z \qquad \text{for } z \in \mathbb{R}, \\ a_i(s) &= |s|^{p-2} s_i \qquad \text{for } s \in \mathbb{R}^d, \quad i = 1, \ldots, d, \\ f(z) &= |z|^{k-1} z \qquad \text{for } z \in \mathbb{R}. \end{aligned}$$

We assume that $m > 1$, $p > 1$ and treat the *slow diffusion case*

$$m(p-1) > 1$$

with *weak absorption* $k \geq m(p-1)$. Concerning the initial condition, we take $u_0(x) \geq 0$ and spt u_0 to be bounded. For simplicity, we restrict ourselves to L^∞–initial data but u_0 can more generally be in L^1, in which case we have to approximate it with L^∞–functions.

The pertinent definition of a weak solution for the problem is

Definition 2.1. We say that $u(x,t)$ is a weak solution of (2.1) if

$$u \in L^\infty(0,T;L^\infty(\mathbb{R}^d)) \;; \quad b(u) \in L^p(0,T;W^{1,p}(\mathbb{R}^d)) \;; \tag{2.2}$$

$$\begin{aligned} -\int_0^T\int_{\mathbb{R}^d} u\,\phi_t\,\mathrm{d}x\mathrm{d}t + \sum_{i=1}^d \int_0^T\int_{\mathbb{R}^d} a_i(\nabla b(u))\,\partial_i\phi\,\mathrm{d}x\mathrm{d}t \\ + \int_0^T\int_{\mathbb{R}^d} f(u)\,\phi\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^d} u_0\,\phi(\cdot,0)\,\mathrm{d}x \;, \end{aligned} \tag{2.3}$$

for all $\phi \in L^p(0,T;W^{1,p}(\mathbb{R}^d)) \cap\ W^{1,1}(0,T;L^1(\mathbb{R}^d))$ such that $\phi(\cdot,T) \equiv 0$.

Remark 2.2. The existence of a weak solution may be proven as in [16], where doubly nonlinear parabolic equations are considered. Existence and uniqueness results concerning related equations are given in [3, 4, 20] and [19, 21], respectively.

Remark 2.3. It is well known that u is Hölder continuous; see [9, 19, 22, 24, 25, 27].

Remark 2.4. As in [16] it can be shown that

$$u_t\,\partial_t b(u) \in L^1(0,T;L^1(\mathbb{R}^d)). \tag{2.4}$$

Since $u_0 \geq 0$, the weak solution is nonnegative. In fact, applying a comparison theorem (see, e.g., [4, 18]) we find that $0 \leq u \leq \|u_0\|_{L^\infty}$. Moreover, spt $u(x,T)$ is bounded. To see this, let us consider the self-similar Barenblatt solutions (cf. [2])

$$u^*(x,t;\alpha,\tau) = (t+\tau)^{-\frac{1}{\mu}}\left[\left\{\alpha - k\left(|x|(t+\tau)^{-\frac{1}{n\mu}}\right)^{\frac{p}{p-1}}\right\}^+\right]^{\frac{p-1}{m(p-1)-1}},$$

where $\mu = m(p-1) - 1 + \frac{p}{d}$, $k = \frac{m(p-1)-1}{mp}(d\mu)^{-\frac{1}{p-1}}$, and $\alpha, \tau > 0$. Clearly, spt $u^*(x,T;\alpha,\tau)$ is bounded for each $T > 0$. Let $0 \leq u_0 \leq u_0^*$ and $T > 0$. From the comparison theorem it follows that $u \leq u^*$ on $\mathbb{R}^d \times [0,T]$. Thus,

$$\text{spt } u(x,t) \subset \text{spt } u^*(x,T;\alpha,\tau) \qquad \text{for all } t \in [0,T].$$

Further, it holds that

$$\text{spt } u(\cdot,t) \subset \text{spt } u(\cdot,s) \qquad \text{for all } t < s.$$

Now let us take a convex polyhedron $\Omega \subset \mathbb{R}^d$ such that spt $u(\cdot,T) \subset\subset \Omega$. It follows that

$$\text{spt } u(\cdot,t) \subset \Omega \qquad \text{for all } t \in [0,T]$$

and, in fact, $u = 0$ on $\partial\Omega \times (0,T]$. Hence, we may rewrite the Cauchy problem as

$$\begin{aligned} u_t &= \sum_{i=1}^d \partial_i\,a_i(\nabla b(u)) - f(u) && \text{in } \Omega \times (0,T], \\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T], \\ u(x,0) &= u_0(x) && \text{in } \Omega. \end{aligned} \tag{2.5}$$

Our main result is as follows.

Theorem 2.5. *Let u be a nonnegative weak solution of* (2.1) *in the sense of the previous definition. Then*

$$u_t \geq -\frac{c}{t}\, u \qquad \text{in } \mathcal{D}'(\mathbb{R}^d \times (0,T)) \ , \tag{2.6}$$

where $c = \frac{1}{m(p-1)-1}$.

As a consequence we get

Corollary 2.6. *Let u be a nonnegative weak solution of* (2.1) *in the sense of the previous definition. For a.e. $t>0$, the following estimate holds:*

$$\int_{\mathbb{R}^d} |u_t(x,t)|\, dx \leq \frac{2c}{t} \int_{\mathbb{R}^d} |u_0(x)|\, dx,$$

where $c = \frac{1}{m(p-1)-1}$.

3. The regularized problem

Now we define a regularization of problem (2.5) as follows. We consider the initial boundary value problem

$$\begin{aligned} u_t^\varepsilon &= \sum_{i=1}^d \partial_i\, a_i(\nabla b(u^\varepsilon)) - f(u^\varepsilon) && \text{in } \Omega \times (0,T], \\ u^\varepsilon(x,t) &= \varepsilon && \text{on } \partial\Omega \times (0,T], \\ u^\varepsilon(x,0) &= \bar{u}_0(x) + \varepsilon \equiv u_0^\varepsilon && \text{in } \Omega, \end{aligned} \tag{3.1}$$

where $\varepsilon \in (0,1)$ is small and $\bar{u}_0$ is a smooth (at least C^1) approximation of u_0, such that $u^\varepsilon(x,0) \geq \varepsilon_0 > 0$ for some $\varepsilon_0 \in (0,\varepsilon)$.

Remark 3.1. As before, it can be shown that there exists a unique weak solution u^ε. Furthermore, a comparison argument provides the inequality $u^\varepsilon(x,t) \geq \varepsilon_0 e^{-t}$, valid for all $(x,t) \in \Omega \times (0,T]$. This implies that

$$b'(u^\varepsilon) \geq c_0 > 0\,, \tag{3.2}$$

for $c_0 = m\varepsilon_0^{m-1} e^{-(m-1)T}$.

Multiplying the equation in (3.1) by a smooth test function ϕ, such that $\phi(\cdot,T) = 0$, and integrating by parts, we obtain

$$\begin{aligned} -\int_0^T\!\!\int_\Omega u^\varepsilon \phi_t + \sum_{i=1}^d \int_0^T\!\!\int_\Omega (b'(u^\varepsilon))^{p-1} a_i(\nabla u^\varepsilon)\, \partial_i \phi \\ + \int_0^T\!\!\int_\Omega f(u^\varepsilon)\phi = \int_\Omega u_0^\varepsilon\, \phi(\cdot,0). \end{aligned} \tag{3.3}$$

In view of (3.2), the coefficient $(b'(u^\varepsilon))^{p-1}$ in (3.3) is bounded from below by a positive constant. Hence, in (3.3) there is no longer a double degeneracy.

A simple but crucial result is

Lemma 3.2. *The solution u^ε of* (3.3) *satisfies*

$$u_t^\varepsilon \in L^2(0,T;L^2(\Omega)). \tag{3.4}$$

Proof. Due to (2.4) we have $u_t^\varepsilon\, \partial_t b(u^\varepsilon) \in L^1(0,T;L^1(\Omega))$. We conclude from inequality (3.2) that

$$\int_0^T\int_\Omega u_t^\varepsilon\, \partial_t b(u^\varepsilon) \geq c_0 \int_0^T\int_\Omega (u_t^\varepsilon)^2,$$

which gives the result. □

We now approximate the regularized problem (3.1) using a Galerkin procedure. We consider a family of decompositions of Ω into closed d-simplices. We get a sequence of finite element spaces V^n such that $V^n \subset V^k$ for $n \leq k$. Here, $V^n := \text{span}\{\varphi_1, \dots, \varphi_n\}$ is a space of continuous functions that are piecewise linear with respect to the triangulation of Ω and vanish on $\partial\Omega$.

Let $\Pi_n g(x)$ be the piecewise linear interpolant of a continuous function $g(x)$. That is, $\Pi_n g(x)$ is the continuous piecewise linear function satisfying $\Pi_n g(x_j) = g(x_j)$ for all nodes x_j of the triangulation of Ω. Further, let us introduce the notation

$$(g_1(x), g_2(x))_n = \int_\Omega \Pi_n(g_1(x)g_2(x))\, dx$$

and the spaces

$$S_i^n(\alpha) = \left\{ f \in H^i(0,T;L^\infty(\Omega)) : f(x,t) = \alpha + \sum_{j=1}^n f_j^n(t)\, \varphi_j(x) \right\}, \quad i = 0, 1,$$

given a constant α and functions $f_j^n \in H^i(0,T) \equiv W^{i,2}(0,T)$. Observe that if $f \in S_i^n(\alpha)$, then $f = \alpha$ on $\partial\Omega \times (0,T)$.

The approximate problem consists in finding a function

$$u^n(x,t) = b^{-1}\left(\varepsilon^m + \sum_{j=1}^n \gamma_j^n(t)\, \varphi_j(x) \right) , \tag{3.5}$$

with $\gamma_j^n \in L^2(0,T)$, such that

$$(u_t^n, \phi^n)_n + \sum_{i=1}^d \int_\Omega a_i(\nabla b(u^n))\, \partial_i \phi^n \,\mathrm{d}x + (f(u^n), \phi^n)_n = 0 , \tag{3.6}$$

for all test functions $\phi^n \in S_0^n(0)$, and

$$u^n(0) \equiv u_0^n = b^{-1}(\Pi_n b(u_0^\varepsilon)) . \tag{3.7}$$

Here, $\Pi_n b(u_0^\varepsilon)$ is the interpolant of $b(u_0^\varepsilon)$.

It follows from classical results that there exists a unique weak solution u^n. Let us state some regularity results for u^n. As above (cf. (2.4) and (3.10)) it can be shown that $\gamma_j^n \in H^1(0,T)$. Due to the structure of u^n this implies that

$$\nabla \partial_t b(u^n) \in L^2(0,T;L^2(\Omega)) . \tag{3.8}$$

Moreover, u^n is an L^∞–function, which follows from a comparison argument similar to the one used in Lemma 3.3 below. Due to the special structure of (3.6), the usual conditions on the acuteness of the triangulation (see [11]) are not needed in order to prove this semi-discrete maximum principle.

Noting that $V^n \subset W_0^{1,\infty}(\Omega)$ we also have

$$\nabla b(u^n) \in L^\infty(0,T;L^\infty(\Omega)) . \tag{3.9}$$

Next, assuming that the family of triangulations is regular, we obtain estimates uniform in n, from which we derive the convergence

$$u^n \longrightarrow u^\varepsilon \quad \text{in } L^1(0,T;L^1(\Omega))\,, \quad \text{as } n \to \infty\,.$$

Finally, let us prove a crucial result in the context of the regularization: that the approximate solution is bounded from below away from 0.

Lemma 3.3. *It holds that*

$$u^n(x,t) \geq \varepsilon_0\, e^{-t}\,, \quad \textit{for a.e. } (x,t) \in \Omega \times [0,T]. \tag{3.10}$$

Proof. The function $v = \varepsilon_0\, e^{-t}$ satisfies the inequality

$$\begin{aligned}(v_t, \phi^n)_n &+ \sum_{i=1}^d \int_\Omega a_i(\nabla b(v))\, \partial_i \phi^n + (f(v), \phi^n)_n \\ &= (\varepsilon_0\, e^{-t}\,(-1 + \varepsilon_0^{k-1}\, e^{-(k-1)t}), \phi^n)_n \\ &\leq 0\,,\end{aligned} \tag{3.11}$$

for all $\phi^n \in S_0^n(0)$ satisfying $\phi^n \geq 0$. We now subtract identity (3.6) from (3.11) and integrate in time from 0 to $\tau \leq T$. Choosing

$$\phi^n = \Pi_n g_\delta(b(v) - b(u^n)) \ \in S_0^n(0)\,,$$

where, for $0 < \delta < 1$,

$$g_\delta(s) = \begin{cases} 1 & \text{if } s > \delta, \\ \delta^{-1} s & \text{if } 0 \leq s \leq \delta, \\ 0 & \text{if } s < 0\,, \end{cases} \tag{3.12}$$

we obtain

$$\begin{aligned}&\int_0^\tau ((v-u^n)_t, g_\delta(b(v) - b(u^n)))_n \\ &\quad\leq \int_0^\tau \int_\Omega |\nabla b(u^n)|^{p-2} \langle \nabla(b(u^n) - b(v)), \nabla \Pi_n g_\delta(b(v) - b(u^n))\rangle \\ &\qquad + \int_0^\tau (f(u^n) - f(v), g_\delta(b(v) - b(u^n)))_n\,.\end{aligned}$$

Observe that the integrals on the right-hand side are negative. In fact, $b(u^n) - b(v)$ is piecewise linear in x, so we have

$$\begin{aligned}&\langle \nabla(b(u^n) - b(v)), \nabla \Pi_n g_\delta(b(v) - b(u^n))\rangle \\ &\quad = \langle \nabla \Pi_n(b(u^n) - b(v)), \nabla \Pi_n g_\delta(b(v) - b(u^n))\rangle \leq 0\,.\end{aligned}$$

Taking the $\lim_{\delta\to 0}$ and using the fact that

$$\lim_{\delta\to 0} g_\delta(b(v) - b(u^n)) = \lim_{\delta \to 0} g_\delta(v - u^n)$$

and $(v-u^n)_t\, g_\delta(v-u^n) = \partial_t G_\delta(v-u^n)$, where $G_\delta(s) = \int_0^s g_\delta(r)\,\mathrm{d}r$, we conclude that

$$\int_\Omega \Pi_n\{v(\cdot,\tau) - u^n(\cdot,\tau)\}^+ \leq \int_\Omega \Pi_n\{v(\cdot,0) - u^n(\cdot,0)\}^+ = 0.$$

Hence, it follows that $v(x_j,\tau) \le u^n(x_j,\tau)$ for all nodes x_j of the triangulation. This implies that $b(v) \le b(u^n)$ in $\Omega \times \{\tau\}$, since $b(v)$ and $b(u^n)$ are piecewise linear functions. Noting that τ is arbitrary the assertion follows. □

4. Proof of the main result

In this section we give the proof of the smoothing property. To begin with we establish a comparison theorem (see Lemma 4.1 below).

Let u^n be the function satisfying (3.5)–(3.7). We now introduce the parabolic operator, defined by

$$\begin{aligned} L^n(v) \quad &= \quad v_t - c_1 \frac{v}{t} - \frac{b''(u^n)}{t(b'(u^n))^2} v^2 + f'(u^n) v \\ & \quad -b'(u^n) \sum_i \partial_i \left[(p-2) |\nabla b(u^n)|^{p-4} \langle \nabla b(u^n), \nabla v \rangle \partial_i b(u^n) \right] \\ & \quad -b'(u^n) \sum_i \partial_i \left[|\nabla b(u^n)|^{p-2} \partial_i v \right], \end{aligned}$$

where the constant c_1 is given by

$$c_1 = \frac{m-1}{m(p-1)-1}\ . \tag{4.1}$$

Further, we set

$$\begin{aligned} L^n(v,\phi^n) \quad &= \quad -(v,\phi^n_t)_n - \left(c_1 \frac{v}{t} + \frac{b''(u^n)}{t(b'(u^n))^2} v^2 - f'(u^n) v, \phi^n \right)_n \\ & \quad +(p-2) \int_\Omega |\nabla b(u^n)|^{p-4} \langle \nabla b(u^n), \nabla v \rangle \langle \nabla b(u^n), \nabla \Pi_n (b'(u^n)\, \phi^n) \rangle \\ & \quad + \int_\Omega |\nabla b(u^n)|^{p-2} \langle \nabla v, \nabla \Pi_n (b'(u^n)\, \phi^n) \rangle - (v(\cdot,0), \phi^n(\cdot,0))_n \end{aligned}$$

for $v \in S^n_0(\alpha)$ and $\phi^n \in S^n_1(0)$ with $\phi^n(\cdot,T) = 0$. The comparison theorem reads as follows.

Lemma 4.1. *Let $v \in S^n_0(\alpha)$, $w \in S^n_0(\alpha')$, $\alpha \ge \alpha'$, and*

$$v(x,0) \ge w(x,0) \quad \textit{in } \Omega.$$

If, for all $\phi^n \in S^n_1(0)$, with $\phi^n(\cdot,T) = 0$ and $\phi^n \ge 0$,

$$\int_0^T L^n(v,\phi^n) \ge \int_0^T L^n(w,\phi^n)$$

and the integrals are well defined, then we have

$$v(x,t) \ge w(x,t)\ , \quad \textit{for a.e. } (x,t) \in \Omega \times [0,T].$$

Proof. We have

$$\begin{aligned} 0 &\leq \int_0^T (L^n(v,\phi^n) - L^n(w,\phi^n)) \\ &= -\int_0^T (v-w,\phi^n_t)_n - c_1 \int_0^T \left(\frac{v-w}{t},\phi^n\right)_n \\ &\quad -\int_0^T \left(\frac{b''(u^n)}{t(b'(u^n))^2}\,(v^2-w^2),\phi^n\right)_n + \int_0^T (f'(u^n)\,(v-w),\phi^n)_n \\ &\quad +(p-2)\int_0^T\!\!\int_\Omega |\nabla b(u^n)|^{p-4}\langle\nabla b(u^n),\nabla(v-w)\rangle\langle\nabla b(u^n),\nabla\Pi_n(b'(u^n)\phi^n)\rangle \\ &\quad +\int_0^T\!\!\int_\Omega |\nabla b(u^n)|^{p-2}\langle\nabla(v-w),\nabla\Pi_n(b'(u^n)\phi^n)\rangle \\ &\quad -(v(\cdot,0)-w(\cdot,0),\phi^n(\cdot,0))_n \\ &=: J_1+\cdots+J_7. \end{aligned}$$

Now we choose an appropriate test function ϕ^n. Let

$$\psi_\sigma(x,t) = \frac{1}{\sigma}\int_t^{t+\sigma}\psi(x,\tau)\,d\tau$$

and let g_δ be the function defined in (3.12). We set

$$\phi^n(x,t) = \frac{1}{\sigma}\int_{t-\sigma}^t \Pi_n g_\delta(w_\sigma(x,\tau)-v_\sigma(x,\tau))\,\{\tau\}^+\{T-\sigma-\tau\}^+e^{\lambda\tau}\,d\tau,$$

where $\lambda<0$ is a constant, and the functions v and w are extended for $t<0$ and $t>T$ in an appropriate way. Let us note that $w_\sigma - v_\sigma \leq 0$ on $\partial\Omega\times(0,T]$. Thus, $\phi^n=0$ on $\partial\Omega\times(0,T]$. Moreover, it is not hard to see that $\phi^n\in S_1^n(0)$, $\phi^n(\cdot,T)=0$, and $\phi^n\geq 0$. Hence, ϕ^n is an admissible test function.

Let us estimate the integrals $J_1,\ldots,J_7$ from above. Let

$$D_t^{\pm\sigma}f(x,t) = \frac{f(x,t\pm\sigma)-f(x,t)}{\sigma}$$

be the difference quotients associated with f and recall the definition of the function

$$G_\delta(s) = \int_0^s g_\delta(r)\,\mathrm{d}r\ .$$

We easily see that

$$\begin{aligned} J_1 &= \int_0^T \left(w-v,-D_t^{-\sigma}[g_\delta(w_\sigma-v_\sigma)\,\{t\}^+\{T-\sigma-t\}^+e^{\lambda t}]\right)_n \\ &= \int_{-\sigma}^{T-\sigma}\left(-D_t^\sigma(w-v),g_\delta(w_\sigma-v_\sigma)\,\{t\}^+\{T-\sigma-t\}^+e^{\lambda t}\right)_n\ . \end{aligned}$$

Due to the fact that $D_t^\sigma(w-v)=\partial_t(w_\sigma-v_\sigma)$, we have

$$D_t^\sigma(w-v)\,g_\delta(w_\sigma-v_\sigma) = \partial_t G_\delta(w_\sigma-v_\sigma)\ ;$$

thus, integrating by parts, we obtain

$$J_1 = \int_{-\sigma}^{T-\sigma}\left(G_\delta(w_\sigma-v_\sigma),\partial_t\left[\{t\}^+\{T-\sigma-t\}^+e^{\lambda t}\right]\right)_n\ .$$

Taking the limit in $\sigma \to 0$, and noting that $\partial_t(t(T-t)e^{\lambda t}) = (\lambda t(T-t)+T-2t)e^{\lambda t}$, we arrive at

$$\lim_{\substack{\sigma\to 0\\ \delta\to 0}} J_1 = \lambda \int_0^T \left(\{w-v\}^+, t(T-t)e^{\lambda t}\right)_n + \int_0^T \left(\{w-v\}^+, (T-2t)e^{\lambda t}\right)_n .$$

Next, we find

$$\begin{aligned}\lim_{\substack{\sigma\to 0\\ \delta\to 0}} J_2 &= \lim_{\delta\to 0}\int_0^T \left((w-v)\, g_\delta(w-v), c_1(T-t)e^{\lambda t}\right)_n \\ &= c_1\int_0^T \left(\{w-v\}^+, (T-t)e^{\lambda t}\right)_n\end{aligned}$$

and

$$\begin{aligned}\lim_{\substack{\sigma\to 0\\ \delta\to 0}} J_3 &= \lim_{\delta\to 0}\int_0^T \left(\frac{b''(u^n)}{t(b'(u^n))^2}\,(v+w)\,(w-v)\,g_\delta(w-v), t(T-t)e^{\lambda t}\right)_n \\ &= \int_0^T \left(\frac{b''(u^n)}{(b'(u^n))^2}\,(v+w)\,\{w-v\}^+, (T-t)e^{\lambda t}\right)_n .\end{aligned}$$

Noting that f is a monotone increasing function we obtain

$$\begin{aligned}\lim_{\substack{\sigma\to 0\\ \delta\to 0}} J_4 &= -\lim_{\delta\to 0}\int_0^T \left(f'(u^n)\,(w-v)\,g_\delta(w-v), t(T-t)e^{\lambda t}\right)_n \\ &= -\int_0^T \left(f'(u^n)\,\{w-v\}^+, t(T-t)e^{\lambda t}\right)_n \\ &\leq 0.\end{aligned}$$

Moreover $\nabla b(u^n)$, $\nabla b'(u^n)$, and $\nabla\Pi_n(b'(u^n)g_\delta(w-v))$ are L^∞–functions, bounded uniform in δ, so we may conclude that

$$\lim_{\substack{\sigma\to 0\\ \delta\to 0}} (J_5+J_6) \leq c\int_0^T \|\nabla(w-v)\|_{L^1(\Omega)}\, t(T-t)e^{\lambda t} .$$

Thus, collecting results and noting that $J_7 \leq 0$, we arrive at

$$0 \leq \lim_{\substack{\sigma\to 0\\ \delta\to 0}} (J_1+\cdots+J_7) \leq \int_0^T \left(\lambda t(T-t)\,\|\Pi_n\{w-v\}^+\|_{L^1(\Omega)} + K(t)\right)\, e^{\lambda t}\, dt,$$

where $K(t) = K_1(t) + K_2(t)$, with

$$\begin{aligned}K_1(t) &= c\left\|\Pi_n\left((1+v+w)\,\{w-v\}^+\right)\right\|_{L^1(\Omega)} \\ &\leq c\,\|1+v+w\|_{H^1(\Omega)}\,\|w-v\|_{H^1(\Omega)} \\ &\leq c\left(1+\|v\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)}\right)\end{aligned}$$

and

$$K_2(t) = c\,\|\nabla(w-v)\|_{L^1(\Omega)} .$$

Now let us assume that $\|\Pi_n\{w-v\}^+\|_{L^1(\Omega\times(0,T))} > 0$. Choosing $\lambda < 0$ such that $|\lambda|$ is sufficiently large we obtain a contradiction. Hence, it follows that $w(x_j,t) \leq v(x_j,t)$ for all nodes x_j of the triangulation and a.e. $t \in (0,T)$. This implies that $w \leq v$ a.e. in $\Omega\times[0,T]$, since v and w are piecewise linear functions. This concludes the proof. ∎

We now begin to state and prove a series of lemmas that will be the building blocks of the proof of our main result.

Lemma 4.2. *Let* u^n *be a solution of* (3.5)–(3.7) *and*

$$z(x,t) = \frac{m(p-2)}{m(p-1)-1}\, \partial_t b(u^n(x,t)) \ .$$

Then

$$\int_0^T L^n(t\partial_t b(u^n), \phi^n) = \int_0^T (z, \phi^n)_n \ ,$$

for all $\phi^n \in S_1^n(0)$ *such that* $\phi^n(\cdot, T) = 0$.

Proof. Let $v(x,t) = t\,\partial_t b(u^n(x,t))$. Note that

$$v(x,t) = t\,\partial_t b(u^n) = t\, b'(u^n)\, u_t^n, \tag{4.2}$$

where $u_t^n = \partial_t u^n$. We have

$$\begin{aligned} J_0 &:= \int_0^T (u_t^n, \partial_t(t\, b'(u^n)\, \phi^n))_n \\ &= \int_0^T (u_t^n, b'(u^n)\, \phi^n)_n + \int_0^T (u_t^n, t\, b''(u^n) u_t^n\, \phi^n)_n + \int_0^T (u_t^n, t\, b'(u^n)\, \phi_t^n)_n \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Let us define the constant $c_0 = \frac{m(p-2)}{m(p-1)-1}$. Note that $1 - c_0 = \frac{m-1}{m(p-1)-1} = c_1$, where c_1 is the constant given in (4.1). Using equation (4.2) we get

$$\begin{aligned} J_1 &= \int_0^T (c_0\, b'(u^n)\, u_t^n, \phi^n)_n + \int_0^T (c_1\, b'(u^n)\, u_t^n, \phi^n)_n \\ &= \int_0^T (z, \phi^n)_n + \int_0^T \left(c_1\, \frac{v}{t}, \phi^n\right)_n \\ &=: J_{11} + J_{12}. \end{aligned}$$

Further, due to the fact that

$$(u_t^n)^2 = \left(\frac{v}{t\, b'(u^n)}\right)^2$$

it follows that

$$J_2 = \int_0^T \left(\frac{b''(u^n)}{t\,(b'(u^n))^2}\, v^2, \phi^n\right)_n .$$

Next, we have

$$J_3 = \int_0^T (v, \phi_t^n)_n \ .$$

In addition, using equation (3.6), we obtain

$$\begin{aligned} J_0 &= \int_0^T (u_t^n, \Pi_n \partial_t(t\, b'(u^n)\, \phi^n))_n \\ &= -\sum_i \int_0^T\!\!\int_\Omega a_i(\nabla b(u^n))\, \partial_i \Pi_n \partial_t(t\, b'(u^n)\, \phi^n) \\ &\quad - \int_0^T (f(u^n), \Pi_n \partial_t(t\, b'(u^n)\, \phi^n))_n \, . \end{aligned}$$

Note that $u_t^n = \frac{v}{t\, b'(u^n)}$. Thus,

$$\partial_t f(u^n) = f'(u^n)\, u_t^n = \frac{f'(u^n) v}{t\, b'(u^n)}.$$

Integration by parts yields

$$\begin{aligned} -\int_0^T \left(f(u^n), \Pi_n \partial_t (t\, b'(u^n)\, \phi^n)\right)_n &= -\int_0^T \left(f(u^n), \partial_t (t\, b'(u^n)\, \phi^n)\right)_n \\ &= \int_0^T \left(f'(u^n)\, v, \phi^n\right)_n . \end{aligned}$$

Moreover, we find

$$\partial_t b(u^n) = b'(u^n)\, u_t^n = \frac{v}{t}$$

and

$$\begin{aligned} \partial_t a_i(\nabla b(u^n)) &= (p-2)|\nabla b(u^n)|^{p-4} \langle \nabla b(u^n), \nabla \partial_t b(u^n)\rangle \partial_i b(u^n) \\ &\quad + |\nabla b(u^n)|^{p-2} \partial_i \partial_t b(u^n) \\ &= (p-2)|\nabla b(u^n)|^{p-4} \langle \nabla b(u^n), \nabla \frac{v}{t}\rangle \partial_i b(u^n) \\ &\quad + |\nabla b(u^n)|^{p-2} \partial_i \frac{v}{t}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} J_0 &= \int_\Omega (p-2)|\nabla b(u^n)|^{p-4} \langle \nabla b(u^n), \nabla v\rangle \langle \nabla b(u^n), \nabla \Pi_n (b'(u^n)\, \phi^n)\rangle \\ &\quad + \int_\Omega |\nabla b(u^n)|^{p-2} \langle \nabla v, \nabla \Pi_n (b'(u^n)\, \phi^n)\rangle + \int_0^T \left(f'(u^n)\, v, \phi^n\right)_n . \end{aligned}$$

Altogether, it follows that

$$\int_0^T L^n(v, \phi^n) = -J_3 - J_{12} - J_2 + J_0 = J_{11} = \int_0^T (z, \phi^n)_n$$

for all $\phi^n \in S_1^n(0)$ with $\phi^n(\cdot, T) = 0$. Thus the assertion is proven. □

Lemma 4.3. *Let* $c_2 = \frac{m}{m(p-1)-1}$ *and* $z(x,t) = \frac{m(p-2)}{m(p-1)-1}\, \partial_t b(u^n(x,t))$. *Then*

$$\int_0^T L^n(-c_2 b(u^n), \phi^n) \leq \int_0^T (z, \phi^n)_n \ ,$$

for all $\phi^n \in S_1^n(0)$ *such that* $\phi^n(\cdot, T) = 0$ *and* $\phi^n \geq 0$.

Proof. Let us compute $L^n(-c_2 b(u^n), \phi^n)$. We have

$$\begin{aligned}
&\int_0^T L^n(-c_2 b(u^n), \phi^n)\\
= \quad & \int_0^T (c_2 b(u^n), \phi_t^n)_n + \int_0^T \left(c_1 \frac{c_2 b(u^n)}{t}, \phi^n \right)_n \\
& - \int_0^T \left(\frac{b''(u^n)}{t (b'(u^n))^2} (c_2 b(u^n))^2, \phi^n \right)_n - \int_0^T (f'(u^n)\, c_2\, b(u^n), \phi^n)_n \\
& -(p-2) \int_0^T\!\!\int_\Omega |\nabla b(u^n)|^{p-4} \langle \nabla b(u^n), c_2 \nabla b(u^n) \rangle \, \langle \nabla b(u^n), \nabla \Pi_n (b'(u^n) \phi^n) \rangle \\
& - \int_0^T\!\!\int_\Omega |\nabla b(u^n)|^{p-2} \langle c_2 \nabla b(u^n), \nabla \Pi_n (b'(u^n) \phi^n) \rangle + (c_2 b(u_0^n), \phi^n(\cdot, 0))_n \\
=: \quad & J_1 + \cdots + J_7.
\end{aligned}$$

Let us estimate $J_1 + J_4 + J_5 + J_6 + J_7$. We find

$$J_1 + J_7 = -c_2 \int_0^T (u_t^n, \Pi_n (b'(u^n) \phi^n))_n \, .$$

In view of the fact that $k \geq m(p-1)$ and $u^n > 0$ we get

$$\begin{aligned}
J_4 &= -c_2 \int_0^T (f'(u^n)\, b(u^n), \phi^n)_n \\
&= -\frac{c_2 k}{m} \int_0^T (f(u^n)\, b'(u^n), \phi^n)_n \\
&\leq -c_2 (p-1) \int_0^T (f(u^n), \Pi_n (b'(u^n) \phi^n))_n \, .
\end{aligned}$$

Further, let us note that

$$J_5 = -c_2 (p-2) \int_0^T\!\!\int_\Omega \sum_i a_i (\nabla b(u^n))\, \partial_i \Pi_n (b'(u^n) \phi^n)$$

and

$$J_6 = -c_2 \int_0^T\!\!\int_\Omega \sum_i a_i (\nabla b(u^n))\, \partial_i \Pi_n (b'(u^n) \phi^n).$$

Using equation (3.6) it follows that

$$(p-1)\,(J_1 + J_7) + J_4 + J_5 + J_6 \leq 0;$$

thus,

$$\begin{aligned}
J_1 + J_4 + J_5 + J_6 + J_7 &\leq (1 - (p-1))\; (J_1 + J_7) \\
&= (p-2)\, c_2 \int_0^T (b'(u^n)\, u_t^n, \phi^n)_n = \int_0^T (z, \phi^n)_n \, .
\end{aligned}$$

Next, let us show that $J_2 + J_3 = 0$. We have

$$\begin{aligned}
J_2 + J_3 &= \int_0^T \left(\frac{c_1 c_2\, b(u^n)}{t}, \phi^n \right)_n - \int_0^T \left(\frac{c_2^2\, b''(u^n)\, (b(u^n))^2}{t\, (b'(u^n))^2}, \phi^n \right)_n \\
&= \int_0^T \left(\frac{c_2^2\, b(u^n)}{t} \left(\frac{c_1}{c_2} - \frac{b''(u^n)\, b(u^n)}{(b'(u^n))^2} \right), \phi^n \right)_n .
\end{aligned}$$

Further, it holds that

$$\frac{c_1}{c_2} = \frac{m-1}{m}$$

and

$$\frac{b''(u^n)\, b(u^n)}{(b'(u^n))^2} = \frac{m-1}{m}.$$

Thus, we obtain

$$J_2 + J_3 = 0.$$

Altogether, the result follows. □

Lemma 4.4. *Let* u^n *be a solution of* (3.5)–(3.7). *The pointwise estimate*

$$u_t^n \geq -\frac{c}{t}\, u^n, \tag{4.3}$$

holds for a.e. $t > 0$, *where* $c = \frac{1}{m(p-1)-1}$.

Proof. Lemma 4.2 and Lemma 4.3 yield

$$\int_0^T\!\!\int_\Omega L^n(t\,\partial_t b(u^n), \phi^n) \geq \int_0^T\!\!\int_\Omega L^n(-c_2 b(u^n), \phi^n)\;,$$

for all $\phi^n \in S_1^n(0)$, with $\phi^n(\cdot, T) = 0$ and $\phi^n \geq 0$, where $c_2 = \frac{m}{m(p-1)-1} > 0$. Let us note that

$$t\,\partial_t b(u^n) = 0 \qquad \text{for } t = 0$$

and

$$-c_2\, b(u_0^n) < 0.$$

Further, we have

$$t\,\partial_t b(u^n) = 0 \qquad \text{on } \partial\Omega \times (0, T]$$

and

$$-c_2\, b(u^n) < 0 \qquad \text{on } \partial\Omega \times (0, T].$$

Thus, applying the comparison theorem (Lemma 4.1) we obtain

$$t\,\partial_t b(u^n) \geq \frac{-m}{m(p-1)-1}\, b(u^n).$$

This implies that

$$u_t^n = \frac{b'(u^n)\, u_t^n}{b'(u^n)} \geq \frac{-m}{[m(p-1)-1]\, t}\, \frac{b(u^n)}{b'(u^n)} = \frac{-1}{[m(p-1)-1]\, t}\, u^n,$$

since $b'(u^n) > 0$ and $b'(u^n)u^n = m|u^n|^{m-1}u^n = mb(u^n)$. □

We conclude the paper with the proofs of Theorem 2.5 and Corollary 2.6.

Proof of Theorem 2.5. We multiply the pointwise estimate (4.3) by a smooth function $0 \leq \varphi \in \mathcal{D}(\Omega \times (0,T))$ and integrate by parts in time. Due to the fact that $u^n \to u^\varepsilon$ and $u^\varepsilon \to u$ in $L^1(\Omega \times (0,T))$, we obtain

$$\int_0^T\!\!\int_\Omega u\,\varphi_t \leq \int_0^T\!\!\int_\Omega \frac{c}{t}\, u\,\varphi\;, \quad \forall\, \varphi \in \mathcal{D}(\Omega \times (0,T)) \;:\; \varphi \geq 0\;,$$

where $c = \frac{1}{m(p-1)-1}$. □

Proof of Corollary 2.6. We multiply equation (3.1) by $\phi = g_\delta(b(u^\varepsilon)-\varepsilon^m)$, where g_δ is defined in (3.12), and integrate in time from 0 to τ. Noting that $\nabla g_\delta(b(u^\varepsilon)-\varepsilon^m) = g'_\delta(b(u^\varepsilon)-\varepsilon^m)\,\nabla b(u^\varepsilon)$, $g'_\delta \geq 0$, and $f(u^\varepsilon)\,g_\delta(b(u^\varepsilon)-\varepsilon^m) \geq 0$ we find

$$\int_0^\tau\int_\Omega u_t^\varepsilon\, g_\delta(b(u^\varepsilon)-\varepsilon^m) \leq 0\,.$$

Using the fact that $\lim_{\delta\to 0} g_\delta(b(u^\varepsilon)-b(\varepsilon)) = \lim_{\delta\to 0} g_\delta(u^\varepsilon-\varepsilon)$ and $u_t^\varepsilon = (u^\varepsilon-\varepsilon)_t$ we obtain

$$\int_\Omega \{u^\varepsilon(\cdot,\tau)-\varepsilon\}^+ \leq \int_\Omega \{u_0^\varepsilon-\varepsilon\}^+.$$

This yields

$$\int_\Omega \{u^\varepsilon(\cdot,\tau)\}^+ \leq \int_\Omega \{u_0^\varepsilon\}^+ + c\varepsilon,$$

that is,

$$\|u^\varepsilon(\cdot,\tau)\|_{L^1(\Omega)} \leq \|u_0^\varepsilon\|_{L^1(\Omega)} + c\varepsilon\,, \qquad \text{a.e. } \tau > 0\,.$$

Taking the limit $\lim_{\varepsilon\to 0}$ and noting that spt u is bounded, we conclude that

$$\|u(\cdot,\tau)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}\,, \qquad \text{a.e. } \tau > 0\,.$$

The proof of the corollary now follows from this and Theorem 2.5, using standard arguments (cf. [14]). □

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