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SUSPENSIONS OF CROSSED AND QUADRATIC COMPLEXES, CO-H-STRUCTURES AND APPLICATIONS

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ABSTRACT. Crossed and quadratic modules are algebraic models of the 2-type and the 3-type of a space, respectively. In this paper we compute a purely algebraic suspension functor from crossed to quadratic modules which sends a 2-type to the 3-type of its suspension. We also give some applications in homotopy theory and group theory.

1. INTRODUCTION

Crossed modules were introduced by J. H. C. Whitehead ([23]) in order to give an algebraic invariant of a CW-complex stronger than the cellular chain complex with local coefficients. Almost at the same time Mac Lane ([18]) used them to describe the third cohomology of a group; moreover, in [19] Whitehead and Mac Lane proved that crossed modules are algebraic models of the 2-type of a CW-complex. Although they are defined by non-abelian groups, crossed modules have Peiffer nilpotency degree 1 (see [22] and [9]), hence they are in some sense abelian-like objects. Their non-abelian analogues in the sense of Peiffer are termed precrossed modules.

In [7] Baues introduced quadratic modules to describe the 3-type of a CW-complex. They are defined by means of non-abelian groups as well, but have Peiffer nilpotency degree 2. Previously Conduché had defined similar objects called 2-crossed modules and showed their relation with the fourth cohomology of a group ([16]). 2-crossed modules have been extensively studied and generalized to other contexts; see for example [21] and [3]. However, despite their name, they are generalizations of precrossed modules rather than crossed modules, because they do not satisfy any Peiffer nilpotency condition.

Whitehead's homotopy systems [23], now usually called crossed complexes, are chain complexes of groups which in low dimensions have a crossed module and in higher dimensions consist of modules over a group. Baues constructs quadratic complexes in a similar way by using quadratic modules, and proves that the categories of crossed and quadratic complexes are homotopy categories ([7]). In fact, they are I-categories in the sense of [6] when we restrict to totally free objects, which are the cofibrant models. A Quillen model category structure was already known for crossed complexes ([14]).

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When we work with simply connected CW-complexes the notion of Peiffer nilpotency corresponds to the usual notion of nilpotency of a group. For example, the 2-type of such a space is described by an abelian group, namely the second homology group. Reduced quadratic modules are a special kind of quadratic modules, describing the 3-types of simply connected CW-complexes (see [7]), which are constructed with groups of nilpotency degree 2.

In this paper we define a suspension functor from crossed modules to reduced quadratic modules (Proposition 3.3) that, topologically speaking, sends a 2-type to the 3-type of its suspension (Corollary 4.8). This functor is extended to crossed and quadratic complexes (Proposition 4.1). A notion of suspension exists in the categories of totally free crossed and quadratic complexes, because they are I-categories. However this categorical suspension applied to a crossed complex gives simply a chain complex of abelian groups. It does not encode the 3-type of the suspension, but only the 2-type. The functor constructed here avoids this problem. Furthermore, we prove that the suspension functor in the category of quadratic complexes is, up to isomorphism, the composition of the canonical functor from quadratic to crossed complexes, which sends a 3-type to its 2-type ([7]), and our suspension functor (Theorem 4.3).

Suspensions in the category of totally free quadratic complexes have a canonical co-H-structure that we study in Section 5. We observe that it is not only a co-H-structure, but a strict cogroup structure, and compute explicitly their structural morphisms in terms of the basis (Theorem 5.6 and Corollary 5.8). These computations are applied in Section 6 to calculate for any CW-complex X the natural homomorphism $H_2X \to \wedge^2 H_1X$ induced in the second homology by the map $h\colon X \to K(H_1X, 1)$ such that $\pi_1 h\colon \pi_1 X \twoheadrightarrow H_1X$ is the Hurewicz homomorphism (Theorem 6.1). As a consequence we give abelian presentations for the second quotient $\Gamma_2 G/\Gamma_3 G$ of the lower central series of a group G with a given presentation (Proposition 6.4), as well as for the second homology of Leedham-Green ([17]) of $G/\Gamma_3 G$ in the variety of groups of nilpotency degree 2 (Corollary 6.5).

In the last section we define for any abelian group A a natural element $c_A \in \mathrm{Ext}(A, S^2A)$ which vanishes if and only if the Moore space of type $(A, 2)$ has a commutative co-H-structure (Proposition 7.2). The results of Section 5 are applied here to compute an explicit formula of the element c_A from a free resolution of A (Theorem 7.5). In this way we show that if A is finitely generated, then c_A is trivial if and only if A has an element of order 2 (Corollary 7.6), generalizing a result of Arkowitz and Golasiński ([2]) for cyclic groups.

Further applications of these results to proper homotopy theory will be given in a future paper ([20]). More precisely, the results of Section 6 will be used to show a proper Moore space of degree 2 which does not admit any co-H-structure at all. See [5] and [4] for the construction and basic properties of proper Moore spaces.

2. Preliminaries

2.1. **Categorical notations.** We use the multiplicative notation for the composition of two morphisms $\varphi\colon X \to Y$, $\psi\colon Y \to Z$ in a category $\mathbf{C}$, $\psi\varphi\colon X \to Z$. The coproduct of two objects X, Y in $\mathbf{C}$ is denoted by $X \vee Y$, the corresponding inclusions are $X \stackrel{i_1}{\hookrightarrow} X \vee Y \stackrel{i_2}{\hookleftarrow} Y$. Given two morphisms $X \stackrel{f}{\to} Z \stackrel{g}{\leftarrow} Y$ in $\mathbf{C}$, $(f, g)\colon X \vee Y \to Z$ is the unique morphism with $(f, g)i_1 = f$ and $(f, g)i_2 = g$. Moreover, 1_X is the identity morphism in X (we shall omit the subindex if it is not relevant), and the arrows "$\twoheadrightarrow$", "$\hookrightarrow$" and "$\rightarrowtail$" mean epimorphism, monomorphism and cofibration, respectively. The symbol $\simeq$ will stand for isomorphisms in any category as well as for the homotopy relation between morphisms in homotopy categories, the meaning of this symbol in each case will be clear from the context.

2.2. **Quadratic functors.** Recall the definition of the following functors in the category **Ab** of abelian groups

$$\Gamma, \wedge^2, S^2 \colon \mathbf{Ab} \longrightarrow \mathbf{Ab}.$$

The first one is *Whitehead's* Γ *functor* ([24]), which carries an abelian group A to the abelian group ΓA generated by $\{\gamma(a); a \in A\}$ with the relations

$$\gamma(a) = \gamma(-a),$$
$$\gamma(a+b+c) - \gamma(b+c) - \gamma(a+c) - \gamma(a+b) + \gamma(a) + \gamma(b) + \gamma(c) = 0,$$

for any $a, b, c \in A$. Let $T \colon A^{\otimes^2} \simeq A^{\otimes^2}$ be the involution $T(a \otimes b) = b \otimes a$, the *exterior square* $\wedge^2 A$ and *symmetric square* $S^2 A$ are defined by the following natural exact sequences

$$\Gamma A \xrightarrow{\tau} A^{\otimes^2} \overset{q}{\twoheadrightarrow} \wedge^2 A, \tag{2.A}$$

$$A^{\otimes^2} \xrightarrow{1-T} A^{\otimes^2} \overset{q'}{\twoheadrightarrow} S^2 A, \tag{2.B}$$

where $\tau\gamma(a) = a \otimes a$, $q(a \otimes b) = a \wedge b$ and $q'(a \otimes b) = ab$. There are short splitting exact sequences

$$A \otimes B \overset{l}{\hookrightarrow} \Gamma(A \oplus B) \twoheadrightarrow \Gamma A \oplus \Gamma B, \tag{2.C}$$

$$A \otimes B \overset{\imath}{\hookrightarrow} \wedge^2(A \oplus B) \twoheadrightarrow \wedge^2 A \oplus \wedge^2 B, \tag{2.D}$$

with $l(a \otimes b) = \gamma(i_1 a + i_2 b) - \gamma(i_1 a) - \gamma(i_2 b)$ and $\imath(a \otimes b) = i_1 a \wedge i_2 b$, that is, $\imath = q(i_1 \otimes i_2)$. The arrows on the right are induced by the projections of the factors of the direct sum; moreover,

$$lT = \Gamma(i_2, i_1)l. \tag{2.E}$$

The sequences (2.C) and (2.D) are natural in A and B. There is an analogous exact sequence for S^2 as well, but we shall not use it. Notice that the failure in the additivity of these functors is measured by a biadditive functor, for this reason they are said to be *quadratic*.

The three functors above are right-exact in the quadratic sense, for $\wedge^2$ this property means that given an exact sequence $C \overset{f}{\to} B \overset{g}{\twoheadrightarrow} A$, then

$$(C \otimes B) \oplus \wedge^2 C \xrightarrow{(q(f\otimes 1), \wedge^2 f)} \wedge^2 B \xrightarrow{\wedge^2 g} \wedge^2 A \tag{2.F}$$

is also exact. Furthermore, if B is free abelian with basis E and we choose a total ordering $\preceq$ in E, then $\{e_1 \wedge e_2\,; e_1 \prec e_2 \in E\}$ is a basis of $\wedge^2 B$, hence an abelian presentation of A and (2.F) yield an abelian presentation of $\wedge^2 A$.

2.3. **A review of some group theory.** The composition operation in a group G will always be written additively $a+b$ $(a,b\in G)$, even when the group G is not commutative, because we shall use the multiplicative notation for the composition of morphisms in a category and for the generators of the symmetric square; see (2.1) and (2.B). The commutator of two elements in a group $a,b\in G$ is $[a,b]=-a-b+a+b$, $\Gamma_n G$ is the normal subgroup of G generated by commutators of weight n. We say that G is of nilpotency degree $\leq n$, or simply a nil(n)-group, if $\Gamma_{n+1}G=0$. The *abelianization* of G is the quotient $G^{ab}=G/\Gamma_2 G$, and the *nilization* is $G^{nil}=G/\Gamma_3 G$; moreover, we shall denote the natural projections by $p\colon G\twoheadrightarrow G^{ab}$ and $\bar{p}\colon G\twoheadrightarrow G^{nil}$. These constructions are functorial in the category of groups.

Let **gr**, **nil**, **ab** be the categories of free objects in the varieties of all groups, nil(2)-groups and abelian groups respectively. The object with basis E is denoted in each case $\langle E\rangle$, $\langle E\rangle^{nil}$ and $\mathbb{Z}\langle E\rangle$. The "nilization" and "abelianization" defined above restrict to functors

$$\mathbf{gr}\xrightarrow{nil}\mathbf{nil}\xrightarrow{ab}\mathbf{ab}.$$

There is a natural central extension

$$\wedge^2\mathbb{Z}\langle E\rangle\overset{s}{\hookrightarrow}\langle E\rangle^{nil}\overset{p}{\twoheadrightarrow}\mathbb{Z}\langle E\rangle \tag{2.G}$$

where $s(e_1\wedge e_2)=[e_1,e_2]$ $(e_1,e_2\in E)$.

More generally, in any nil(2)-group G commutators are central, and the commutator bracket is bilinear, hence there is a natural central homomorphism

$$w\colon G^{ab}\otimes G^{ab}\longrightarrow G \tag{2.H}$$

defined by $w(p(g)\otimes p(g'))=[g,g']$.

Furthermore, the homomorphism set $\mathrm{Hom}(\langle E\rangle^{nil},G)$ is also a nil(2)-group. The sum of two homomorphisms $\varphi+\psi$ is the unique one such that $(\varphi+\psi)(e)=\varphi(e)+\psi(e)$ for any $e\in E$. In this way morphism sets in **nil** are nil(2)-groups. The sum and the composition of morphisms in **nil** satisfy the following rules (see [10]):

- (i) $\psi(\varphi_1+\varphi_2)=\psi\varphi_1+\psi\varphi_2$,
- (ii) the symbol $(\psi_1|\psi_2)_\varphi=(\psi_1+\psi_2)\varphi-\psi_2\varphi-\psi_1\varphi$ is linear in ψ_1 and ψ_2; moreover, it only depends on ψ_1^{ab} and ψ_2^{ab} and lies in the commutator subgroup of the corresponding homomorphism group,
- (iii) $(-\psi)\varphi=-\psi\varphi+(\psi|\psi)_\varphi$.

By using (ii) and (iii) one can check that

- (iv) if $\psi_1^{ab}=\psi_2^{ab}$, then $(\psi_1-\psi_2)\varphi=\psi_1\varphi-\psi_2\varphi$.

Given any two groups π, G we say that G is a π*-group* if π acts (on the right) on G by automorphisms, the action is written exponentially a^α $(a\in G,\alpha\in\pi)$. Notice that G is canonically a G group acting on itself by conjugation $a^b=-b+a+b$ $(a,b\in G)$. If H is a π'-group and $\upsilon\colon\pi\to\pi'$, $\psi\colon G\to H$ are homomorphisms, we say that ψ is υ*-equivariant* if $\psi(a^\alpha)=\psi(a)^{\upsilon(\alpha)}$ for any $a\in G,\alpha\in\pi$. When $\pi=\pi'$ and $\upsilon=1$ we simply say that ψ is π*-equivariant*.

If we have two groups G, H acting one on each other, their *non-abelian tensor product* $G\otimes H$ is the group generated by the symbols $g\otimes h$ with relations

$$(g+g')\otimes h=g^{g'}\otimes h^{g'}+g'\otimes h\ ,\ g\otimes(h+h')=g\otimes h'+g^{h'}\otimes h^{h'}$$

for any $g,g'\in G$ and $h,h'\in H$. Here the relations are slightly different from those in [15] since we consider right-actions. This translation will be needed whenever we use this construction and related results in [15]. If the actions are both trivial, then $G \otimes H = G^{ab} \otimes H^{ab}$ is the usual tensor product of the abelianized groups. Moreover, one can easily check that this tensor product is functorial with respect to homomorphisms $\varphi\colon G \to G'$, $\psi\colon H \to H'$ such that φ is ψ-equivariant and ψ is φ-equivariant, $\varphi \otimes \psi\colon G \otimes H \to G' \otimes H'\colon g \otimes h \mapsto \varphi(g) \otimes \psi(h)$. Furthermore, there is a canonical homomorphism induced by the commutator bracket

$$w'\colon G \otimes G \longrightarrow G\colon g \otimes g' \mapsto [g, g'].$$

3. A suspension functor from crossed modules to reduced quadratic modules

In this section we are going to construct a functor from the category of crossed modules to the category of reduced quadratic modules which we shall call suspension functor. We shall justify the name "suspension" in the next section. We also give here a brief account of the definitions and basic properties of these algebraic objects; see [7] for more details.

Recall that a *precrossed module* ∂ is an N-group M together with an N-equivariant homomorphism $\partial\colon M \to N$. A morphism of precrossed modules $(\psi, \upsilon)\colon \partial \to \partial'$ is a commutative square of homomorphisms

$$\begin{array}{ccc} M & \xrightarrow{\partial} & N \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\upsilon} \\ M' & \xrightarrow{\partial'} & N' \end{array}$$

such that ψ is υ-equivariant. The *Peiffer commutator* of two elements $x, y \in M$ is $[\![x, y]\!] = -x - y + x + y^{\partial x}$, and $\mathrm{P}_n\partial \subset M$ is the subgroup generated by Peiffer commutators of weight n. We can define the nilpotency degree of a precrossed module just as in the case of groups by using here $\mathrm{P}_n\partial$ instead of $\Gamma_n G$; see [9]. We call nil(n)-modules to precrossed modules of nilpotency degree $\leq n$, nil(1)-modules are usually termed *crossed modules*. The crossed module ∂^{cr} associated to a precrossed module ∂ is the quotient

$$\partial^{cr}\colon M^{cr} = M/\mathrm{P}_2\partial \to N,$$

and the related nil(2)-module is

$$\partial^{nil}\colon M/P_3\partial \to N.$$

Remark 3.1. Notice that a precrossed module ∂ with $N = 0$ is just a group M, in this case $\mathrm{P}_n\partial = \Gamma_n M$. In general, $\Gamma_n \mathrm{Ker}\,\partial \subset \mathrm{P}_n\partial$, hence if ∂ is a nil(n)-module $\mathrm{Ker}\,\partial$ is a nil(n)-group.

If $\partial\colon M \to N$ is a crossed module and we consider M acting on N via ∂ and conjugation, then one easily checks that the actions are compatible in the sense of [15], and hence the following homomorphism

$$M \otimes N \xrightarrow{\varpi} M\colon m \otimes n \mapsto -m + m^n \tag{3.A}$$

is well defined and natural, and verifies

- $\varpi(1 \otimes \partial) = w'\colon M \otimes M \to M$,
- $\partial\varpi = w'(\partial \otimes 1)\colon M \otimes N \to N$.

In [7] a *quadratic module* $(\omega, \delta, \partial)$ is defined as a diagram of N-groups

$$C \otimes C \xrightarrow{\omega} L \xrightarrow{\delta} M \xrightarrow{\partial} N$$

where ∂ is a nil(2)-module, $\phi\colon M \twoheadrightarrow (M^{cr})^{ab} = C$ is the natural projection (which induces a unique action of N on C such that ϕ is N-equivariant), $C \otimes C$ has the diagonal action, and the following equalities hold ($a, b \in L; x, y \in M$):

(i) $\partial\delta = 0$,
(ii) $[\![x, y]\!] = \delta\omega(\phi(x) \otimes \phi(y))$,
(iii) $a^{\partial x} = a + \omega(1 + T)(\phi\delta(a) \otimes \phi(x))$,
(iv) $[a, b] = \omega(\phi\delta(a) \otimes \phi\delta(b))$.

Remark 3.2. Notice that ω is central and L is a nil(2)-group by (iv).

A morphism of quadratic modules $(\zeta, \psi, \upsilon)\colon (\omega, \delta, \partial) \to (\omega', \delta', \partial')$ is a commutative diagram

$$\begin{array}{ccccccc} C \otimes C & \xrightarrow{\omega} & L & \xrightarrow{\delta} & M & \xrightarrow{\partial} & N \\ \downarrow{\scriptstyle \psi_* \otimes \psi_*} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \upsilon} \\ C' \otimes C' & \xrightarrow{\omega'} & L' & \xrightarrow{\delta'} & M' & \xrightarrow{\partial'} & N' \end{array}$$

of υ-equivariant homomorphisms with $\psi_*\colon C \to C'$ induced by ψ.

We are mainly interested in *reduced quadratic modules*, which are those with $N = 0$. In this case M is also a nil(2)-group, $C = M^{ab}$ and conditions above reduce to the following:

(i) $[x, y] = \delta\omega(\phi(x) \otimes \phi(y))$,
(ii) $\omega(1 + T)(\phi\delta(a) \otimes \phi(x)) = 0$,
(iii) $[a, b] = \omega(\phi\delta(a) \otimes \phi\delta(b))$.

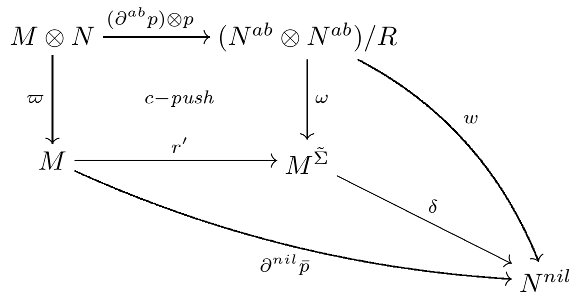
We shall also use the *central push-out* P of a diagram of groups $H \xleftarrow{g} G \xrightarrow{f} A$ with A abelian. Namely, P is a group which fits into a commutative square

$$\begin{array}{ccc} G & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & c\text{-}push & \downarrow{\scriptstyle \bar{g}} \\ H & \xrightarrow[\bar{f}]{} & P \end{array}$$

with $\bar{g}$ central, and satisfies the following universal property: for any pair of homomorphisms $\varphi\colon H \to K$, $\psi\colon A \to K$ with ψ central and $\varphi g = \psi f$ there exists a unique homomorphism $\phi\colon P \to K$ with $\varphi = \phi\bar{f}$ and $\psi = \phi\bar{g}$. The group P can be constructed as the quotient of $H \times A$ by the normal subgroup generated by the elements $(g(x), -f(x))$, $x \in G$.

Proposition 3.3. *There exists a suspension functor $\tilde{\Sigma}$ from crossed to reduced quadratic modules which sends $\partial\colon M \to N$ to the reduced quadratic module $\tilde{\Sigma}\partial =$*

(ω, δ) given by the following commutative diagram:



Here R is the image of $(1+T)(\partial^{ab} \otimes 1)\colon M^{ab} \otimes N^{ab} \to N^{ab} \otimes N^{ab}$.

Proof. The natural homomorphism w defined in (2.H) factors through R, since commutators are anticommutative. Moreover, δ exists and (ω, δ) is a reduced quadratic module by the properties of ϖ in (3.A) and central push-outs. Furthermore, the diagram above is clearly natural in the category of crossed modules, hence $\tilde{\Sigma}$ is a functor and the proof is finished. □

We say that a precrossed module $\partial\colon M \to N$ is *totally free* with basis $\varphi\colon \langle E\rangle \to \langle F\rangle$ if $N = \langle F\rangle$ and a homomorphism $r\colon \langle E\rangle \to M$ is given such that if ∂' is another precrossed module, then any commutative diagram of groups

$$\begin{array}{ccc} \langle E\rangle & \xrightarrow{\varphi} & \langle F\rangle \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \upsilon} \\ M' & \xrightarrow{\partial'} & N' \end{array}$$

extends to a unique morphism $(\psi, \upsilon)\colon \partial \to \partial'$ of precrossed modules with $\psi r = \psi'$. This object is constructed in the following way: $M = \operatorname{Ker}[(0,1)\colon \langle E\rangle * N \to N]$ with N acting on M by conjugation in the free product $\langle E\rangle * N$, and

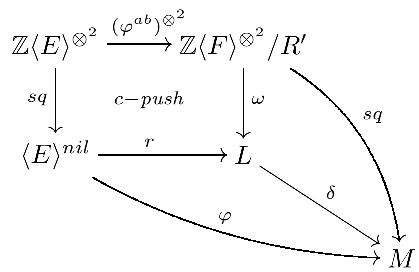
$$\partial\colon M \subset \langle E\rangle * N \overset{(\varphi,1)}{\longrightarrow} N. \tag{3.B}$$

The totally free crossed module (resp. nil(2)-module) with basis φ is ∂^{cr} (resp. ∂^{nil}).

Similarly, a quadratic module $(\omega, \delta, \partial)$ is totally free with basis $\langle D\rangle \to M$ if the nil(2)-module ∂ is totally free and there is a homomorphism $r\colon \langle D\rangle \to L$ satisfying the universal property analogue to the case of precrossed modules.

A reduced quadratic module $M^{ab} \otimes M^{ab} \overset{\omega}{\to} L \overset{\delta}{\to} M$ is *totally free* with basis $\varphi\colon \langle E\rangle^{nil} \to \langle F\rangle^{nil}$ if $M = \langle F\rangle^{nil}$ and a homomorphism $r\colon \langle E\rangle^{nil} \to L$ is given satisfying the obvious universal property. This object is constructed by the following diagram:

(3.C)



Here R' is the image of $(1+T)(\varphi^{ab}\otimes 1)\colon \mathbb{Z}\langle E\rangle\otimes\mathbb{Z}\langle F\rangle\to\mathbb{Z}\langle F\rangle^{\otimes^2}$.

The next proposition shows that the suspension functor defined in (3.3) preserves totally free objects.

Proposition 3.4. *If the crossed module $\partial\colon M\to N$ is totally free with basis $\varphi\colon\langle E\rangle\to\langle F\rangle$, then the reduced quadratic module $\tilde{\Sigma}\partial=(\omega,\delta)$ is totally free with basis φ^{nil}.*

Proof. Write $\bar{\partial}\colon\bar{M}\to N$ for the totally free precrossed module with basis φ, then $\bar{M}=\underset{f\in\langle F\rangle}{*}\langle E^f\rangle\subset\langle E\rangle*\langle F\rangle$ and $M^{\tilde{\Sigma}}$ in Proposition 3.3 is the quotient of $\bar{M}\times(\mathbb{Z}\langle F\rangle^{\otimes^2}/R)$ by the normal subgroup generated by the elements

$$(-e'-e+e'+e^{\varphi e'},0)\,,(-e+e^f,-\varphi^{ab}p(e)\otimes p(f))$$

for any $e,e'\in\langle E\rangle$, $f\in\langle F\rangle$. Hence in the quotient we have the equality $(e^f,0)=(e,\varphi^{ab}p(e)\otimes p(f))$, and therefore $M^{\tilde{\Sigma}}$ is also the quotient of $\langle E\rangle\times(\mathbb{Z}\langle F\rangle^{\otimes^2}/R)$ by the elements $([e',e],\varphi^{ab}p(e)\otimes\varphi^{ab}p(e'))$ for any $e,e'\in\langle E\rangle$. In particular, $([e,[e',e'']],0)$ is a relation for every $e,e',e''\in\langle E\rangle$, then $M^{\tilde{\Sigma}}$ is the quotient of $\langle E\rangle^{nil}\times(\mathbb{Z}\langle F\rangle^{\otimes^2}/R)$ by the relations $(-[e,e'],\varphi^{ab}p(e)\otimes\varphi^{ab}p(e'))$ for any $e,e'\in\langle E\rangle^{nil}$, since $[e',e]=-[e,e']$. Moreover, by an easy computation one can check that $R=R'$, hence $M^{\tilde{\Sigma}}$ coincides with L in the central push-out diagram (3.C). The proof is now complete. ☐

4. Crossed and quadratic complexes. Suspensions

Crossed (resp. quadratic) complexes are special chain complexes of groups with a crossed (resp. quadratic) module structure in low dimensions (see [7]). These algebraic objects are used to encode homotopical information of spaces, such as the 2-type and the 3-type. In this section we extend to those complexes the suspension functor defined in the previous section, and give a topological interpretation of this algebraic functor.

A *crossed complex* is a positive chain complex of groups $\rho=(\rho_*,d_*)$ such that $d_2\colon\rho_2\to\rho_1$ is a crossed module, ρ_n $(n>2)$ is a (right) module over $\operatorname{Coker}d_2$ (and hence over ρ_1 through the natural projection $\rho_1\twoheadrightarrow\operatorname{Coker}d_2$), and the differential d_* is ρ_1-equivariant. Similarly, a *quadratic complex* $\sigma=(\sigma_*,d_*,\omega)$ is a positive chain complex of groups (σ_*,d_*) and a homomorphism $\omega\colon C_2\otimes C_2\to\sigma_3$ such that (ω,d_3,d_2) is a quadratic module, σ_n $(n>3)$ is a module over $\operatorname{Coker}d_2$, and the differential d_* is σ_1-equivariant. The *homotopy groups* of ρ and σ are

(4.A) $$\pi_n\rho=H_n(\rho_*,d_*),\ \pi_n\sigma=H_n(\sigma_*,d_*).$$

A morphism of crossed complexes is a morphism of chain complexes

$$f = \{f_n\}_{n\geq 1} \colon \rho \to \rho'$$

such that f_n is f_1-equivariant ($n \geq 1$), and morphisms of quadratic complexes $f = \{f_n\}_{n\geq 1} \colon \sigma \to \sigma'$ are morphisms of chain complexes with $(f_3, f_2, f_1) \colon (\omega, d_3, d_2) \to (\omega', d_3', d_2')$ a morphism of quadratic modules and f_n an f_1-equivariant homomorphism ($n \geq 1$). A morphism of crossed or quadratic complexes is said to be a *weak equivalence* if it induces isomorphisms in homotopy groups.

We say that a crossed or quadratic complex is *reduced* if the 1-dimensional group is trivial. Notice that reduced crossed complexes are simply chain complexes of abelian groups, while reduced quadratic complexes have a reduced quadratic module in low dimensions with a chain complex of abelian groups attached in dimensions > 3.

In the next proposition we extend the suspension functor defined in Proposition 3.3 to a functor between complexes.

Proposition 4.1. *There is a suspension functor $\tilde{\Sigma}$ from crossed to reduced quadratic complexes which sends $\rho = (\rho_*, d_*)$ to the reduced quadratic complex $\tilde{\Sigma}\rho = ((\tilde{\Sigma}\rho)_*, d_*^{\tilde{\Sigma}}, \omega)$ with $(\omega, d_3^{\tilde{\Sigma}}) = \tilde{\Sigma} d_2$, $(\tilde{\Sigma}\rho)_{n+1} = \rho_n \otimes_{\pi_1\rho} \mathbb{Z}$ for $n > 2$, where $\mathbb{Z}$ is the trivial $\pi_1\rho$-module, and $d_{n+1}^{\tilde{\Sigma}} = d_n \otimes_{\pi_1\rho} \mathbb{Z}$ for $n > 3$. Moreover, the homomorphism $d_4^{\tilde{\Sigma}}$ is the unique one which extends commutatively the following diagram:*

$$\begin{array}{ccc}
\rho_2 \otimes \rho_1 & \xrightarrow{(d_2^{ab}p)\otimes p} & (\rho_1^{ab} \otimes \rho_1^{ab})/R \\
\downarrow{\scriptstyle \varpi} & c-push & \downarrow{\scriptstyle \omega} \\
\rho_2 & \xrightarrow{r'} & (\tilde{\Sigma}\rho)_3 \\
\uparrow{\scriptstyle d_3} & & \uparrow{\scriptstyle d_4^{\tilde{\Sigma}}} \\
\rho_3 & \xrightarrow{z} & \rho_3 \otimes_{\pi_1\rho} \mathbb{Z}
\end{array}$$

where z is the natural projection $z(x) = x \otimes 1$ $(x \in \rho_3)$.

Proof. The filler $d_4^{\tilde{\Sigma}}$ exists and is unique provided $d_3(\operatorname{Ker} z) \subset \operatorname{Ker} r'$. This inclusion is satisfied, because $\operatorname{Ker} z$ is generated by the elements $-x + x^\alpha$ $(x \in \rho_3,\ \alpha \in \rho_1)$, and

$$\begin{aligned}
r' d_3(-x + x^\alpha) &= r'(-d_3(x) + d_3(x)^\alpha) \\
&= r'\varpi(d_3(x) \otimes \alpha) \\
&= \omega(d_2^{ab} p d_3(x) \otimes p\alpha) \\
&= \omega(p d_2 d_3(x) \otimes p\alpha) \\
&= 0.
\end{aligned}$$

It is readily checked that $d^{\tilde{\Sigma}} d^{\tilde{\Sigma}} = 0$, since $dd = 0$, hence $\tilde{\Sigma}\rho$ is a reduced quadratic complex. Furthermore, $\tilde{\Sigma}$ is a functor because the diagram of the statement is natural in the category of crossed complexes. □

A crossed (resp. quadratic) complex ρ (resp. σ) is said to be *totally free* if d_2 (resp. (ω, d_3, d_2)) is a totally free crossed (resp. quadratic) module and ρ_n (resp. σ_n) is a free module over $\operatorname{Coker} d_2$ for $n > 2$ (resp. $n > 3$).

We say that $\{E_n\}_{n\geq 1}$ is the *basis* of a totally free quadratic complex $\sigma = (\sigma_*, d_*, \omega)$ if $E_n \subset \sigma_n$ is a subset such that

(i) $\sigma_1 = \langle E_1 \rangle$,
(ii) d_2 induces a group homomorphism $\langle E_2 \rangle \to \langle E_1 \rangle$ which is the basis of the nil(2)-module d_2,
(iii) the homomorphism $\langle E_3 \rangle \to \sigma_2$ induced by d_3 is the basis of the quadratic module (ω, d_3, d_2),
(iv) E_n $(n \geq 4)$ is the basis of the free $\pi_1\sigma$-module σ_n.

If σ is reduced, then $E_1 = \emptyset$ is the empty set and conditions above reduce to the following:

(i) $\sigma_2 = \langle E_2 \rangle^{nil}$,
(ii) d_3 induces a homomorphism $\langle E_3 \rangle^{nil} \to \langle E_2 \rangle^{nil}$ which is the basis of the reduced quadratic module (ω, d_3),
(iii) E_n $(n \geq 4)$ is the basis of the free abelian group σ_n.

Remark 4.2. The basis of crossed complexes is defined in a similar way, and has analogous properties. Moreover, a totally free crossed or quadratic complex is completely determined by the basis sets and the values of the differential over the basis. Furthermore, if $f\colon \sigma \to \sigma'$ is a morphism of quadratic complexes and σ is totally free, f is also determined by its values on the basis. The same happens in the crossed case.

The categories $\mathbf{H}$, $\mathbf{Q}$ of totally free crossed and quadratic complexes, respectively, are homotopy categories; in fact, I-categories with the structure given in [7]. The zero object in both categories is that with the trivial group in each dimension, and the basis of the coproduct of two objects is the coproduct (disjoint union) of the basis. Moreover, the inclusion of a factor into a coproduct is induced by the inclusion of its basis; see also the construction of homotopy push-outs in these categories given in [7], III.4.17 and IV.5.5.

The basis of the cylinder $I\sigma$ of a quadratic complex σ with basis $\{E_n\}_{n\geq 1}$ is given by the sets $\{E'_n\}_{n\geq 1}$ with

$$E'_n = i_0 E_n \vee sE_{n-1} \vee i_1 E_n \quad (n \geq 2),$$
$$E'_1 = i_0 E_1 \vee i_1 E_1.$$

Here $i_t E_n, sE_n$ are copies of E_n $(n \geq 1, t = 0, 1)$ and $\vee$ is the disjoint union of sets. The natural top and bottom inclusions of σ into the cylinder $i_t\colon \sigma \rightarrowtail I\sigma$ $(t = 0, 1)$ are induced by the basis inclusions $E_n = i_t E_n \subset E'_n$, and the natural projection $p\colon I\sigma \to \sigma$ by $pi_t = 1$ and $p_{n+1}(sx) = 0$ $(t = 0, 1; x \in E_n, n \geq 1)$. The differential in the cylinder is defined by the differential of σ and the following equalities $(x \in E_n, n \geq 1, t = 1, 2)$:

$$d_n i_t(x) = i_t d_n(x),\ d_{n+1}(sx) = -i_0(x) + i_1(x) - S_{n-1} d_n(x).$$

Here $S_0 d_1(x) = 0$ and $S_n\colon \sigma_n \to (I\sigma)_{n+1}$ $(n \geq 1)$ is the function defined by $S_n x = sx$ $(x \in E_n)$ and the fact that it is an i_0-equivariant homomorphism for $n \geq 3$, an i_0-crossed homomorphism for $n = 1$, and an (S_1, i_0, i_1)-quadratic operator for $n = 2$.

Recall that, if G is a π'-group and $\varphi\colon \pi \to \pi'$ is a homomorphism, $\psi\colon \pi \to G$ is a *φ-crossed homomorphism* if $\psi(a + b) = \psi(a)^{\varphi b} + \psi(b)$ for any $a, b \in \pi$. The definition of *quadratic operator* is more complicated, we refer to [7], IV.4.3.

Given a morphism $f\colon \sigma \to \sigma'$ in $\mathbf{Q}$, the induced morphism $If\colon I\sigma \to I\sigma'$ is the unique one with $(If)i_t = i_t f$ $(t = 0, 1)$ and $(If)_{n+1}(sx) = S_n f_n(x)$ $(x \in E_n, n \geq 1)$.

The *suspension* functor in $\mathbf{Q}$, as in any I-category, is defined by $\Sigma\sigma = I\sigma/\sigma \vee \sigma$, the cofiber of the natural morphism $(i_0, i_1)\colon \sigma \vee \sigma \rightarrowtail I\sigma$. In order to relate this suspension functor with that defined in Proposition 4.1 recall that there is a functor λ from quadratic complexes to crossed complexes which sends $\sigma = (\sigma_*, d_*, \omega)$ to the crossed complex $\lambda\sigma = ((\lambda\sigma)_*, d_*^\lambda)$ such that $((\lambda\sigma)_*, d_*^\lambda)$ coincides with (σ_*, d_*) in dimensions > 3,

$$d_4^\lambda\colon (\lambda\sigma)_4 = \sigma_4 \stackrel{d_4}{\to} \sigma_3 \twoheadrightarrow \operatorname{Coker}\omega = (\lambda\sigma)_3,$$
$$d_2^\lambda = d_2^{cr}\colon (\lambda\sigma)_2 = \sigma_2^{cr} \to \sigma_1 = (\lambda\sigma)_1,$$

and d_3^λ fits into the following commutative square:

$$\begin{array}{ccc} \sigma_3 & \xrightarrow{d_3} & \sigma_2 \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \operatorname{Coker}\omega & \xrightarrow{d_3^\lambda} & \sigma_2^{cr} \end{array}$$

(see IV.3.2 in [7]). This functor preserves totally free complexes; moreover, if $\{E_n\}_{n\geq 1}$ is the basis of σ, then it is also the basis of $\lambda\sigma$, and $\lambda I\sigma = I\lambda\sigma$.

Suspensions in $\mathbf{H}$ are just chain complexes of free abelian groups which are trivial in dimensions ≤ 1, so they are not of much interest. The main result of this section is the following.

Theorem 4.3. *There is a natural equivalence between the functors*

$$\mathbf{Q} \xrightarrow{\Sigma} \mathbf{Q} \quad \textit{and} \quad \mathbf{Q} \xrightarrow{\lambda} \mathbf{H} \xrightarrow{\tilde{\Sigma}} \mathbf{Q}.$$

In order to prove this theorem we first state several lemmas.

The following lemma follows directly from Proposition 3.4 and the construction of $\tilde{\Sigma}$ in Proposition 4.1.

Lemma 4.4. *If ρ is a totally free crossed module with basis $\{E_n\}_{n\geq 1}$, then the reduced quadratic module $\tilde{\Sigma}\rho$ is totally free with basis $\{E_n^{\tilde{\Sigma}}\}_{n\geq 1}$, $E_1^{\tilde{\Sigma}} = \emptyset$, $E_n^{\tilde{\Sigma}} = E_{n-1}$ $(n \geq 2)$.*

By observing the explicit constructions of the cylinder functor, the suspension functor Σ, and homotopy push-outs in $\mathbf{Q}$ ([7]) one easily checks that

Lemma 4.5. *If $\{E_n\}_{n\geq 1}$ is the basis of the quadratic complex σ, then the basis of $\Sigma\sigma$ is $\{E_n^\Sigma\}_{n\geq 1}$ with $E_1^\Sigma = \emptyset$ and $E_n^\Sigma = E_{n-1}$ for $n \geq 2$. Moreover, if $\varrho\colon I\sigma \to I\sigma/\sigma \vee \sigma = \Sigma\sigma$ is the natural quotient morphism, then the restriction of ϱ_n induces the identity $\varrho_n\colon sE_{n-1} = E_{n-1} = E_n^\Sigma$ $(n \geq 2)$.*

In the next lemma we determine the low-dimensional reduced quadratic module of a suspended quadratic complex.

Lemma 4.6. *If $\bar{\varphi}\colon \langle E_2\rangle \to \langle E_1\rangle$ is the basis of the* $\mathrm{nil}(2)$*-module $d_2\colon \sigma_2 \to \sigma_1$ and $\varphi = \bar{\varphi}^{nil}$, then $-\varphi\colon \langle E_2\rangle^{nil} \to \langle E_1\rangle^{nil}$ is the basis of the low-dimensional reduced quadratic module of $\Sigma\sigma$,*

$$\mathbb{Z}\langle E_1\rangle^{\otimes 2} \xrightarrow{\omega^\Sigma} (\Sigma\sigma)_3 \xrightarrow{d_3^\Sigma} (\Sigma\sigma)_2 = \langle E_1\rangle^{nil}.$$

Proof. By (4.5) we only need to prove that the composition

$$E_2 = sE_2 \subset (I\sigma)_3 \stackrel{d_3}{\to} (I\sigma)_2 \stackrel{\varrho_2}{\to} (\Sigma\sigma)_2 = \langle E_1\rangle^{nil}$$

sends $x \in E_2$ to $-\varphi(x) \in \langle E_1\rangle^{nil}$. By construction of the differential of $I\sigma$ and the suspension functor we have that

$$\varrho_2 d_3(sx) = -\varrho_2 S_1\bar{\varphi}(x).$$

Moreover, since $\varrho i_0 = 0$, $\varrho_2 S_1$ is in fact a group homomorphism, and given $y \in E_1$, $\varrho_2 S_1(y) = y$, so $\varrho_2 S_1 = \bar{p}\colon \langle E_1\rangle \twoheadrightarrow \langle E_1\rangle^{nil}$ is the natural projection and $\varrho_2 d_3(sx) = -\varrho_2 S_1\bar{\varphi}(x) = -\bar{p}\bar{\varphi}(x) = -\varphi(x)$. □

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Let $\sigma = (\sigma_*, d_*, \omega)$ be a quadratic complex. We claim that $(\Sigma\sigma)_n = (\tilde{\Sigma}\lambda\sigma)_n$ and the natural equivalence is given by the morphism $\chi\colon \Sigma\sigma \to \tilde{\Sigma}\lambda\sigma$ with $\chi_n = (-1)^n$ for $n \neq 3$, $\chi_3\omega = \omega$ and $\chi_3 r = r(-1)$.

Given $\varphi \in$ **nil**, the reduced quadratic modules with basis $-\varphi$ and φ, (ω_1, δ_1) and (ω_2, δ_2) respectively, are defined by the same groups, $\omega_1 = \omega_2$ but $\delta_1 \neq \delta_2$ in general, here we use (3.C) and the fact that $(-\varphi^{ab})^{\otimes^2} = (\varphi^{ab})^{\otimes^2}$. Moreover, $(\chi_3, 1, 0)\colon (\omega_1, \delta_1) \to (\omega_2, \delta_2)$ is an isomorphism of quadratic modules. Therefore, the equalities $(\Sigma\sigma)_n = (\tilde{\Sigma}\lambda\sigma)_n$ and $\chi_2 d_3^{\Sigma} = d_3^{\tilde{\Sigma}\lambda}\chi_3$ follow from Proposition 3.4 and Lemmas 4.4, 4.5 and 4.6.

Since $\varrho i_t = 0$ $(t = 0, 1)$, for $n \geq 5$ we have that $\varrho_n S_{n-1}\colon \sigma_{n-1} \to (\Sigma\sigma)_n = (\tilde{\Sigma}\lambda\sigma)_n = \sigma_{n-1} \otimes_{\pi_1\sigma} \mathbb{Z}$ is the natural projection $x \mapsto x \otimes 1$ and $d_{n+1}^{\Sigma}(x) = -\varrho_n S_{n-1} d_n(x)$, so $d_{n+1}^{\Sigma} = -d_n \otimes_{\pi_1\sigma} \mathbb{Z} = -d_{n+1}^{\tilde{\Sigma}\lambda}$ and $\chi_n d_{n+1}^{\Sigma} = d_{n+1}^{\tilde{\Sigma}\lambda}\chi_{n+1}$. Furthermore, $\varrho_4 S_3\colon \sigma_3 \to (\Sigma\sigma)_4 = (\tilde{\Sigma}\lambda\sigma)_4 = (\operatorname{Coker}\omega) \otimes_{\pi_1\sigma} \mathbb{Z}$ is also the projection homomorphism onto the quotient, and if $x \in E_4$, then $d_5^{\Sigma}(x) = -\varrho_4 S_3 4 d_4(x) = -d_5^{\tilde{\Sigma}\lambda}(x)$, hence $\chi_4 d_5^{\Sigma} = d_5^{\tilde{\Sigma}\lambda}\chi_5$. The map S_2 is an (S_1, i_0, i_1)-quadratic operator in the sense of [7], IV.4.3 (see [7], IV.4.10 (5)); in addition $\varrho i_0 = 0 = \varrho i_1$ and $\varrho_2 S_1 = \bar{p}$ (for this last equality see the proof of Lemma 4.6). Then $\varrho_3 S_2$ is a $(\bar{p}, 0, 0)$-quadratic operator. Let ξ be the composite $\sigma_2 \twoheadrightarrow \sigma_2^{cr} = (\lambda\sigma)_2 \stackrel{r'}{\to} (\tilde{\Sigma}\lambda\sigma)_3$; see Proposition 3.3. By using property (iii) in the definition of a reduced quadratic module (see Section 3) and Proposition 3.3 it is easy to see that the map $-\chi_3\xi : \sigma_2 \to (\tilde{\Sigma}\lambda\sigma)_3 = (\Sigma\sigma_3) : x \mapsto -\chi_3\xi(x)$ is also a $(\bar{p}, 0, 0)$-quadratic operator. Moreover, given $e \in E_2$, $\varrho_3 S_2(e) = e = -\chi_3\xi(e)$ and therefore by [7], IV.4.5, $\varrho_2 S_1 = -\chi_3\xi$. In particular by Proposition 4.1 we have that for $x \in E_3$, $d_4^{\tilde{\Sigma}\lambda}\chi_4(x) = d_4^{\tilde{\Sigma}\lambda}x = \xi d_3(x) = \chi_3\chi_3\xi d_3(x) = \chi_3(-\varrho S_2 d_3(x)) = \chi_3 d_4^{\Sigma}(x)$. Here we use the obvious fact that $\chi_3\chi_3 = 1$. The proof is now finished. □

Let **CW** be the category of CW-complexes with trivial 0-skeleton and cellular maps. From the homotopy point of view this category is equivalent to the category of all connected spaces, therefore in this paper "space" will mean CW-complex as above, and maps between them will always be cellular.

Given a space X there are defined objects $\rho(X)$, $\sigma(X)$ in **H** and **Q**, respectively, such that $\rho(X) = \lambda\sigma(X)$; see [7]. Moreover, the basis of these objects is the set of cells of X. In fact, they are functors $\rho\colon \mathbf{CW} \to \mathbf{H}$ and $\sigma\colon \mathbf{CW} \to \mathbf{Q}/\stackrel{0}{\simeq}$. Here $\stackrel{0}{\simeq}$ is a natural equivalence relation in **Q** given by 0-homotopies, which are filtration-preserving homotopies with respect to skeleta. More precisely, two morphisms

$f, g\colon \sigma \to \sigma'$ are 0-*homotopic* if $f_n = g_n$ for $n = 1$ and $n \geq 4$, and there is a $\pi_1 f$-equivariant homomorphism $\alpha\colon C_2 \to C_2' \otimes C_2'$ such that if $\phi\colon \sigma_2 \twoheadrightarrow C_2$ is the natural projection, then

$$-f_2(a) + g_2(a) = d_3'\omega'\alpha\phi(a), \quad -f_3(b) + g_3(b) = \omega'\alpha\phi d_3(b),$$

for any $a \in \sigma_2$, and $b \in \sigma_3$. The functor ρ (resp. σ) is full and faithful when we restrict to the full subcategory of spaces of dimension 2 (resp. 3) and take the quotient by the homotopy relation, in particular, there are natural isomorphisms $\pi_n\rho(X) \simeq \pi_n X$ (resp. $\pi_n\sigma(X) \simeq \pi_n X$) for $n \leq 2$ (resp. $n \leq 3$). Moreover, the functor λ factors trough $\mathbf{Q}/\overset{0}{\simeq}$ and there is a natural equivalence $\rho = \lambda\sigma$. Furthermore, σ and ρ preserve cylinders and homotopy push-outs, hence the following result follows directly from Theorem 4.3.

Corollary 4.7. *For any space X there is a natural isomorphism $\tilde{\Sigma}\rho(X) \simeq \sigma(\Sigma X)$.*

The 2-*type* of a crossed complex $\rho = (\rho_*, d_*)$ is the crossed module $P_2\rho = d_2\colon \rho_2/d_3\rho_3 \to \rho_1$ and the 3-*type* of a quadratic complex $\sigma = (\sigma_*, d_*, \omega)$ is the quadratic module $P_3\sigma$ given by

$$C_2 \otimes C_2 \xrightarrow{\omega} \sigma_3/d_4\sigma_4 \xrightarrow{d_3} \sigma_2 \xrightarrow{d_2} \sigma_1.$$

For any space X the algebraic objects $P_2\rho(X)$ and $P_3\sigma(X)$ are models of its topological 2-type P_2X and 3-type P_3X in the following sense: a map $f\colon X \to Y$ induces a weak equivalence $P_2\rho(f)\colon P_2\rho(X) \to P_2\rho(Y)$ if and only if $P_2f\colon P_2X \to P_2Y$ is a homotopy equivalence, and the same for $P_3\sigma$; see [7].

The next corollary is a direct consequence of Corollary 4.7, and shows that the suspension functor from crossed to quadratic modules defined in Proposition 3.3 is equivalent to the topological functor which sends the 2-type of X to the 3-type of ΣX.

Corollary 4.8. *Given any space X, there is a natural isomorphism $\tilde{\Sigma}P_2\rho(X) \simeq P_3\sigma(\Sigma X)$.*

5. The co-H-structure of a suspended quadratic complex

As in any I-category, suspensions in $\mathbf{Q}$ are equipped with a natural co-H-group structure given by morphisms

- the co-H-multiplication $\mu_\sigma\colon \Sigma\sigma \to \Sigma\sigma \vee \Sigma\sigma$,
- and the co-H-inversion $\nu_\sigma\colon \Sigma\sigma \to \Sigma\sigma$,

such that

(5.A) $(1 \vee \mu_\sigma)\mu_\sigma \simeq (\mu_\sigma \vee 1)\mu_\sigma, (1,0)\mu_\sigma \simeq 1 \simeq (0,1)\mu_\sigma, (1,\nu_\sigma)\mu_\sigma \simeq 0 \simeq (\nu_\sigma, 1)\mu_\sigma,$

here the symbol $\simeq$ denotes the homotopy relation. These morphisms are well defined up to homotopy; see [6], II.6. In this section we shall give explicit formulas for these morphisms. Moreover, we shall prove that the co-H-structure of $\Sigma\sigma$ is a strict cogroup structure in $\mathbf{Q}$, that is, the homotopies above are in fact equalities.

Remark 5.1. The quadratic complex of a wedge of circles $\sigma(\vee_E S^1)$ is naturally isomorphic to $\pi_1(\vee_E S^1) = \langle E \rangle$ concentrated in degree 1 and trivial otherwise. Therefore the canonical co-H-structure of S^1 induces the co-H-structure in σS^1

given by

- the co-H-multiplication $\mu\colon \sigma S^1 \to \sigma S^1 \vee \sigma S^1$, which is $\mu_1\colon \langle e\rangle \to \langle e_1, e_2\rangle$, $\mu_1(e) = e_1 + e_2$, in dimension 1, here $e_k = i_k e$ $(k = 1, 2)$,
- and the co-H-inversion $\nu\colon \sigma S^1 \to \sigma S^1$, given by $\nu_1\colon \langle e\rangle \to \langle e\rangle$, $\nu_1(e) = -e$.

One readily checks that this is in fact a strict cogroup structure in $\mathbf{Q}$, that is, the homotopies in (5.A) are strict equalities in $\mathbf{Q}$.

In [7] Baues constructs a tensor product of quadratic complexes $\sigma \otimes \sigma'$ such that $\sigma(X) \otimes \sigma(Y)$ is isomorphic to the quadratic complex of the product space $\sigma(X \times Y)$. This isomorphism is natural in Y if we fix X and vice versa. Moreover, there is a natural inclusion (actually a cofibration) of the coproduct of two quadratic complexes into their tensor product $\jmath\colon \sigma \vee \sigma' \rightarrowtail \sigma \otimes \sigma'$, which corresponds to the natural inclusion $X \vee Y \subset X \times Y$; see [7], IV.12.2 (3). Hence if we define the *smash product* of quadratic complexes as the cofiber of $\jmath$,

$$\sigma \wedge \sigma' = \sigma \otimes \sigma'/\sigma \vee \sigma',$$

we get an induced isomorphism

$$\sigma(X \wedge Y) \simeq \sigma(X) \wedge \sigma(Y).$$

Remark 5.2. Notice that smash products are always reduced quadratic complexes, since the natural inclusion $\jmath$ is an isomorphism in dimension 1. Moreover, the smash product is naturally distributive with respect to the coproduct

$$(\sigma \wedge \sigma'') \vee (\sigma' \wedge \sigma'') \simeq (\sigma \vee \sigma') \wedge \sigma''.$$

The isomorphism is $(i_1 \wedge 1, i_2 \wedge 1)$.

In the following proposition we give a further characterization of the suspension functor in $\mathbf{Q}$ which will allow us to compute explicit formulas for the co-H-structures.

Proposition 5.3. *The suspension functor in the I-category $\mathbf{Q}$ coincides with the smash product with the quadratic complex of the circle S^1,*

$$\Sigma = \sigma S^1 \wedge -\colon \mathbf{Q} \longrightarrow \mathbf{Q},$$

up to natural equivalence. Furthermore, under this equivalence the co-H-structure of a suspension is induced by the canonical co-H-structure of σS^1, that is,

$$\mu_\sigma = \mu \wedge 1_\sigma, \ \ \nu_\sigma = \nu \wedge 1_\sigma,$$

and therefore it is a strict cogroup structure in $\mathbf{Q}$.

Proof. Suppose we have checked that $\Sigma = \sigma S^1 \wedge -$, $\mu_\sigma = \mu \wedge 1_\sigma$ and $\nu_\sigma = \nu \wedge 1_\sigma$. Then μ_σ, ν_σ is a strict cogroup structure in $\mathbf{Q}$ because μ, ν is a strict cogroup structure for σS^1 and $- \wedge \sigma$ is a functor from $\mathbf{Q}$ to $\mathbf{Q}$.

If σ is a quadratic complex with $\sigma_n = 0$ for $n > 4$, one can construct a 4-dimensional space X with $\sigma = \sigma X$ by using [7], IV.7.5, compare with IV.8.5 in the same reference. Since σ preserves cylinders and its structural maps, it also preserves suspensions and its co-H-structures. Moreover, it is well known that $S^1 \wedge -$ coincides with the suspension functor in the category of spaces, and the canonical co-H-structure of a suspended space is induced by the usual co-H-structure of S^1. Therefore, the proposition is true for a quadratic complex σ with $\sigma_n = 0$ for $n > 4$. For a general quadratic complex it is a consequence of the previous particular case and the fact that chain complexes admit a unique co-H-structure up to homotopy; see the next lemma. □

Lemma 5.4. *A chain complex (of abelian groups) admits a unique co-H-structure up to homotopy.*

Proof. This category of chain complexes is an additive category. We can define a co-H-structure on a chain complex C_* by $\mu = i_1 + i_2$ and $\nu = -1$. Consider another co-H-structure given by μ' and ν'. There are chain homotopies $\{\alpha_n\}_{n\in\mathbb{Z}}\colon (1,0)\mu' \simeq 1$, $\{\beta_n\}_{n\in\mathbb{Z}}\colon (0,1)\mu' \simeq 1$. One readily checks that $\{i_1\alpha_n + i_2\beta_n\}_{n\in\mathbb{Z}}$ is a chain homotopy $\mu' \simeq \mu$. Similarly, if $\{\gamma_n\}_{n\in\mathbb{Z}}\colon (1,\nu')\mu \simeq 0$ is a chain homotopy, then $\{\gamma_n\}_{n\in\mathbb{Z}}$ is a chain homotopy $\nu' \simeq \nu$ as well. □

A direct consequence of Proposition 5.3 is the following.

Corollary 5.5. *The morphism set* $\mathbf{Q}(\Sigma\sigma,\sigma')$ *has a group structure such that the natural projection onto the group of homotopy classes* $\mathbf{Q}(\Sigma\sigma,\sigma') \twoheadrightarrow [\Sigma\sigma,\sigma']$ *is a homomorphism.*

In order to compute the cogroup structure of a suspended quadratic complex we introduce the following notation. Given a homomorphism $\varphi\colon \langle E\rangle^{nil} \to \langle F\rangle^{nil}$, $\nabla\varphi$ is the unique homomorphism

$$\nabla\varphi\colon \mathbb{Z}\langle E\rangle \to \mathbb{Z}\langle F\rangle^{\otimes^2}$$

such that $s\imath(\nabla\varphi)p = (i_2|i_1)_\varphi$; moreover,

$$\Delta\colon \mathbb{Z}\langle E\rangle \to \mathbb{Z}\langle E\rangle^{\otimes^2}$$

is the non-natural diagonal homomorphism defined on generators by $\Delta(a) = a\otimes a$, $(a\in E)$.

The comultiplication of the suspension of a quadratic complex σ such that $\bar\varphi\colon \langle E_2\rangle \to \langle E_1\rangle$ is the basis of the nil(2)-module $d_2\colon \sigma_2 \to \sigma_1$, is completely determined by the formulas of the following theorem, where $\varphi = \bar\varphi^{nil}$. The morphisms r and ω are the structural morphisms of any totally free reduced quadratic module in the central push-out diagram (3.C).

Theorem 5.6. *The comultiplication* μ_σ *in* $\mathbf{Q}$ *verifies the following equalities:*

1. $(\mu_\sigma)_n = i_1 + i_2$ *if* $n \neq 3$,
2. $(\mu_\sigma)_3 r = r(i_1+i_2) + \omega(i_2\otimes i_1)(\nabla\varphi + \Delta\varphi^{ab})p$,
3. $(\mu_\sigma)_3\omega = \omega(i_1+i_2)^{\otimes^2}$.

Before proving this result we shall state some consequences.

Corollary 5.7. *The group structure in* $\mathbf{Q}(\Sigma\sigma,\sigma')$ *is given by the following formulas* $(f,g\in\mathbf{Q}(\Sigma\sigma,\sigma'))$:

1. $(f+g)_n = f_n + g_n$ *if* $n\neq 3$,
2. $(f+g)_3 r = f_3 r + g_3 r + \omega(g_2^{ab}\otimes f_2^{ab})(\nabla\varphi + \Delta\varphi^{ab})p$,
3. $(f+g)_3\omega = \omega(f_2^{ab} + g_2^{ab})^{\otimes^2}$,

and

4. $(-f)_n = -f_n$ *if* $n\neq 3$,
5. $(-f)_3 r = -f_3 r + \omega(f_2^{ab}\otimes f_2^{ab})(\nabla\varphi + \Delta\varphi^{ab})p$,
6. $(-f)_3\omega = \omega(f_2^{ab}\otimes f_2^{ab})$.

Proof. Recall that σ_3' and $\operatorname{Ker}[d_2\colon \sigma_2' \to \sigma_1']$ are nil(2)-groups (see (3.1) and (3.2)), so the formulas above make sense. Equality (1) is trivial, (3) follows from (1), and (4), (5), (6) can be obtained from (1), (2), (3) by imposing $f+(-f)=0$. Finally, (2) follows from the equalities:

$$\begin{aligned}(f+g)_3r = (f,g)_3(\mu_\sigma)_3r &= (f,g)_3r(i_1+i_2)+(f,g)_3\omega(i_2\otimes i_1)(\nabla\varphi+\Delta\varphi^{ab})p\\ \text{(a)}\qquad &= (f_3r,g_3r)(i_1+i_2)\\ &\quad +\omega(f_2^{ab},g_2^{ab})(i_2\otimes i_1)(\nabla\varphi+\Delta\varphi^{ab})p\\ &= f_3r+g_3r+\omega(g_2^{ab}\otimes f_2^{ab})(\nabla\varphi+\Delta\varphi^{ab})p.\end{aligned}$$

In (a) we use that the basis of the coproduct of totally free quadratic complexes is the coproduct of the basis. □

Since $\nu_\sigma=-1\in\mathbf{Q}(\Sigma\sigma,\Sigma\sigma)$, we also derive the following.

Corollary 5.8. *The coinversion ν_σ is determined by the following equalities:*

1. $(\nu_\sigma)_n=-1$ *if* $n\neq 3$,
2. $(\nu_\sigma)_3r=-r+\omega(\nabla\varphi+\Delta\varphi^{ab})p$,
3. $(\nu_\sigma)_3\omega=\omega$.

In order to prove Theorem 5.6 we first give some technical results.

Lemma 5.9. *If $a_i\in E$, $\epsilon_i\in\mathbb{Z}$ and $x=\epsilon_1a_1+\cdots+\epsilon_na_n\in\langle E\rangle^{nil}$, then the following equalities hold in $\langle E\rangle^{nil}\vee\langle E\rangle^{nil}$:*

$$\begin{aligned}(i_2+i_1)x-i_1x-i_2x &= \sum_{1\le i<j\le n}\epsilon_i\epsilon_j\,[i_1a_i,i_2a_j]\\ &\quad+\sum_{i=1}^n\binom{|\epsilon_i|}{2}[i_1a_i,i_2a_i]-\sum_{\epsilon_i<0}\epsilon_i[i_1a_i,i_2a_i]\\ &= s\imath\Bigg(\sum_{1\le i<j\le n}\epsilon_i\epsilon_j a_i\otimes a_j\\ &\quad+\sum_{i=1}^n\binom{|\epsilon_i|}{2}a_i\otimes a_i-\sum_{\epsilon_i<0}\epsilon_i a_i\otimes a_i\Bigg).\end{aligned}$$

Here $\binom{m}{2}=\frac{m(m-1)}{2}$ *for any* $m\geq 0$.

Proof. The second equality is obvious. We proceed by induction on the number of summands. Suppose $n=1$, $m>0$ and $a=a_1$. The result is trivial for $\epsilon_1=m=1$. If $\epsilon_1=m+1$ and it is true for $\epsilon_1=m$, then

$$\begin{aligned}(i_2+i_1)x-i_1x-i_2x &= (m+1)(i_2a+i_1a)-(m+1)i_1a-(m+1)i_2a\\ &= m(i_2a+i_1a)+i_2a+i_1a-i_1a-mi_1a-i_2a-mi_2a\\ &= m(i_2a+i_1a)-mi_1a+[-mi_1a,-i_2a]-mi_2a\\ &= m(i_2a+i_1a)-mi_1a-mi_2a+[mi_1a,i_2a]\\ \text{(a)}\qquad &= \binom{m}{2}[i_1a,i_2a]+m[i_1a,i_2a]\\ &= \binom{m+1}{2}[i_1a,i_2a].\end{aligned}$$

The induction hypothesis is applied in (a). If $\epsilon_1 = -m$ for some $m > 0$, then

$$\begin{aligned}(i_2+i_1)(-ma) - i_1(-ma) - i_2(-ma) &= -m(i_2a + i_1a) + mi_1a + mi_2a \\ &= m(-i_1a - i_2a) + mi_1a + mi_2a.\end{aligned}$$

Now we check by induction on m that this is equal to $\binom{m}{2}[i_1a, i_2a] + m[i_1a, i_2a]$. It is clear for $m = 1$. If it is true for m, then

$$\begin{aligned}&(m+1)(-i_1a - i_2a) \\ &+(m+1)i_1a + (m+1)i_2a = m(-i_1a - i_2a) - i_1a - i_2a \\ &\qquad + i_1a + mi_1a + i_2a + mi_2a \\ &= m(-i_1a - i_2a) + [i_1a, i_2a] \\ &\qquad + [i_2a, -mi_1a] + mi_1a + mi_2a \\ &= m(-i_1a - i_2a) + mi_1a + mi_2a + (m+1)[i_1a, i_2a] \\ \text{(b)}\quad &= \binom{m}{2}[i_1a, i_2a] + m[i_1a, i_2a] + (m+1)[i_1a, i_2a] \\ &= \binom{m+1}{2}[i_1a, i_2a] + (m+1)[i_1a, i_2a].\end{aligned}$$

The induction step is given in (b).

Suppose now that the lemma is true for $\leq n$ summands ($n \geq 1$). Let us check it for $n+1$ summands. For this, we set $y = \epsilon_1 a_1 + \cdots + \epsilon_n a_n$, $\epsilon = \epsilon_{n+1}$ and $a = a_{n+1}$. Then

$$\begin{aligned}(i_2+i_1)x - i_1x - i_2x &= (i_2+i_1)y + (i_2+i_1)(\epsilon a) - i_1(\epsilon a) - i_1y - i_2(\epsilon a) - i_2y \\ &= (i_2+i_1)y + (i_2+i_1)(\epsilon a) - i_1(\epsilon a) - i_2(\epsilon a) \\ &\quad +[-i_2(\epsilon a), i_1y] - i_1y - i_2y \\ \text{(c)}\quad &= (i_2+i_1)y - i_1y - i_2y \\ &\quad +(i_2+i_1)(\epsilon a) - i_1(\epsilon a) - i_2(\epsilon a) + [i_1y, i_2(\epsilon a)] \\ \text{(d)}\quad &= \sum_{1\leq i<j\leq n} \epsilon_i\epsilon_j\,[i_1a_i, i_2a_j] + \sum_{i=1}^{n}\binom{|\epsilon_i|}{2}[i_1a_i, i_2a_i] \\ &\quad - \sum_{\epsilon_i<0, i\leq n} \epsilon_i[i_1a_i, i_2a_i] + \binom{|\epsilon|}{2}[i_1a, i_2a] \\ &\quad +\mathrm{sign}(\epsilon)\epsilon[i_1a, i_2a] + \sum_{i=1}^{n}\epsilon_i\epsilon[i_1a_i, i_2a] \\ &= \sum_{1\leq i<j\leq n+1} \epsilon_i\epsilon_j\,[i_1a_i, i_2a_j] + \sum_{i=1}^{n+1}\binom{|\epsilon_i|}{2}[i_1a_i, i_2a_i] \\ &\quad - \sum_{\epsilon_i<0}\epsilon_i[i_1a_i, i_2a_i].\end{aligned}$$

Here $\mathrm{sign}(\epsilon)$ is -1 if $\epsilon < 0$ and 0 otherwise. In (c) we use that $(i_2+i_1)(\epsilon a) - i_1(\epsilon a) - i_2(\epsilon a)$ belongs to the commutator subgroup, by the case $n = 1$, and commutators are central in any nil(2)-group. We apply the induction hypothesis in (d). Now the proof is finished. □

Lemma 5.10. *For any $\varphi \in$ **nil** we have $\nabla\varphi + \nabla(-\varphi) = (\varphi^{ab} \otimes \varphi^{ab})\Delta$.*

Proof. In fact, it follows from Lemma 5.9 that if $a_i \in E$, $\epsilon_i \in \mathbb{Z}$ and $x = \epsilon_1 a_1 + \cdots + \epsilon_n a_n \in \langle E\rangle^{nil}$, then

$$\begin{array}{rcl} (i_2+i_1)x - i_1x - i_2x & & \\ +(i_2+i_1)(-x) - i_1(-x) - i_2(-x) & = & \sum_{i,j=1}^{n} \epsilon_i\epsilon_j[i_1a_i, i_2a_j] = s\imath(p(x)\otimes p(x)), \end{array}$$

hence the lemma holds. □

Now we are ready to prove Theorem 5.6.

Proof of Theorem 5.6. The first formula can be easily checked by using the description of the suspension functor in **Q** and the cogroup structure of a suspension given in Proposition 5.3, together with the formulas in [7], IV.12.10, IV.12.3 (3) and IV.12.3 (5). The third one is a consequence of (1) for $n = 2$.

In order to prove (2) consider $b \in E$ with $d_2(b) = \bar{\varphi}(b) = \epsilon_1 a_1 + \cdots \epsilon_n a_n$, for some $a_i \in F$, $\epsilon_i \in \mathbb{Z}$. Then $(\mu_\sigma)_3 r(b)$ is the projection of $(e_1+e_2)\overline{\overline{\otimes}}b \in ((\sigma S^1 \vee \sigma S^1)\otimes\sigma)_3$ to the quotient $(\Sigma\sigma \vee \Sigma\sigma)_3 = ((\sigma S^1 \vee \sigma S^1)\otimes\sigma/(\sigma S^1\vee\sigma S^1)\vee\sigma)_3$; see [7], IV.12.10 and IV.12.5 (2). In the following equations the formulas are in $((\sigma S^1 \vee \sigma S^1)\otimes\sigma)_3$ and the symbol $\approx$ means that they coincide when projecting onto $(\Sigma\sigma\vee\Sigma\sigma)_3 = ((\sigma S^1 \vee \sigma S^1)\otimes\sigma/(\sigma S^1\vee\sigma S^1)\vee\sigma)_3$,

$$\begin{array}{llcl} \text{(a)} & (e_1+e_2)\overline{\overline{\otimes}}b & = & -S_2^{e_1+e_2}(b) \\ \text{(b)} & & = & \omega\Theta(e_1+e_2, \bar{\varphi}(b)) + S(b)(e_1+e_2) \\ \text{(c)} & & \approx & -\omega\widetilde{T}(\vartheta^{\sigma S^1\vee\sigma S^1}(e_1+e_2)\otimes\vartheta^\sigma\bar{\varphi}(b)) + e_2\times b + e_1\times b. \end{array}$$

Here $\widetilde{T}$ is the same as T in [7], IV.12.4(3). We have changed the name of this homomorphism because in this paper T is the interchange of factors in the tensor square; see (2.2). We have used [7], IV.12.5 (13) for (a), IV.12.5 (12) for (b), and IV.12.4 (6) together with IV.12.5 (9) for (c). Moreover, by [7], IV.12.4 (1), we have that $\vartheta^\sigma(0) = \vartheta^\sigma(0+0) \approx \vartheta^\sigma(0) + \vartheta^\sigma(0)$, and given $a \in F$,

$$0 \approx \vartheta^\sigma(0) = \vartheta^\sigma(a+(-a)) \approx \vartheta^\sigma(a) + \vartheta^\sigma(-a) - a\otimes a = \vartheta^\sigma(-a) - a\otimes a,$$

so $\vartheta^\sigma(-a) \approx a\otimes a$. By using this relation one can check inductively, as in the proof of Lemma 5.9, that the following relation holds:

$$\vartheta^\sigma\bar{\varphi}(b) \approx \sum_{1\leq i<j\leq n} \epsilon_i\epsilon_j a_i\otimes a_j + \sum_{i=1}^{n}\binom{|\epsilon_i|}{2} a_i\otimes a_i - \sum_{\epsilon_1<0}\epsilon_i a_i\otimes a_i,$$

and thus by using [7], IV.12.4 (3), and the fact that ω is central we see that

$$\begin{array}{lcl} \text{(c)} & \approx & e_2\times b + e_1\times b - \sum_{1\leq i<j\leq n}\epsilon_i\epsilon_j\omega(e_2\otimes a_j\otimes e_1\otimes a_i) \\ & & -\sum_{i=1}^{n}\binom{|\epsilon_i|}{2}\omega(e_2\otimes a_i\otimes e_1\otimes a_i) + \sum_{\epsilon_i<0}\epsilon_i\omega(e_2\otimes a_i\otimes e_1\otimes a_i). \end{array}$$

This proves, by projecting onto $(\Sigma\sigma \vee \Sigma\sigma)_3$, that

$$\begin{aligned}(\mu_\sigma)_3 r(b) &= ri_2(b) + ri_1(b) - \omega(i_2 \otimes i_1)\Bigg(\sum_{1\le j<i\le n} \epsilon_i \epsilon_j a_i \otimes a_j \\ &\qquad + \sum_{i=1}^{n} \binom{|\epsilon_i|}{2} a_i \otimes a_i - \sum_{\epsilon_1<0} \epsilon_i a_i \otimes a_i \Bigg) \\ \text{(d)} \qquad &= ri_2(b) + ri_1(b) - \omega(i_2 \otimes i_1)\Big(\nabla(-\varphi)(b) - \Delta\varphi^{ab}(b) \Big).\end{aligned}$$

In (d) we use Lemma 5.9. By Lemma 5.10 this implies that

$$\text{(e)} \qquad (\mu_\sigma)_3 r = r(i_2 + i_1) + \omega(i_2 \otimes i_1)(\nabla\varphi - (\varphi^{ab} \otimes \varphi^{ab})\Delta + \Delta\varphi^{ab})p.$$

Recall that the basis of the low-dimensional reduced quadratic module of the coproduct $\Sigma\sigma \vee \Sigma\sigma$ is $(-\varphi) \vee (-\varphi) \in$ **nil**, hence by using (3.C) we obtain

$$\omega(i_2 \otimes i_1)(\varphi^{ab} \otimes \varphi^{ab})\Delta p = \omega(\varphi^{ab} \oplus \varphi^{ab})^{\otimes^2}(i_2 \otimes i_1)\Delta p = rsq(i_2 \otimes i_1)\Delta p.$$

Furthermore, for any $b \in E$ we have that $rsq(i_2 \otimes i_1)\Delta p(b) = r[i_2 b, i_1 b]$ and therefore

$$(e) = r(i_1 + i_2) + \omega(i_2 \otimes i_1)(\nabla\varphi + \Delta\varphi^{ab})p.$$

□

6. The natural homomorphism $H_2X \to \wedge^2 H_1 X$

The Pontrjagin product in the homology of an abelian group A induces an isomorphism $\wedge^2 A \simeq H_2 A$; see [13]. In [15] it is proven that the James-Hopf invariant $\gamma_2 \colon \pi_3 \Sigma K(A,1) \to A^{\otimes^2}$ is an isomorphism; moreover, the Hurewicz homomorphism h_3 corresponds with the natural projection q in the following diagram:

$$\text{(6.A)} \qquad \begin{array}{ccc} \pi_3\Sigma K(A,1) & \xrightarrow[\gamma_2]{\simeq} & A^{\otimes^2} \\ {\scriptstyle h_3}\downarrow & & \downarrow{\scriptstyle q} \\ H_3 \Sigma K(A,1) & = & \wedge^2 A \end{array}$$

Given any space X, consider the natural homotopy class $h\colon X \to K(H_1X, 1)$ such that $\pi_1 h = h_1 \colon \pi_1 X \twoheadrightarrow H_1 X$ is the Hurewicz homomorphism. The naturality of the James-Hopf invariant γ_2, the Hurewicz homomorphism, the suspension isomorphism in homology, and (6.A) imply that the following diagram also commutes

$$\text{(6.B)} \qquad \begin{array}{ccc} \pi_3\Sigma X & \xrightarrow{\gamma_2} & (H_1X)^{\otimes^2} \\ {\scriptstyle h_3}\downarrow & & \downarrow{\scriptstyle q} \\ H_2 X & \xrightarrow{h_*} & \wedge^2 H_1 X \end{array}$$

In this section we shall compute the natural homomorphism h_* by using the results of Section 5. The results concerning the co-H-structure of a suspension are useful because of the existent relation between the James-Hopf invariant and this co-H-structure; see (6.C) below.

We can suppose, without loss of generality, that the 2-skeleton X^2 of the space X is the mapping cone of a based map between wedges of circles

$$f\colon \vee_{E_2} S^1 \longrightarrow \vee_{E_1} S^1.$$

This map carries the same homotopical information as the induced homomorphism in π_1,

$$\bar{\varphi} = \pi_1 f\colon \langle E_2 \rangle \longrightarrow \langle E_1 \rangle,$$

which is the basis of the low-dimensional nil(2)-module of the quadratic complex $\sigma(X)$.

If we write C_*X for the (reduced) cellular chain complex of X, it is clear that $E_i \subset C_iX$ are bases $(i = 1, 2)$; moreover, $\bar{\varphi}^{ab} = d_2\colon C_2X \to C_1X$.

Theorem 6.1. *Choose a basis of* 2*-cycles* $Z_2 \subset \operatorname{Ker} d_2$ *and a homomorphism* $\psi\colon \langle Z_2 \rangle^{nil} \to \langle E_2 \rangle^{nil}$ *such that* $\psi^{ab}\colon \operatorname{Ker} d_2 \hookrightarrow C_2X$ *is the inclusion. The natural homomorphism* $h_*\colon H_2X \to \wedge^2 H_1X$ *is induced by the unique homomorphism* $\alpha\colon \operatorname{Ker} d_2 \to \wedge^2 C_1X$ *verifying* $\varphi\psi = s\alpha p$*, where* $\varphi = \bar{\varphi}^{nil}$*, and* p *and* s *are part of the central extension in* (2.G).

Before beginning with the proof of this result we shall give some homotopical considerations. The following natural identifications are obtained by using Hurewicz's theorem, the suspension isomorphism in homology and Künneth's formula,

$$\pi_3 \Sigma X \wedge X = H_3 \Sigma X \wedge X = H_2 X \wedge X = (H_1 X)^{\otimes^2}.$$

If $h_1\colon \pi_1 X \twoheadrightarrow H_1 X$ is the Hurewicz homomorphism, then the natural identification $(H_1X)^{\otimes^2} = \pi_3 \Sigma X \wedge X$ sends a generator $h_1(f) \otimes h_1(g)$ to the homotopy class of the map $\Sigma(f \wedge g)\colon S^3 = \Sigma S^1 \wedge S^1 \to \Sigma X \wedge X$; compare Section 5 in [12] and [1], 6.3.16. By using the naturality properties of the Whitehead product element $[i_1, i_2]_Y \in [\Sigma Y \wedge Y, \Sigma(Y \vee Y)]$ (see [8]), we observe that the following diagram commutes:

$$\begin{array}{ccc} S^3 & \xrightarrow{\Sigma(f\wedge g)} & \Sigma X \wedge X \\ \downarrow{\scriptstyle [i_1,i_2]_{S^1}} & & \downarrow{\scriptstyle [i_1,i_2]_X} \\ S^2 \vee S^2 & \xrightarrow[\Sigma(f\vee g)]{} & \Sigma(X \vee X) \end{array}$$

Hence the homotopy left distributivity formula in [8], A.10.2 (b), proves that the following diagram commutes:

(6.C)
$$\begin{array}{ccc} \pi_3\Sigma X & \xrightarrow{\gamma_2} & \pi_2\Sigma X \otimes \pi_2\Sigma X \\ & {\scriptstyle \mu_{X*}-i_{2*}-i_{1*}} \searrow \quad \swarrow {\scriptstyle [i_{1*},i_{2*}]} & \\ & \pi_3 \Sigma X \vee \Sigma X & \end{array}$$

Here the bracket $[\,,\,]$ is the usual Whitehead product operation in homotopy groups, induced by $[i_1, i_2]_{S^1}$, and $[i_{1*}, i_{2*}](f \otimes g) = [i_1 f, i_2 g]$, $(f, g \in \pi_2 \Sigma X)$. It is a well-known fact that $[i_{1*}, i_{2*}]$ is a splitting injection whose cokernel is $\pi_3\Sigma X \oplus \pi_3 \Sigma X$.

Diagram (6.C) and formulas in Theorem 5.6 will help us to compute the James-Hopf invariant γ_2 in certain cases, which allow us to prove Theorem 6.1 by using the commutativity of (6.B).

We also need the translation of the Whitehead product operation and the Hurewicz homomorphism h_3 to the language of quadratic complexes. For this, note that, when σ is a reduced quadratic complex, the second homotopy group $\pi_2\sigma$ is a quotient of σ_2 and the natural projection $\sigma_2 \twoheadrightarrow \pi_2\sigma$ factors through the abelianization $\sigma_2 \twoheadrightarrow \sigma_2^{ab} = C_2$.

The Whitehead product homomorphism $[i_{1*}, i_{2*}]$ for a suspended quadratic complex $\Sigma\sigma$ can be computed by passing to the quotient the following homomorphism

$$\text{(6.D)} \qquad \omega(i_1 \otimes i_2 + i_2 \otimes i_1 T)\colon (\Sigma\sigma)_2^{ab} \otimes (\Sigma\sigma)_2^{ab} \to \operatorname{Ker} d_3 \subset (\Sigma\sigma \vee \Sigma\sigma)_3,$$

compare IV.3.7 and I.4.4 in [7]. Moreover, if $\sigma = \sigma(X)$, then $C_*\Sigma X$ is $\lambda\Sigma\sigma$. This crossed complex is in fact a chain complex of free abelian groups because a suspended crossed complex is reduced. There is a natural projection of chain complexes of groups $\Sigma\sigma \twoheadrightarrow \lambda\Sigma\sigma = C_*\Sigma X$; in fact, for any quadratic complex σ one can easily construct a canonical projection $\sigma \twoheadrightarrow \lambda\sigma$ from the construction of λ; see Section 4 or [7], IV.3.3. The Hurewicz homomorphism h_3 coincides with the morphism induced by this projection in the third homology; see IV.C.10 and IV.3.7 in [7].

Proof of Theorem 6.1. Consider $\sigma = \sigma(X)$. In order to carry out the necessary computations we choose a total ordering $\preceq$ in E_1 and define the homomorphism $\eta\colon \wedge^2\mathbb{Z}\langle E_1\rangle \to \mathbb{Z}\langle E_1\rangle^{\otimes^2}$ as $\eta(e_1 \wedge e_2) = e_1 \otimes e_2$ ($e_1 \prec e_2 \in E_1$). This homomorphism is a splitting of q since it satisfies $q\eta = 1$.

Let $\beta : \operatorname{Ker} d_2 \to \wedge^2 C_1X$ be the unique homomorphism satisfying $(-\varphi)\psi = s\beta_p$; by (2.3) (iii)

$$\begin{aligned} s(\alpha+\beta)p &= (\varphi \mid \varphi)_\psi \\ &= (\varphi+\varphi)\psi - \varphi\psi - \varphi\psi \\ &= (\varphi,\varphi)((i_2+i_1)\psi - i_1\psi - i_2\psi) \\ &= (\varphi,\varphi)s\iota(\nabla\psi)p \\ &= s(\wedge^2\varphi^{ab}q(\nabla\psi)p \end{aligned}$$

so $\alpha + \beta = (\wedge^2 d_2)q(\nabla\psi)$. In the last equality we use that, given $a, b \in E_2$,

$$\begin{aligned} (\varphi,\varphi)s\iota(a\otimes b) &= (\varphi,\varphi)s(i_1a \wedge i_2b) = (\varphi,\varphi)[i_1a, i_2b] \\ &= [\varphi(a), \varphi(b)] = s(\wedge^2\varphi^{ab})q(a\otimes b). \end{aligned}$$

The image of the homomorphism

$$\text{(6.E)} \qquad r\psi - \omega\eta\beta p\colon \langle Z_2\rangle^{nil} \to (\Sigma\sigma)_3$$

lies on $\operatorname{Ker} d_3$, since, by Lemma 4.6, $d_3(r\psi - \omega\eta\beta p) = (-\varphi)\psi - sq\eta\beta p = s\beta p - s\beta p = 0$.

Let us begin with the proof of the statement of the theorem. Given a 2-cycle $x \in \mathbb{Z}\langle Z_2\rangle$, if $\bar{x} \in \langle Z_2\rangle^{nil}$ is an element such that $p(\bar{x}) = x$, then

$$y = (r\psi - \omega\eta\beta p)(\bar{x}) \in (\Sigma\sigma)_3$$

represents an element of $\pi_3\Sigma X$ such that $h_3\{y\} = \{x\}$, here $\{\cdot\}$ denotes the equivalence class in the corresponding quotient group, so by (6.C),

$$(\mu_\sigma)_3(y) - (i_2)_3(y) - (i_1)_3(y) \in (\Sigma\sigma \vee \Sigma\sigma)_3$$

represents $[i_{1*}, i_{2*}]\gamma_2\{y\} \in \pi_3(\Sigma X \vee \Sigma X)$.

Let us compute this element and $\gamma_2\{y\}$. For this recall that $(-\varphi) \vee (-\varphi) \in$ **nil** is the basis of the low-dimensional reduced quadratic module of $\Sigma\sigma \vee \Sigma\sigma$, and the inclusions of the factors of the coproduct $i_k\colon \Sigma\sigma \to \Sigma\sigma \vee \Sigma\sigma$ $(k = 1, 2)$ satisfy $(i_k)_n = i_k$ $(n \neq 3)$ and $(i_k)_3 r = r i_k$.

$$\begin{array}{llcl}
\text{(a)} & (\mu_\sigma)_3(r\psi - \omega\eta\beta p) & = & r(i_1+i_2)\psi + \omega(i_2\otimes i_1)(\nabla\varphi + \Delta\varphi^{ab})p\psi \\
& & & -\omega(i_1+i_2)^{\otimes^2}\eta\beta p, \\
\text{(b)} & (i_1)_3(r\psi - \omega\eta\beta p) & = & r i_1\psi - \omega(i_1\otimes i_1)\eta\beta p, \\
\text{(c)} & (i_2)_3(r\psi - \omega\eta\beta p) & = & r i_2\psi - \omega(i_2\otimes i_2)\eta\beta p, \\
\text{(d)} & \text{(a)} - \text{(c)} - \text{(b)} & = & r(i_1|i_2)_\psi + \omega(i_2\otimes i_1)(\nabla\varphi + \Delta\varphi^{ab})p\psi \\
& & & -\omega(i_1\otimes i_2 + i_2\otimes i_1)\eta\beta p, \\
\text{(e)} & r(i_1|i_2)_\psi & = & r(i_2,i_1)(i_2|i_1)_\psi \\
& & = & r(i_2,i_1)s\imath(\nabla\psi)p \\
\text{(f)} & & = & rs\wedge^2(i_2,i_1)\imath(\nabla\psi)p \\
\text{(g)} & & = & rsq(i_2\otimes i_1)(\nabla\psi)p \\
\text{(h)} & & = & \omega(\varphi^{ab}\oplus\varphi^{ab})^{\otimes^2}(i_2\otimes i_1)(\nabla\psi)p \\
& & = & \omega(i_2\otimes i_1)(\varphi^{ab}\otimes\varphi^{ab})(\nabla\psi)p.
\end{array}$$

Here we use that ω is central. Moreover, in (a) we apply Theorem 5.6, in (f) the naturality of (2.G), in (h) (3.C), and in (g) we use that for any $a, b \in C_2X$,

$$\wedge^2(i_2,i_1)\imath(a\otimes b) = \wedge^2(i_2,i_1)(i_1 a\wedge i_2 b) = i_2 a\wedge i_1 b = q(i_2\otimes i_1)(a\otimes b).$$

Furthermore, $\varphi^{ab}p\psi = \varphi^{ab}\psi^{ab}p = 0p = 0$, therefore by (e) we get

$$\begin{array}{llcl}
\text{(i)} & \text{(d)} & = & -\omega(i_1\otimes i_2)\eta\beta p \\
& & & -\omega(i_2\otimes i_1)(\eta\beta - (\nabla\varphi)\psi^{ab} - (\varphi^{ab}\otimes\varphi^{ab})(\nabla\psi))p.
\end{array}$$

We have the following equalities:

$$\begin{array}{llcl}
\text{(j)} & s\imath(\nabla\varphi)\psi^{ab}p & = & ((i_2+i_1)\varphi - i_1\varphi - i_2\varphi)\psi \\
& & = & (i_2+i_1)\varphi\psi - (i_2\varphi + i_1\varphi)\psi, \\
\text{(k)} & s\imath(\varphi^{ab}\otimes\varphi^{ab})(\nabla\psi)p & = & s\wedge^2(\varphi^{ab}\oplus\varphi^{ab})\imath(\nabla\psi)p \\
\text{(l)} & & = & (\varphi\vee\varphi)s\imath(\nabla\psi)p \\
& & = & (\varphi\vee\varphi)((i_2+i_1)\psi - i_1\psi - i_2\psi) \\
& & = & (i_2\varphi + i_1\varphi)\psi - i_1\varphi\psi - i_2\varphi\psi, \\
\text{(m)} & \text{(j)} + \text{(k)} & = & (i_2+i_1)\varphi\psi - i_1\varphi\psi - i_2\varphi\psi \\
& & = & (i_2+i_1)s\beta p - i_1 s\beta p - i_2 s\beta p \\
\text{(n)} & & = & s(\wedge^2(i_2+i_1) - \wedge^2 i_1 - \wedge^2 i_2)\beta p \\
\text{(o)} & & = & s\imath(1-T)\eta\beta p.
\end{array}$$

In (k) we apply the naturality of (2.D), and in (l) and (n) the naturality of (2.G). In (o) we use that given $e_1 \prec e_2 \in E_1$,

$$\begin{array}{rcl}
(\wedge^2(i_2+i_1) - \wedge^2 i_1 - \wedge^2 i_2)(e_1\wedge e_2) & = & (i_1e_1 + i_2e_1)\wedge(i_1e_2 + i_2e_2) \\
& & -i_1e_1\wedge i_1e_1 - i_2e_2\wedge i_2e_2 \\
& = & i_1e_1\wedge i_2e_2 - i_1e_2\wedge i_2e_1 \\
& = & \imath(1-T)\eta(e_1\wedge e_2).
\end{array}$$

Hence by (m) we get

$$\begin{aligned}(\mathrm{i}) &= -\omega(i_1\otimes i_2)\eta\beta p-\omega(i_2\otimes i_1)(\eta\beta-(1-T)\eta\beta)p\\ &= -\omega(i_1\otimes i_2+i_2\otimes i_1T)\eta\beta p.\end{aligned}$$

So by (6.C) and (6.D) $\gamma_2\{y\}$ is represented by $-\eta\beta(x)\in(C_1X)^{\otimes^2}$, and by using also (6.B) and $q\eta=1$ we see that either $-\beta(x)=\alpha(x)-(\wedge^2 d_2)q(\nabla\psi)(x)$ or $\alpha(x)\in\wedge^2 C_1X$ represents $h_*\{x\}$. The proof is now finished. □

In the following remark we make some observations about the proof of Theorem 6.1 which will be useful in applications of this result to proper homotopy theory.

Remark 6.2. If we consider diagram (6.B) for the 2-skeleton X^2 we get

(6.F)
$$\begin{array}{ccc}\pi_3\Sigma X^2 & \xrightarrow{\gamma_2} & (H_1X)^{\otimes^2}\\ {\scriptstyle h_3}\big\downarrow & & \big\downarrow{\scriptstyle q}\\ \operatorname{Ker} d_2 & \xrightarrow{h_*} & \wedge^2 H_1X\end{array}$$

since $\operatorname{Ker} d_2=H_2X^2$. This group is known to be free abelian, hence h_3 in (6.F) admits a section homomorphism. In fact a concrete section is determined by the homomorphism $r\psi-\omega\eta(-\alpha+(\wedge^2 d_2)q(\nabla\psi))p$ in (6.E). For the definition of this homomorphism we use the homomorphism ψ chosen in the statement of Theorem 6.1, the section $\eta\colon\ \wedge^2 C_1X\hookrightarrow(C_1X)^{\otimes^2}$ of the natural projection q induced by a total ordering $\preceq$ in the set of 1-cells of X, and the homomorphism φ induced by the attaching map of 2-cells in X. Let us call ζ_X to this section of h_3. Moreover, in the proof of Theorem 6.1 we check that, if $t_X\colon C_1X\twoheadrightarrow H_1X$ is the natural projection, then the composition of this section with the James-Hopf invariant is $\gamma_2\zeta_X=(t_X)^{\otimes^2}\eta(\alpha-(\wedge^2 d_2)q(\nabla\psi))$. Recall from the statement of Theorem 6.1 that α is the unique homomorphism such that $\varphi\psi=s\alpha p$.

If $Y\subset X$ is a subcomplex, the homomorphism φ' induced by the attaching map of 2-cells in Y is completely determined by φ; in fact, φ' is a restriction of φ in the target and the source to the subgroups with basis the cells of X which lie in Y. Moreover, the section η induces a section $\eta'\colon\ \wedge^2 C_1Y\hookrightarrow(C_1Y)^{\otimes^2}$ which is the section defined by the restriction of the total ordering $\preceq$ to the subset of 1-cells in Y. Furthermore, suppose that we choose the basis set Z_2 of the free abelian group $\operatorname{Ker} d_2=H_2X^2$ considered in the statement of Theorem 6.1 in such a way that Z_2 contains a basis Z_2' of H_2Y^2, and the homomorphism ψ such that the image of Z_2' lies in the subgroup generated by the 2-cells of Y. In this case there exists a homomorphism ψ' from $\langle Z_2'\rangle^{nil}$ to the free nil(2)-group with basis the set of 2-cells in Y which is a restriction of ψ in the same way as φ restricts to φ'. This homomorphism ψ' satisfies the conditions of the statement of Theorem 6.1. Let α' be the unique homomorphism such that $\varphi'\psi'=s\alpha'p$, this α' is also determined by α, in the same way as φ' and ψ', and there is a section of the Hurewicz homomorphism $\zeta_Y\colon H_2Y^2\hookrightarrow\pi_3\Sigma Y^2$ induced by α', ψ' and η', as in the case of ζ_X. This section ζ_Y is compatible with ζ_X in the sense that the following diagram is commutative:

(6.G)
$$\begin{array}{ccc}\pi_3\Sigma Y^2 & \longrightarrow & \pi_3\Sigma X^2\\ {\scriptstyle \zeta_Y}\big\uparrow & & \big\uparrow{\scriptstyle \zeta_X}\\ H_2Y^2 & \longrightarrow & H_2X^2\end{array}$$

Moreover, we also have that $\gamma_2\zeta'_X = (t_Y)^{\otimes 2}\eta'(\alpha' - (\wedge^2 d_2)q(\nabla\psi'))$, and notice that the homomorphisms $\eta(\alpha - (\wedge^2 d_2)q(\nabla\psi))$ and $\eta'(\alpha' - (\wedge^2 d_2)q(\nabla\psi'))$ are also compatible, that is, the next diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}\langle Z_2'\rangle & \xrightarrow{\eta'(\alpha'-(\wedge^2 d_2)q(\nabla\psi'))} & (C_1Y)^{\otimes 2} \\ \downarrow & & \downarrow \\ \mathbb{Z}\langle Z_2\rangle & \xrightarrow{\eta(\alpha-(\wedge^2 d_2)q(\nabla\psi))} & (C_1X)^{\otimes 2} \end{array} \tag{6.H}$$

Now we shall give two applications of Theorem 6.1 in group theory, based on the following observation.

Remark 6.3. The image of $h_*\colon H_2X \to \wedge^2 H_1X$ coincides with that of the homomorphism $H_2\pi_1X \to H_2H_1X = \wedge^2 H_1X$ induced by the abelianization $\pi_1X \twoheadrightarrow (\pi_1X)^{ab} = H_1X$, here we use the Hurewicz theorem and the Hopf exact sequence in [13], II.5.2. Hence, by the classical 5-term exact sequence in the low-dimensional homology of groups, the cokernel of h_* is the homomorphism $\tilde{w}\colon \wedge^2 (\pi_1X)^{ab} \twoheadrightarrow \Gamma_2\pi_1X/\Gamma_3\pi_1X$ induced by the commutator bracket. In fact, $\tilde{w}$ is a factorization of w in (2.H) through the natural projection q in (2.A).

A presentation of a group G can be regarded as a pair of homomorphisms

$$\langle E_2\rangle \overset{\varphi_G}{\to} \langle E_1\rangle \overset{p_G}{\twoheadrightarrow} G \tag{6.I}$$

such that $\operatorname{Ker} p_G \subset \langle E_1\rangle$ is the normal subgroup generated by the image of φ_G. In the next proposition we construct a presentation of Γ_2G/Γ_3G as an abelian group from a given presentation of G.

Proposition 6.4. *Let G be a group presented by* (6.I)*. If Z_2 is a basis of $\operatorname{Ker}\varphi_G^{ab}$, $\psi\colon \langle Z_2\rangle^{nil} \to \langle E_2\rangle^{nil}$ is a homomorphism such that $\psi^{ab}\colon \operatorname{Ker}\varphi_G^{ab} \hookrightarrow \mathbb{Z}\langle E_2\rangle$ is the inclusion, and $\alpha\colon \mathbb{Z}\langle Z_2\rangle \to \wedge^2\mathbb{Z}\langle E_1\rangle$ satisfies $s\alpha p = \varphi_G^{nil}\psi$, then the following sequence is exact:*

$$\mathbb{Z}\langle Z_2\rangle \oplus \mathbb{Z}\langle E_2\rangle \otimes \mathbb{Z}\langle E_1\rangle \oplus \wedge^2\mathbb{Z}\langle E_2\rangle \overset{\zeta}{\longrightarrow} \wedge^2\mathbb{Z}\langle E_1\rangle \overset{\tilde{w}(\wedge^2 p_G^{ab})}{\twoheadrightarrow} \Gamma_2G/\Gamma_3G.$$

Here $\zeta = (\alpha, q(\varphi_G^{ab} \otimes 1), \wedge^2\varphi_G^{ab})$.

Proof. If we take X to be the 2-dimensional CW-complex with the attaching map of 2-cells given by φ_G, then, by Theorem 6.1 and Remark 6.3, we get an exact sequence

$$\mathbb{Z}\langle Z_2\rangle \overset{(\wedge^2 p_G^{ab})\alpha}{\longrightarrow} \wedge^2 G^{ab} \overset{\tilde{w}}{\twoheadrightarrow} \Gamma_2G/\Gamma_3G. \tag{6.J}$$

Therefore this proposition follows by (2.F). □

The kernel of $\tilde{w}$ in (6.J) is the second homology group of G^{nil} in the variety of groups of nilpotency degree 2, as defined in [17]; see Theorem 5.1 in [11]. This abelian group is denoted by $\mathfrak{B}_1(G^{nil},\mathbb{Z})$ in [17] and $H_2^{Nil}G^{nil}$ in [11]. Thus if

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} : \mathbb{Z}\langle F\rangle \hookrightarrow \mathbb{Z}\langle Z_2\rangle \oplus \mathbb{Z}\langle E_2\rangle \otimes \mathbb{Z}\langle E_1\rangle \oplus \wedge^2\mathbb{Z}\langle E_2\rangle$$

is the kernel of ζ, then by (2.F) one easily checks that

Corollary 6.5. *There is an exact sequence*

$$\mathbb{Z}\langle F\rangle \xrightarrow{\beta_1} \mathbb{Z}\langle Z_2\rangle \twoheadrightarrow H_2^{Nil}G^{nil}.$$

This is in fact an abelian presentation of the homology group $H_2^{Nil}G^{nil}$. This presentation and the universal coefficient exact sequences for H_*^{Nil} in [11] yield an explicit computation of $H_n^{Nil}G^{nil}$ for any finitely presented group G and $n \geq 1$.

7. Commutative co-H-structures on degree 2 Moore spaces

A *Moore space* of type (A, n) is a simply connected space $M = M(A, n)$ whose unique non-trivial reduced homology group is $H_nM = A$, this space is unique up to homotopy type. The number n is the *degree* of the Moore space. The quadratic complex σ_A of the Moore space $M(A, 2)$ is the suspension of the quadratic complex concentrated in degrees 1 and 2 whose nil(2)-module has as a basis a homomorphism

$$\bar{\varphi}_A \colon \langle A_2\rangle \to \langle A_1\rangle$$

such that $\bar{\varphi}_A^{ab}$ is injective and $\operatorname{Coker}\bar{\varphi}_A^{ab} = A$, the natural projection is denoted by

$$\hat{p}_A \colon \mathbb{Z}\langle A_1\rangle \twoheadrightarrow A.$$

Hence, by Lemma 4.6, σ_A is the reduced quadratic complex concentrated in degrees 2 and 3 such that the basis of its reduced quadratic module is $-\varphi_A$, where $\varphi_A = \bar{\varphi}_A^{nil}$. This suspended quadratic complex is constructed as in (3.C) and carries a canonical cogroup structure μ_A, ν_A, which is determined by φ_A; see Theorem 5.6 and Corollary 5.8.

The goal of this section is constructing and computing an extension element

$$c_A \in \operatorname{Ext}(A, S^2A)$$

such that $c_A = 0$ if and only if $M(A, 2)$ admits a commutative co-H-structure. We shall see that this element is natural with respect to homomorphisms $f \colon A \to B$, that is,

$$(S^2f)_*c_A = f^*c_B \in \operatorname{Ext}(A, S^2B); \tag{7.A}$$

therefore it represents an element in the cohomology of the category **Ab** of abelian groups with coefficients in the bifunctor $\operatorname{Ext}(-, S^2)$ (see for example [7], VI.1),

$$c \in H^0(\mathbf{Ab}, \operatorname{Ext}(-, S^2)).$$

Moreover, we shall show that c is a non-trivial element of order 2.

Since Moore spaces of degree 2 are 3-dimensional, the homotopy category of these spaces coincides with that of their quadratic complexes. There is a well-known natural central extension

$$\operatorname{Ext}(A, \Gamma B) \overset{j}{\hookrightarrow} [\sigma_A, \sigma_B] \overset{H_2}{\twoheadrightarrow} \operatorname{Hom}(A, B). \tag{7.B}$$

Here $H_2 = \pi_2$ as defined in (4.A), by the Hurewicz theorem. Moreover, given an element $\beta \in \operatorname{Ext}(A, \Gamma B)$ represented by $\bar{\beta} \colon \mathbb{Z}\langle A_2\rangle \to \Gamma B$, $j\beta$ is the morphism of quadratic complexes with $(j\beta)_n = 0$ if $n \neq 3$,

$$(j\beta)_3 r = \bar{\beta}p \colon \langle A_2\rangle^{nil} \to \Gamma B = \pi_3 M(B, 2) = \operatorname{Ker} d_3 \subset (\sigma_B)_3,$$

and $(j\beta)_3\omega = 0$. The natural extension (7.B) satisfies the following linear distributivity law (see [8]): given two morphisms $\sigma_A \xrightarrow{f} \sigma_B \xrightarrow{g} \sigma_C$ and elements $\beta \in \mathrm{Ext}(A, \Gamma B)$, $\zeta \in \mathrm{Ext}(B, \Gamma C)$, then

$$(g + j\zeta)(f + j\beta) = gf + j(\Gamma H_2 g)_*\beta + j(H_2 f)^*\zeta \in [\sigma_A, \sigma_C]. \tag{7.C}$$

Remark 7.1. In this section we often identify a morphism in $\mathbf{Q}$ with its homotopy class in $\mathbf{Q}/\simeq$, hence the symbol "=" sometimes means that two homomorphisms in $\mathbf{Q}$ are homotopic, but may not be strictly equal. However, it will be clear from the context when "=" stands for equality and when for homotopy.

If we combine (7.B) with (2.C), we easily check that the following sequence, where $p_1 = (1, 0)$ and $p_2 = (0, 1)$, is a central extension as well:

$$\mathrm{Ext}(A, B^{\otimes^2}) \overset{jl_*}{\hookrightarrow} [\sigma_A, \sigma_B \vee \sigma_B] \xrightarrow{(p_{1*}, p_{2*})} [\sigma_A, \sigma_B] \times [\sigma_A, \sigma_B]\ . \tag{7.D}$$

Here if $\beta \in \mathrm{Ext}(A, B^{\otimes^2})$ is represented by $(\hat{p}_B \otimes \hat{p}_B)\tilde{\beta}$ for some $\tilde{\beta}\colon \mathbb{Z}\langle A_2\rangle \to \mathbb{Z}\langle B_1\rangle^{\otimes^2}$, then $jl_*\beta$ is the morphism with $(jl_*\beta)_n = 0$ for $n \neq 3$ and

$$(jl_*\beta)_3 r = \omega(i_1 \otimes i_2 + i_2 \otimes i_1 T)\tilde{\beta}p\colon \langle A_2\rangle^{nil} \to (\sigma_B \vee \sigma_B)_3. \tag{7.E}$$

Here we use (6.D) and the well-known fact that the Whitehead product $[i_{1*}, i_{2*}]\colon \pi_2 M \otimes \pi_2 M \to \pi_3(M \vee M)$ coincides with l in (2.D) for a Moore space $M = M(B, 2)$.

From (7.D) it is easy to observe that there is an effective and transitive action of $\mathrm{Ext}(A, A^{\otimes^2})$ in the set of homotopy classes of co-H-multiplications for $M(A, 2)$; see [2]. Recall that this set is formed by the elements $\mu \in [\sigma_A, \sigma_A \vee \sigma_A]$ such that $(1, 0)\mu = (0, 1)\mu = 1$. The homotopy class of μ_A is such a co-H-multiplication and any other co-H-multiplication verifies $\mu = \mu_A + jl_*\beta$ for a unique $\beta \in \mathrm{Ext}(A, A^{\otimes^2})$, in particular, there is a unique $\hat{c}_A \in \mathrm{Ext}(A, A^{\otimes^2})$ such that $\mu_A + jl_*\hat{c}_A = (i_2, i_1)\mu_A$. By using the canonical morphism q' in (2.B) we get an element

$$c_A = q'_*\hat{c}_A \in \mathrm{Ext}(A, S^2 A)$$

which satisfies the following obstruction property.

Proposition 7.2. *The element $c_A \in \mathrm{Ext}(A, S^2 A)$ is zero if and only if $M(A, 2)$ admits a commutative co-H-multiplication. Moreover, it is natural in the category of abelian groups, in the sense of* (7.A)*.*

Recall that a co-H-multiplication μ is *commutative* if μ is homotopic to $(i_2, i_1)\mu$.

Proof of Proposition 7.2. As we have seen, $M(A, 2)$ admits a commutative co-H-multiplication if and only if there exists $\beta \in \mathrm{Ext}(A, A^{\otimes^2})$ such that

$$\begin{aligned} \mu_A + jl_*\beta &= (i_2, i_1)(\mu_A + jl_*\beta) \\ &= (i_2, i_1)\mu_A + j(\Gamma(i_2, i_1))_* l_*\beta && \text{(a)} \\ &= (i_2, i_1)\mu_A + jl_* T_*\beta. && \text{(b)} \end{aligned}$$

In (a) we use (7.C) and in (b) (2.E). Therefore $\hat{c}_A = (1 - T)_*\beta$ and, by the construction of S^2 in (2.B), we get $c_A = q'_*\hat{c}_A = q'_*(1 - T)_*\beta = 0$.

On the other hand, the functor $\mathrm{Ext}(A, -)$ is known to be right-exact, so that $0 = c_A = q'_*\hat{c}_A$ if and only if there exists $\beta \in \mathrm{Ext}(A, A^{\otimes^2})$ such that $\hat{c}_A = (1-T)_*\beta$; see (2.B). In this case one can check as above that $\mu_A + jl_*\beta$ is a commutative co-H-multiplication for $M(A, 2)$.

Now we prove the naturality of c_A. Given a homomorphism $f\colon A \to B$ we can choose a morphism $F\colon \sigma_A \to \sigma_B$ such that $H_2F = f$, then

$$\begin{aligned}
(i_2, i_1)(F \vee F)\mu_A &= (F \vee F)(i_2, i_1)\mu_A \\
&= (F \vee F)(\mu_A + jl_*\hat{c}_A) \\
\text{(c)} \qquad &= (F \vee F)\mu_A + j(\Gamma(f \oplus f))_* l_* \hat{c}_A \\
\text{(d)} \qquad &= (F \vee F)\mu_A + jl_*(f \otimes f)_*\hat{c}_A, \\
(i_2, i_1)\mu_B F &= (\mu_B + jl_*\hat{c}_B)F \\
\text{(e)} \qquad &= \mu_B F + jl_* f^* \hat{c}_B
\end{aligned}$$

In (c) and (e) we apply (7.C), and in (d) (2.E). Hence

(f) $\quad (i_2, i_1)((F \vee F)\mu_A - \mu_B F) = (F \vee F)\mu_A - \mu_B F + jl_*(f \otimes f)_*\hat{c}_A - jl_* f^*\hat{c}_B.$

By (7.D) there exists a unique $\zeta \in \mathrm{Ext}(A, B^{\otimes^2})$ such that $\mu_B F + jl_*\zeta = (F \vee F)\mu_A$, therefore by (7.C) and (2.E) $(i_2, i_1)\mu_B F + jl_* T_*\zeta = (i_2, i_1)(F \vee F)\mu_A$, and hence (f) is equivalent to the following equality in $\mathrm{Ext}(A, B^{\otimes^2})$:

$$T_*\zeta = \zeta + (f \otimes f)_*\hat{c}_A - f^*\hat{c}_B.$$

Thus

$$f^* c_B - (S^2 f)_* c_A = q'_*(f^*\hat{c}_B - (f \otimes f)_*\hat{c}_A) = q'_*(1 - T)_*\zeta = 0.$$

Here we use the naturality and exactness of (2.B). The proof is now finished. □

Corollary 7.3. *The extension element $c_A \in \mathrm{Ext}(A, S^2A)$ does not depend on the choice of the homomorphism $\bar{\varphi}_A$ for the construction of σ_A and μ_A.*

Proof. Take $\bar{\varphi}'_A$ in the same conditions as $\bar{\varphi}_A$. Let σ'_A be the Moore space quadratic complex of type $(A, 2)$ constructed by using $\bar{\varphi}'_A$ instead of $\bar{\varphi}_A$, and $c'_A \in \mathrm{Ext}(A, S^2A)$ the extension element related to $\bar{\varphi}'_A$, constructed in the same way as c_A. Choose $B = A$, $f\colon A \to B$ the identity homomorphism $f = 1_A$, and $F\colon \sigma_A \to \sigma'_A$ a quadratic complex morphism inducing 1_A in H_2. Then the same technique used in Proposition 7.2 to prove the naturality of c_A shows that $c_A = c'_A \in \mathrm{Ext}(A, S^2A)$. □

Corollary 7.4. *For any abelian group $2c_A = 0$. Therefore if $\mathrm{Ext}(A, S^2A)$ has no element of order 2, then $M(A, 2)$ admits a commutative co-H-multiplication.*

Proof. In general $S^2(n1_A) = n^2 1_{S^2A}$, then by the naturality property of c_A in (7.A),

$$0 = (S^2(1_A + 1_A))_* c_A - (1_A + 1_A)^* c_A = 4c_A - 2c_A = 2c_A.$$

□

In the next theorem we compute a formula for c_A from a free resolution of A.

Theorem 7.5. *If $\mathbb{Z}\langle A_2\rangle \overset{\psi_A}{\hookrightarrow} \mathbb{Z}\langle A_1\rangle \overset{\hat{p}_A}{\twoheadrightarrow} A$ is a free resolution of A such that $\psi_A(b) = \sum_{i=1}^n \epsilon_i a_i$ for $b \in A_2$, $a_i \in A_1$ and $\epsilon_i \in \mathbb{Z}$, then the extension element $c_A \in \mathrm{Ext}(A, S^2A)$ is represented by the homomorphism $\check{c}_A\colon \mathbb{Z}\langle A_2\rangle \to S^2A$ defined by*

$$\check{c}_A(b) = \sum_{i=1}^n \binom{|\epsilon_i|}{2} \hat{p}_A(a_i)^2.$$

Proof. We can take the basis $-\varphi_A = -\varphi \in \mathbf{nil}$ of the reduced quadratic complex σ_A of the Moore space $M(A,2)$ to be defined by $\varphi(b) = \epsilon_1 a_1 + \cdots + \epsilon_n a_n \in \langle A_1\rangle^{nil}$, hence $\varphi^{ab} = \psi_A$. The following equalities hold:

$$
\begin{array}{llcl}
\text{(a)} & (\mu_\sigma - (i_2,i_1)\mu_\sigma)_2 & = & i_1 + i_2 - i_1 - i_2 \\
 & & = & [i_1, i_2] \\
\text{(b)} & & = & s\imath\Delta p \\
\text{(c)} & & = & sq(i_1\otimes i_2)\Delta p \\
\text{(d)} & & = & d_3\omega(i_1\otimes i_2)\Delta p.
\end{array}
$$

Here we use Theorem 5.6 in (a), Lemma 5.10 in (b), the definition of $\imath$ (2.A) in (c), and (3.C) in (d). Hence the homomorphism $(i_1\otimes i_2)\Delta\colon \mathbb{Z}\langle A_1\rangle \to (\mathbb{Z}\langle A_1\rangle\oplus\mathbb{Z}\langle A_1\rangle)^{\otimes^2}$ determines a 0-homotopy between $\mu_\sigma - (i_2,i_1)\mu_\sigma$ and $f\colon \Sigma\sigma\to\Sigma\sigma\vee\Sigma\sigma$ with $f_n = 0$ for $n\neq 3$, $f_3\omega = 0$ and

$$
\text{(e)} \qquad f_3 r = (\mu_\sigma - (i_2,i_1)\mu_\sigma)_3 r - \omega(i_1\otimes i_2)\Delta p(-\varphi).
$$

Moreover,

$$
\begin{array}{llcl}
\text{(f)} & r[i_1,i_2] & = & rsq(i_1\otimes i_2)\Delta p \\
\text{(g)} & & = & \omega(\varphi^{ab}\oplus\varphi^{ab})^{\otimes^2}(i_1\otimes i_2)\Delta p \\
 & & = & \omega(i_1\otimes i_2)(\varphi^{ab}\otimes\varphi^{ab})\Delta p, \\
\text{(h)} & ((i_2,i_1)\mu_\sigma)_3 r & = & r(i_2+i_1) + \omega(i_1\otimes i_2)(\nabla\varphi + \Delta\varphi^{ab})p.
\end{array}
$$

Here we use (b) and (c) in (f), (3.C) in (g), and Theorem 5.6 in (h). Hence by Theorem 5.6 and Corollary 5.7

$$
\begin{array}{rcl}
(-(i_2,i_1)\mu_\sigma)_3 r & = & -r(i_2+i_1) - \omega(i_1\otimes i_2)(\nabla\varphi+\Delta\varphi^{ab})p \\
 & & +\omega(i_1+i_2)^{\otimes^2}(\nabla\varphi+\Delta\varphi^{ab})p,
\end{array}
$$

and

$$
\begin{array}{lcl}
\text{(i)} \quad \text{(e)} & = & r(i_1+i_2) + \omega(i_2\otimes i_1)(\nabla\varphi+\Delta\varphi^{ab})p \\
 & & -r(i_2+i_1) - \omega(i_1\otimes i_2)(\nabla\varphi+\Delta\varphi^{ab})p \\
 & & +\omega(i_1+i_2)^{\otimes^2}(\nabla\varphi+\Delta\varphi^{ab})p \\
 & & +\omega((i_1+i_2)\otimes(-i_1-i_2))(\nabla\varphi+\Delta\varphi^{ab})p - \omega(i_1\otimes i_2)\Delta(-\varphi^{ab})p \\
\text{(j)} & = & r[i_1,i_2] - \omega(i_1\otimes i_2)(\nabla\varphi+\Delta\varphi^{ab}+\Delta(-\varphi^{ab}))p \\
 & & +\omega(i_2\otimes i_1)(\nabla\varphi+\Delta\varphi^{ab})p \\
\text{(k)} & = & \omega(i_1\otimes i_2)((\varphi^{ab}\otimes\varphi^{ab})\Delta - \nabla\varphi) + \omega(i_2\otimes i_1)(\nabla\varphi+\Delta\varphi^{ab}).
\end{array}
$$

In (j) we use that ω is central, and in (k) we apply (f).

By using Lemma 5.9, one can prove the equality $T\nabla(-\varphi) = \nabla\varphi + \Delta\varphi^{ab}$ by an easy computation, therefore by Lemma 5.10

$$
T(\nabla\varphi + \Delta\varphi^{ab}) = (\varphi^{ab}\otimes\varphi^{ab})\Delta - \nabla\varphi,
$$

and hence by (i)

$$
f_3 r = \omega(i_1\otimes i_2 + i_2\otimes i_1 T)T(\nabla\varphi+\Delta\varphi^{ab})p.
$$

So, by (7.E), $\hat{c}_A \in \operatorname{Ext}(A, A^{\otimes^2})$ is represented by $-(\hat{p}_A\otimes\hat{p}_A)T(\nabla\varphi+\Delta\varphi^{ab})$, and then $c_A\in\operatorname{Ext}(A, S^2A)$ is represented by

$$
-q'(\hat{p}_A\otimes\hat{p}_A)T(\nabla\varphi+\Delta\varphi^{ab}) = -(S^2\hat{p}_A)q'T(\nabla\varphi+\Delta\varphi^{ab}) = -(S^2\hat{p}_A)(\nabla\varphi+\Delta\varphi^{ab}).
$$

Here we use the naturality of q' and the fact that $q'T = q'$; see (2.B).

From these computations we see that we can take $\check{c}_A = -(S^2\hat{p}_A)(\nabla\varphi + \Delta\varphi^{ab})$. Let us check now that $\check{c}_A$ satisfies the formula of the statement. The first equality in the next formula is a consequence of Lemma 5.9, in the third we apply the fact that $\hat{p}_A\psi_A = 0$

$$\begin{aligned}
-\check{c}_A(b) &= \sum_{1\leq i<j\leq n} \epsilon_i\epsilon_j \hat{p}_A(a_i)\hat{p}_A(a_j) + \sum_{\epsilon_i>0} \frac{\epsilon_i(\epsilon_i+1)}{2}\hat{p}_A(a_i)^2 \\
&\quad + \sum_{\epsilon_i<0} \frac{\epsilon_i(\epsilon_i-1)}{2}\hat{p}_A(a_i)^2 \\
&= \sum_{0<\epsilon_i \text{ even}} \left[(\epsilon_i\hat{p}_A(a_i))\Big(\frac{\epsilon_i}{2}\hat{p}_A(a_i) + \sum_{j\neq i}\epsilon_j\hat{p}_A(a_j)\Big) + \frac{\epsilon_i}{2}\hat{p}_A(a_i)^2\right] \\
&\quad + \sum_{0<\epsilon_i \text{ odd}} (\epsilon_i\hat{p}_A(a_i))\Big(\frac{\epsilon_i+1}{2}\hat{p}_A(a_i) + \sum_{j\neq i}\epsilon_j\hat{p}_A(a_j)\Big) \\
&\quad + \sum_{0>\epsilon_i \text{ even}} \left[(\epsilon_i\hat{p}_A(a_i))\Big(\frac{\epsilon_i}{2}\hat{p}_A(a_i) + \sum_{j\neq i}\epsilon_j\hat{p}_A(a_j)\Big) - \frac{\epsilon_i}{2}\hat{p}_A(a_i)^2\right] \\
&\quad + \sum_{0>\epsilon_i \text{ odd}} (\epsilon_i\hat{p}_A(a_i))\Big(-\frac{\epsilon_i+1}{2}\hat{p}_A(a_i) + \sum_{j\neq i}\epsilon_j\hat{p}_A(a_j)\Big) \\
&= \sum_{0<\epsilon_i \text{ even}} \left[(\epsilon_i\hat{p}_A(a_i))\left(-\frac{\epsilon_i}{2}\hat{p}_A(a_i)\right) + \frac{\epsilon_i}{2}\hat{p}_A(a_i)^2\right] \\
&\quad + \sum_{0<\epsilon_i \text{ odd}} (\epsilon_i\hat{p}_A(a_i))\left(-\frac{\epsilon_i-1}{2}\hat{p}_A(a_i)\right) \\
&\quad + \sum_{0>\epsilon_i \text{ even}} \left[(\epsilon_i\hat{p}_A(a_i))\left(-\frac{\epsilon_i}{2}\hat{p}_A(a_i)\right) - \frac{\epsilon_i}{2}\hat{p}_A(a_i)^2\right] \\
&\quad + \sum_{0>\epsilon_i \text{ odd}} (\epsilon_i\hat{p}_A(a_i))\left(-\frac{\epsilon_i+1}{2}\hat{p}_A(a_i)\right) \\
&= \sum_{\epsilon_i>0} -\frac{\epsilon_i^2-\epsilon_i}{2}\hat{p}_A(a_i)^2 + \sum_{\epsilon_i<0} -\frac{\epsilon_i^2+\epsilon_i}{2}\hat{p}_A(a_i)^2 \\
&= -\sum_{i=1}^{n}\binom{|\epsilon_i|}{2}\hat{p}_A(a_i)^2.
\end{aligned}$$

□

Corollary 7.6. *If A is a direct sum of cyclic groups, then $M(A,2)$ admits a commutative co-H-multiplication if and only if A has no elements of order* 2.

Proof. Suppose that $A = \bigoplus_{i\in I}\mathbb{Z}/n_i$ where $n_i \in \mathbb{Z}$ is either 0 or a prime power. We take the following free resolution of A:

$$\mathbb{Z}\langle\{a_{2,i};\ \ i\in I, n_i\neq 0\}\rangle \overset{\psi_A}{\hookrightarrow} \mathbb{Z}\langle\{a_{1,i};\ \ i\in I\}\rangle \overset{\hat{p}_A}{\twoheadrightarrow} A,$$

here $\psi_A(a_{2,i}) = n_i a_{1,i}$ and $\hat{p}_A(a_{1,i})$ is the generator of the direct summand $\mathbb{Z}/n_i \subset A$. If we choose a total ordering $\preceq$ in I the symmetric square of A is $S^2A =$

$\bigoplus_{i \preceq j} \mathbb{Z}/(n_i, n_j)$, where (n_i, n_j) is the greatest common divisor and the direct summand $\mathbb{Z}/(n_i, n_j)$ is generated by $\hat{p}_A(a_{1,i})\hat{p}_A(a_{1,j})$. The element $c_A \in \operatorname{Ext}(A, S^2A)$ is represented by the homomorphism $\check{c}_A$ with $\check{c}_A(a_{2,i}) = \binom{n_i}{2}\hat{p}_A(a_{1,i})^2$. Therefore, if n_i is odd, then

$$\check{c}_A(a_{2,i}) = (n_i \hat{p}_A(a_{1,i})) \left(\frac{n_i - 1}{2} \hat{p}_A(a_{1,i}) \right) = 0;$$

otherwise $n_i = 2^m$ and $\check{c}_A(a_{2,i}) = 2^{m-1}\hat{p}_A(a_{ii})^2$ is the unique element of order 2 in $\mathbb{Z}/2^m = \mathbb{Z}/(n_i, n_i) \subset S^2A$. If A has no elements of order 2, then n_i is always odd or zero, and $\check{c}_A = 0$ is the trivial homomorphism, hence $c_A = 0$. On the other hand, if A has order 2 elements, then $n_i = 2^m$ for some $i \in I$, and the usual computation of $\operatorname{Ext}(\mathbb{Z}_{2^m}, \mathbb{Z}_{2^m}) \simeq \mathbb{Z}_{2^m}$ shows that $c_A \neq 0$. □

Remark 7.7. Corollary 7.6 is no longer true if A is not a direct sum of cyclic groups. Consider $A = \mathbb{Q}/\mathbb{Z}$. It is well known that $(\mathbb{Q}/\mathbb{Z})^{\otimes^2} = 0$, hence $S^2(\mathbb{Q}/\mathbb{Z}) = 0$ and $\operatorname{Ext}(A, S^2A) = 0$, in particular, $c_A = 0$. But the class of $1/2$ in $\mathbb{Q}/\mathbb{Z}$ is a non-trivial element of order 2.

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