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DEGENERATION OF LINEAR SYSTEMS THROUGH FAT POINTS ON $K3$ SURFACES

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Abstract. In this paper we introduce a technique to degenerate $K3$ surfaces and linear systems through fat points in general position on $K3$ surfaces. Using this degeneration we show that on generic $K3$ surfaces it is enough to prove that linear systems with one fat point are non-special in order to obtain the non-speciality of homogeneous linear systems through $n = 4^u 9^w$ fat points in general position. Moreover, we use this degeneration to obtain a result for homogeneous linear systems through $n = 4^u 9^w$ fat points in general position on a general quartic surface in $\mathbb{P}^3$.

1. Introduction

In this paper we assume the ground field is the field of the complex numbers.

Let S be a smooth projective generic $K3$ surface (i.e. $\mathrm{Pic}\,(S) \cong \mathbb{Z}$) and let H be the generator of $\mathrm{Pic}\,(S)$.

Consider n points in general position on S, to each one of them associate a natural number m_i called the *multiplicity* of the point and let n_j be the number of points with multiplicity m_j.

For a linear system of curves in $|dH|$ with n_j general base points of multiplicity m_j for $j = 1, \ldots, k$, define its *virtual dimension* v as $\dim |dH| - \sum n_i m_i(m_i + 1)/2$ and its *expected dimension* by $e = \max\{v, -1\}$. If the dimension of the linear system is l, then $v \leq e \leq l$.

Observe that it is possible to have $e < l$, since the conditions imposed by the points may be dependent. In this case we say that the system is *special*.

Linear systems through general fat points on rational surfaces have been studied by many authors (see e.g. [AH00, BZ03, CM98, CM01, Eva99]), but, as far as we know, on $K3$ surfaces, no results on the non-speciality of such systems are known.

In Section 3 we develop a technique to degenerate a $K3$ surface and linear systems through fat points on $K3$ surfaces, this degeneration is based on a degeneration of the projective plane which was developed by C. Ciliberto and R. Miranda and used by A. Buckley and M. Zompatori in [BZ03].

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In the subsequent section, we use this degeneration to prove that, a homogeneous linear system of curves in $|dH|$ with n general base points of multiplicity m is non-special if all linear systems of curves in $|dH|$ with one multiple base point are non-special.

Finally, in Section 5, we prove conjecture [DL03, Conjecture 2.3 (i)] for homogeneous linear systems of curves in $|dH|$ with 4^u9^w fat points of multiplicity m, if either $v \geq 0$ or $v \leq -1$ and $u > 0$ or $2d \neq 1 \mod 3$.

2. Preliminaries

Let S be a smooth projective generic $K3$ surface and let H be the generator of $\mathrm{Pic}\,(S)$. Then H is ample, $H^2 = 2g - 2 \geq 2$ and $h^0(H) = g + 1$; moreover, H is very ample if $g \geq 3$ and if $g = 2$, H defines a double covering of $\mathbb{P}^2$ branched at an irreducible sextic (see [May72, Proposition 3]).

Consider $Q_1, \cdots, Q_n$ points in general position on S, for each one of these points fix a multiplicity $m_1, \ldots, m_n$. Define $\mathcal{L} = \mathcal{L}^\gamma(d, m_1, \ldots, m_n)$, with $\gamma = H^2 = 2g-2$, as the linear system of curves in $|dH|$ with multiplicity at least m_i at Q_i for all i. By abuse of notation, let $\mathcal{L}^\gamma(d, m_1, \ldots, m_n)$ also denote the associated sheaf on S. Let v denote the virtual dimension of $\mathcal{L}$. Then, using $\dim |dH| = \frac{\gamma d^2}{2} + 1$, we obtain that

$$v = \frac{\gamma d^2}{2} + 1 - \sum_{i=1}^{n} \frac{m_i(m_i + 1)}{2}. \tag{2.a}$$

Let S' be the blowing-up of S along the points $Q_1, \ldots, Q_n$, let $\pi : S' \to S$ be the projection map, and let E_i be the exceptional divisor corresponding to Q_i. The linear system $\mathcal{L}$ then corresponds to the system associated to the line bundle $\pi^*(\mathcal{O}_S(dH)) \otimes \mathcal{O}_{S'}(-m_1E_1 - \cdots - m_nE_n)$. By abuse of notation, we will denote this linear system and its associated line bundle on the blowing-up also by $\mathcal{L}^\gamma(d, m_1, \ldots, m_n)$.

Similary, by $\mathcal{L}^\gamma(d, m_1^{n_1}, \ldots, m_y^{n_y})$, we denote the linear system of curves in $|dH|$ with n_i points (in general position) of multiplicity at least m_i (for all $i = 1, \ldots, y$) as well as the associated sheaf on S and the corresponding linear system and its associated line bundle on the blowing-up of S along those $n_1 + \cdots + n_y$ general points.

Let $Z = \sum_{i=1}^n m_iQ_i$ be the 0-dimensional scheme of S defined by the multiple points and consider the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{O}_S(dH) \otimes \mathcal{I}_Z \longrightarrow \mathcal{O}_S(dH) \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z. Taking cohomology we obtain

$$v = h^0(\mathcal{O}_S(dH) \otimes \mathcal{I}(Z)) - h^1(\mathcal{O}_S(dH) \otimes \mathcal{I}(Z)) - 1, \tag{2.b}$$

because $h^1(\mathcal{O}_S(dH)) = 0$ (see e.g. [May72]).

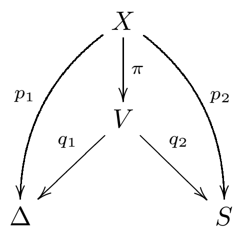
Let e denote the expected dimension of the linear system $\mathcal{L}$, i.e. $e = \max\{-1, v\}$, and let l denote the dimension of $\mathcal{L}$. Then obviously we have that $v \leq e \leq l$.

If $e < l$ (resp. $e = l$), we say the linear system $\mathcal{L}$ is special (resp. non-special). Note that equation (2.b) shows that a non-empty linear system $\mathcal{L}$ is non-special if and only if $h^1(\mathcal{L}) = 0$.

Analogously, on $\mathbb{P}^2$, consider n general points $Q_1, \ldots, Q_n$. Let $\mathcal{L}(d, m_1, \ldots, m_n)$ denote the linear system of plane curves of degree d, having multiplicity at least m_i at Q_i for all $i = 1, \ldots, n$. And let $\mathcal{L}(d, m_1^{n_1}, \ldots, m_y^{n_y})$ denote the linear system of plane curves of degree d having n_i points (in general position) of multiplicity at least m_i (for all $i = 1, \dots, y$). By abuse of notation, we also denote the associated sheafs on $\mathbb{P}^2$ and the corresponding linear systems and their associated line bundles on the blowings-up of $\mathbb{P}^2$ by $\mathcal{L}(d, m_1, \dots, m_n)$ and $\mathcal{L}(d, m_1^{n_1}, \dots, m_y^{n_y})$.

3. Degeneration of linear systems on $K3$ surfaces

3.A. **The degeneration of a $K3$ surface.** Let S be a $K3$ surface and Δ a complex disc around the origin. Consider the product $V = S \times \Delta$ and its two projections $q_1 : V \to \Delta$ and $q_2 : V \to S$. Let V_t denote $S \times \{t\}$. Consider b general points in V_0 and blow up V along those b points. We then get a new threefold X and the maps $\pi : X \to V$, $p_1 = q_1 \circ \pi : X \to \Delta$ and $p_2 = q_2 \circ \pi : X \to S$; i.e. we obtain the following commutative diagram:



Let X_t be the fiber of p_1 over $t \in \Delta$. If $t \neq 0$, then $X_t \cong V_t$ is our $K3$ surface S. But, X_0 is the union of the proper transform $\widetilde{S}$ of V_0 and the b exceptional divisors $\mathbf{P}_i$. Obviously each $\mathbf{P}_i$ is isomorphic to $\mathbb{P}^2$ and $\widetilde{S}$ is the blowing-up of our $K3$ surface S at the b general points with projection map $\mathfrak{b} : \widetilde{S} \to S$.

Every $\mathbf{P}_i$ intersects $\widetilde{S}$ transversally along a curve R_i, which is a line in $\mathbf{P}_i$ and an exceptional divisor on $\widetilde{S}$. When we want to indicate that we consider R_i in $\widetilde{S}$, resp. $\mathbf{P}_i$, we denote it by E_i, resp. L_i.

Note that the map p_1 gives a flat family of surfaces over Δ, so X_0 can be seen as a degeneration of S.

3.B. **The degeneration of a linear system on a $K3$ surface S.** Let $\mathcal{L}$ be a line bundle on S, and, for $k \in \mathbb{Z}$, define the line bundle $\mathcal{O}_X(\mathcal{L}, k)$ on X by

$$\mathcal{O}_X(\mathcal{L}, k) = p_2^*(\mathcal{L}) \otimes \mathcal{O}_X(k\widetilde{S}).$$

The restriction of $\mathcal{O}_X(\mathcal{L}, k)$ to X_t, for $t \neq 0$, is then isomorphic to $\mathcal{L}$.

The restriction of $\mathcal{O}_X(\mathcal{L}, k)$ to X_0, which we denote by $\mathcal{X}(\mathcal{L}, k)$, is a flat limit of the line bundle $\mathcal{L}$ on the general fiber X_t, so $\mathcal{X}(\mathcal{L}, k)$ can be seen as a degeneration of the line bundle $\mathcal{L}$.

On any $\mathbf{P}_i$, we have that $\mathcal{X}(\mathcal{L}, k)|_{\mathbf{P}_i} = \mathcal{O}_X(\mathcal{L}, k)|_{\mathbf{P}_i} \cong \mathcal{O}_{\mathbb{P}^2}(k)$, since $\widetilde{S}$ intersects $\mathbf{P}_i$ along a line.

On $\widetilde{S}$ we obtain that

$$\mathcal{X}(\mathcal{L}, k)|_{\widetilde{S}} = \mathcal{O}_X(\mathcal{L}, k)|_{\widetilde{S}} = p_2^*(\mathcal{L})|_{\widetilde{S}} \otimes \mathcal{O}_X(k\widetilde{S})|_{\widetilde{S}}.$$

But, since $\widetilde{S} \sim X_t - \sum_{i=1}^{l} \mathbf{P}_i$ as divisors on X and $\mathbf{P}_i$ intersects $\widetilde{S}$ along R_i, this means that

$$\mathcal{X}(\mathcal{L}, k)|_{\widetilde{S}} \cong \mathfrak{b}^*(\mathcal{L}) \otimes \mathcal{O}_{\widetilde{S}}(-\sum_{i=1}^{b} kE_i).$$

Let $Q_1, \dots, Q_n$ be general points on S, consider a zero-dimensional subscheme $Z = m_1Q_1 + \cdots + m_nQ_n$ and let $\mathcal{M}$ be the sheaf $\mathcal{L} \otimes \mathcal{I}_Z$, where $\mathcal{I}_Z$ denotes the ideal sheaf that defines Z. Choose positive integers $a_1, \dots, a_b$ such that $a_1 + \cdots + a_b \leq n$. Now, for all $i \in \{1, \dots, b\}$, consider a_i general points on $\mathbf{P}_i$; and take $n - \sum_{i=1}^{b} a_i$ general points on $\widetilde{S}$. Denote those n points (on the $\mathbf{P}_i$ and $\widetilde{S}$) by Q'_i (any order will do), and let Z' be the zero-dimensional subscheme of X_0 given by $m_1Q'_1 + \cdots + m_nQ'_n$.

Then we obtain that $\mathcal{X}(\mathcal{L}, k) \otimes \mathcal{I}_{Z'}$ is a degeneration of $\mathcal{M}$.

3.C. **Homogeneous linear systems on generic $K3$ surfaces.** Let S be a generic $K3$ surface and consider a homogeneous linear system $\mathcal{L} = \mathcal{L}^{\gamma}(d, m^n)$ on S. By abuse of notation, $\mathcal{L}$ also denotes the corresponding sheaf.

Choose positive integers b and a such that $ab \leq n$, let X and $\mathcal{X}(\mathcal{O}_S(dH), k)$ be as constructed before. For all $1 \leq i \leq b$ and $1 \leq j \leq a$, let $Q'_{i,j}$ be a general point on $\mathbf{P}_i$ and for $1 \leq j \leq n - ab$ let Q'_i be a general point on $\widetilde{S}$. Now consider the zero-dimensional subscheme

$$Z' = \sum_{i=1}^{n-ab} mQ'_i + \sum_{\substack{i=1,\dots,b \\ j=1,\dots,a}} mQ'_{i,j},$$

On X_0, resp. $\mathbf{P}_i$ and $\widetilde{S}$, define the sheaf $\mathcal{L}_0 := \mathcal{X}(\mathcal{O}_S(dH), k) \otimes \mathcal{I}_{Z'}$, resp. $\mathcal{L}_i := \mathcal{L}_0|_{\mathbf{P}_i}$ and $\mathcal{L}_{\widetilde{S}} := \mathcal{L}_0|_{\widetilde{S}}$. By abuse of notation, we also denote the corresponding linear systems on resp. X_0, $\mathbf{P}_i$ and $\widetilde{S}$, by resp. $\mathcal{L}_0$, $\mathcal{L}_i$ and $\mathcal{L}_{\widetilde{S}}$. Then we see that a divisor in the linear system $\mathcal{L}_0$ on X_0 consists of a divisor $D_{\widetilde{S}} \in \mathcal{L}_{\widetilde{S}}$ and divisors $D_i \in \mathcal{L}_i$ such that $D_{\widetilde{S}}|_{R_i} = D_i|_{R_i}$ for all i.

By $\mathcal{R}_{\widetilde{S}}$, resp. $\mathcal{R}_i$, we denote the linear system on $\bigcup_{i=1}^{b} E_i$, resp. L_i, induced by $\mathcal{L}_{\widetilde{S}}$, resp. $\mathcal{L}_i$.

Define the sheafs $\widehat{\mathcal{L}}_{\widetilde{S}} = \mathcal{L}_{\widetilde{S}} \otimes \mathcal{O}_{\widetilde{S}}(-\sum_{i=1}^{b} E_i)$ and $\widehat{\mathcal{L}}_i = \mathcal{L}_i \otimes \mathcal{O}_{\mathbf{P}_i}(-L_i)$, and, again by abuse of notation, let $\widehat{\mathcal{L}}_{\widetilde{S}}$, resp. $\widehat{\mathcal{L}}_i$, also denote the corresponding linear system on $\widetilde{S}$, resp. $\mathbf{P}_i$.

Notation 3.1.

$l = \dim \mathcal{L}$, $l_0 = \dim \mathcal{L}_0$,
$l_{\widetilde{S}} = \dim \mathcal{L}_{\widetilde{S}}$, $l_{\mathbf{P}} = \dim \mathcal{L}_i$,
$r_{\widetilde{S}} = \dim \mathcal{R}_{\widetilde{S}}$, $r_{\mathbf{P}} = \dim \mathcal{R}_i$,
$\hat{l}_{\widetilde{S}} = \dim \widehat{\mathcal{L}}_{\widetilde{S}}$, $\hat{l}_{\mathbf{P}} = \dim \widehat{\mathcal{L}}_i$.

Obviously we have the following:

$$l_{\widetilde{S}} = r_{\widetilde{S}} + \hat{l}_{\widetilde{S}} + 1, \tag{3.c}$$

$$l_{\mathbf{P}} = r_{\mathbf{P}} + \hat{l}_{\mathbf{P}} + 1. \tag{3.d}$$

Let $\mathcal{R}_{\bigcup L_i}$ denote the linear system on $\bigcup_{i=1}^{b} R_i$ which consists of $\mathcal{R}_j$ on R_j for all $j = 1, \dots, b$, and denote $r_{\bigcup L_i} = \dim \mathcal{R}_{\bigcup L_i}$. Then we have the following equality:

$$l_0 = \dim(\mathcal{R}_{\widetilde{S}} \cap \mathcal{R}_{\bigcup L_i}) + b(\hat{l}_{\mathbf{P}} + 1) + \hat{l}_{\widetilde{S}} + 1. \tag{3.e}$$

But, proceeding as in [BZ03, § 2], we obtain the transversality of $\mathcal{R}_{\widetilde{S}}$ and $\mathcal{R}_{\bigcup L_i}$; i.e.

$$\begin{aligned}\dim(\mathcal{R}_{\widetilde{S}} \cap \mathcal{R}_{\bigcup L_i}) &= \max\{-1\,,\, r_{\widetilde{S}} + r_{\bigcup L_i} - b\,k\}\\ &= \max\{-1\,,\, r_{\widetilde{S}} + b\,r_{\mathbf{P}} - b\,k\}.\end{aligned} \tag{3.f}$$

For the virtual dimensions of the systems we introduce the following.

Notation 3.2.

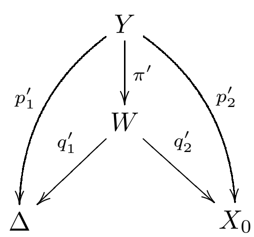
$v = \operatorname{vdim} \mathcal{L}$,
$v_{\mathbf{P}} = \operatorname{vdim} \mathcal{L}_i$, $v_{\widetilde{S}} = \operatorname{vdim} \mathcal{L}_{\widetilde{S}}$,
$\hat{v}_{\widetilde{S}} = \operatorname{vdim} \widehat{\mathcal{L}}_{\widetilde{S}}$, $\hat{v}_{\mathbf{P}} = \operatorname{vdim} \widehat{\mathcal{L}}_i$.

Since $\mathcal{L} = \mathcal{L}^{\gamma}(d, m^n)$, $\mathcal{L}_{\widetilde{S}} \cong \mathcal{L}^{\gamma}(d, k^b, m^{n-ab})$, $\widehat{\mathcal{L}}_{\widetilde{S}} \cong \mathcal{L}^{\gamma}(d, (k+1)^b, m^{n-ab})$, $\mathcal{L}_i \cong \mathcal{L}(k, m^a)$ and $\widehat{\mathcal{L}}_i \cong \mathcal{L}(k-1, m^a)$, a simple calculation shows that

$$\begin{aligned} v &= v_{\widetilde{S}} + b\,\hat{v}_{\mathbf{P}} + b \,= v_{\widetilde{S}} + b\,(v_{\mathbf{P}} - k)\\ &= \hat{v}_{\widetilde{S}} + b\,v_{\mathbf{P}} + b \,= \hat{v}_{\widetilde{S}} + b\,(\hat{v}_{\mathbf{P}} + k + 2).\end{aligned} \tag{3.g}$$

3.D. **Remark.** Let S be a $K3$ surface, $\mathcal{L}$ a line bundle on S and let X and $\mathcal{X}(\mathcal{L}, k)$ be as before.

Now do the construction of §3.A and §3.B using X_0 instead of S and $\mathcal{X}(\mathcal{L}, k)$ (or $\mathcal{X}(\mathcal{L}, k) \otimes \mathcal{I}_{Z'}$) instead of $\mathcal{L}$ (or $\mathcal{L} \otimes \mathcal{I}_Z$); i.e. consider $W = X_0 \times \Delta$, blow up W along b' general points on $\widetilde{S} \times \{0\} \subset W_0 = X_0 \times \{0\}$ and obtain the following commutative diagram:



So we obtain a degeneration of X_0, and a degeneration of $\mathcal{X}(\mathcal{L}, k)$ (or $\mathcal{X}(\mathcal{L}, k) \otimes \mathcal{I}_{Z'}$).

We call this a *double degeneration* of S and $\mathcal{L}$ (or $\mathcal{L} \otimes \mathcal{I}_Z$), and, continuing in the same way, we can obtain an η-uple degeneration of S and $\mathcal{L}$ (or $\mathcal{L} \otimes \mathcal{I}_Z$), for any $\eta \geq 2$.

4. Applying the degeneration to homogeneous linear systems on generic $K3$ surfaces

From now on we assume that we work on a generic $K3$ surface S, with $H^2 = \gamma$, where H is the generator of $\operatorname{Pic} S$.

The main result of this section is the following.

Theorem 4.1. *If $\mathcal{L}^{\gamma}(d, \mu)$ is non-special for all μ, then $\mathcal{L} = \mathcal{L}^{\gamma}(d, m^n)$ with $n = 4^u 9^w$ is non-special for all positive integers m, u and w.*

To make the proof of this theorem more transparent we first fix some notation and state a few auxiliary results.

Let $c \in \{4, 9\}$ such that $c|n$ and consider the degeneration $(X_0, \mathcal{L}_0)$ of $(S, \mathcal{L})$ obtained by using the construction explained in § 3 with $a = c$, $b = n/c$ and $k \in \mathbb{N}$. Note that, since $\mathcal{L}_0$ is a degeneration of $\mathcal{L}$, $l_0 \geq l \geq e \geq v$.

On $\mathbb{P}^2$, let $\mathcal{L}(\delta, \mu^\nu)$ denote the linear system of plane curves of degree δ having multiplicity μ at ν points (in general position).

The following is a well-known result, and can easily be checked using, for instance, the results of [Har85].

Lemma 4.2. *On $\mathbb{P}^2$ the linear system $\mathcal{L}(\delta, \mu^c)$ with $c \in \{4, 9\}$ is non-special for all δ and μ.* □

Remark 4.3. Lemma 4.2 implies, in particular, that $\mathcal{L}_i$ and $\widehat{\mathcal{L}}_i$ are non-special linear systems on $\mathbf{P}_i$.

Claim 4.4. *If $v \geq -1$, $v_{\widetilde{S}} \geq -1$, $v_{\mathbf{P}} \geq -1$ and $\mathcal{L}_{\widetilde{S}}$ and $\widehat{\mathcal{L}}_{\widetilde{S}}$ are non-special systems; then* $\dim(\mathcal{R}_{\widetilde{S}} \cap \mathcal{R}_{\bigcup L_i}) = r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k$.

Proof. Note that the conditions of the theorem immediately imply that $l_{\widetilde{S}} = v_{\widetilde{S}}$ and $l_{\mathbf{P}} = v_{\mathbf{P}}$.

Since $\dim(\mathcal{R}_{\widetilde{S}} \cap \mathcal{R}_{\bigcup L_i}) = \max\{-1, r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k\}$, it suffices to show that $r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k \geq -1$.

Because of (3.c) and (3.d), we have that

$$r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k = l_{\widetilde{S}} - \hat{l}_{\widetilde{S}} - 1 + b\,(l_{\mathbf{P}} - \hat{l}_{\mathbf{P}} - 1) - b\, k.$$

If $\hat{v}_{\widetilde{S}} \geq -1$ and $\hat{v}_{\mathbf{P}} \geq -1$, then $\hat{l}_{\widetilde{S}} = \hat{v}_{\widetilde{S}}$ and $\hat{l}_{\mathbf{P}} = \hat{v}_{\mathbf{P}}$. So

$$\begin{aligned} r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k &= v_{\widetilde{S}} - \hat{v}_{\widetilde{S}} - 1 + b\,(v_{\mathbf{P}} - \hat{v}_{\mathbf{P}} - 1) - b\, k \\ &= b\,(k+1) - 1 \geq -1. \end{aligned}$$

If $\hat{v}_{\widetilde{S}} \leq -2$ and $\hat{v}_{\mathbf{P}} \geq -1$, then $\hat{l}_{\widetilde{S}} = -1$ and $\hat{l}_{\mathbf{P}} = \hat{v}_{\mathbf{P}}$. So

$$\begin{aligned} r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k &= v_{\widetilde{S}} + b\,(v_{\mathbf{P}} - \hat{v}_{\mathbf{P}} - 1) - b\, k \\ &= v_{\widetilde{S}} \geq -1. \end{aligned}$$

If $\hat{v}_{\widetilde{S}} \geq -1$ and $\hat{v}_{\mathbf{P}} \leq -2$, then $\hat{l}_{\widetilde{S}} = \hat{v}_{\widetilde{S}}$ and $\hat{l}_{\mathbf{P}} = -1$. So

$$\begin{aligned} r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k &= v_{\widetilde{S}} - \hat{v}_{\widetilde{S}} - 1 + b\, v_{\mathbf{P}} - b\, k \\ &= b\,(1 + v_{\mathbf{P}}) - 1 \geq -1. \end{aligned}$$

If $\hat{v}_{\widetilde{S}} \leq -2$ and $\hat{v}_{\mathbf{P}} \leq -2$, then $\hat{l}_{\widetilde{S}} = -1$ and $\hat{l}_{\mathbf{P}} = -1$. So

$$\begin{aligned} r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k &= v_{\widetilde{S}} + b\,(v_{\mathbf{P}} - k) \\ &\overset{(3.g)}{=} v \geq -1. \end{aligned}$$

□

Lemma 4.5. *If $v \geq -1$, $v_{\widetilde{S}} \geq -1$, $v_{\mathbf{P}} \geq -1$ and $\mathcal{L}_{\widetilde{S}}$ and $\widehat{\mathcal{L}}_{\widetilde{S}}$ are non-special systems; then $\mathcal{L}$ is non-special.*

Proof. Because $\mathcal{L}_0$ is a degeneration of $\mathcal{L}$, we know that $v \leq l \leq l_0$; so it suffices to prove that $l_0 = v$.

Using (3.e), (3.f) and Claim 4.4 we obtain that

$$l_0 = r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k + b\,(\hat{l}_{\mathbf{P}} + 1) + \hat{l}_{\widetilde{S}} + 1.$$

So, using (3.c) and (3.d), we see that

$$l_0 = l_{\widetilde{S}} + b\,(l_{\mathbf{P}} - k) = v_{\widetilde{S}} + b\,(v_{\mathbf{P}} - k) \overset{(3.g)}{=} v.$$

□

Lemma 4.6. *Let $\mathcal{L} = \mathcal{L}^\gamma(d, m^n)$ with $n = 4^u 9^w$, $d, m, u, w \in \mathbb{N}$, $u + w > 0$ and $v \geq -1$. Take c, b, X_0 and $\mathcal{L}_0$ as before. Then $\exists\, k \in \mathbb{N}$ such that $v_{\widetilde{S}} \geq -1$ and $v_{\mathbf{P}} \geq -1$.*

Proof. Because $v_{\widetilde{S}} = \frac{\gamma}{2}d^2 + 1 - b\frac{k(k+1)}{2}$ and $v_{\mathbf{P}} = \frac{k(k+3)}{2} - c\frac{m(m+1)}{2}$, the inequalities $v_{\widetilde{S}} \geq -1$ and $v_{\mathbf{P}} \geq -1$ are equivalent to

$$k^2 + k - \alpha \leq 0, \quad \text{with } \alpha = \frac{1}{b}(\gamma d^2 + 4), \tag{4.h}$$

$$k^2 + 3k - \beta \geq 0, \quad \text{with } \beta = cm(m+1) - 2. \tag{4.i}$$

But this is the same as

$$k \in \left[\frac{-1 - \sqrt{1 + 4\alpha}}{2}, \frac{-1 + \sqrt{1 + 4\alpha}}{2}\right], \tag{4.j}$$

$$k \in \left]-\infty, \frac{-3 - \sqrt{9 + 4\beta}}{2}\right] \cup \left[\frac{-3 + \sqrt{9 + 4\beta}}{2}, \infty\right[. \tag{4.k}$$

So proving the statement is equivalent to proving that there exists a positive integer k such that both (4.j) and (4.k) are satisfied, i.e. it is enough to show that

$$\frac{-3 + \sqrt{9 + 4\beta}}{2} + 1 \leq \frac{-1 + \sqrt{1 + 4\alpha}}{2}.$$

A simple calculation shows that the previous inequality is equivalent to $\alpha \geq \beta + 2$, which is in turn equivalent to $v \geq -1$. $\square$

Lemma 4.7. *If $v \leq -1$, $\hat{v}_{\widetilde{S}} \leq -1$, $\hat{v}_{\mathbf{P}} \leq -1$ and $\mathcal{L}_{\widetilde{S}}$ and $\widehat{\mathcal{L}}_{\widetilde{S}}$ are non-special systems, then $\mathcal{L}$ is non-special and thus empty.*

Proof. Since $v_{\widetilde{S}} = \hat{v}_{\widetilde{S}} + b(k+1)$ and $v_{\mathbf{P}} = \hat{v}_{\mathbf{P}} + k + 1$, we see that $v_{\widetilde{S}} \leq b(k+1) - 1$ and $v_{\mathbf{P}} \leq k$.

If $v_{\widetilde{S}} \leq -1$ and $v_{\mathbf{P}} \leq -1$, then $r_{\widetilde{S}} = r_{\mathbf{P}} = -1$. If $v_{\widetilde{S}} \leq -1$ and $v_{\mathbf{P}} \geq -1$, then $r_{\widetilde{S}} = -1$ and $r_{\mathbf{P}} = v_{\mathbf{P}} \leq k$. If $v_{\widetilde{S}} \geq -1$ and $v_{\mathbf{P}} \leq -1$, then $r_{\widetilde{S}} = v_{\widetilde{S}} \leq b(k+1) - 1$ and $r_{\mathbf{P}} = -1$. If $v_{\widetilde{S}} \geq -1$ and $v_{\mathbf{P}} \leq -1$, then $r_{\widetilde{S}} = v_{\widetilde{S}}$ and $r_{\mathbf{P}} = v_{\mathbf{P}}$. So in any case we obtain that $r_{\widetilde{S}} + b\, r_{\mathbf{P}} - b\, k \leq -1$, i.e. $\dim(\mathcal{R}_{\widetilde{S}} \cap \mathcal{R}_{\bigcup L_i}) = -1$.

Using (3.e) and $\hat{l}_{\widetilde{S}} = \hat{l}_{\mathbf{P}} = -1$, we see that $l_0 = \hat{l}_{\widetilde{S}} + k(\hat{l}_{\mathbf{P}} + 1) = -1$. $\square$

Lemma 4.8. *Let $\mathcal{L} = \mathcal{L}^\gamma(d, m^n)$ with $n = 4^u 9^w$, $d, m, u, w \in \mathbb{N}$, $u + w > 0$ and $v \leq -1$. Take c, b, X_0 and $\mathcal{L}_0$ as before. Then $\exists\, k \in \mathbb{N}$ such that $\hat{v}_{\widetilde{S}} \leq -1$ and $\hat{v}_{\mathbf{P}} \leq -1$.*

Proof. Because $\hat{v}_{\widetilde{S}} = \frac{\gamma}{2}d^2 + 1 - b\frac{(k+1)(k+2)}{2}$ and $\hat{v}_{\mathbf{P}} = \frac{(k-1)(k+2)}{2} - c\frac{m(m+1)}{2}$, the inequalities $\hat{v}_{\widetilde{S}} \leq -1$ and $\hat{v}_{\mathbf{P}} \leq -1$ are equivalent to

$$k^2 + 3k - \alpha \geq 0, \quad \text{with } \alpha = \frac{1}{b}(\gamma d^2 + 4) - 2,$$
$$k^2 + k - \beta \leq 0, \quad \text{with } \beta = cm(m+1).$$

Proceeding as in the proof of Lemma 4.6, we obtain that it is sufficient to prove that

$$\frac{-1 + \sqrt{1 + 4\beta}}{2} \geq \frac{-3 + \sqrt{9 + 4\alpha}}{2} + 1,$$

which is equivalent to $\beta \geq \alpha + 2$. This last inequality is equivalent to $v \geq -1$. $\square$

Proof of Theorem 4.1. Using induction, the result follows immediately from Lemmas 4.5 and 4.6 if $v \geq -1$, and from Lemmas 4.7 and 4.8 if $v \leq -1$. $\square$

4.A. **Using a higher order degeneration.** Using the η-uple degeneration explained in Remark 3.D, we can, proceeding as above, prove that the non-speciality of $\mathcal{L}^\gamma(d,\mu_1,\dots,\mu_\eta)$ for all $\mu_1,\dots,\mu_\eta$ implies the non-speciality of

$$\mathcal{L}=\mathcal{L}^\gamma(d,m_1^{n_1},\dots,m_\eta^{n_\eta})$$

with $n_i=4^{u_i}9^{w_i}$ for all $m_1,\dots,m_\eta,u_1,\dots,u_\eta,w_1,\dots,w_\eta$.

As we will see in Proposition 5.3, proving the non-speciality of systems $\mathcal{L}^\gamma(d,\mu)$ is already rather complex. In fact, in Proposition 5.3 we only prove when such a system is non-special for general generic $K3$ surfaces with $\gamma=4$.

4.B. **Remark.** As proved in [CM01, Theorem 6.1], the Segre conjecture on planar linear systems through fat points implies that the only special homogeneous linear systems $\mathcal{L}(k,m^n)$ are the ones with $n\in\{2,3,5,6,7,8\}$. So if the Segre conjecture is true, we can do the degeneration using, not only $a=4$ or 9, but $a\in\{4\}\cup\mathbb{Z}_{\geq 9}$. In this way we can then prove Theorem 4.1, for any n which can be written as a product of powers of numbers in $\{4\}\cup\mathbb{Z}_{\geq 9}$, and thus, in particular, for any $n\geq 9$. Note that the idea of using the results on $\mathbb{P}^2$ to reduce the problem of linear systems through fat points to a problem involving a single fat point has previously come up in the closely related context of Seshadri constants (see [Roe03, Theorem 3 and Corollary 8])

5. The non-speciality of linear systems $\mathcal{L}^\gamma(d,m^n)$ with $n=4^u9^w$ on general generic $K3$ surfaces with $\gamma=4$

The main result of this section is the following.

Theorem 5.1. *Let S be a general generic $K3$ surface with $\gamma=4$ and let $\mathcal{L}=\mathcal{L}^4(d,m^n)$, with $n=4^u9^w$, $u,w\in\mathbb{Z}_{\geq 0}$ and $d\in\mathbb{Z}_{>0}$.*

(1) *If $v(\mathcal{L})\geq 0$, then the linear system $\mathcal{L}$ is non-special.*

(2) *If $v(\mathcal{L})\leq -1$ and either $u>0$ or $2d\neq 1 \mod 3$, then $\mathcal{L}$ is non-special unless $n=1$, $m=2d$ and $d\geq 2$. In the latter case* $\dim(\mathcal{L}^4(d,2d))=0>\operatorname{edim}(\mathcal{L}^4(d,2d))=-1$ *and $\mathcal{L}^4(d,2d)=dC$, with C the unique element of $\mathcal{L}^4(1,2)$.*

Remark 5.2. Note that the previous theorem implies that [DL03, Conjecture 2.3 (i)] is true for homogeneous linear systems with $n=4^u9^w$ fat points on S in the following cases:

(i) $v\geq 0$, or

(ii) $v\leq -1$ and $u>0$, or

(iii) $v\leq -1$ and $2d\neq 1 \mod 3$.

We will prove Theorem 5.1 by using the degeneration as introduced in §4 and the following.

Proposition 5.3. *Let S be a general generic $K3$ surface with $\gamma=4$. Then $\mathcal{L}=\mathcal{L}^4(d,\mu)$, with $d,\mu\in\mathbb{Z}_{>0}$, is non-special unless $\mu=2d$ and $d\geq 2$. In the latter case* $\dim(\mathcal{L}^4(d,2d))=0>\operatorname{vdim}(\mathcal{L}^4(d,2d))=-1$ *and $\mathcal{L}^4(d,2d)=dC$, with C the unique element of $\mathcal{L}^4(1,2)$.*

Proof. The map $\phi:S\to\mathbb{P}^3$ corresponding to $|H|$ (with H the generator of $\operatorname{Pic}S$) is an embedding, so we may look at S as a quartic surface in $\mathbb{P}^3$. The unique element C of $\mathcal{L}^4(1,2)$ is then the divisor on $S\subset\mathbb{P}^3$ defined by the tangent plane to S at this general point P. So C is an irreducible plane curve of degree 4 having a node at P (and no other singularities).

Blow up S along the point P, and recall that, by abuse of notation, $\mathcal{L}^4(d,\mu)$ also denotes the complete linear system and the line bundle on this blowing-up corresponding to $\mathcal{L}^4(d,\mu)$ on S. Let $\widetilde{C}$ denote the strict transform of C on this blowing-up; then $g(\widetilde{C})=2$.

Assume $\mu=2d$ ($d\geq 2$) and consider the following sequence:

$$0\longrightarrow\mathcal{L}^4(d-1,2d-2)\longrightarrow\mathcal{L}^4(d,2d)\longrightarrow\mathcal{L}^4(d,2d)\otimes\mathcal{O}_{\widetilde{C}}\longrightarrow 0.$$

Because $\deg(\mathcal{L}^4(d,2d)\otimes\mathcal{O}_{\widetilde{C}})=0=2g(\widetilde{C})-4$, we know that $h^0(\mathcal{L}^4(d,2d)\otimes\mathcal{O}_{\widetilde{C}})=0$ unless $\mathcal{L}^4(d,2d)\otimes\mathcal{O}_{\widetilde{C}}=\mathcal{O}_{\widetilde{C}}$. But $\mathcal{L}^4(d,2d)\otimes\mathcal{O}_{\widetilde{C}}=|dg^2_4-2d(P_1+P_2)|$, where P_1 and P_2 are the intersection points of $\widetilde{C}$ with the exceptional curve. So this would mean that $|dg^2_4-2d(P_1+P_2)|=\mathcal{O}_{\widetilde{C}}$. But, since P is a general point on the general quaritc S, we know that $g^2_4=|K_{\widetilde{C}}+P_1+P_2|$; so we would obtain that $|dK_{\widetilde{C}}-d(P_1+P_2)|=\mathcal{O}_{\widetilde{C}}$, or thus that $|dK_{\widetilde{C}}|=|d(P_1+P_2)|$, which is not true (because P is a general element of S). So $h^0(\mathcal{L}^4(d,2d))=h^0(\mathcal{L}^4(d-1,2(d-1)))$ for all $d\geq 2$. Using this a number of times we thus obtain that $h^0(\mathcal{L}^4(d,2d))=h^0(\mathcal{L}^4(1,2))=1$, so dC is the only divisor in $\mathcal{L}^4(d,2d)$.

It now follows immediately that $\dim\mathcal{L}=-1$ if $\mu\geq 2d+1$, since the only divisor in $\mathcal{L}^4(d,2d)$ has multiplicity exactly $2d$ in P.

Now consider the case $\mu\leq 2d-1$. If we can prove that $\mathcal{L}$ is non-special for $\mu=2d-1$, then the non-speciality follows for all $\mu\leq 2d-1$ (since $\operatorname{vdim}\mathcal{L}^4(d,2d-1)=d+1$). For $d=1$, there is nothing to prove, since P is a general point of S. So we may assume that $d\geq 2$ and that the $\mathcal{L}^4(d',2d'-1)$ is non-special for $d'\leq d-1$. Now consider the following sequence:

$$0\longrightarrow\mathcal{L}^4(d-1,2d-3)\longrightarrow\mathcal{L}^4(d,2d-1)\longrightarrow\mathcal{L}^4(d,2d-1)\otimes\mathcal{O}_{\widetilde{C}}\longrightarrow 0.$$

Because

$$\deg(\mathcal{L}^4(d,2d-1)\otimes\mathcal{O}_{\widetilde{C}})=2=2g(\widetilde{C})-2,$$

we know that $h^1(\mathcal{L}^4(d,2d-1)\otimes\mathcal{O}_{\widetilde{C}})=0$ unless $\mathcal{L}^4(d,2d-1)\otimes\mathcal{O}_{\widetilde{C}}=K_{\widetilde{C}}$ (the canonical class on $\widetilde{C}$). But $\mathcal{L}^4(d,2d)\otimes\mathcal{O}_{\widetilde{C}}=|dg^2_4-(2d-1)(P_1+P_2)|$ and $g^2_4=|K_{\widetilde{C}}+P_1+P_2|$ (where P_1 and P_2 are, as before, the intersection points of $\widetilde{C}$ with the exceptional curve). Then this would mean that $|(d-1)K_{\widetilde{C}}|=|(d-1)(P_1+P_2)|$, which is not true (because P is a general element of S). So $h^1(\mathcal{L}^4(d,2d-1)\otimes\mathcal{O}_{\widetilde{C}})=0$ and, by hypotheses, $h^1(\mathcal{L}^4(d-1,2d-3))=0$, thus also $h^1(\mathcal{L}^4(d,2d-1))=0$. □

Proof of Theorem 5.1. (1) In case $n=1$ we are done because of Proposition 5.3. So we assume $n>1$, or equivalently $u+w\geq 1$. Consider the degeneration $(X_0,\mathcal{L}_0)$ of $(S,\mathcal{L})$ obtained by using the construction explained in §3 with $a=n$, $b=1$ and $k=2d-1$. Then we have that $\mathcal{L}_{\widetilde{S}}=\mathcal{L}^4(d;2d-1)$, $\widehat{\mathcal{L}}_{\widetilde{S}}=\mathcal{L}^4(d;2d)$, $\mathcal{L}_1=\mathcal{L}(2d-1;m^n)$ and $\widehat{\mathcal{L}}_1=\mathcal{L}(2d-2;m^n)$. So, because of Proposition 5.3, we know that $l_{\widetilde{S}}=v_{\widetilde{S}}$ $(=d+1)$ and $\hat{l}_{\widetilde{S}}=0$ $(>\hat{v}_{\widetilde{S}})$. A simple calculation shows that $v_{\mathbf{P}}=v+d-2\geq v\geq 0$. Moreover, because of [BZ03, Theorem 4], we know that $\mathcal{L}_1$ and $\widehat{\mathcal{L}}_1$ are non-special. Proceeding as in the proof of Claim 4.4, one can easily check (using $v\geq 0$) that $\dim(\mathcal{R}_{\widetilde{S}}\cap\mathcal{R}_{\cup L_i})=r_{\widetilde{S}}+r_{\mathbf{P}}-2d+1$. So, using (3.e), we obtain that

$$l_0=r_{\widetilde{S}}+r_{\mathbf{P}}-2d+1+\hat{l}_{\widetilde{S}}+l_{\mathbf{P}}+2=l_{\widetilde{S}}+l_{\mathbf{P}}-2d+1=v_{\widetilde{S}}+v_{\mathbf{P}}-k=v.$$

(2) Again, if $n = 1$ we are done because of Proposition 5.3. So we assume $n > 1$, or equivalently $u + w \geq 1$. As in the proof of Theorem 4.1, we can use Lemmas 4.7 and 4.8, up to the last step. For this last step, we have $c = n$ and $b = 1$, and we will take $k = 2d$. In this case, we can no longer use Lemma 4.7 since $\mathcal{L}_{\widetilde{S}}$ is special. If we can prove, however, that $\dim(\mathcal{R}_{\widetilde{S}} \cap \mathcal{R}_{\bigcup L_i}) = \hat{l}_{\widetilde{S}} = \hat{l}_{\mathbf{P}} = -1$, then we still obtain $l_0 = -1$, and thus also $l = -1$. Since $\dim(\mathcal{R}_{\widetilde{S}} \cap \mathcal{R}_{\bigcup L_i}) = \max\{-1, r_{\widetilde{S}} + (r_{\mathbf{P}} - 2d)\}$, $r_{\widetilde{S}} = 0$ and $r_{\mathbf{P}} = l_{\mathbf{P}}$ it is enough to prove that, for $k = 2d$, $v_{\mathbf{P}} \leq 2d - 1$ and $\hat{v}_{\mathbf{P}} \leq -1$. These inequalities are equivalent to respectively $v \leq -d$ and $v \leq 1 - d$, so it is sufficient to have $v \leq -d$.

In case $u > 0$, we can make sure that in the last step $c = 4$, and we obtain

$$v \leq -1 \iff m \geq d \iff v \leq 1 - 2d.$$

In case $u = 0$, we have $c = 9$ in the last step, and using the fact that $2d \neq 1 \mod 3$, a simple calculation shows that

$$v \leq -1 \iff m \geq \frac{2d}{3} \iff v \leq 1 - 3d. \qquad \square$$

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