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RENORMING JAMES TREE SPACE

PETR HÁJEK AND JAN RYCHTÁŘ

ABSTRACT. We show that the James tree space JT can be renormed to be Lipschitz separated. This negatively answers the question of J. Borwein, J. Giles and J. Vanderwerff as to whether every Lipschitz separated Banach space is an Asplund space.

1. INTRODUCTION

The main result of the present paper, Theorem 5, states the existence of an equivalent 2-WUR renorming (see Definition 2) of the James tree space JT. As a corollary, and this was in fact the motivation of our work, we answer in the negative a question of J. Borwein, J. Giles and J. Vanderwerff, whether every Lipschitz separated Banach space is an Asplund space.

Let us explain the situation in more detail. In [1] authors investigate properties of the Clarke subdifferential of a typical Lipschitz function on a given Banach space. In the course of their work, they study extensions of (bounded) Lipschitz functions from subspaces to the whole space, which preserve the Lipschitz constant. The results have implications for the behavior of the Clarke subdifferential. More precisely, they call a Banach space $(X, \|.\|)$ *Lipschitz separated*, if for every closed convex set $C \subset X$ and every bounded 1-Lipschitz real valued function f on C and $x \notin C$ there exist 1-Lipschitz extensions of f on the whole X, say f_1, f_2, satisfying $f_1(x) \neq f_2(x)$. This property depends heavily on the norm $\|.\|$. Let $B_X = \{x \in X; \|x\| \leq 1\}$ be a unit ball. In [1] the following characterization is proved:

Theorem 1. *For a given Banach space $(X, \|.\|)$ the following are equivalent:*

(1) *X is Lipschitz separated.*
(2) *For every pair of sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset B_X$ such that $\lim_{n,m\to\infty} \|x_n + y_m\| = 2$, there is no $\phi \in X^*$ such that $\limsup_{n\to\infty} \phi(x_n) < 0 < \liminf_{n\to\infty} \phi(y_n)$.*

It is observed that the WUR property of $\|.\|$ implies (2) and so does the 2-WUR (defined below), and on the other hand (2) implies that $\|.\|^{**}$ is rotund. The last fact implies that ℓ_1 is not isomorphic to any subspace of X.

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Definition 2. Let $(X, \|.\|)$ be a Banach space and let B_X be a closed unit ball. We say that the norm $\|.\|$ is WUR (*weakly uniformly rotund*) if for all $f \in X^*$

$$\lim_{n\to\infty} f(x_n - y_n) = 0,$$

whenever $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset B_X$ are such that $\lim_{n\to\infty} \|x_n + y_n\| = 2$. We say that $\|.\|$ is 2-WUR (2-*weakly uniformly rotund*) if for every $f \in X^*$

$$\lim_{n,m\to\infty} f(x_n - y_m) = 0,$$

whenever $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset B_X$ are such that $\lim_{n,m\to\infty} \|x_n + y_m\| = 2$.

Many examples of various renormings are presented in [1], supporting the natural conjecture that Lipschitz separated Banach spaces, although not necessarily WUR, should be WUR renormable or at least Asplund.

In our paper we construct a 2-WUR renorming $\|.\|$ of JT, a classical example of a separable Banach space not containing ℓ_1 and having non-separable dual. Thus $(JT, \|.\|)$ is Lipschitz separated, but is neither Asplund nor WUR renormable (see [5]). Recall that by [5], the space JH of Hagler, which also does not contain ℓ_1, does not admit an equivalent norm $\|.\|$ such that $\|.\|^{**}$ is rotund. Therefore JH does not admit a Lipschitz separated renorming. Thus separable spaces with 2-WUR renorming (or Lipschitz separating renorming) cut in between Asplund spaces and spaces not containing ℓ_1. Thus, for separable Banach spaces, the property of being Lipschitz separated for some equivalent norm is strictly weaker than having a separable dual and strictly stronger than not containing ℓ_1.

In this connection it would be interesting to find a rotundity renorming characterization of (separable) Banach spaces not containing ℓ_1, similar to cases of superreflexive (UR), reflexive (2-R) or Asplund (WUR) (see [3], [7], [5]).

We also do not know whether X admits an equivalent 2-WUR norm if X is Lipschitz separated.

The organization of this paper is as follows. In Section 2 we introduce our notation and prove two lemmas that will be often used in the sequel. In Section 3 we state the main theorem and present the core of its proof. For the reader's convenience, proofs of auxiliary lemmas will be presented separately in Section 4.

2. Preliminaries and notation

The James tree space JT was introduced by J. Lindenstrauss and C. Stegall in [6]. Let us summarize the notation we use here. Let T be an infinite dyadic tree, that is,

$$T = \{(n, i); n \in \mathbb{N}, i \in \{0, \dots, 2^n - 1\}\}.$$

We define a partial ordering $>$ on T by letting $(m, j) > (n, i)$ if and only if $m > n$ and there is a sequence of integers $\{i_k\}_{k=0}^{m-n}$ such that $i_0 = i, i_{m-n} = j$ and $i_k \in \{2i_{k-1}, 2i_{k-1} + 1\}$, for $k \le m - n$. A maximal linearly ordered subset of T will be called a *branch*. The set of all branches will be denoted $\mathcal{B}$. An interval $[s, t]$ is the maximal linearly ordered subset of T with s as a minimal element and t as a maximal element. Similarly we define intervals $(s, t], (s, t)$, $[s, t)$ and $[s, \infty)$. For every $t = (n, i) \in T$ we define the height of t as $|t| = n$. Denote $T_n = \{t \in T; |t| \le n\}$. For a non-empty and finite set $A \subset T$ define $\min(A) = \min\{|t|; t \in A\}$ and

$\max(A) = \max\{|t|; t \in A\}$. For every real bounded function $x : T \to \mathbb{R}$ define a Hilbertian norm

$$|||x|||^2 = \sum_{t \in T} 2^{-4|t|} |x(t)|^2.$$

We say that $\mathcal{S} = (S_j)_{j=1}^k$ is an *admissible collection* if it is a collection of pairwise disjoint intervals of T. If $x : T \to \mathbb{R}$ is a real bounded function and $\mathcal{S} = (S_j)_{j=1}^k$ is an admissible collection, we define

$$\|x\|_{\mathcal{S}}^2 = \sum_{j=1}^k \left(\sum_{t \in S_j} x(t) \right)^2.$$

Observe that $\|.\|_{\mathcal{S}}$ is a Hilbertian seminorm for every admissible collection $\mathcal{S}$. The James tree space JT is defined in [6] as the space of all bounded functions $x : T \to \mathbb{R}$ such that $\|x\|_{JT} < \infty$, where

$$\|x\|_{JT}^2 = \sup \Big\{ \|x\|_{\mathcal{S}}^2 ; \mathcal{S} \text{ is an admissible collection} \Big\}. \tag{1}$$

Let us define an equivalent renorming $\|.\|$ of JT by means of the formula

$$\|x\|^2 = \|x\|_{JT}^2 + |||x|||^2. \tag{2}$$

Let $B_{JT} = \{x \in JT; \|x\| \leq 1\}$. For $x \in JT$ we define the *support* of x, $\mathrm{supp}(x)$, by

$$\mathrm{supp}(x) = \{t \in T; x(t) \neq 0\}.$$

For $\varepsilon > 0$ and $x \in JT$ define

$$\mathcal{A}(x, \varepsilon) = \{\mathcal{S}; \mathcal{S} \text{ is an admissible collection, } \|x\|_{\mathcal{S}}^2 + |||x|||^2 > \|x\|^2 - \varepsilon^2\}.$$

Observe that $\mathcal{A}(x, \varepsilon) \neq \emptyset$ for every $x \in JT$ and every $\varepsilon > 0$.

If $\emptyset \neq S \subset T$ is a set, then we define $f_S \in JT^*$ by

$$f_S(x) = \sum_{t \in S} x(t), \text{ for every } x \in JT.$$

The set S will not always be finite but will be such that f_S will always make sense. In case $S = \emptyset$ we set $f_S \equiv 0$. Note that if S is an interval, then $\|f_S\| \leq 1$.

Let $x \in JT$ and $T' \subset T$. We denote by $x|_{T'}$ the element of JT such that $x|_{T'}(t) = x(t)$, for $t \in T'$, and $x|_{T'}(t) = 0$ for $t \notin T'$.

Lemma 3. *Let $\varepsilon_0 > 0, x, y \in B_{JT}, \|x + y\|^2 > 4 - \varepsilon_0^2$ and $\mathcal{S} \in \mathcal{A}(x + y, \varepsilon_0)$. Then*

(i) $\mathcal{S} \in \mathcal{A}(x, \varepsilon_0) \cap \mathcal{A}(y, \varepsilon_0)$,
(ii) $\|x - y\|_{\mathcal{S}}^2 < 2\varepsilon_0^2$.

Proof. By contradiction, let us assume that $\mathcal{S} \notin \mathcal{A}(x, \varepsilon_0)$. This implies that $\|x\|_{\mathcal{S}}^2 + |||x|||^2 \leq 1 - \varepsilon_0^2$. Using the parallelogram identity,

$$\begin{aligned} 4 - 2\varepsilon_0^2 &< \|x + y\|^2 - \varepsilon_0^2 < \|x + y\|_{\mathcal{S}}^2 + |||x + y|||^2 \\ &\leq \|x + y\|_{\mathcal{S}}^2 + |||x + y|||^2 + \|x - y\|_{\mathcal{S}}^2 + |||x - y|||^2 \\ &= 2\big(\|x\|_{\mathcal{S}}^2 + |||x|||^2\big) + 2\big(\|y\|_{\mathcal{S}}^2 + |||y|||^2\big) \leq 2(1 - \varepsilon_0^2) + 2, \end{aligned}$$

a contradiction. Hence $\|x\|^2 - \varepsilon_0^2 < \|x\|_{\mathcal{S}}^2 + |||x|||^2$ and by the same argument $\|y\|^2 - \varepsilon_0^2 < \|y\|_{\mathcal{S}}^2 + |||y|||^2$. Thus (i) is satisfied. Moreover

$$\begin{aligned} 0 &\leq \|x-y\|_{\mathcal{S}}^2 + |||x-y|||^2 \\ &= 2\big(\|x\|_{\mathcal{S}}^2 + |||x|||^2\big) + 2\big(\|y\|_{\mathcal{S}}^2 + |||y|||^2\big) - \big(\|x+y\|_{\mathcal{S}}^2 + |||x+y|||^2\big) \\ &\leq 2\|x\|^2 + 2\|y\|^2 - \big(\|x+y\|^2 - \varepsilon_0^2\big) \\ &< 2+2-(4-2\varepsilon_0^2) = 2\varepsilon_0^2, \end{aligned}$$

and (ii) is satisfied. □

Lemma 4. *Let $x \in JT$, $\varepsilon_0 > 0$ and $\mathcal{S} \in \mathcal{A}(x,\varepsilon_0)$. Let $S \subset T$ be an interval such that $\mathcal{S}' = \mathcal{S} \cup \{S\}$ is an admissible collection. Then*

$$\big|f_S(x)\big| < \varepsilon_0.$$

Proof.

$$\begin{aligned} \|x\|^2 &\geq \|x\|_{\mathcal{S}'}^2 + |||x|||^2 = \|x\|_{\mathcal{S}}^2 + |||x|||^2 + \big|f_S(x)\big|^2 \\ &> \|x\|^2 - \varepsilon_0^2 + \big|f_S(x)\big|^2. \end{aligned}$$

□

3. Main theorem

Theorem 5. *The norm $\|.\|$ on JT defined above is 2-WUR.*

Proof. The proof proceeds by contradiction. Assume that there is $\varepsilon' > 0$, $\{x'_n\}_{n=1}^\infty$, $\{y'_m\}_{m=1}^\infty \in B_{JT}$ and $\varphi \in JT^*$ such that

$$\lim_{n,m\to\infty} \|x'_n + y'_m\| = 2$$

and

$$\limsup_{n\to\infty} \varphi(x'_n) + \varepsilon' < \liminf_{m\to\infty} \varphi(y'_m). \tag{3}$$

In order to proceed faster to the core of the proof of Theorem 5, proofs of the following two facts are presented in the next section.

In our first step we replace $x'_n, y'_m \in JT$ and $\varphi \in JT^*$ by $x_n, y_n \in JT$ and $f_B \in JT^*$ having additional properties.

Fact 6. *There is $\varepsilon \in (0, 200^{-8})$, and there are $x_0 \in JT$, sequences $\{x_k\}_{k=1}^\infty$, $\{y_l\}_{l=1}^\infty \in B_{JT}$, $n_0 \in \mathbb{N}$ and $B \in \mathcal{B}$ such that for all $k, l \in \mathbb{N}$*

$$f_B(x_k) + \sqrt[8]{\varepsilon} \leq f_B(y_l), \tag{4}$$

and moreover

(a) $\|x_k + y_l\| > 2 - 2\varepsilon$,
(b) $x_k|_{T_{n_0}} = y_l|_{T_{n_0}} = x_0$,
(c) $n_0 < \mathrm{mintail}(x_k) < \mathrm{maxtail}(x_k) < \mathrm{mintail}(y_k)$, *for all $k \geq 1$,*
(d) $\mathrm{mintail}(y_l) < \mathrm{maxtail}(y_l) < \mathrm{mintail}(x_{l+1})$, *for all $l \geq 1$,*

where

$$\mathrm{mintail}(x) = \min\Big\{\mathrm{supp}(x - x_0)\Big\},$$

$$\mathrm{maxtail}(x) = \max\Big\{\mathrm{supp}(x - x_0)\Big\}.$$

From now on, B will be the branch provided by Fact 6. In the rest of the proof we will show that the statement of Fact 6 (i.e. the estimate (4)) is contradicting. Note that x_k and y_l are finitely supported for all $k, l \in \mathbb{N}$.

Fact 7. *Upon passing to subsequences and keeping the original notation, there are sequences* $\{x_k\}_{k=1}^{\infty}, \{y_l\}_{l=1}^{\infty}$ *and an admissible collection* $\mathcal{S}_{k,l} \in \mathcal{A}(x_k + y_l, 9\sqrt{\varepsilon})$ *such that the conclusion of Fact* 6 *still holds, and for all* $k \in \mathbb{N}$*:*

(a) $\min(S) \leq n_k$, *for all* $S \in \mathcal{S}_{k,l}$ *and all* $l \geq k$, *where* $n_k = \operatorname{maxtail}(x_k)$.
(b) *For every* $l \geq k$, *every* $S_l \in \mathcal{S}_{k,l}$ *and every* $l' \geq k$, *there is* $S_{l'} \in \mathcal{S}_{k,l'}$ *such that* $S_l \cap T_{n_k} = S_{l'} \cap T_{n_k}$. *Moreover, if* S_l *is such that* $\max(S_l) \leq n_k$, *then* $S_l = S_{l'}$.
(c) *If* $P \subset T_{n_k}$ *is a fixed set and* $\mathcal{S}_{k,l}^P \subset \mathcal{S}_{k,l}$ *is the collection of intervals starting at points of* P, *then traces*

$$\mathcal{S}_{k,l}^P \cap T_{n_k} := \{S \cap T_{n_k}; S \in \mathcal{S}_{k,l}^P\}$$

are, for fixed $k \in \mathbb{N}$, *independent of* $l \geq k$. *In particular,* $\|x_k\|_{\mathcal{S}_{k,l}^P}$ *are, for fixed* $k \in \mathbb{N}$, *independent of* $l \geq k$.

Similarily, for all $l \in \mathbb{N}$*:*

(d) $\min(S) \leq m_l$ *for all* $S \in \mathcal{S}_{k,l}$ *and all* $k > l$, *where* $m_l = \operatorname{maxtail}(y_l)$.
(e) *For every* $k > l$, *every* $S_k \in \mathcal{S}_{k,l}$ *and every* $k' > l$, *there is* $S_{k'} \in \mathcal{S}_{k',l}$ *such that* $S_k \cap T_{m_l} = S_{k'} \cap T_{m_l}$. *Moreover, if* $\max(S_k) \leq m_l$, *then* $S_k = S_{k'}$.
(f) *If* $P \subset T_{m_l}$ *is a fixed set and* $\mathcal{S}_{k,l}^P \subset \mathcal{S}_{k,l}$ *is the collection of intervals starting at points of* P, *then traces*

$$\mathcal{S}_{k,l}^P \cap T_{m_l} := \{S \cap T_{m_l}; S \in \mathcal{S}_{k,l}^P\}$$

are independent of $k > l$. *In particular,* $\|y_l\|_{\mathcal{S}_{k,l}^P}$ *are independent of* $k > l$.

Thus, by Lemma 3,

$$\|x_k - y_l\|_{\mathcal{S}_{k,l}}^2 < 2.9^2\varepsilon = 162\varepsilon, \tag{5}$$

and consequently,

$$\left|f_S(x_k - y_l)\right|^2 \leq \|x_k - y_l\|_{\mathcal{S}_{k,l}}^2 < 162\varepsilon, \tag{6}$$

for all $k, l \in \mathbb{N}$ and all $S \in \mathcal{S}_{k,l}$. The problem is that the branch B need not be covered by intervals $S \in \mathcal{S}_{k,l}$ in a nice way to use (6) directly for estimating $|f_B(x_k - y_l)|$. Thus the following case analysis is needed.

For all $k, l \in \mathbb{N}$ we define subcollections $\mathcal{S}_{k,l}^B(i) \subset \mathcal{S}_{k,l}$, for $i = 1, 2, 4, 5$, of intervals starting on B as follows:

$$\begin{aligned}
\mathcal{S}_{k,l}^B(1) &= \big\{S \in \mathcal{S}_{k,l}; \min(S) \leq n_0, \max(S \cap B) < \operatorname{mintail}(x_k)\big\},\\
\mathcal{S}_{k,l}^B(2) &= \big\{S \in \mathcal{S}_{k,l}; \min(S) \leq n_0, \max(S \cap B) \geq \operatorname{mintail}(x_k)\big\},\\
\mathcal{S}_{k,l}^B(4) &= \big\{S \in \mathcal{S}_{k,l}; \min(S) \geq \operatorname{mintail}(x_k),\\
&\qquad \max(S \cap B) \leq \operatorname{maxtail}(x_k)\big\},\\
\mathcal{S}_{k,l}^B(5) &= \big\{S \in \mathcal{S}_{k,l}; \min(S) \geq \operatorname{mintail}(x_k),\\
&\qquad \max(S \cap B) > \operatorname{maxtail}(x_k)\big\}.
\end{aligned}$$

Note that the above represents all intervals in $\mathcal{S}_{k,l}$ starting on B. In the following steps, we will define a partition of B into at most six pieces $B^0_{k,l}, \dots, B^5_{k,l}$ such that

$$f_B(x_k - y_l) = \sum_{i=0}^{5} f_{B^i_{k,l}}(x_k - y_l),$$

and we will estimate each term $|f_{B^i_{k,l}}(x_k - y_l)|$ separately.

The following proof is like an algorithm in the shape of a tree with seven levels. In first six levels, there is a branching in the proof. There are places where the proof ends, and there are two places where a certain estimate holds for all but finitely many $k \in \mathbb{N}$. This should be understood in the following way. If we come during the proof to such an estimate, we have to skip those finitely many $k \in \mathbb{N}$ for which that estimate does not hold, and after that we have to return to the beginning and start again at the Level 0. If we come to the same estimate once more, it will already hold and we can proceed futher. For initiating the process, set $k = l = 1$.

Level 0.
Set $m_l = \text{maxtail}(y_l)$ and define $B^0_{k,l} \subset B \cap T_{m_l}$ as the maximal interval containing $(0,0)$ and such that $B^0_{k,l} \cap S = \emptyset$, for all $S \in \mathcal{S}_{k,l}$. Because $\mathcal{S}_{k,l} \cup B^0_{k,l}$ is an admissible collection, by Lemma 4

$$\left|f_{B^0_{k,l}}(x_k - y_l)\right| \le \left|f_{B^0_{k,l}}(x_k)\right| + \left|f_{B^0_{k,l}}(y_l)\right| < 18\sqrt{\varepsilon}. \tag{7}$$

If $B^0_{k,l} = B \cap T_{m_l}$, then the above inequality is a contradiction of (4) and the proof is finished. If $B^0_{k,l} \ne B \cap T_{m_l}$ we have to proceed to the next level.

Level 1.
Provided $S^B_{k,l}(1) \ne \emptyset$, let $B^1_{k,l} = [a_1, b_1] \subset B$ be an interval such that a_1 is the minimal and b_1 is the maximal element of $\{t \in B \cap S; S \in \mathcal{S}^B_{k,l}(1)\}$. Set $B^1_{k,l} = \emptyset$ if $S^B_{k,l}(1) = \emptyset$. Because $x_k(t) = y_l(t)$ for all $t \in B^1_{k,l}$, we have that

$$\left|f_{B^1_{k,l}}(x_k - y_l)\right| = 0. \tag{8}$$

Level 2.
As $\mathcal{S}_{k,l}$ is an admissible collection, there is at most one interval in $\mathcal{S}^B_{k,l}(2)$. By Fact 7, the existence of such an interval does not depend on $l \ge k$. We will distinguish the following cases:

Case I. There is $S_{k,l} \in \mathcal{S}^B_{k,l}(2)$ such that $\max(S_{k,l} \cap B) \ge \text{maxtail}(x_k)$. Put

$$S^1_{k,l} = (B \setminus S_{k,l}) \cap \{t \in T; |t| \ge \text{mintail}(y_l)\},$$

$$S^2_{k,l} = (S_{k,l} \setminus B) \cap \{t \in T; |t| \ge \text{mintail}(y_l)\}.$$

By Fact 7 no interval of $\mathcal{S}_{k,l}$ starts after $\text{maxtail}(x_k)$ for $l \ge k$. Thus

$$\mathcal{S} = \mathcal{S}_{k,l} \cup \{S^1_{k,l}\}$$

is an admissible collection and by Fact 7 and Lemma 4

$$\left|f_{S^1_{k,l}}(x_k - y_l)\right| = \left|f_{S^1_{k,l}}(y_l)\right| < 9\sqrt{\varepsilon}.$$

By Lemma 11, there is $l \ge k$ such that

$$\left|f_{S^2_{k,l}}(x_k - y_l)\right| = \left|f_{S^2_{k,l}}(y_l)\right| < 20\sqrt[4]{\varepsilon}.$$

Finally, by (7) and (8)

$$\begin{aligned}\left|f_B(x_k-y_l)\right| &\le \left|f_{B^0_{k,l}}(x_k-y_l)\right|+\left|f_{B^1_{k,l}}(x_k-y_l)\right|+\left|f_{S_{k,l}}(x_k-y_l)\right|\\ &\quad+\left|f_{S^1_{k,l}}(x_k-y_l)\right|+\left|f_{S^2_{k,l}}(x_k-y_l)\right|\\ &\le 18\sqrt{\varepsilon}+0+9\sqrt{\varepsilon}+9\sqrt{\varepsilon}+20\sqrt[4]{\varepsilon},\end{aligned}$$

a contradiction with (4). Thus the proof is finished.

Case II. There is $S_{k,l}\in\mathcal{S}^B_{k,l}(2)$ such that

$$\max(S_{k,l}\cap B)<\operatorname{maxtail}(x_k)\ \text{and}\ \max(S_{k,l})\ge\operatorname{mintail}(y_l).$$

Let $B^2_{k,l}=S_{k,l}\cap B$ and $C_k=B^2_{k,l}\cap\{t\in T;|t|\ge\operatorname{mintail}(x_k)\}$. By Fact 7, C_k does not depend on $l\ge k$ and, by Lemma 10,

$$\left|f_{B^2_{k,l}}(x_k-y_l)\right|=\left|f_{C_k}(x_k)\right|<10\sqrt[4]{\varepsilon}, \tag{9}$$

for all but finitely many $k\in\mathbb{N}$

Case III. There is $S_{k,l}\in\mathcal{S}^B_{k,l}(2)$ such that

$$\max(S_{k,l}\cap B)<\operatorname{maxtail}(x_k)\ \text{and}\ \max(S_{k,l})<\operatorname{mintail}(y_l).$$

Let $B^2_{k,l}=S_{k,l}\cap B$, $A_k=S_{k,l}\cap T_{n_0}$, $C_k=B^2_{k,l}\cap\{t\in T;|t|\ge\operatorname{mintail}(x_k)\}$. By Fact 7, A_k and C_k do not depend on $l\ge k$. By (6)

$$\left|f_{S_{k,l}}(x_k-y_l)\right|=\left|f_{S_{k,l}\setminus A_k}(x_k)\right|<9\sqrt{2\varepsilon}.$$

Thus, by Lemma 8

$$\left|f_{B^2_{k,l}}(x_k-y_l)\right|=\left|f_{C_k}(x_k)\right|<10\sqrt[4]{\varepsilon}. \tag{10}$$

Case IV. If $\mathcal{S}^B_{k,l}(2)=\emptyset$ set $B^2_{k,l}=(B\cap T_{n_0})\setminus(B^0_{k,l}\cup B^1_{k,l})$. By Fact 6

$$\left|f_{B^2_{k,l}}(x_k-y_l)\right|=0. \tag{11}$$

In the summary, after Level 2, the proof is either finished or we define $B^2_{k,l}\subset B$ such that, by (9), (10), and (11),

$$\left|f_{B^2_{k,l}}(x_k-y_l)\right|<10\sqrt[4]{\varepsilon}. \tag{12}$$

Level 3.

Set $m_l=\operatorname{maxtail}(y_l)$ and define $B^3_{k,l}\subset B\cap T_{m_l}$ as the maximal interval disjoint with $(B^0_{k,l}\cup B^1_{k,l}\cup B^2_{k,l})$ and such that no interval of $\mathcal{S}_{k,l}$ starts on $B^3_{k,l}$. As in Level 0,

$$\left|f_{B^3_{k,l}}(x_k-y_l)\right|<18\sqrt{\varepsilon}. \tag{13}$$

If $B\cap T_{m_l}=\bigcup_{i=0}^3 B^i_{k,l}$ we have arrived at a contradiction with (4), otherwise we have to proceed further.

Level 4.

Provided $\mathcal{S}^B_{k,l}(4)\ne\emptyset$, set $B^4_{k,l}=[a_4,b_4]\subset B$ to be the interval such that b_4 is the maximal element of $\{t\in S\cap B;S\in\mathcal{S}^B_{k,l}(4)\}$ and

$$B^4_{k,l}=[(0,0),b_4]\setminus\bigcup_{i=0}^3 B^i_{k,l}.$$

If $S^B_{k,l}(4) = \emptyset$, set $B^4_{k,l} = \emptyset$. Assume that $J \in \mathcal{S}^B_{k,l}(4)$, $J' \in \mathcal{S}^B_{k',l}(4)$ are such that $J \cap J' \neq \emptyset$. Then necessarily $J \cap J' \cap B \neq \emptyset$. Thus $k = k'$ and $J = J'$. Hence, by Lemma 9, for all but finitely many $k \in \mathbb{N}$ and for all $l \geq k$,

$$\|x_k\|^2_{\mathcal{S}^B_{k,l}(4)} < 2 \cdot 18^2 \varepsilon.$$

Hence

$$\mathcal{S}_{k,l} \setminus \mathcal{S}^B_{k,l}(4) \in \mathcal{A}\big(x_k, (9^2\varepsilon + 2 \cdot 18^2\varepsilon)^{1/2}\big),$$

and by Lemma 4

$$\big|f_{B^4_{k,l}}(x_k - y_l)\big| = \big|f_{B^4_{k,l}}(x_k)\big| \leq (9^2\varepsilon + 2 \cdot 18^2\varepsilon)^{1/2} < 100\sqrt{\varepsilon}. \tag{14}$$

Level 5.
This level is almost identical to Level 2, Case I. By the admissibility of set $\mathcal{S}_{k,l}$, there is at most one interval in $\mathcal{S}^B_{k,l}(5)$. By Fact 7, its existence does not depend on $l \geq k$. If there is $S_{k,l} \in \mathcal{S}^B_{k,l}(5)$, put

$$S^1_{k,l} = (B \setminus S_{k,l}) \cap \{t \in T; |t| \geq \operatorname{mintail}(y_l)\},$$

$$S^2_{k,l} = (S_{k,l} \setminus B) \cap \{t \in T; |t| \geq \operatorname{mintail}(y_l)\}.$$

By Fact 7, no interval of $\mathcal{S}_{k,l}$ starts after $\operatorname{maxtail}(x_k)$ for $l \geq k$. Thus

$$\mathcal{S} = \mathcal{S}_{k,l} \cup \{S^1_{k,l}\}$$

is an admissible collection and by Fact 7 and Lemma 4

$$\big|f_{S^1_{k,l}}(x_k - y_l)\big| = \big|f_{S^1_{k,l}}(y_l)\big| < 9\sqrt{\varepsilon}.$$

By Lemma 11, there is $l \geq k$ such that

$$\big|f_{S^2_{k,l}}(x_k - y_l)\big| = \big|f_{S^2_{k,l}}(y_l)\big| < 11\sqrt[4]{\varepsilon}.$$

Thus by (7), (8), (12), (13), and (14)

$$\begin{aligned}\big|f_B(x_k - y_l)\big| &\leq \sum_{i=0}^{4}\big|f_{B^i_{k,l}}(x_k - y_l)\big| + \big|f_{S_{k,l}}(x_k - y_l)\big| \\ &\qquad + \big|f_{S^1_{k,l}}(x_k - y_l)\big| + \big|f_{S^2_{k,l}}(x_k - y_l)\big| \\ &\leq \sqrt{\varepsilon}(18 + 0 + 18 + 100 + 9 + 9) + 10\sqrt[4]{\varepsilon} + 11\sqrt[4]{\varepsilon},\end{aligned}$$

a contradiction of (4). Thus the proof for this case is finished.

If $\mathcal{S}^B_{k,l}(5) = \emptyset$, we have to go to the last level.

Level 6.
Define $B^5_{k,l} \subset B$ as the maximal interval disjoint with $\bigcup_{i=0}^4 B^i_{k,l}$ and such that $\max(B^5_{k,l}) = \operatorname{maxtail}(y_l)$. By Fact 7, $\mathcal{S}_{k,l} \cup B^5_{k,l}$ is an admissible collection and thus, by Lemma 4,

$$\big|f_{B^5_{k,l}}(x_k - y_l)\big| < 9\sqrt{\varepsilon}. \tag{15}$$

Thus by (7), (8), (12), (13), (14), and (15)

$$\big|f_B(x_k - y_l)\big| \leq \sum_{i=0}^{5}\big|f_{B^i_{k,l}}(x_k - y_l)\big| \leq 50\sqrt[4]{\varepsilon},$$

a final contradiction of (4). □

4. Auxiliary lemmas

Proof of Fact 6. As $\{x'_n\}_{n=1}^{\infty}, \{y'_m\}_{m=1}^{\infty} \subset B_{JT}$ are such that

$$\lim_{n,m\to\infty} \|x'_n + y'_m\| = 2,$$

by [2, Fact II.2.3 (ii)],

$$\lim_{n\to\infty} f_{\{t\}}(x'_n) = \lim_{n\to\infty} x'_n(t) = \lim_{m\to\infty} y'_m(t) = \lim_{m\to\infty} f_{\{t\}}(y'_m),$$

for all $t \in T$. Recall that JT_*, the predual of JT, can be represented as

$$JT_* = \overline{\text{span}}\,\{f_{\{t\}}; t \in T\}.$$

Thus there is $x'_0 \in JT$ such that

$$w^* - \lim_{n\to\infty} x'_n = x'_0 = w^* - \lim_{m\to\infty} y'_m. \tag{16}$$

Since

$$JT^* = \overline{\text{span}}\,\Big\{\{f_{\{t\}}; t \in T\} \cup \{f_A; A \in \mathcal{B}\}\Big\},$$

by (3), there is $B \in \mathcal{B}$ such that

$$\lim_{n\to\infty} f_B(x'_n) \neq \lim_{m\to\infty} f_B(y'_m).$$

After a possible passing to subsequences and keeping the original notation, we may assume that there is $\varepsilon'' > 0$ such that

$$\varepsilon'' + f_B(x'_n) \le f_B(y'_m), \tag{17}$$

for all $n, m \in \mathbb{N}$. Pick

$$\varepsilon < \min\{10^{-12}(\varepsilon'')^8, 200^{-8}\},$$

and assume $\|x'_n + y'_m\| \ge 2 - \varepsilon/3$, for all $m, n \in \mathbb{N}$. Set

$$n_0 = \min\big\{n \in \mathbb{N}; \|x'_0|_{T_n}\| > \|x'_0\| - \varepsilon/3\big\} + 1,$$

and define

$$x_0 = x'_0|_{T_{n_0}}.$$

By (16) and (17) it is possible to find sequences $\{\widetilde{x}_k\}_{k=1}^{\infty}, \{\widetilde{y}_l\}_{l=1}^{\infty} \subset JT$ and sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_l\}_{l=1}^{\infty}$ of integers such that

(a) $\|\widetilde{x}_k - x'_{n_k}\| < \varepsilon/3$,
(b) $\|\widetilde{y}_l - y'_{m_l}\| < \varepsilon/3$,
(c) $\widetilde{x}_k|_{T_{n_0}} = x_0 = \widetilde{y}_l|_{T_{n_0}}$,
(d) $n_0 < \text{mintail}(\widetilde{x}_k) < \text{maxtail}(\widetilde{x}_k) < \text{mintail}(\widetilde{y}_k)$ for all $k \in \mathbb{N}$,
(e) $\text{mintail}(\widetilde{y}_l) < \text{maxtail}(\widetilde{y}_l) < \text{mintail}(\widetilde{x}_{l+1})$ for all $l \in \mathbb{N}$.

Clearly, $\|\widetilde{x}_k + \widetilde{y}_l\| \ge 2 - \varepsilon$, and $\|\widetilde{x}_k\| \le 1 + \varepsilon/3, \|\widetilde{y}_l\| \le 1 + \varepsilon/3$, for all $k, l \in \mathbb{N}$. Set $x_k = (1 + \varepsilon/3)^{-1}\widetilde{x}_k$, and $y_l = (1 + \varepsilon/3)^{-1}\widetilde{y}_l$. Thus $\|x_k\| \le 1, \|y_l\| \le 1$, and

$$\|x_k + y_l\| \ge (2 - \varepsilon)(1 + \varepsilon/3)^{-1} > 2 - 2\varepsilon,$$

for all $k, l \in \mathbb{N}$. Moreover, by (17),

$$f_B(x_k) + \sqrt[8]{\varepsilon} < f_B(x_k) + (\varepsilon'' - 2\varepsilon/3)(1 + \varepsilon/3)^{-1} \le f_B(y_l).$$

□

Proof of Fact 7. Note that $\|x_k + y_l\|^2 > (2-2\varepsilon)^2 > 4 - 10\varepsilon$. For $k, l \in \mathbb{N}$ choose an admissible collection $\widetilde{\mathcal{S}}_{k,l} \in \mathcal{A}(x_k + y_l, 4\sqrt{\varepsilon})$ and define

$$\mathcal{S}_{k,l} = \{S \in \widetilde{\mathcal{S}}_{k,l}; S \cap \operatorname{supp}(x_k) \neq \emptyset \text{ and } S \cap \operatorname{supp}(y_l) \neq \emptyset\},$$
$$\mathcal{S}^x_{k,l} = \{S \in \widetilde{\mathcal{S}}_{k,l}; S \cap \operatorname{supp}(x_k) \neq \emptyset \text{ and } S \cap \operatorname{supp}(y_l) = \emptyset\},$$
$$\mathcal{S}^y_{k,l} = \{S \in \widetilde{\mathcal{S}}_{k,l}; S \cap \operatorname{supp}(y_l) \neq \emptyset \text{ and } S \cap \operatorname{supp}(x_k) = \emptyset\}.$$

We may and do assume that

$$\widetilde{\mathcal{S}}_{k,l} = \mathcal{S}_{k,l} \cup \mathcal{S}^x_{k,l} \cup \mathcal{S}^y_{k,l}.$$

By Lemma 3(ii) applied for $\varepsilon_0 = 4\sqrt{\varepsilon}$, $\|x_k - y_l\|^2_{\widetilde{\mathcal{S}}_{k,l}} < 32\varepsilon$. Thus the following estimate holds for both $\mathcal{S} = \mathcal{S}^x_{k,l}$ and $\mathcal{S} = \mathcal{S}^y_{k,l}$:

$$\begin{aligned}\sum_{S \in \mathcal{S}} \left(\sum_{t \in S} (x_k + y_l)(t) \right)^2 &= \sum_{S \in \mathcal{S}} \left(\sum_{t \in S} (x_k - y_l)(t) \right)^2 \\ &= \|x_k - y_l\|^2_{\mathcal{S}} \leq \|x_k - y_l\|^2_{\widetilde{\mathcal{S}}_{k,l}} < 32\varepsilon.\end{aligned}$$

Hence

$$\begin{aligned}\|x_k + y_l\|^2_{\mathcal{S}_{k,l}} + |||x_k + y_l|||^2 &\geq \|x_k + y_l\|^2_{\widetilde{\mathcal{S}}_{k,l}} + |||x_k + y_l|||^2 - 64\varepsilon \\ &\geq \|x_k + y_l\|^2 - 16\varepsilon - 64\varepsilon.\end{aligned}$$

This shows that

$$\mathcal{S}_{k,l} \in \mathcal{A}(x_k + y_l, 9\sqrt{\varepsilon}). \tag{18}$$

Clearly, $\mathcal{S}_{k,l}$ satisfies properties (a) and (d). Denote $E_1 = \mathbb{N}$ and $k_1 = \min(E_1)$. Set $n_1 = \max(\operatorname{supp}(x_1))$. There are only finitely many possibilities of what collections

$$\mathcal{S}_{k_1,l} \cap T_{n_1} = \{S \cap T_{n_1}; S \in \mathcal{S}_{k_1,l}\}$$

can look like for $l \in E_1$. Thus there is an infinite set $E_1' \subset E_1$ such that the system $\mathcal{S}_{k_1,l} \cap T_{n_1}$ does not depend on $l \in E_1'$ and the first part of the property (b) is satisfied. For every $l \geq k_1$ and every interval $I \in \mathcal{S}_{k_1,l} \cap T_{n_1}$, there is a unique interval $I_l \in \mathcal{S}_{k_1,l}$ such that $I = I_l \cap T_{n_1}$. There are exactly two posibilities: either $\max(I_l)$ is less than $\operatorname{mintail}(y_l)$ or it is not. Thus there is an infinite set $E_1'' \subset E_1'$ such that the above property is independent of $l \in E_1''$. Take $l_1 = \min(E_1'')$. Assume that k_i, l_i, E_i and E_i'' have been defined for $i = 1, \ldots, N$. Define $E_{N+1} = E_N \setminus \{k_i, l_i; i = 1, \ldots, N\}$, $k_{N+1} = \min(E_{N+1})$ and repeat the above procedure. This will define sets of indexes $K = \{k_i; i \in \mathbb{N}\}$ and $L = \{l_i; i \in \mathbb{N}\}$ such that the properties (a) and (b) are satisfied for $k \in K$ and $l \in L$, $l \geq k$. Clearly, (c) follows from (b).

By the similar procedure, we get the properties (d), (e) and (f). □

Lemma 8. *Let $x \in B_{JT}, \varepsilon > 0$ and $\mathcal{S} \in \mathcal{A}(x, 9\sqrt{2\varepsilon})$. Assume $S \in \mathcal{S}$ and an interval $A \subset S$ are such that*

$$\left| f_{S \setminus A}(x) \right| < 18\sqrt{2\varepsilon}.$$

Then

$$\left| f_C(x) \right| < 10\sqrt[4]{\varepsilon},$$

for any interval $C \subset S \setminus A$.

Proof. Clearly $\mathcal{S}' = \mathcal{S} \cup \{A, C\} \setminus \{S\}$ is an admissible collection. Thus

$$\begin{aligned}
\|x\|^2 &\geq \|x\|_{\mathcal{S}'}^2 + |||x|||^2 \\
&= \|x\|_{\mathcal{S}}^2 + |||x|||^2 + \big|f_A(x)\big|^2 + \big|f_C(x)\big|^2 - \big|f_S(x)\big|^2 \\
&= \|x\|_{\mathcal{S}}^2 + |||x|||^2 + \big|f_A(x)\big|^2 + \big|f_C(x)\big|^2 - \big|f_{S\setminus A}(x) + f_A(x)\big|^2 \\
&\geq \|x\|^2 - (9\sqrt{\varepsilon})^2 + \big|f_C(x)\big|^2 - 2\big|f_{S\setminus A}(x)\big|\big|f_A(x_k)\big| - \big|f_{S\setminus A}(x)\big|^2 \\
&\geq \|x\|^2 - 81\varepsilon - 2.18\sqrt{2\varepsilon} - 2.18^2\varepsilon + \big|f_C(x)\big|^2.
\end{aligned}$$

□

Lemma 9. *Let $\mathcal{S}_{k,l}^0 \subset \mathcal{S}_{k,l}$, for $k, l \in \mathbb{N}$, be collections of intervals such that*

$$\mathcal{S} = \bigcup_{k \leq M} \mathcal{S}_{k,l}^0$$

is an admissible collection for any $M \in \mathbb{N}$ and any $l \geq M$. Then

$$\operatorname{card}\left\{k \in \mathbb{N}; \|x_k\|_{\mathcal{S}_{k,l}^0} > 18\sqrt{2\varepsilon} \text{ for some } l \geq k\right\} < (\varepsilon^{-1/2} + 1)^2.$$

Proof. Put $N = (\varepsilon^{-1/2} + 1)^2$. Note that $\|x_k\|_{\mathcal{S}_{k,l}^0}$ is independent of l for $l \geq k$ by Fact 7. Assume, by a contradiction, that the statement is not true. Without loss of generality we may assume that

$$\|x_k\|_{\mathcal{S}_{k,l}^0} > 18\sqrt{2\varepsilon}$$

for all $k \leq N$ and $l = N$. By (5) and the triangle inequality,

$$9\sqrt{2\varepsilon} \geq \|x_k - y_l\|_{\mathcal{S}_{k,l}} \geq \|x_k - y_l\|_{\mathcal{S}_{k,l}^0} \geq \left|\|x_k\|_{\mathcal{S}_{k,l}^0} - \|y_l\|_{\mathcal{S}_{k,l}^0}\right|,$$

for all $k \leq N$. Therefore

$$\|y_l\|_{\mathcal{S}_{k,l}^0} > 9\sqrt{2\varepsilon},$$

for all $k \leq N$. Hence

$$1 \geq \|y_l\|^2 \geq \|y_l\|_{\mathcal{S}}^2 = \sum_{k=1}^{N} \|y_l\|_{S_{k,l}^0}^2 > N(9\sqrt{2\varepsilon})^2 > 9^2,$$

a contradiction. □

Lemma 10. *Let $S_{k,l} \in \mathcal{S}_{k,l}^B(2)$ be such that*

$$\max(S_{k,l} \cap B) < \operatorname{maxtail}(x_k) \text{ and } \max(S_{k,l}) \geq \operatorname{mintail}(y_l).$$

Let $A_k = S_{k,l} \cap T_{n_0}$ and $C_k = S_{k,l} \cap B \cap \{t \in T; |t| \geq \operatorname{mintail}(x_k)\}$. Denote

$$Q = \left\{k \in \mathbb{N}; \big|f_{S_{k,l}\setminus A_k}(x_k)\big| > 18\sqrt{2\varepsilon}, \text{ for some } l \geq k\right\}.$$

Then

$$\operatorname{card}(Q) < (\varepsilon^{-1/2} + 1)^2,$$

and for $k \notin Q$,

$$|f_{C_k}(x_k)| < 10\sqrt[4]{\varepsilon}.$$

Proof. Note that, by Fact 7, A_k, C_k, and $f_{S_{k,l}\setminus A_k}(x_k)$ do not depend on $l \geq k$. Put $N = (\varepsilon^{-1/2}+1)^2$. Assume, by contradiction, that $\operatorname{card}(Q) \geq N$ and without loss of generality assume that $\{1,\dots,N\} \subset Q$. Set $l = N+1$. Let $R_{k,l} = S_{k,l} \cap \{t \in T; |t| \geq \operatorname{mintail}(y_l)\}$. Thus by (6)

$$9\sqrt{2\varepsilon} \geq \left|f_{S_{k,l}}(x_k - y_l)\right| = \left|f_{S_{k,l}\setminus A_k}(x_k) - f_{R_{k,l}}(y_l)\right|,$$

and consequently

$$\left|f_{R_{k,l}}(y_l)\right| > 9\sqrt{2\varepsilon}.$$

As $\mathcal{S} = \bigcup_{k=1}^N R_{k,l}$ is an admissible collection we have

$$1 \geq \|y_l\|^2 \geq \|y_l\|^2_{\mathcal{S}} = \sum_{k=1}^N \left|f_{R_{k,l}}(y_l)\right|^2 \geq N(9\sqrt{2\varepsilon})^2 > 9^2,$$

a contradiction. Finally, by Lemma 8,

$$|f_{C_k}(x_k)| < 10\sqrt[4]{\varepsilon},$$

for $k \notin Q$. □

Lemma 11. *Let $k_0 \in \mathbb{N}$ be fixed. Upon passing to a subsequence of $\{y_l\}$,*

$$|f_{S^2_{k_0,l}}(y_l)| < 11\sqrt[4]{\varepsilon},$$

for all $l \geq k_0$, where $S^2_{k_0,l} = (S_{k_0,l}\setminus B)\cap\{t \in T; |t| \geq \operatorname{mintail}(y_l)\}$ and $S_{k_0,l} \in \mathcal{S}^B_{k_0,l}(2)$ or $S_{k_0,l} \in \mathcal{S}^B_{k_0,l}(5)$.

Proof. The proof will be divided into several steps. For simplicity of notation we will omit the index k_0 and write $L_l = S^2_{k_0,l}$. Notice that the family $\{L_l; l \geq k_0\}$ is disjoint. Moreover

$$\mathcal{S} := \bigcup_{l \geq k_0}^{M} \{S \in \mathcal{S}_{k,l}; S \text{ starts on } \ L_l\}$$

is an admissible collection for every $M \geq k_0$ and every $k > M$. Set $\mathcal{S}_{k,l}(L_l) = \{S \in \mathcal{S}_{k,l};\ S \text{ starts on } L_l\}$. By repeating the proof of Lemma 9 we get

$$\operatorname{card}\Big\{l \geq k_0; \|y_l\|_{S_{k,l}(L_l)} > 18\sqrt{2\varepsilon} \text{ for some } k > l\Big\} < (\varepsilon^{-1/2}+1)^2.$$

Thus we may assume that

$$\|y_l\|_{S_{k,l}(L_l)} \leq 18\sqrt{2\varepsilon},$$

for all $l \in \mathbb{N}$ and all $k > l$. Hence $\mathcal{S}_{k,l} \setminus \mathcal{S}_{k,l}(L_l) \in \mathcal{A}(y_l, \sqrt{809\varepsilon})$. Define $L^2_l \subset L_l$ as the maximal interval such that, for all $k > l$, all intervals of $\mathcal{S}_{k,l}(L_l)$ start at some point of L^2_l. Notice that, by Fact 7, the definition of L^2_l does not depend on $k > l$ and $\mathcal{S}_{k,l} \setminus \mathcal{S}_{k,l}(L_l) \cup \{L^2_l\}$ is an admissible collection. By Lemma 4, we have

$$\left|f_{L^2_l}(y_l)\right| \leq \sqrt{809\varepsilon} < 30\sqrt{\varepsilon}. \tag{19}$$

Set $L^1_l = L_l \setminus L^2_l$. We claim that upon passing to a subsequence of $\{y_l\}$

$$\left|f_{L^1_l}(y_l)\right| < 10\sqrt[4]{\varepsilon}, \tag{20}$$

for all $l \in \mathbb{N}$. Indeed, if $L^1_l = \emptyset$, the above inequality is trivial. If $L^1_l \neq \emptyset$, by the maximality of L^2_l, there has to be an interval $I_{k,l} \in \mathcal{S}_{k,l}$ such that $L^1_l \subset I_{k,l}$. We will distinguish the following two cases. By Fact 7, if one of them occurs for one $k > l$, then it occurs for all $k > l$.

Case I. $\max(I_{k,l}) \le \operatorname{maxtail}(y_l)$. By Fact 7, $I_{k,l}$ does not depend on $k > l$. Put $I_l^0 = I_{k,l} \cap T_{n_0}$. Thus, by (6), for $k > l+1$,

$$\left|f_{I_{k,l}\setminus I_l^0}(y_l)\right| = \left|f_{I_{k,l}}(x_k - y_l)\right| < 9\sqrt{2\varepsilon}.$$

Since $L_l^1 \subset I_{k,l} \setminus I_l^0$, by Lemma 8,

$$\left|f_{L_l^1}(y_l)\right| < 10\sqrt[4]{\varepsilon}.$$

Case II. $\max(I_{k,l}) \ge \operatorname{mintail}(x_k)$. In this case the interval $I_{k,l}$ depends on $k > l$. Define $I_{k,l}^0 = I_{k,l} \cap T_{n_0}$, $I_{k,l}^1 = I_{k,l} \setminus I_{k,l}^0$, and $I_{k,l}^2 = I_{k,l} \cap \{t \in T; \operatorname{mintail}(x_k) \le |t| \le \operatorname{maxtail}(x_k)\}$. Then

$$f_{I_{k,l}}(y_l) = f_{I_{k,l}^0}(x_0) + f_{I_{k,l}^1}(y_l),$$

$$f_{I_{k,l}}(x_k) = f_{I_{k,l}^0}(x_0) + f_{I_{k,l}^2}(x_k).$$

Thus, by (6),

$$\left|f_{I_{k,l}^1}(y_l) - f_{I_{k,l}^2}(x_k)\right| = \left|f_{I_{k,l}}(x_k - y_l)\right| < 9\sqrt{2\varepsilon}.$$

We claim that

$$\left|f_{I_{k,l}^1}(y_l)\right| < 18\sqrt{2\varepsilon}, \tag{21}$$

for all $l > k_0$ and any $k > l$. Note that, by Fact 7, the above is independent of $k > l$. Indeed, if, for some $l_0 > k_0$, the inequality (21) does not hold, then

$$\left|f_{I_{k,l}^2}(x_k)\right| > 9\sqrt{2\varepsilon},$$

for all $k > l$. Set $k_1 = l_0 + 1 + (\varepsilon^{-1/2} + 1)^2$. The family

$$\mathcal{S} = \bigcup_{k=l_0+1}^{k_1} I_{k,l}^2$$

is an admissible collection and

$$1 \ge \|x_{k_1}\|^2 \ge \|x_{k_1}\|_{\mathcal{S}}^2 = \sum_{k=l_0+1}^{k_1} \left|f_{I_{k,l}^2}(x_k)\right|^2 = (\varepsilon^{-1/2}+1)^2(9\sqrt{\varepsilon})^2 > 9^2,$$

a contradiction.

Since $L_l^1 \subset I_{k,l}^1$, by Lemma 8

$$\left|f_{L_l^1}(y_l)\right| < 10\sqrt[4]{\varepsilon}.$$

Thus by (19) and (20)

$$\left|f_{L_l}(y_l)\right| \le \left|f_{L_l^1}(y_l)\right| + \left|f_{L_l^2}(y_l)\right| \le 11\sqrt[4]{\varepsilon}.$$

□

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Mathematical Institute, AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic
E-mail address: hajek@math.cas.cz

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1
E-mail address: jrychtar@math.ualberta.ca
Current address: Department of Mathematical Sciences, University of North Carolina, Greensboro, North Carolina 27402