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L^p IMPROVING ESTIMATES FOR SOME CLASSES OF RADON TRANSFORMS

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ABSTRACT. In this paper, we give $L^p - L^q$ estimates and the L^p regularizing estimate of Radon transforms associated to real analytic functions, and we also give estimates of the decay rate of the L^p operator norm of corresponding oscillatory integral operators. For $L^p - L^q$ estimates and estimates of the decay rate of the L^p operator norm we obtain sharp results except for extreme points; however, for L^p regularity we allow some restrictions on the phase function.

1. INTRODUCTION

In this paper, we consider oscillatory integral operators of the form

$$T_\lambda f(x) = \int e^{i\lambda S(x,y)} f(y)\chi(x,y)dy$$

and Radon transforms of the form

$$Rf(x_1,x_2) = \int f(y_1, x_2 + S(x_1,y_1))\tilde{\chi}(x_1,x_2,y_1)dy_1,$$

where $x, y, x_1, x_2, y_1 \in \mathbb{R}$, S is real-analytic near the origin and χ and $\tilde{\chi}$ are smooth cut-off functions supported in small neighborhoods of the origin of $\mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{R}$, respectively. For T_λ, we are interested in the decay rate of the L^p operator norm as $\lambda \to \infty$. For R we are interested in the L^p regularizing property and $L^p - L^q$ boundedness.

There have been several results on the decay of the L^2 operator norm of T_λ ([PSt2], [PSt3], [R], [S2]). When S is real-analytic, Phong and Stein proved sharp results on the decay rate of $||T_\lambda||_{L^2\to L^2}$ depending on the Newton polygon of S''_{xy} [PSt2]. In [S1] Seeger obtained nearly optimal results when S is a C^∞ real function. In [R] Rychkov developed ideas from [PSt2] and [S1] to obtain optimal estimates of $||T_\lambda||_{L^2\to L^2}$. For the L^p operator norm of T_λ, Greenleaf and Seeger obtained the sharp decay estimates in the case of two-sided fold singularities, including uniform estimates [GS].

Sharp L^p regularity and $L^p - L^q$ boundedness of R were obtained by Phong and Stein [PSt3] when S is a homogeneous polynomial. L^p regularity and $L^p - L^q$ boundedness of more general Radon transforms $\mathcal{R}$ of the form

$$\mathcal{R}f(x_1,x_2) = \int f(y_1, \mathfrak{S}(x_1,x_2,y_1))\chi(x_1,x_2,y_1)dy_1$$

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were considered by Seeger ([S1], [S2]) when $\mathfrak{S}$ is a C^∞ real function. In [S1], he proved that $\mathcal{R}$ is bounded from L^p to $L^p_{1/p}$ with $m < p < \infty$ if $\mathcal{M}$ satisfies a 'left finite type' condition of order $m \geq 3$ and with $1 < p < m/(m-1)$ if $\mathcal{M}$ satisfies a 'right finite type' condition of order $m \geq 3$, where $\mathcal{M} = \{(x, y) : y_2 = \mathfrak{S}(x_1, x_2, y_1)\}$. Moreover, he obtained strong estimates in the exponent range $3/2 < p < 3$ with the assumption of two-sided fold singularities [S1]. In [S2], finite type conditions and optimal conditions for L^p regularity and $L^p - L^q$ boundedness of Radon transforms are formulated. In the special case where $\mathfrak{S}(x_1, x_2, y_1) = x_2 + S(x_1, x_2)$ the conditions reduce to the following:

Let $\mathcal{A}$ be the closed convex hull of the points in $\{(u, \beta) : 0 \leq u \leq 1, \beta \leq 0\}$ and the points $\left(\frac{\mu}{\mu+\nu}, \frac{1}{\mu+\nu}\right)$, where μ and ν are positive integers such that

$$\frac{\partial^{\mu'+\nu'} S}{\partial^{\mu'} x \partial^{\nu'} y}(0,0) = 0$$

if either $\mu' \leq \mu$ and $\nu' < \nu$ or $\mu' < \mu$ and $\nu' \leq \nu$, and

$$\frac{\partial^{\mu+\nu} S}{\partial^{\mu} x \partial^{\nu} y}(0,0) \neq 0.$$

Let $\mathcal{B}$ be the closed convex hull of the points in $\{(u, v) : 0 \leq u \leq 1, v \geq u\}$ and the points $\left(\frac{\mu+1}{\mu+\nu+1}, \frac{\mu}{\mu+\nu+1}\right)$, where μ and ν are positive integers which satisfy the above conditions.

Remark 1.1. We can use the 'reduced Newton polygon' of S to formulate the conditions (see Remark 2.3 below).

Then in [S2] the following has been obtained:

(1) $\mathcal{R}$ is bounded from L^p_s to $L^p_{s+\alpha}$ for all $s \in \mathbb{R}$ if $(1/p, \alpha)$ belongs to the interior of $\mathcal{A}$.
(2) If $\mathcal{R}$ is bounded from L^p_s to $L^p_{s+\alpha}$ for all $s \in \mathbb{R}$, then $(1/p, \alpha)$ belongs to $\mathcal{A}$.
(3) $\mathcal{R}$ is bounded from L^p to L^q if $(1/p, 1/q)$ belongs to the interior of $\mathcal{B}$.
(4) If $\mathcal{R}$ is bounded from L^p to L^q, then $(1/p, 1/q)$ belongs to $\mathcal{B}$.

In this paper, we shall give the analogous results of (1) and (2) for T_λ including estimates on the boundary of $\mathcal{A}$. For R, we shall extend (1) and (3) to some part of the boundaries of $\mathcal{A}$ and $\mathcal{B}$. To describe the results, we need some terminology. The boundary of $\mathcal{A}$ contains at least two compact faces. Let l and r be compact faces of $\mathcal{A}$ containing $(0,0)$ and $(1,0)$, respectively. Similarly we denote by l' and r' the compact faces of $\mathcal{B}$ containing $(0,0)$ and $(1,1)$, respectively. Let $e(\mathcal{A})$ be the set of all vertices of the boundary of $\mathcal{A}$ except $(0,0)$ and $(1,0)$. Let $e(\mathcal{B})$ be the set of all vertices of the boundary of $\mathcal{B}$ except $(0,0)$ and $(1,1)$.

Theorem 1.2. *Let $S(x, y)$ be a real-analytic function near the origin.*

(1) *If T_λ is bounded on L^p with norm $O(\lambda^{-\alpha})$ as $\lambda \to \infty$, then $(1/p, \alpha)$ belongs to $\mathcal{A}$.*
(2) *If $(1/p, \alpha)$ belongs to $\mathcal{A} \setminus e(\mathcal{A})$, then T_λ is bounded on L^p with norm $O(\lambda^{-\alpha})$ as $\lambda \to \infty$.*
(3) *R is bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ if $(1/p, 1/q)$ belongs to $\mathcal{B} \setminus e(\mathcal{B})$.*

Remark 1.3. We have learned that S. Lee [L] has independently obtained sharp $L^p - L^q$ estimates using the multilinear arguments in [B] and [BOS].

In addition to this, we shall prove L^p regularity estimates with some restrictions on S''_{xy}. To describe the results in a precise way, we may need more terminology.

Definition 1.4. Let P be a real-analytic function near the origin with Taylor expansion at $(0,0)$,

$$P(x,y) = \sum_{i=m}^{\infty} P_i(x,y), \tag{1.1}$$

where $P_i(x,y)$ is a homogeneous polynomial of degree i with real coeffecients.

(1) P_m is said to be the *leading term* of P if P_m is the first nonzero term in (1.1).
(2) Let P_m be the leading term of P. P is said to be *almost translation invariant* if $\frac{\partial^m P_m}{\partial x^m}(0,0) \neq 0$, $\frac{\partial^m P_m}{\partial y^m}(0,0) \neq 0$ and $P_m(x,1) = 0$ has only one root with multiplicity m.
(3) P is said to be *separate* if P is not almost translation invariant.
(4) Let α and β be nonnegative integers. Then P is said to have (α,β) factors if
$$P(x,y) = x^\alpha y^\beta Q(x,y),$$
where $Q(x,y)$ is an analytic function near the origin, which contains both nonzero pure x^m- and y^n-terms in its Taylor series expansion for some $m \geq 0$ and $n \geq 0$.

Theorem 1.5. *Let S be a real analytic function near the origin. Suppose that S''_{xy} is separate and does not have $(1,1)$ factors. Then R is bounded from $L^p_s(\mathbb{R}^2)$ to $L^p_{s+\alpha}(\mathbb{R}^2)$ for all $s \in \mathbb{R}$ if $(1/p,\alpha)$ belongs to $\mathcal{A} \setminus (e(\mathcal{A}) \cup l \cup r)$.*

If $\mathcal{M} = \{y_2 = x_2 + S(x_1,y_1)\}$ satisfies a certain type of conditions, we have weak type estimates at some vertices of $\mathcal{A}$ and $\mathcal{B}$. More precisely, we define left and right finite type conditions as follows:

Definition 1.6. (1) $\mathcal{M}$ is said to satisfy a left finite type condition of order k if
$$\frac{\partial^k S}{\partial x \partial y^{k-1}}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial^{k'} S}{\partial x \partial y^{k'-1}}(0,0) = 0$$
for $k' < k$.

(2) $\mathcal{M}$ is said to satisfy a right finite type condition of order l if
$$\frac{\partial^l S}{\partial x^{l-1} \partial y}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial^{l'} S}{\partial x^{l'-1} \partial y}(0,0) = 0$$
for $l' < l$.

Let e_l and e'_l be in $e(\mathcal{A})$ and $e(\mathcal{B})$, respectively, which are closest to the vertical axes among points in $e(\mathcal{A})$ and $e(\mathcal{B})$, respectively. Let e_r and e'_r be in $e(\mathcal{A})$ and $e(\mathcal{B})$, respectively, which are furthest from the vertical axes. With these definitions, we have the following theorem.

Theorem 1.7. *Suppose that S''_{xy} is separate and that $\mathcal{M}$ satisfies a left (or right) finite type condition. Then*

1. *T_λ is bounded from $L^{p,1/(1-\alpha)} \to L^p$ (or from L^p to $L^{p,1/\alpha}$) with operator norm $O(\lambda^{-\alpha})$ as $\lambda \to \infty$ if $(1/p, \alpha)$ is e_l (or e_r).*
2. *R is bounded from $L^{p,q'} \to L^q$ (or from L^p to $L^{q,p'}$) if $(1/p, 1/q)$ is e'_l (or e'_r) with $1/p + 1/p' = 1/q + 1/q' = 1$.*
3. *R is bounded from $L^{p,1/(1-\alpha)} \to L^p_\alpha$ (or from L^p to $L^{p,1/\alpha}_\alpha$) if $(1/p, \alpha)$ is e_l (or e_r).*

In Section 2, we give a brief review on the Newton polyhedron of S and the factorization of S''_{xy} in [PSt2]. We then describe the relation between the reduced Newton polyhedron of S and domains $\mathcal{A}$ and $\mathcal{B}$ and prove several preliminary lemmas. Section 3 will be devoted to obtaining $L^{\frac{\mu+\nu-2}{\mu-1}}$ estimates of operators of the form

$$T_{\lambda,-\frac{1}{\mu+\nu-2}} f(x) = \int e^{i\lambda S(x,y)} |S''_{xy}(x,y)|^{-1/(\mu+\nu-2)} \chi(x,y) f(y) dy,$$

where (μ, ν) is in a part of the boundary of the reduced Newton polyhedron of S. When (μ, ν) is contained in the interior of the compact face of the reduced Newton diagram, we shall show the strong $L^{\frac{\mu+\nu-2}{\mu-1}}$ boundedness. If (μ, ν) is contained in the noncompact faces or vertices of the reduced Newton diagram, we do not have the $L^{\frac{\mu+\nu-2}{\mu-1}}$ estimates. However, if S satisfies some suitable conditions, we shall obtain weak type estimates at some vertices. At the end of the section, we shall show that the strong $L^p(\mathbb{R})$ estimate of $T_{\lambda,-\mu}$ implies the strong $L^p(\mathbb{R}^2)$ estimate of $R_{-\mu}$ defined as

$$R_{-\mu} f(x_1, x_2) = \int f(y_1, x_2 + S(x_1, y_1)) |S''_{xy}(x,y)|^{-\mu} \chi(x_1, y_1) dy_1.$$

In Section 4, we shall prove theorems stated in this section by using interpolation theorems and the results in Section 3.

2. Preliminaries

We will need to understand the geometry of the singular set $\{S''_{xy}(x,y) = 0\}$. It has been studied in [PSt2] in terms of the Newton polyhedron and Puiseux series. Let $\sum c_{pq} x^p y^q$ be the Taylor series of S and let $K = \{(p,q) \in \mathbb{Z}^2_+ | c_{pq} \neq 0\}$. The Newton polyhedron $\Gamma_+(S)$ of S is defined by the closed convex hull of $\bigcup_{n \in K} (n + \mathbb{R}^2_+)$. The Newton diagram $\Gamma(S)$ of S is defined by the boundary of $\Gamma_+(S)$. The reduced Newton polyhedron $\widetilde{\Gamma}_+(S)$ of S is defined as

$$\widetilde{\Gamma}_+(S) = \Gamma_+(S''_{xy}) + (1,1).$$

The reduced Newton diagram $\widetilde{\Gamma}(S)$ is defined as the boundary of $\widetilde{\Gamma}_+(S)$. In view of the Weierstrass preparation theorem and factorization by Puiseux series, we can write S''_{xy} as

$$S''_{xy}(x,y) = U(x,y) x^A y^B \prod_{l=1}^{n} \prod_{i=1}^{N_l} (y - r_{l,i}(x)) \tag{2.1}$$

$$= \tilde{U}(x,y) x^A y^B \prod_{m=1}^{n} \prod_{j=1}^{M_m} (x - \tilde{r}_{l,i}(y)), \tag{2.2}$$

where $r_{l,i}(x) = c_{l,i}x^{a_l} + \cdots$, $\tilde{r}_{m,j}(y) = \tilde{c}_{m,j}y^{b_m} + \cdots$ with $0 < a_1 < \cdots < a_n$ and $0 < b_n < \cdots < b_1$, and U and $\tilde{U}$ are smooth functions with $U(0,0) \neq 0$ and $\tilde{U}(0,0) \neq 0$ (see [PSt2]). We set

$$A_k = A + \sum_{i=1}^{k} a_i N_i = A + \sum_{i=1}^{k} M_i, \tag{2.3}$$

$$B_k = B + \sum_{i=k+1}^{n} N_i = B + \sum_{i=k+1}^{n} b_i M_i. \tag{2.4}$$

We can immediately see that $\{(A_k+1, B_k+1)\}_{k=0}^{n}$ is the set of the vertices of $\widetilde{\Gamma}(S)$. We set $\mu_k = \frac{1}{A_k+B_k}$, $p_k = \frac{A_k+B_k}{A_k}$ and $q_k = \frac{A_k+B_k}{B_k}$.

Remark 2.1. It may be useful to paraphrase Definition 1.4(3) and 1.4(4) in terms of (2.1) and (2.2).

1. $S''_{xy}(x,y)$ is separate if and only if it is not of the form

$$S''_{xy}(x,y) = U(x,y)\prod_{i=1}^{N}(y - r_i(x)),$$

where $r_i(x) = cx + \cdots$ with nonzero real number c.

2. S''_{xy} has (α,β) factors if and only if $A = \alpha$ and $B = \beta$ in (2.1) and (2.2).

To understand the relation between the reduced Newton polyhedron and $\mathcal{A}$, $\mathcal{B}$ in a precise way, we need the following lemma.

Lemma 2.2. *Let* $\varphi(x,y) = (\frac{x}{x+y}, \frac{1}{x+y})$ *and* $\psi(x,y) = (\frac{x+1}{x+y+1}, \frac{x}{x+y+1})$.

(1) *The map* φ *from* $(0,\infty)\times(0,\infty)$ *to* $(0,1)\times(0,\infty)$ *is a diffeomorphism and it maps line segments to line segments.*

(2) ψ *is a diffeomorphism from* $(0,\infty)\times(0,\infty)$ *to* $\{(x,y) \mid 0 < x, y < 1,\ y < x\}$ *and it maps line segments to line segments.*

Proof. It is clear that φ and ψ are diffeomorphisms on the given domains. To show that φ maps line segments to line segments, it suffices to show that it maps the midpoint of two points A and B to a point on a line segment connecting $\varphi(A)$ and $\varphi(B)$. We take $A(x_1,y_1)$ and $B(x_2,y_2)$ contained in $(0,\infty)\times(0,\infty)$. The midpoint of A and B is $C((x_1+x_2)/2, (y_1+y_2)/2)$ and we have $\varphi(A) = (\frac{x_1}{x_1+y_1}, \frac{1}{x_1+y_1})$, $\varphi(B) = (\frac{x_2}{x_2+y_2}, \frac{1}{x_2+y_2})$ and $\varphi(C) = (\frac{x_1+x_2}{x_1+x_2+y_1+y_2}, \frac{2}{x_1+x_2+y_1+y_2})$. Here we can observe that $\varphi(C) = \theta\varphi(A) + (1-\theta)\varphi(B)$ with $\theta = \frac{x_1+y_1}{x_1+x_2+y_1+y_2}$. This shows that φ maps line segments to line segments. The argument for ψ is parallel to the above with $\theta = \frac{x_1+y_1+1}{x_1+x_2+y_1+y_2+2}$. □

Remark 2.3. By Lemma 2.2, we observe that $\mathcal{A}\cap(0,1)\times(0,\infty) = \varphi(\widetilde{\Gamma}_+(S))$ and $\mathcal{B}\cap\{(x,y)|\, 0 < x, y < 1,\ y < x\} = \psi(\widetilde{\Gamma}_+(S))$.

Now, we shall prove elementary facts which will be used later. The first one is a variant of Schur's lemma.

Lemma 2.4. *If* $K(x,y) \geq 0$ *is a kernel of an operator* T *and* $K(x,y)$ *satisfies*

$$\int K(x,y)y^{-\frac{1}{p}}dy \leq Cx^{-\frac{1}{p}}, \quad \int K(x,y)x^{-\frac{1}{q}}dx \leq Cy^{-\frac{1}{q}},$$

where $1/p + 1/q = 1$, then

$$Tf(x) = \int K(x,y)f(y)dy$$

is bounded in L^p.

Proof. For $f \in L^p$ and $g \in L^q$ ($\frac{1}{p} + \frac{1}{q} = 1$) with $||f||_p = ||g||_q = 1$,

$$\begin{aligned} |f(y)g(x)| &= |f(y)x^{-\frac{1}{pq}}y^{\frac{1}{pq}}g(x)y^{-\frac{1}{pq}}x^{\frac{1}{pq}}| \\ &\leq \frac{1}{p}|f(y)|^p x^{-\frac{1}{q}}y^{\frac{1}{q}} + \frac{1}{q}|g(x)|^q y^{-\frac{1}{p}}x^{\frac{1}{p}}. \end{aligned}$$

We therefore obtain

$$\begin{aligned} &\left|\int\int K(x,y)f(y)g(x)dydx\right| \\ &\leq \int\int K(x,y)\frac{1}{p}|f(y)|^p x^{-\frac{1}{q}}y^{\frac{1}{q}}dydx \\ &\quad + \int\int K(x,y)\frac{1}{q}|g(x)|^q y^{-\frac{1}{p}}x^{\frac{1}{p}}dydx \\ &= \frac{1}{p}\int |f(y)|^p\left[\int K(x,y)x^{-\frac{1}{q}}dx\right]y^{\frac{1}{q}}dy \\ &\quad + \frac{1}{q}\int |g(x)|^q\left[\int K(x,y)y^{-\frac{1}{p}}dy\right]x^{\frac{1}{p}}dx \\ &\leq C/p + C/q. \end{aligned}$$

This completes the proof. □

Lemma 2.5. *Let S''_{xy} be as in* (2.1) *and* (2.2)*. If either $a_l \geq 1$ and $N_l\mu < 1$ or $a_l < 1$ and $M_l\mu < 1$, then*

$$\int_{c_l x^{a_l}}^{C_l x^{a_l}} \left|S''_{xy}(x,y)\right|^{-\mu} y^{-1/p}dy \leq Cx^{-(A_l+B_l)\mu - a_l/p + a_l}. \tag{2.5}$$

Proof. (1) $a_l \geq 1$. In view of (2.1)

$$|S''_{xy}(x,y)| \sim x^{A_{l-1}+B_l}\prod_{i=1}^{N_l}|y - r_{l,i}(x)|$$

when $c_l x^{a_l} \leq y \leq C_l x^{a_l}$. We therefore obtain

$$\int_{c_l x^{a_l}}^{C_l x^{a_l}} \left|S''_{xy}(x,y)\right|^{-\mu} y^{-1/p}dy \leq Cx^{-(A_{l-1}+B_l)\mu - a_l/p}\int_{c_l x^{a_l}}^{C_l x^{a_l}} \left|\prod_{i=1}^{N_l}(y - r_{l,i}(x))\right|^{-\mu} dy.$$

By using the Hölder's inequality, we obtain

$$\begin{aligned} \int_{c_l x^{a_l}}^{C_l x^{a_l}} \left|\prod_{i=1}^{N_l}(y - r_{l,i}(x))\right|^{-\mu} dy &\leq C\prod_{i=1}^{N_l}\left[\int_{c_l x^{a_l}}^{C_l x^{a_l}} |y - r_{l,i}(x)|^{-N_l\mu}dy\right]^{1/N_l} \\ &\leq C\prod_{i=1}^{N_l} x^{(-N_l a_l\mu + a_l)/N_l} \\ &= Cx^{-N_l a_l\mu + a_l}, \end{aligned}$$

which proves (2.5) in the case where $a_l \geq 1$.

(2) $a_l < 1$. In this case, we need to use (2.2) to obtain

$$\int_{c_l x^{a_l}}^{C_l x^{a_l}} \left|S''_{xy}(x,y)\right|^{-\mu} y^{-1/p} dy \le C x^{-(A_{l-1}+B_l)\mu - a_l/p} \int_{c_l x^{a_l}}^{C_l x^{a_l}} \left|\prod_{j=1}^{M_l}(x - \tilde{r}_{l,j}(y))\right| dy.$$

By applying Hölder's inequality again, we obtain (2.5). □

3. L^p ESTIMATES

In this section, we obtain L^p estimates of oscillatory integral operators with factor $|S''_{xy}|^{-\mu}$. Precisely, we shall consider the Newton diagram $\Gamma(S''_{xy})$ of S''_{xy}. We have two noncompact faces which are parallel to the coordinate axes. From (2.3) and (2.4), we know that $\{(A_k, B_k)\}_{k=0}^n$ is the set of vertices of $\Gamma(S''_{xy})$. If a point (a,b) in $\Gamma(S''_{xy})$ is contained in neither the noncompact faces nor the set of vertices of $\Gamma(S''_{xy})$, we can find two vertices (A_{k-1}, B_{k-1}), (A_k, B_k) for $k = 1, \cdots, n$ and θ $(0 < \theta < 1)$ such that

$$\begin{aligned} a &= \theta A_{k-1} + (1-\theta)A_k = A_k - \theta a_k N_k, \\ b &= \theta B_{k-1} + (1-\theta)B_k = B_k + \theta N_k. \end{aligned}$$

Now we set $A_\theta = A_k - \theta a_k N_k$, $B_\theta = B_k + \theta N_k$ and

$$\begin{aligned} \mu_\theta &= \frac{1}{A_\theta + B_\theta} = \frac{1}{A_k + B_k + \theta(1-a_k)N_k}, \\ p_\theta &= \frac{A_\theta + B_\theta}{A_\theta} = \frac{A_k + B_k + \theta(1-a_k)N_k}{A_k - \theta a_k N_k}, \\ q_\theta &= \frac{A_\theta + B_\theta}{B_\theta} = \frac{A_k + B_k + \theta(1-a_k)N_k}{B + \theta N_k}. \end{aligned}$$

Remark 3.1. Since (A_θ, B_θ) is not in either the noncompact faces or the set of vertices of $\Gamma(S''_{xy})$, we always assume $0 < \theta < 1$.

Now we are going to obtain L^{p_θ} boundedness of

$$T_{\lambda,-\mu_\theta} f(x) = \int_{-\infty}^{+\infty} e^{i\lambda S(x,y)} |S''_{xy}(x,y)|^{-\mu_\theta} \chi(x,y) f(y) dy.$$

Since the purpose is to get the boundedness without a decay rate, the oscillating factor plays no role. Therefore, instead of considering $T_{\lambda,-\mu_\theta}$, we shall consider the operator of the form

$$|T|_{-\mu_\theta} f(x) = \int_I |S''_{xy}(x,y)|^{-\mu_\theta} f(y) dy,$$

where I is a small interval near the origin in $\mathbb{R}$ such that the support of χ is contained in $I \times I$. Moreover, by considering the four quadrants in the xy-plane separately, we may assume that x and y are small and positive. Therefore, we assume that I is a small interval containing the origin in $\mathbb{R}_+$.

Proposition 3.2. *If S''_{xy} is separate, then for $0 < \theta < 1$*

$$|T|_{-\mu_\theta} f(x) = \int_0^{|I|} f(y) \left|S''_{xy}(x,y)\right|^{-\mu_\theta} dy$$

is bounded on $L^{p_\theta}(I)$.

Proof. By Lemma 2.4, it suffices to show that

$$\int_0^{|I|} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta} dy \le C x^{-1/p_\theta} \tag{3.1}$$

and

$$\int_0^{|I|} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} x^{-1/q_\theta} dx \le C y^{-1/q_\theta}. \tag{3.2}$$

By using a symmetric argument, one can easily observe that it suffices to prove (3.1). To do this, we divide $I \times I$ into regions of the forms: $0 < y < c_n x^{a_n}$, $c_l x^{a_l} < y < C_l x^{a_l}$, $C_{l+1} x^{a_{l+1}} < y < c_l x^{a_l}$, and $C_1 x^{a_1} < y < |I|$, where c_l and C_l are chosen so that $0 < c_l < C_l$, $c_l x^{a_l} < r_l(x) < C_l x^{a_l}$, and $C_{l+1} x^{a_{l+1}} < c_l x^{a_l}$. Now we treat each region separately.

(1) $0 < y < c_n x^{a_n}$. In this region one observes

$$|S''_{xy}(x,y)| \sim x^{A_n},$$

which implies

$$\int_0^{c_n x^{a_n}} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta} dy \le C \int_0^{c_n x^{a_n}} x^{-A_n/(A_\theta+B_\theta)} y^{-(B+A_\theta)/(A_\theta+B_\theta)} dy.$$

By Remark 3.1, we observe that $(B + A_\theta)/(A_\theta + B_\theta) < 1$. Therefore

$$\begin{aligned} &\int_0^{c_n x^{a_n}} x^{-A_n/(A_\theta+B_\theta)} y^{-(B+A_\theta)/(A_\theta+B_\theta)} dy \\ &\quad\le C x^{(-A_n - a_n(B+A_\theta))/(A_\theta+B_\theta)+a_n} \\ &\quad= C x^{(-A_n - a_n B + a_n B_\theta)/(A_\theta+B_\theta)}. \end{aligned}$$

Here a simple computation yields

$$\begin{aligned} &\frac{-A_n - a_n B + a_n B_\theta}{A_\theta + B_\theta} + \frac{A_\theta}{A_\theta + B_\theta} \\ &\quad= \frac{\sum_{i=k+1}^{n} (a_n - a_i) N_i + \theta(a_n - a_k) N_k}{A_\theta + B_\theta} \ge 0. \end{aligned}$$

This implies

$$\int_0^{c_n x^{a_n}} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta} dy \le C x^{-\frac{A_\theta}{A_\theta+B_\theta}} = C x^{-1/p_\theta},$$

which is the desired estimate.

(2) $c_l x^{a_l} < y < C_l x^{a_l}$. In this region we consider three cases: $a_l > 1$, $a_l < 1$ and $a_l = 1$.

(i) $a_l > 1$. First, by the definition of A_θ and B_θ and the fact that $a_l > 1$, we know $A_\theta + B_\theta \ge N_l$. The equality occurs only when the number of vertices of $\Gamma(S''_{xy})$ is two, $A = B = 0$, and $\theta = 1$. However this cannot happen by Remark 3.1. Hence $N_l \mu_\theta < 1$. By applying Lemma 2.5, we obtain

$$\int_{c_l x^{a_l}}^{C_l x^{a_l}} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta} dy \le C x^{-(A_l + a_l B_l + a_l A_\theta)/(A_\theta+B_\theta)+a_l}. \tag{3.3}$$

If $l \geq k$, then

$$\begin{aligned}
(3.4) \qquad & \frac{-(A_l + a_l B_l + a_l A_\theta)}{A_\theta + B_\theta} + a_l + \frac{A_\theta}{A_\theta + B_\theta} \\
&= \frac{\sum_{i=k+1}^{l}(a_l - a_i)N_i + (a_l - a_k)\theta N_k}{A_\theta + B_\theta} \geq 0.
\end{aligned}$$

If $l < k$, then

$$\begin{aligned}
(3.5) \qquad & \frac{-(A_l + a_l B_l + a_l A_\theta)}{A_\theta + B_\theta} + a_l + \frac{A_\theta}{A_\theta + B_\theta} \\
&= \frac{\sum_{i=l+1}^{k-1}(a_i - a_l)N_i + (a_k - a_l)(1-\theta)N_k}{A_\theta + B_\theta} \geq 0.
\end{aligned}$$

Equations (3.4) and (3.5) show that the right-hand side of (3.3) is dominated by $Cx^{-A_\theta/(A_\theta+B_\theta)}$ and this completes the proof of the case $a_l > 1$.

(ii) $a_l < 1$. The argument for this case is the same as that for the case $a_l > 1$. First we observe $M_l \mu_\theta < 1$. By applying Lemma 2.5 again, we obtain the desired inequality.

(iii) $a_l = 1$. Similarly to (i), we know that $A_\theta + B_\theta \geq N_l$ and that the equality occurs only when S''_{xy} is of the form

$$S''_{xy}(x,y) = \prod_{i=1}^{N_1}(y - r_i(x)),$$

where the $r_i(x)$'s are Puiseux series of the form $c_{i,1}x^{d_{i,1}} + c_{i,2}x^{d_{i,2}} + \cdots$ where $d_{i,1} = 1$ and $d_{i,j} < d_{i,j+1}$.

If $A_\theta + B_\theta > N_l$, the argument is parallel to that of (i). Therefore it suffices to consider the case where $A_\theta + B_\theta = N_l$. Since we assume that S''_{xy} is separate, $c_{i,1} \neq c_{i',1}$ for some $i, i' \in \{1, \cdots, N_1\}$. Then there exist a constant c and $k_0 \in \mathbb{N}$ with rearrangement of r_i if necessary such that

$$r_i(x) < cx < r_j(x), \quad 1 \leq i \leq k_0 < j \leq N_1.$$

We can divide the region $c_l x < y < C_l x$ into two regions of the form $c_l x < y < cx$ and $cx < y < C_l x$. Then by applying Hölder's inequality we obtain

$$\begin{aligned}
& \int_{c_l x}^{C_l x} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta}\, dy \\
= \quad & \int_{c_l x}^{cx} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta}\, dy + \int_{cx}^{C_l x} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta}\, dy \\
\leq \quad & Cx^{-(N_1-k_0)/N_1 - 1/p_\theta} \int_{c_l x}^{cx} \left|\prod_{i=1}^{k_0}(y - r_i(x))\right|^{-1/N_1} dy \\
& + Cx^{-k_0/N_1 - 1/p_\theta} \int_{cx}^{C_l x} \left|\prod_{i=k_0+1}^{N_1}(y - r_i(x))\right|^{-1/N_1} dy \\
\leq \quad & Cx^{-1/p_\theta}.
\end{aligned}$$

(3) $C_{l+1}x^{a_{l+1}} < y < c_l x^{a_l}$. Since we assume $0 < \theta < 1$, $(B_l + A_\theta)/(A_\theta + B_\theta) \neq 1$. We therefore obtain

$$\begin{aligned} &\int_{C_{l+1}x^{a_{l+1}}}^{c_l x^{a_l}} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta}\, dy \\ &\leq \quad Cx^{-A_l/(A_\theta+B_\theta)} \int_{C_{l+1}x^{a_{l+1}}}^{c_l x^{a_l}} y^{-(B_l+A_\theta)/(A_\theta+B_\theta)} dy \\ &\leq \quad Cx^{-\frac{A_l}{A_\theta+B_\theta}} \left(x^{-\frac{(B_l+A_\theta)a_l}{A_\theta+B_\theta}+a_l} + x^{-\frac{(B_l+A_\theta)a_{l+1}}{A_\theta+B_\theta}+a_{l+1}} \right) \\ &= \quad Cx^{-\frac{A_l+(B_l+A_\theta)a_l}{A_\theta+B_\theta}+a_l} + Cx^{-\frac{A_{l+1}+B_{l+1}+A_\theta a_{l+1}}{A_\theta+B_\theta}+a_{l+1}}. \end{aligned} \tag{3.6}$$

By (3.4) and (3.5), each term of (3.6) is bounded by x^{-1/p_θ}.

(4) $C_1 x^{a_1} < y < |I|$. In this region, by using the fact $B_0 > B_\theta$, we obtain

$$\begin{aligned} \int_{C_1 x^{a_1}}^{|I|} \left|S''_{xy}(x,y)\right|^{-\mu_\theta} y^{-1/p_\theta} dy \quad &\leq \quad Cx^{-\frac{A}{A_\theta+B_\theta}} \int_{C_1 x^{a_1}}^{|I|} y^{-\frac{A_\theta+B_0}{A_\theta+B_\theta}} \\ &\leq \quad Cx^{-\frac{(A_\theta+B_0)a_1+A}{A_\theta+B_\theta}+a_1} \\ &\leq \quad Cx^{-1/p_\theta}, \end{aligned}$$

and this completes the proof. □

We do not have strong bounds on the noncompact faces or vertices of $\Gamma(S''_{xy})$ in general. However, if $\mathcal{M} = \{(x,y)|y_2 = x_2 + S(x_1,y_1)\}$ satisfies a left or right finite type condition, we obtain weak type estimate at some vertices of $\Gamma(S''_{xy})$.

Proposition 3.3. (1) *If $\mathcal{M}$ satisfies a left finite type condition, that is, $A = 0$ in* (2.1)*, then $(|T|_{-1/B_0})^*$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$.*

(2) *If $\mathcal{M}$ satisfies a right finite type condition, that is, $B = 0$ in* (2.1)*, then $|T|_{-1/A_n}$ is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$.*

Proof. To obtain a weak type (1,1) bound of $|T|_{-1/A_n}$, we consider the following operators:

(1) $|T|_{1,n} f(x) = \int_{c_n x^{a_n}}^{C_1 x^{a_1}} \left|S''_{xy}(x,y)\right|^{-1/A_n} f(y)dy$.

(2) $|T|_{n,\infty} f(x) = \int_0^{x^{c_n a_n}} \left|S''_{xy}(x,y)\right|^{-1/A_n} f(y)dy$.

(3) $|T|_{0,1} f(x) = \int_{C_1 x^{a_1}}^{|I|} \left|S''_{xy}(x,y)\right|^{-1/A_n} f(y)dy$.

First, it is easy to see that the same argument in the proof of Proposition 3.2 yields

$$\int_{c'_1 y^{b_1}}^{C'_n y^{b_n}} |S''_{xy}(x,y)|^{-1/A_n} dx \leq C,$$

where C is a constant independent of y, and this immediately implies that $|T|_{1,n}$ is bounded on $L^1(\mathbb{R})$. For $|T|_{n,\infty}$, we observe

$$|T|_{n,\infty} f(x) = \int_0^{x^{c_n a_n}} \left|S''_{xy}(x,y)\right|^{-1/A_n} f(y)dy \leq \frac{C}{x} \int f(y)dy,$$

which implies a weak type (1,1) bound of $|T|_{n,\infty}$. This can be applied to obtain a weak type (1,1) bound of $|T|_{0,1}$. These prove the second part of the proposition. We can prove the first part of the proposition by exchanging the roles of x and y. □

To obtain estimates of Radon transforms, we shall need the following lemma.

Lemma 3.4. *Define*

$$R_{-\mu}f(x_1,x_2)=\int_{\mathbb{R}}|S''_{x_1y_1}(x_1,y_1)|^{-\mu}f(y_1,x_2+S(x_1,y_1))\chi(x_1,x_2,y_1)dy_1.$$

If $|T|_{-\mu}$ is bounded on $L^p(\mathbb{R})$, then $R_{-\mu}$ is bounded on $L^p(\mathbb{R}^2)$.

Proof. Take $f\in L^p(\mathbb{R}^2)$. Then we have

$$\begin{aligned}
&||R_{-\mu}f||_{L^p(\mathbb{R}^2)}\\
&\le C\left[\int_I\int_I\left[\int_I|S''_{x_1y_1}(x_1,y_1)|^{-\mu}f(y_1,x_2+S(x_1,y_1))dy_1\right]^p dx_1dx_2\right]^{1/p}\\
&\le C\left[\int_I\left[\int_I|S''_{x_1y_1}(x_1,y_1)|^{-\mu}\left[\int_I|f(y_1,x_2+S(x_1,y_1))|^p\,dx_2\right]^{1/p}dy_1\right]^p dx_1\right]^{1/p}\\
&\le C\left[\int_I\left[\int_I|S''_{x_1y_1}(x_1,y_1)|^{-\mu}||f(y_1,\cdot)||_{L^p(\mathbb{R})}dy_1\right]^p dx_1\right]^{1/p}\\
&\le C\left[\int_I\Big||T|_{-\mu}||f(*,\cdot)||^p_{L^p(\mathbb{R})}(x_1)\Big|^p dx_1\right]^{1/p}\\
&\le C||f||_{L^p(\mathbb{R}^2)}.
\end{aligned}$$

□

4. Proofs of the theorems

In this section, we shall prove Theorems 1.2, 1.5, and 1.7. The proofs will be based on the interpolation argument in [CSWW], [PSt3], and [StW]. In the proof of Theorem 1.2(1), we use examples constructed in [PSt2]. Phong and Stein used those examples to show the sharpness of the L^2 bounds of T_λ. We observe here that they also give the sharpness of L^p bounds of T_λ. In what follows, $\Re\alpha$ and $\Im\alpha$ denote the real and the imaginary parts of α, respectively.

Proof of Theorem 1.2. (1) To see that the domain $\mathcal{A}$ is the optimal domain for the decay rate of the L^p operator norm of T_λ, we need to show that for every point (C,D) in the reduced Newton diagram, there exists $C_0>0$ such that

$$||T_\lambda||_{L^{\frac{C+D}{C}}\to L^{\frac{C+D}{C}}}\ge C_0\lambda^{\frac{-1}{C+D}} \tag{4.1}$$

as $\lambda\to\infty$. Here, without loss of generality, we may assume that $S(x,0)=0$ and $S(0,y)=0$. If not, we may use $e^{-i\lambda S(0,y)}f_\lambda$ and $e^{-i\lambda S(x,0)}g_\lambda$ instead of f_λ and g_λ below. Therefore, we may assume that there is no pure x^m or y^n term in the Taylor series $\sum_{p,q}c_{pq}x^py^q$ of S. We consider the two cases depending on whether (C,D) is contained in the compact face of the reduced Newton diagram or not. First, let us consider the case that (C,D) is in the compact face. Let $(\alpha,0)$ and $(0,\beta)$ be the intersections of the extended line of the face with p- and q-axes. Then, we clearly have

$$\frac{C}{\alpha}+\frac{D}{\beta}=1. \tag{4.2}$$

Now, define f_λ, g_λ by

$$f_\lambda(y) = \begin{cases} 1 & \text{if} \quad 1 \le \lambda y^\beta \le 1 + c_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_\lambda(x) = \begin{cases} 1 & \text{if} \quad 1 \le \lambda x^\alpha \le 1 + c_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$||f_\lambda||_{\frac{C+D}{C}} \sim \lambda^{-\frac{1}{\beta}\frac{C}{C+D}} \quad \text{and} \quad ||g_\lambda||_{\frac{C+D}{D}} \sim \lambda^{-\frac{1}{\alpha}\frac{D}{C+D}}. \tag{4.3}$$

For every $\epsilon > 0$, we can take $c_1 > 0$ and λ_0 such that

$$|\lambda S(x,y) - S_0| \le \epsilon \quad \text{if} \quad \lambda \ge \lambda_0,$$

where (x, y) is in the support of $f_\lambda(y)g_\lambda(x)$ and $S_0 = \sum' c_{pq}$ when the sum runs through all (p, q) which are on the compact face. Now, we have

$$|\langle T_\lambda f_\lambda, g_\lambda\rangle| \ge c \int f_\lambda dy \int g_\lambda dx \sim \lambda^{-\frac{1}{\alpha}-\frac{1}{\beta}}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we see that there exists $C_0 > 0$ such that

$$\frac{|\langle T_\lambda f_\lambda, g_\lambda\rangle|}{||f||_{\frac{C+D}{C}}||g||_{\frac{C+D}{D}}} \ge C_0 \lambda^{-(\frac{C}{\alpha}+\frac{D}{\beta})\frac{1}{C+D}} = C_0\lambda^{-\frac{1}{C+D}}$$

as $\lambda \to \infty$. This gives (4.1) when (C, D) is not in the noncompact faces. Now, let us consider the case that (C, D) is not contained in a compact face. (C, D) can be contained in the vertical face or in the parallel face. We assume that (C, D) is contained in the vertical face. In this case, put $\alpha = C$ and $\beta = 0$. Take f_λ as a characteristic function of small interval about the origin. Then the argument is parallel to the above. We can use the same argument for the case where (C, D) is in the parallel face.

(2) First, we shall consider the case where S''_{xy} is separate. In this case, we imbed T_λ in an analytic family of operators $T_{\lambda,\alpha}$ of the form

$$T_{\lambda,\alpha} f(x) = \int e^{iS(x,y)}|S_{xy}(x,y)|^\alpha f(y)\chi(x,y)dy.$$

If $\Re\alpha = 1/2$, the result in [PSt1] gives

$$||T_{\lambda,\alpha}||_{L^2\to L^2} \le C(1 + |\Im\alpha|)^2\lambda^{-1/2}. \tag{4.5}$$

If $\Re\alpha = -1/(A_\theta + B_\theta)$, we have $L^{A_\theta/(A_\theta+B_\theta)}$ boundedness of $T_{\lambda,\alpha}$ by Proposition 3.2. By the interpolation theorem [StW], we obtain $L^{(A_\theta+1)/(A_\theta+B_\theta+2)}$ boundedness of $T_{\lambda,0} = T_\lambda$ with norm $O(\lambda^{-1/(A_\theta+B_\theta+2)})$ as $\lambda \to \infty$. This treats the case where S''_{xy} is separate. Now we shall prove a restricted weak type inequality at e_l and e_r and this finishes the proof by real interpolation. We start with decomposing T_λ by using smooth cut-off functions. Let $\eta \in C_0^\infty(-1, 1)$ so that $\eta(s) = 1$ for $|s| \le 1/2$. Set

$$\beta_l(x,y) = \eta(2^l|S''_{xy}(x,y)|) - \eta(2^{l+1}|S''_{xy}(x,y)|).$$

Now, define T^l_λ as

$$T^l_\lambda f(x) = \int e^{i\lambda S(x,y)}\beta_l(x,y)\chi(x,y)f(y)dy.$$

The results of Phong and Stein in [PSt1] yield

$$||T^l_\lambda||_{L^2\to L^2} \le C\lambda^{-1/2}2^{l/2}. \tag{4.6}$$

We shall obtain estimates at e_l. The duality argument can be applied to obtain estimates at e_r. In view of (2.3) and (2.4), (A_0, B_0) is the extreme point of $\Gamma(S''_{xy})$ which is closest to the vertical axis and

$$e_l = ((A_0+1)/(A_0+B_0+2), 1/(A_0+B_0+2)).$$

We claim that

$$||T^l_\lambda||_{L^{p_0}\to L^{p_0,\infty}} \le C2^{-l\gamma}, \tag{4.7}$$

where $p_0 = (A_0+B_0)/A_0$ and $\gamma = 1/(A_0+B_0)$. To prove the claim, we define $A_l(x)$ as

$$A_l(x) = \{y : |S''_{xy}(x,y)| \le 2^{-l+1}\}.$$

The van der Corput type lemma of M. Christ in [Ch1] gives

$$\mathrm{meas}(A_l(x)) \le C2^{-l/B_0}x^{-A_0/B_0}.$$

By Hölder's inequality, we obtain

$$|T^l_\lambda f(x)| \le C2^{-l\gamma}x^{-1/p_0}||f||_{L^{p_0}},$$

which implies (4.7). By interpolation of (4.7) with (4.6) (cf. [CSWW]), we obtain the restricted weak type estimate at e_l.

(3) To obtain $L^p - L^q$ boundedness, we shall first consider the case that $S''_{x_1y_1}$ is separate. We shall follow the procedure in [PSt3]. We shall omit the details but let us recall the procedure in a brief way. First, define an analytic family of operators $R_{\alpha,\beta}$ of two parameters, α and β,

$$\begin{aligned} &R_{\alpha,\beta}f(x_1,x_2) \\ &= \beta\int_0^1\int_{-\infty}^{\infty} f(y_1, x_2 - s + S(x_1,y_1))|S''_{x_1y_1}|^\alpha s^{-1+\beta}\chi(x,y_1)dy_1ds, \end{aligned}$$

where α and β are complex numbers. By using the same argument as in [PSt3], we obtain the boundedness of $R_{\alpha,0}$ from $L^{3/2}(\mathbb{R}^2)$ to $L^3(\mathbb{R}^2)$ when $\Re(\alpha) = 1/3$, with $||R_{\alpha,0}||_{L^{3/2}\to L^3} \le C(1+|\Im\alpha|)^N$ for some $N \in \mathbb{N}$. By interpolation of this with Proposition 3.2, we obtain the desired estimate. Now it remains to show the restricted weak type estimates at e'_l and e'_r. We define R^l as

$$R^l f(x_1,x_2) = \int_{-\infty}^{\infty} f(y_1, x_2 + S(x_1,y_1))\chi(x_1,x_2,y_1)\beta_l(x_1,y_1)dy_1.$$

Set

$$A_l(x_1,x_2) = \{y_1 : S''_{x_1y_1}(x_1,y_1) \le 2^{-l+1}\}.$$

By the van der Corput type lemma of M. Christ in [Ch1], we obtain

$$\mathrm{meas}(A_l(x_1,x_2)) \le C2^{-l/B_0}x_1^{-A_0/B_0}.$$

We consider

$$D_\alpha = \{(x_1,x_2) : |R^l f(x_1,x_2)| > \alpha\}.$$

Hölder's inequality gives

$$|R^l f(x_1,x_2)| \le C2^{-l\gamma}|x_1|^{-1/p_0}||f(\cdot, x_2 + S(x_1,\cdot))||_{L^{p_0}}.$$

This implies that if $(x_1,x_2) \in D_\alpha$, then

$$|x_1| \le C2^{-l\gamma p}\alpha^{-p}||f(\cdot, x_2+S(x_1,\cdot))||^p_{L^p}.$$

Integrating in x_2 yields

$$|x_1| \le C2^{-l\gamma p}\alpha^{-p}||f||^p_{L^p}.$$

We therefore obtain

$$\text{meas}(D_\alpha)^{1/p}\alpha \le C2^{-l\gamma}||f||_{L^p},$$

which immediately implies

$$||R^l||_{L^{p_0}\to L^{p_0,\infty}} \le C2^{-l\gamma}. \tag{4.8}$$

Now we define $R_{l,\beta}$ by

$$\begin{aligned} &R^l_\beta f(x_1,x_2)\\ &= \beta\int_0^1\int_{-\infty}^{\infty} f(y_1,x_2-s+S(x_1,y_1))s^{-1+\beta}\beta_l(x,y)\chi(x,y_1)dy_1ds.\end{aligned}$$

The argument in [PSt3] yields

$$||R^l_\beta||_{L^{3/2}\to L^3} \le C(1+|\Im\beta|)2^{l/3}, \tag{4.9}$$

when $\Re\beta = 0$. By combining (4.9) with (4.8), we have the boundedness of R from $L^{p,1}$ to $L^{q,\infty}$ when $e'_l = (1/p, 1/q)$. The duality argument gives the estimate at e'_r. □

Now, we discuss L^p regularizing properties of R. L^p regularity of R follows from Proposition 3.2 and L^2 regularity of Radon transforms with a damping factor $|S''_{x_1y_1}|^{1/2}$. However, in this paper, we allow two restrictions on $S''_{x_1y_1}$ for the L^p regularity of R as stated above. We do not have strong estimates in Proposition 3.2 if $S''_{x_1y_1}$ is almost translation invariant. In the case of L^p bounds of T_λ and $L^p - L^q$ estimates of R, we could get rid of this restriction using the interpolation theorem in [CSWW]. However this method cannot be applied to L^p regularity because the real interpolation space of Sobolev spaces is usually not a Sobolev space. One more restriction comes from Lemma 4.1 below.

Proof of Theorem 1.5. We imbed R in the analytic family of operators of the form

$$R_\alpha(f)(x_1,x_2) = \int_{-\infty}^{\infty} f(y_1,x_2+S(x_1,y_1))|S''_{x_1y_1}|^\alpha\chi(x_1,y_1)dy_1.$$

We claim that if $\Re\alpha = 1/2$ and $S''_{x_1y_1}$ does not have (1,1) factors, then R_α is bounded from $L^2(\mathbb{R}^2)$ to $L^2_{1/2}(\mathbb{R}^2)$. In what follows $\hat{f}$ denotes the Fourier transform of f in the second variable. Then we have

$$\widehat{R_\alpha(f)}(x_1,\lambda) = T_{\lambda,\alpha}(\hat{f}(\cdot,\lambda))(x_1). \tag{4.10}$$

By (4.10) and (4.5), we know that $(1+|D_{x_2}|^2)^{1/4}R_\alpha$ is bounded on $L^2(\mathbb{R}^2)$ for $\Re\alpha = 1/2$, with norm of polynomial growth in $\Im\alpha$. Now, to prove the claim, it suffices to show that if $\Re\alpha = 1/2$, then $D_{x_1}T_{\lambda,\alpha}$ is bounded on $L^2(\mathbb{R})$ when $A \ne 1$ in (2.1) with norm $O((1+|\Im\alpha|)^N(1+|\lambda|)^{1/2})$. By the argument in [PSt3], it suffices to prove the following lemma.

Lemma 4.1. *The operator $\mathcal{H}$ whose kernel is of the form*

$$|S''_{xy}(x,y)|^{-1/2}|S'''_{xxy}(x,y)||\chi(x,y)| \tag{4.11}$$

is bounded on $L^2(\mathbb{R})$ with norm $O(1)$ if $A \ne 1$ in (2.1).

Proof of Lemma 4.1. We shall use the argument in the proof of Proposition 3.2 with suitable modification. We may assume that $x > 0$ and $y > 0$. We divide the first quadrant into regions of the forms: $\{0 < y < c_n x^{a_n}\}$, $\{c_l x^{a_l} < y < C_l x^{a_l}\}$, $\{C_{l+1} x^{a_{l+1}} < y < c_l x^{a_l}\}$, and $\{C_1 x^{a_1} < y < |I|\}$, and denote these by $E_{\infty,n}$, $E_{l,l}$, $E_{l+1,l}$, and $E_{1,0}$, respectively. Now, we consider operators $\mathcal{H}_{i,j}$ whose kernels are of the form

$$|S''_{xy}(x,y)|^{-1/2}|S'''_{xxy}(x,y)|\chi(x,y)\chi_{E_{i,j}}(x,y),$$

where $\chi_{E_{i,j}}(x,y)$ are characteristic functions of the region $E_{i,j}$. By (2.1), we obtain

$$\begin{aligned}
&|S''_{xy}(x,y)|^{-1/2}|S'''_{xxy}(x,y)| \\
&\leq \quad Cx^{A/2-1}y^{B/2}\prod_{l=1}^{n}\prod_{i=1}^{N_l}|y-r_{l,i}(x)|^{1/2} \\
&\qquad + C\sum_{i'=1}^{n}\sum_{l'=1}^{N_n} x^{A/2}y^{B/2}\prod_{l=1}^{n}\prod_{i=1}^{N_l}|y-r_{l,i}(x)|^{1/2}/(|y-r_{l',i'}(x)|x^{1-a_{l'}}) \\
&= \quad S_1(x,y)+S_2(x,y).
\end{aligned}$$

By (2.2), we obtain

$$S_2(x,y) \sim \sum_{i'=1}^{n}\sum_{l'=1}^{M_n} x^{A/2}y^{B/2}\prod_{l=1}^{n}\prod_{i=1}^{M_l}|x-\tilde{r}_{l,i}(y)|^{1/2}/|x-\tilde{r}_{l',i'}(y)|.$$

Here, we observe that if $A = 0$, then $S_1 \equiv 0$ and if $A \geq 2$, (4.11) is integrable in both x and y with uniformly bounded norm. This implies that the operator $\mathcal{H}$ is bounded on $L^p(\mathbb{R})$ for $1 \leq p \leq \infty$ if $A \geq 2$. Therefore, to prove the lemma, it suffices to show that $\mathcal{H}$ is bounded on $L^2(\mathbb{R})$ when $A = 0$.

Case 1. ($i = \infty$ and $j = n$) In this case

$$\begin{aligned}
&\int_0^{c_n x^{a_n}} \left[|S''_{xy}(x,y)|^{-1/2}|S'''_{xxy}(x,y)|\right]^2 dy \\
&\leq \quad C\int_0^{c_n x^{a_n}} [S_2(x,y)]^2 dy \\
&\leq \quad Cx^{A_n+a_nB-2+a_n}.
\end{aligned}$$

Since $A_n + a_n B - 2 + a_n > -1$, $x^{A_n+a_nB-2+a_n}$ is integrable. This yields L^2 boundedness of $\mathcal{H}_{\infty,n}$.

Case 2. ($i = l$ and $j = l$) In this case

$$\begin{aligned}
&\int_{c_l x^{a_l}}^{C_l x^{a_l}} y^{-1/2}|S''_{xy}(x,y)|^{-1/2}|S'''_{xxy}(x,y)|dy \\
&\leq \quad C\int_{c_l x^{a_l}}^{C_l x^{a_l}} y^{-1/2}S_2(x,y)dy \\
&\leq \quad Cx^{(A_l+a_lB_l+a_l)/2-1} \leq Cx^{-1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{c'_l y^{b_l}}^{C'_l y^{b_l}} x^{-1/2}S_2(x,y)dx \\
&\leq \quad Cy^{(A_lb_l+B_l-b_l)/2} \leq Cy^{-1/2}.
\end{aligned}$$

L^2 boundedness of $\mathcal{H}_{l,l}$ is implied by Lemma 2.4 with $p = q = 2$.

Case 3. ($i = l + 1$ and $j = l$) In this case

$$\begin{aligned}&\int_{C_{l+1}x^{a_{l+1}}}^{c_l x^{a_l}} \left[|S''_{xy}(x,y)|^{-1/2}|S'''_{xxy}(x,y)|\right]^2 dy\\ &\quad\leq \quad Cx^{A_l+a_lB_l+a_l-2} + Cx^{A_{l+1}+a_{l+1}B_{l+1}+a_{l+1}-2}.\end{aligned}$$

Since $A_k + a_kB_k + a_k - 2 > -1$, $x^{A_l+a_lB_l+a_l-2}$ and $x^{A_{l+1}+a_{l+1}B_{l+1}+a_{l+1}-2}$ are integrable. This proves L^2 boundedness of $\mathcal{H}_{l+1,l}$.

Case 4. ($i = 0$ and $j = 1$) In this case

$$\begin{aligned}&\int_0^{c'_1 y^{b_1}} \left[|S''_{xy}(x,y)|^{-1/2}|S'''_{xxy}(x,y)|\right]^2 dx\\ &\leq \quad C\sum_{i'=1}^{n}\sum_{l'=1}^{M_{i'}}\int_0^{c'_1 y^{b_1}} y^{B_0-2b_{l'}}dx\\ &\leq \quad C\sum_{i'=1}^{n}\sum_{l'=1}^{M_{i'}} y^{B_0+b_1-2b_{l'}}.\end{aligned}$$

Since $B_0 + b_1 - 2b_{l'} > -1$, $y^{B_0+b_1-2b_{l'}}$ is integrable in y. This shows that $\mathcal{H}_{0,1}$ is bounded on L^2.

The treatment of all cases gives L^2 boundedness of the operator $\mathcal{H}$ in the case $A = 0$ and this completes the proof. □

Now, we have boundedness of R_α from $L^2(\mathbb{R}^2)$ to $L^2_{1/2}(\mathbb{R}^2)$ for $\Re\alpha = 1/2$ with norm $O((1 + |\Im\alpha|)^N)$. We obtain the desired estimate by applying the complex interpolation theorem in [StW] with Proposition 3.2. □

Now we prove Theorem 1.7.

Proof of Theorem 1.7. We prove the case that $\mathcal{M}$ satisfies a left finite type condition because the other case can be proved by the duality argument.

(1) This is an easy consequence of Proposition 3.3 and complex interpolation with (4.5).

(2) By using Proposition 3.3, we obtain

$$\begin{aligned}&|\{(x_1,x_2); |R_{-1/A_n}f(x_1,x_2)| \geq \beta\}|\\ &= \quad \int |\{x_1; |R_{-1/A_n}f(x_1,x_2)| \geq \beta\}|dx_2\\ &\leq \quad \int \frac{1}{\beta}\int\int |f(y_1, x_2 + S(x_1,y_1)|dy_1dx_1dx_2\\ &\leq \quad \frac{C}{\beta}||f||_{L^1(\mathbb{R}^2)},\end{aligned} \tag{4.12}$$

which implies that R_{-1/A_n} is of weak type (1,1). Now we use the argument in the proof of Theorem 1.2(3) to obtain the desired estimate.

(3) From the proof of Theorem 1.5, we know that $R_{1/2}$ is bounded from L^2 to $L^2_{1/2}$. By (4.12), we know that R_{-1/A_n} is of weak type (1,1). It is easy to see that the complex interpolation theorem implies the desired estimate. □

We conclude this paper with some remarks.

Remark 4.2. (1) Bak obtained sharp $L^p - L^q$ estimates when S is a homogeneous polynomial with left and right finite type conditions [B]. During the preparation of this paper, we learned that Bak, Oberlin and Seeger obtained endpoint $L^p - L^q$ estimates of generalized Radon transforms with the assumption of both left and right finite type conditions [BOS], and that Lee obtained sharp $L^p - L^q$ estimates at all extreme points when S is a real analytic function [L].

(2) Strong endpoint L^p Sobolev estimates at extreme points may not be true (see [Ch2] for a counterexample of the translation invariant case), but there has been evidence which has indicated that we may have strong endpoint decay estimates of the L^p operator norm of T_λ [GS].

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