FIRST COUNTABLE, COUNTABLY COMPACT SPACES AND THE CONTINUUM HYPOTHESIS

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ABSTRACT. We build a model of ZFC+CH in which every first countable, countably compact space is either compact or contains a homeomorphic copy of ω_1 with the order topology. The majority of the paper consists of developing forcing technology that allows us to conclude that our iteration adds no reals. Our results generalize Saharon Shelah's iteration theorems appearing in Chapters V and VIII of *Proper and improper forcing* (1998), as well as Eisworth and Roitman's (1999) iteration theorem. We close the paper with a ZFC example (constructed using Shelah's club–guessing sequences) that shows similar results do not hold for closed pre–images of ω_2.

1. INTRODUCTION

In general topology, the study of the relationship between compactness ("every open cover has a finite subcover") and countable compactness ("every infinite set has a point of accumulation") has a distinguished history reaching back to the earliest days of the subject. It has long been known that these two concepts are not the same, even in the class of first countable spaces (i.e., spaces in which every point has a countable neighborhood base). The simplest example of this is the topological space obtained by taking ω_1, the first uncountable ordinal, and giving it the natural order topology. The space is easily seen to be first countable, and every infinite subset has a limit point essentially because strictly increasing sequences of countable length converge to their supremum in this topology. The space ω_1 is definitely not compact, as the sets of the form $[0, \alpha + 1)$ form an open cover with no finite subcover.

This natural example brings to mind the question of to what extent must a non–compact first countable, countably compact space resemble ω_1. A more precise formulation of this is "is it true that every first countable, countably compact space is either compact, or contains a closed subset homeomorphic to ω_1?". This question first appeared in print at least 15 years ago — it was explicitly asked by the second author in the article [14] because it has some bearing on the theory of non–metrizable manifolds.

This answer to this question was found to be independent of ZFC, the usual axioms of set theory. The second author showed that a very weak form of the axiom $\Diamond$ (so weak that it is compatible with Martin's Axiom) yields a counterexample (see

Fremlin's article [11] for a proof), while the consistency of a positive answer was a byproduct of the torrent of consistency results obtained in the 1980's by Balogh, Dow, Fremlin, and Nyikos ([1], [2], [11]) concerning the Proper Forcing Axiom (PFA) and its consequences for topology.

The goal of this paper is to show that the Continuum Hypothesis does not decide the answer to the question — we produce a model of ZFC in which the Continuum Hypothesis holds and in which first countable, countably compact spaces are either compact or contain a closed subset homeomorphic to ω_1. The results and method contained in this paper have had some impact on current research in set–theoretic topology. For example, Gruenhage [13] has shown that in the model we produce, all countably compact spaces with small diagonal are compact. The notion of forcing we define in the paper has been further generalized by the first author [7] to show that PFA implies that countably tight perfect pre–images of ω_1 contain copies of ω_1, while the iteration technology developed here is crucial for the first author's work on CH and the Moore–Mrówka problem (see [8]).

Our argument is organized along the following lines. In section 2, we show that first countable closed pre–images of ω_1 of size ω_1 behave a lot like ω_1 itself — there are natural analogs of the closed unbounded filter and a lot of the properties of the club filter carry over into this more general context. In section 3, we define a notion of forcing that shoots a copy of ω_1 through a given first countable closed pre–image of ω_1. In sections 4 and 5 we define weakly $< \omega_1$–proper notions of forcing, verify that the notion of forcing defined in section 3 is weakly $< \omega_1$–proper, and prove that this property is preserved by countable support iterations. In section 6 we use weakly $< \omega_1$–properness to prove an iteration theorem useful in constructing models of ZFC+CH. In section 7 we construct a model of ZFC+CH+"first countable, countably compact spaces are either compact or contain a copy of ω_1". Finally, in the last section we show that ω_1 is different from ω_2 in that there is a closed two–to–one pre–image of ω_2 that does not contain a copy of ω_2. The proof uses one of the basic results of Shelah's theory of guessing clubs.

We make some demands regarding the reader's background. In particular, we assume that the reader is comfortable with proper forcing and elementary submodels, as well as their use in topology. Those who are familiar with Dow's papers [3], [4], and [5] should have no trouble with the topological portion of the paper, while Goldstern's [12] provides an excellent background for the theory of iterated forcing. A detailed treatment of totally proper forcing can be found in [9]. A reader needing more background on the topological concepts involved should check out Engelking's encyclopedic reference [10], or Vaughan's article [20] on countable compactness and related ideas.

2. Basics

In this section, we study the structure of first countable topological spaces that can be mapped by a closed continuous function onto the space ω_1. In particular, we show that such spaces have much in common with ω_1 — one can define analogs of "closed unbounded" and "stationary".

Let X be a first countable topological space for which there is a closed (continuous) mapping $\pi : X \to \omega_1$, and let $\mathcal{U}$ be a maximal filter of closed sets that contains $\pi^{-1}(C)$ for every closed unbounded $C \subseteq \omega_1$.

We assume that X has size $\aleph_1$. If CH holds, then every first countably closed pre–image of ω_1 contains a closed subspace with the same properties that is also of size $\aleph_1$, so this assumption causes no loss of generality for our purposes.

We will show that the filter $\mathcal{U}$ on X behaves much as the filter of closed unbounded subsets of ω_1 does. It is not hard to see that $\mathcal{U}$ is closed under countable intersections (this is done in Proposition 2.2 below); we also show that it is possible to define an analog of diagonal intersections, and show that $\mathcal{U}$ is closed under such intersections. The combinatorics exposed in this section are the key to proving that the notion of forcing defined in the next section is proper and does not add reals.

Proposition 2.1. *If $\{x_n : n \in \omega\} \subseteq X$ is such that $\{\pi(x_n) : n \in \omega\}$ is increasing, then $\{x_n : n \in \omega\}$ has a limit point x with $\pi(x) = \sup\{\pi(x_n) : n \in \omega\}$*

Proof. Immediate, because the mapping π is closed. $\qquad\square$

Proposition 2.2. *$\mathcal{U}$ is countably closed.*

Proof. Let $\{A_n : n \in \omega\} \subseteq \mathcal{U}$ be given. We may assume that $A_n \supseteq A_{n+1}$. Choose $x_n \in A_n$ such that the sequence $\{\pi(x_n) : n \in \omega\}$ is strictly increasing. The sequence of x_n's has a limit point x (by the previous proposition), and x witnesses that $\bigcap\{A_n : n \in \omega\}$ is non–empty. A similar proof shows that this intersection is not disjoint to any set in $\mathcal{U}$, so by maximality of $\mathcal{U}$ it must be a member of $\mathcal{U}$. $\quad\square$

Our next project is to define a version of diagonal intersection that is appropriate to this context, and prove that our filter $\mathcal{U}$ is closed under such intersections.

Definition 2.3. Let N be a countable elementary submodel of $H(\lambda)$ that contains X and $\mathcal{U}$. We define the trace of N, denoted $\mathrm{Tr}(N)$, by

$$(2.1) \qquad \mathrm{Tr}(N) = \bigcap_{A \in N \cap \mathcal{U}} \mathrm{cl}(A \cap N).$$

Proposition 2.4. *Let N be a countable elementary submodel of $H(\lambda)$ containing X and $\mathcal{U}$.*

   (1) *$\mathrm{Tr}(N)$ is a non–empty closed set, and $\pi \restriction \mathrm{Tr}(N)$ is constant with value $N \cap \omega_1$.*
   (2) *If $\{N_\alpha : \alpha < \omega_1\}$ is a continuously increasing $\in$–chain of countable elementary submodels of $H(\lambda)$, then*

$$(2.2) \qquad \bigcup_{\alpha < \omega_1} \mathrm{Tr}(N_\alpha) \in \mathcal{U}.$$

Proof. For the first part, let $\{A_n : n \in \omega\}$ enumerate $N \cap \mathcal{U}$. For each $n \in \omega$, select a point $x_n \in N \cap A_0 \cap \cdots \cap A_n$ in such a way that $\pi(x_n) < \pi(x_{n+1})$. The sequence $\{x_n : n \in \omega\}$ has a limit point x that is in $\mathrm{Tr}(N)$ by definition. $\mathrm{Tr}(N)$ is closed by its definition, and the statement about $\pi \restriction \mathrm{Tr}(N)$ also follows easily.

To establish the second part, it suffices to show that the set in (2.2) is closed, and that it is not disjoint to any set in $\mathcal{U}$.

Let Z be the set defined in (2.2). Since X is first countable, to show that Z is closed it suffices to prove that if $\{x_n : n \in \omega\}$ is a sequence of points from Z that converge to a point x, then $x \in Z$.

Suppose we are given such a sequence $\{x_n : n \in \omega\}$. By passing to a subsequence, we may assume that the sequence $\{\pi(x_n) : n \in \omega\}$ is either constant, or strictly

increasing. In the former case, we are done by the first part of the proposition, so assume the latter case occurs.

Let β_n be the unique ordinal with $x_n \in \mathrm{Tr}(N_{\beta_n})$, and let $\beta = \sup\{\beta_n : n \in \omega\}$. It suffices to show that $x \in \mathrm{Tr}(N_\beta)$, i.e., if $A \in N_\beta \cap \mathcal{U}$ is arbitrary, then $x \in \mathrm{cl}(N_\beta \cap A)$.

Note that β is a limit ordinal and so $N_\beta = \bigcup\{N_\alpha : \alpha < \beta\}$. Since the sequence of N_α's is increasing, we have that $A \in N_{\beta_n} \cap \mathcal{U}$ for all but finitely many n. This means that for all large enough n

$$(2.3) \qquad x_n \in \mathrm{cl}(N_{\beta_n} \cap A) \subseteq \mathrm{cl}(N_\beta \cap A),$$

and so $x \in \mathrm{cl}(N_\beta \cap A)$ as required.

Now that we have shown that Z is closed, to show that $Z \in \mathcal{U}$ it suffices to take an arbitrary $B \in \mathcal{U}$ and prove that $Z \cap B \neq \emptyset$. Given such a B, choose countable $M \prec H(\lambda)$ containing X, Z, $\mathcal{U}$, B, and $\{N_\alpha : \alpha < \omega_1\}$.

If $\delta = M \cap \omega_1$, then $\delta = N_\delta \cap \omega_1$ as well. Note also that $N_\delta \subseteq M$ by the properties of the sequence $\{N_\alpha : \alpha < \omega_1\}$, and clearly since $|X| = \aleph_1$, for $A \in N_\delta \cap \mathcal{U}$ we have $N_\delta \cap A = M \cap A$. Thus we have $\mathrm{Tr}(M) \subseteq \mathrm{Tr}(N_\delta)$. Since $\mathrm{Tr}(M)$ is a non–empty subset of B, any $x \in \mathrm{Tr}(M) \subseteq \mathrm{Tr}(N_\delta)$ witnesses that $Z \cap B$ is non–empty. $\qquad\square$

Definition 2.5. Call a subset Y of X *large* if it meets every set in $\mathcal{U}$. Otherwise, Y is said to be small.

Note that in light of the fact that $\mathcal{U}$ is countably complete, we have that a countable union of small sets is small.

Definition 2.6. A promise is a function f such that

- $\mathrm{dom}\, f$ is a large subset of X,
- $f(x)$ is an open neighborhood of x for $x\, \mathrm{dom}\, f$.

Definition 2.7. A point $y \in X$ is banned by a promise f if $\{x \in \mathrm{dom}\, f : y \in f(x)\}$ is small. Let $\mathrm{Ban}\, f$ be the set of all points y that are banned by f.

Proposition 2.8. *If y is not banned by a promise f, then there is an open neighborhood U of y for which*

$$(2.4) \qquad \{x \in \mathrm{dom}\, f : U \subseteq f(x)\} \text{ is large.}$$

Thus $\mathrm{Ban}\, f$ is closed.

Proof. Let $\{U_n : n \in \omega\}$ be a neighborhood base for y. If $y \in f(x)$, then there is an n for which $U_n \subseteq f(x)$. Since a countable union of small sets is small, there must be a single n for which $\{x \in \mathrm{dom}\, f : U_n \subseteq f(x)\}$ is large, as desired. $\qquad\square$

Proposition 2.9. $\mathrm{Ban}\, f$ *is small.*

Proof. If not, then we know $\mathrm{Ban}\, f \in \mathcal{U}$ as it is a closed set. For each $y \in \mathrm{Ban}\, f$, choose a set $A_y \in \mathcal{U}$ disjoint to $\{x \in \mathrm{dom}\, f : y \in f(x)\}$. Let $\{N_\alpha : \alpha < \omega_1\}$ be a tower of models as in the second part of Proposition 2.4, chosen so that both f and the function sending $y \in \mathrm{Ban}\, f$ to A_y are in N_0.

Since $\mathrm{dom}\, f$ is large, by Proposition 2.4 there is an α and $x \in \mathrm{dom}\, f$ such that $x \in \mathrm{Tr}(N_\alpha)$. This means that $x \in A_y$ for all $y \in N_\alpha \cap \mathrm{Ban}\, f$ and $x \in \mathrm{cl}(\mathrm{Ban}\, f \cap N_\alpha)$. This latter point means that $f(x)$ must contain a point $y \in \mathrm{Ban}\, f \cap N_\alpha$ and this contradicts the fact that $x \in A_y$ for this y. $\qquad\square$

3. A NOTION OF FORCING

In this section, we will define a notion of forcing that will shoot a copy of ω_1 into a given first countable closed pre–image of ω_1. This notion of forcing is proper, and it adds no reals. Before we begin, we need some definitions from [9] and [6].

Definition 3.1. Let P be a notion of forcing, and let N be a countable elementary submodel of $H(\lambda)$ for some large enough regular λ. We say a condition $q \in P$ is totally (N, P)–generic if whenever D is a dense open subset of P that is in N, we can find a condition $p \in N \cap D$ with $q \leq p$. We say that P is totally proper if, given N as above, every $p \in N \cap P$ has a totally (N, P)–generic extension q.

Definition 3.2. Define a notion of forcing $P = P_X$ by putting p into P if and only if $p = ([p], \Phi_p)$, where
   (1) $[p]$ is a countable closed subset of X,
   (2) Φ_p is a countable collection of promises,
   (3) $\pi \upharpoonright [p]$ is one–to–one.
A condition q extends p if
   (4) $[q]$ "end extends" $[p]$, i.e., if $x \in [q] \setminus [p]$, then $\pi(x) > \max \operatorname{ran}(\pi \upharpoonright [q])$,
   (5) $\Phi_q \supseteq \Phi_p$,
   (6) for each promise $f \in \Phi_p$ the set

$$(3.1) \qquad Y(f, q, p) = \{x \in \operatorname{dom} f : [q] \setminus [p] \subseteq f(x)\}$$

   is large, and $f \upharpoonright Y(f, q, p) \in \Phi_q$.

Definition 3.3. Given $p \in P$ and $D \subseteq P$ dense open, we say a point $x \in X$ is *bad for p and D* if x has an open neighborhood U_x for which there is no $q \leq p$ with $q \in D$ and $[q] \setminus [p] \subseteq U_x$. We let $\operatorname{Bad}(p, D)$ be the set of all points x that are bad for p and D.

Proposition 3.4. *If $p \in P$ and $D \subseteq P$ is dense open, then $\operatorname{Bad}(p, D)$ is small.*

Proof. Suppose not. Then the function f with domain $\operatorname{Bad}(p, D)$ that sends x to the neighborhood U_x from the previous definition is a promise. Let $q = ([q], \Phi_p \cup \{f\})$; clearly q extends p in P. Since D is dense, there is an extension $r \leq q$ with $r \in D$. By definition, this means

$$(3.2) \qquad Y(f, r, q) = \{x \in \operatorname{dom} f : [r] \setminus [q] \subseteq f(x)\}$$

is large. In particular, it is non–empty and this is a contradiction, for if x is in $Y(f, r, q)$, then $[r] \setminus [q] = [r] \setminus [p]$ is contained in $f(x) = U_x$. $\qquad\square$

Theorem 1. *Let N be a countable elementary submodel of $H(\lambda)$ containing X and P, and let $p \in P$ be arbitrary. Further assume $\bar{x} \in \operatorname{Tr}(N)$ and U is a neighborhood of $\bar{x}$. Then there is a totally (N, P)–generic $q \leq p$ with $[q] \setminus [p] \subseteq U$. In particular, P is totally proper.*

Proof. Let $\{D_n : n \in \omega\}$ list the dense open subsets of P that are in N. Let $\{U_n : n \in \omega\}$ be a decreasing neighborhood base for $\bar{x}$ with $U_0 = U$. We define by induction on n a sequence $\{p_n : n \in \omega\}$ and function $h \in {}^\omega \omega$ such that
   (a) $p_0 = p$, $h(0) = 0$,
   (b) $p_{n+1} \leq p_n$,
   (c) $p_{n+1} \in N \cap D_n$,
   (d) $h(n) < h(n + 1)$,

(e) $[p_{n+1}] \setminus [p_n] \subseteq U_{h(n+1)}$,

(f) if f is a promise in Φ_i for some i, then there is a stage $n \geq i$ for which

$$(3.3) \qquad K(f, n, i) = \{x \in Y(f, p_n, p_i) : U_{h(n+1)} \subseteq f(x)\}$$

is large.

Claim 3.5. *If we can carry out the above construction, then $\{p_n : n < \omega\}$ has a lower bound q with $[q] \setminus [p] \subseteq U$.*

Proof of Claim. Let $[q] = \bigcup_{n \in \omega} [p_n] \cup \{\bar{x}\}$.

Subclaim 1. $[q]$ is closed.

Proof of Subclaim 1. Let $\{z_k : k \in \omega\} \subseteq [q]$ converge to a point z. If there is an n so that $z_k \in [p_n]$ for infinitely many k, then $z \in [p_n] \subseteq [q]$ as well because $[p_n]$ is closed. If no such n exists, then conditions (d) and (e) of our construction guarantee that the sequence of z_k's converges to $\bar{x}$, so $z = \bar{x} \in [q]$ as required. $\square$

Clearly we have that $\pi \restriction [q]$ is one–to–one, and $[q] \setminus [p] \subseteq U$ as well.

Subclaim 2. If $f \in \Phi_{p_i}$ for some i, then

$$(3.4) \qquad Y(f, q, p_i) := \{x \in \operatorname{dom} f : [q] \setminus [p_i] \subseteq f(x)\}$$

is large.

Proof of Subclaim 2. Let $n \geq i$ be as in condition (f) for the promise f. We will show that $K(f, n, i) \subseteq Y(f, q, p_i)$. To do this, fix $x \in K(f, n, i)$; we must verify that $[q] \setminus [p_i] \subseteq f(x)$. This is trivial, as

$$
\begin{aligned}
[q] \setminus [p_i] &= [q] \setminus [p_n] \cup [p_n] \setminus [p_i] \\
&\subseteq U_{h(n+1)} \cup [p_n] \setminus [p_i] \qquad \text{by (e)} \\
&\subseteq U_{h(n+1)} \cup f(x) \qquad \text{as } x \in Y(f, p_n, p_i) \\
&\subseteq f(x) \cup f(x) \qquad \text{by (f), as } x \in K(f, n, i) \\
&= f(x).
\end{aligned}
$$

$\square$

Now if we define

$$(3.5) \qquad \Phi_q = \bigcup_{n \in \omega} \Phi_{p_n} \cup \bigcup_{n \in \omega} \{f \restriction Y(f, q, p_n) : n \in \omega\},$$

then $q := ([q], \Phi_q)$ is as advertised. $\square$

Now how do we carry out the construction? We assume that some bookkeeping procedure has been fixed at the start so that at each stage we will be handed a promise f for which we must ensure condition (f), and such that every promise appearing along the way will be so considered.

The case where $n = 0$ is trivial, so assume we are given p_n and $h \restriction n + 1$ and that our bookkeeping hands us $f \in \Phi_{p_i}$ for some $i \leq n$.

Let $f' = f \restriction Y(f, p_n, p_i)$. Clearly f' is a promise and $f' \in N$. Since $\operatorname{Ban} f' \in N$ is small, we know that $\bar{x} \notin \operatorname{Ban} f'$ — it is a consequence of the fact that $\bar{x} \in \operatorname{Tr}(N)$. Proposition 2.8 tells us we can choose $h(n+1) > h(n)$ large enough so that

$$(3.6) \qquad K(f, n, i) = \{x \in Y(f, p_n, p_i) : U_{h(n+1)} \subseteq f(x)\}$$

is large. Thus we have taken care of (f) for the promise f.

Now the set $\operatorname{Bad}(p_n, D_n)$ is a small set in N (by Proposition 3.4), so we can find $A \in N \cap \mathcal{U}$ disjoint to it. Since $x \in \operatorname{Tr}(N)$, we know that $x \in \operatorname{cl}(N \cap A)$, and

so there is a point $z \in N \cap U_{h(n+1)}$ such that $z \notin \mathrm{Bad}(p_n, D_n)$. Since X is first countable, z has a neighborhood V *in N* with $V \subseteq U_{h(n+1)}$. Thus we can apply the definition of $z \notin \mathrm{Bad}(p_n, D_n)$ inside of N to get $p_{n+1} \leq p_n$ in $N \cap D_n$ with

$$(3.7) \qquad\qquad [p_{n+1}] \setminus [p_n] \subseteq V \subseteq U_{h(n+1)}.$$

This choice of p_{n+1} satisfies conditions (b)–(e), so we have successfully carried out the construction at stage $n + 1$. In light of Claim 3.5, we have now given a proof of the theorem. $\qquad\square$

4. Weak $< \omega_1$–properness

In this section, we begin dealing with some rather technical conditions that will guarantee that we are able to use iterated forcing to produce a model of ZFC in which the Continuum Hypothesis holds.

The following definition pins down some notation that we will be utilizing throughout the rest of the paper.

Definition 4.1. Let λ be some large regular cardinal, and let $x \in H(\lambda)$ be arbitrary. A sequence $\mathfrak{N} = \langle N_\beta : \beta \leq \alpha \rangle$ is said to be an x–suitable α–tower of models if

- N_β is a countable elementary submodel of $H(\lambda)$ for $\beta \leq \alpha$,
- $x \in N_0$,
- $\langle N_\gamma : \gamma \leq \beta \rangle \in N_{\beta+1}$ for $\beta < \alpha$,
- $\mathfrak{N}$ is continuous at limit ordinals.

We will usually suppress reference to the parameter x in the above definition — typically x will be the notion of forcing under discussion.

Definition 4.2. Let P be a notion of forcing, and let $\mathfrak{N}$ be a relevant tower of models. Let $\mathrm{Spec}(\mathfrak{N}, \dot{G}_P)$ be some fixed P–name for the set (in the generic extension) of ordinals $i \leq \alpha$ for which $N_i[\dot{G}_P] \cap \mathrm{Ord} = N_i \cap \mathrm{Ord}$. If $G \subseteq P$ is generic, then $\{i \leq \alpha : N_i[G] \cap \mathrm{Ord} = N_i \cap \mathrm{Ord}\}$ is called the *spectrum of genericity of G with respect to $\mathfrak{N}$*.

Definition 4.3. Given $\alpha < \omega_1$, a notion of forcing P is said to be weakly α–proper if whenever $\mathfrak{N}$ is a suitable α–tower and $p \in N_0 \cap P$, there is a $q \leq p$ such that q is (N_α, P)–generic, and

$$(4.1) \qquad\qquad q \Vdash \mathrm{Spec}(\mathfrak{N}, \dot{G}) \text{ has order–type } \alpha + 1.$$

Such a q is said to be weakly $(\mathfrak{N}, P)$–generic. If P is weakly β–proper for all $\beta \leq \alpha$, then we say P is weakly $\leq \alpha$–proper. We say P is weakly $< \omega_1$–proper if it is weakly α–proper for all $\alpha < \omega_1$.

Note that (4.1) is significantly weaker than the statement

$$(4.2) \qquad\qquad \{\beta \leq \alpha : q \text{ is } (N_\beta, P)\text{–generic}\} \text{ has order–type } \alpha + 1$$

as it is possible for a condition q to satisfy (4.1) without being (N_β, P)–generic for some specific $\beta < \alpha$. We use (4.1) instead of (4.2) in our formulation of weak α–properness so that we can prove iteration theorems — the corresponding formulation of weak α–properness based on (4.2) is not even preserved by iterations of length two.

Definition 4.4. Let $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$ be a suitable tower. We say that $I \subseteq \alpha + 1$ is an $\mathfrak{N}$–suitable index set if I is closed in α and for $\beta \in I$,

$$(4.3) \qquad\qquad\qquad\qquad I \cap \beta \in N_{\beta+1}.$$

If the tower $\mathfrak{N}$ is clear from the context, then we will often say only that I is a suitable index set.

Proposition 4.5. Let P be a notion of forcing, and let $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$ be a suitable tower.

(1) If $I \subseteq \alpha$ is a suitable index set, then $\mathfrak{N} \upharpoonright I = \langle N_i : i \in I \rangle$ is a suitable tower.
(2) $\Vdash_P$ "$\mathrm{Spec}(\mathfrak{N}, \dot{G}_P)$ is an $\mathfrak{N}[G]$–suitable index set".

Proof. The first statement follows immediately from the definition. Regarding the notation of the second statement, if $G \subseteq P$ is generic, then

$$(4.4) \qquad\qquad\qquad\qquad \mathfrak{N}[G] := \langle N_i[G] : i \leq \alpha \rangle.$$

Let G be an arbitrary generic subset of P. Previously known results (see Theorem 2.11 of Chapter III in [19]) tell us that $\mathfrak{N}[G]$ is a suitable tower in $V[G]$. Let I denote the interpretation of the P–name $\mathrm{Spec}(\mathfrak{N}, \dot{G}_P)$. Since $\mathfrak{N}[G]$ is continuous at limit ordinals, it follows that I is a closed subset of $\alpha + 1$. For any $\beta \leq \alpha$, $I \cap \beta$ is definable from $\mathfrak{N} \upharpoonright \beta$ and G. Since both G and $\mathfrak{N} \upharpoonright \beta$ are elements of $N_{\beta+1}[G]$, it follows that $I \cap \beta \in N_{\beta+1}[G]$ as required. $\qquad\square$

The following lemma has a fairly easy proof that is left to the reader — it is simply a version of III.3.3.G in [19] and it is proved in the same fashion.

Lemma 4.6. Let $P * \dot{Q}$ be a forcing iteration of length two, and let $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$ be a suitable tower. A condition $p * \dot{q} \in P * \dot{Q}$ is weakly $(\mathfrak{N}, P * \dot{Q})$–generic if and only if p is weakly $(\mathfrak{N}, P)$–generic and

$$(4.5) \qquad p \Vdash \dot{q} \text{ is weakly } (\mathfrak{N}[G] \upharpoonright \mathrm{Spec}(\mathfrak{N}, \dot{G}_P), \dot{Q})\text{–generic}.$$

Notice that (4.5) is stronger than requiring that p forces $\dot{q}$ to be weakly $(\mathfrak{N}[G], \dot{Q})$–generic — it can be shown that the lemma is false once this change is made. However, the lemma is strong enough to prove the following corollary.

Corollary 4.7. If P is weakly α–proper and $\dot{Q}$ is a P–name for a weakly α–proper notion of forcing, then $P * \dot{Q}$ is weakly α–proper.

In the next section, we will prove a full–fledged iteration theorem for weakly $< \omega_1$–proper notions of forcing.

We now show that the concept of weak $< \omega_1$–properness has some relevance to our topological question by verifying that the notion of forcing from section 3 is weakly $< \omega_1$–proper. In the introduction, we mentioned that Fremlin's paper [11] contains a simple construction (assuming the axiom $\diamondsuit$) of a first countable perfect pre–image of ω_1 that contains no copy of ω_1; for this particular space, the notion of forcing we have been looking at fails to be ω–proper.

Definition 4.8. We call an ordinal $\alpha < \omega_1$ *good* if whenever $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$ is a relevant α–tower, there is a point $\bar{x} \in \mathrm{Tr}(N_\alpha)$ such that for every $p \in N_\alpha \cap P$ and open neighborhood U of $\bar{x}$, there is a $q \leq p$ such that

- q is totally (N_α, P)–generic,

- q is weakly $(\mathfrak{N}, P)$–generic,
- $[q] \setminus [p] \subseteq U$.

Lemma 4.9. *Let $\alpha < \omega_1$ be an infinite ordinal that is closed under addition. Then there is a sequence $\{\alpha_n : n \in \omega\}$ of ordinals such that*

(1) $\alpha = \sup_{n \in \omega} \alpha_n$,
(2) $\alpha_n < \alpha_{n+1}$,
(3) $\mathrm{otp}(\alpha_{n+1} \setminus \alpha_n)$ *is closed under addition.*

Proof. Let $C \subseteq \omega_1$ be the set of countable ordinals that are closed under addition (including the ordinal 1). Note that C is closed unbounded in ω_1. If α happens to be a limit point of C, then the lemma is easy, so assume this does not happen. In this case, there is a maximal $\beta < \alpha$ that is closed under addition. The ordinal $\omega \cdot \beta$ is also closed under addition hence $\alpha \leq \omega \cdot \beta$. Since α is closed under addition, we must have $\alpha = \omega \cdot \beta$ and we are done if we set $\alpha_n = n \cdot \beta$. $\square$

Theorem 2. *If $\alpha < \omega_1$ is closed under addition, then α is good.*

Proof. The proof is by induction, with the case $\alpha = 1$ already handled by Theorem 1. By Lemma 4.9, we can fix an increasing sequence $\{\alpha_n : n \in \omega\}$ cofinal in α such that for each n (letting α_{-1} denote 0), the ordinal $\beta_n := \mathrm{otp}(\alpha_n \setminus \alpha_{n-1})$ is closed under addition.

For each n, we define

$$(4.6) \qquad \mathfrak{N}_n = \langle N_i : \alpha_{n-1} < i \leq \alpha_n \rangle.$$

Note that $\mathfrak{N}_n$ is a relevant β_n–tower, and β_n is good by the induction hypothesis.

For each i, there is a point $x_i \in \mathrm{Tr}(N_{\alpha_i}) \cap N_{\alpha_i+1}$ that witnesses that β_i is good. There is an infinite $I \subseteq \omega$ such that the sequence $\{x_i : i \in I\}$ converges in X to a point $\bar{x} \in \mathrm{Tr}(N_\alpha)$. We will show that this point $\bar{x}$ witnesses that α is good.

To do this, let $p \in N_0 \cap P$ and let an open neighborhood U of $\bar{x}$ be given. By throwing away finitely many members of I, we can assume $\{x_i : i \in I\} \subseteq U$. Let $\{B_n : n \in \omega\}$ be a decreasing neighborhood base for $\bar{x}$.

By induction on $n \in \omega$ we will produce objects p_n, $k(n)$, I_n, and U_n so that

(1) $p_0 = p$, $I_0 = I$, $U_0 = U$,
(2) $k_n = \min(I_{n+1})$,
(3) $p_{n+1} \leq p_n$, $k(n+1) > k(n)$, $I_{n+1} \subseteq I_n$, $U_{n+1} \subseteq U_n \cap B_n$,
(4) p_{n+1} is totally $(N_{\alpha_{k(n)}}, P)$–generic,
(5) p_{n+1} is weakly $(\mathfrak{N}_{k(n)}, P)$–generic,
(6) U_n is an open neighborhood of $\bar{x}$,
(7) $[p_{n+1}] \setminus [p_n] \subseteq U_{n+1}$,
(8) I_n is an infinite subset of ω,
(9) $\{x_i : i \in I_n\} \subseteq U_n$,
(10) for each promise f appearing in Φ_{p_m} for some $m < \omega$, there is a stage n such that

$$(4.7) \qquad \{x \in Y(f, p_n, p_m) : U_{n+1} \subseteq f(x)\} \text{ is large.}$$

At a stage of the construction, we will have p_n, I_n, U_n, and a promise $f \in \Phi_{p_i}$ for some $i \leq n$ that we must take care of. The first thing we do is find $U_{n+1} \subseteq U_n \cap B_n$ such that

$$(4.8) \qquad \{x \in Y(f, p_n, p_i) : U_{n+1} \subseteq f(x)\} \text{ is large.}$$

This is done just as in the proof of Theorem 1. Once this is done, we thin out I_n to I_{n+1} so that $\{x_i : i \in I_{n+1}\} \subseteq U_{n+1}$ and $k(n) > k(n-1)$. Now in the model $N_{\alpha_{k(n)}+1}$ we find a neighborhood V of $x_{k(n)}$ with $V \subseteq U_{n+1}$. Finally, inside the model $N_{\alpha_{k(n)}+1}$ we apply our induction hypothesis to p_n and the open set V to obtain p_{n+1}. Now the verification that the sequence $\{p_n : n \in \omega\}$ has a lower bound proceeds just as in the proof of Theorem 1. $\qquad\square$

Corollary 4.10. *P is weakly $< \omega_1$–proper.*

Proof. We prove this by induction on $\alpha < \omega_1$; the case where $\alpha = 1$ is handled by Theorem 1.

Let $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$ be a relevant α–tower for some $\alpha < \omega_1$, and let $p \in N_0 \cap P$ be arbitrary.

If α is closed under addition, then Theorem 2 tells us there is a totally (N_α, P)–generic $q \leq p$ that is weakly $(\mathfrak{N}, P)$–proper.

If $\alpha = \beta + \gamma$ for some strictly smaller β and γ, then we apply the induction hypothesis for β in the model $N_{\beta+1}$ to get a condition $q_0 \leq p$ that is weakly $(\mathfrak{N} \upharpoonright \beta + 1, P)$–generic, and then we apply the induction hypothesis for γ to the tower $\mathfrak{N} \upharpoonright (\beta, \gamma]$ to get the required condition q. $\qquad\square$

5. Iterations of weakly $< \omega_1$–proper forcings

We now prove a theorem that shows weak $< \omega_1$–properness is preserved by countable support iterations — this will be needed when we prove our main iteration theorem in the next section.

Before embarking on a proof, we set up our conventions regarding iterated forcing. For the most part, we follow the conventions of [19] and [12], though our notation differs in that we take "$q \leq p$" to mean "q extends p". We view conditions in the limit of a countable support iteration as functions with countable domains, so in particular we speak about domains of conditions as opposed to supports of conditions. We also assume that each partial order has a maximum element.

Theorem 3. *Let $\mathbb{P} = \langle P_\xi, \dot{Q}_\xi : \xi < \kappa \rangle$ be a countable support iteration of forcings such that for each $\xi < \kappa$,*

$$(5.1) \qquad\qquad \Vdash_{P_\xi} \dot{Q}_\xi \text{ is weakly} < \omega_1\text{–proper.}$$

If we are given objects α, β, $\mathfrak{N}$, ξ, ζ, and p such that

(1) $\beta \leq \alpha < \omega_1$,
(2) $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$ *is a suitable tower,*
(3) $\xi < \zeta$ *in $N_0 \cap (\kappa + 1)$,*
(4) $p \in N_0 \cap P_\zeta$,

then there is a condition $r \in P_\zeta$ such that

(5) $\operatorname{dom} r \subseteq N_\alpha \cap [\xi, \zeta)$,
(6) *whenever $q \in P_\xi$ satisfies*
   (a) $q \leq p \upharpoonright \xi$,
   (b) $q \Vdash \operatorname{Spec}(\mathfrak{N}, \dot{G}_\xi)$ *is of order–type $\beta + 1$,*
  then
   (c) $q \cup r \leq p$ *in P_ζ,*
   (d) $q \cup r \Vdash \operatorname{Spec}(\mathfrak{N}, \dot{G}_\zeta)$ *is of order–type $\beta + 1$.*

Proof. The proof requires lots of nested induction arguments, so we will take a little time and invest in notation that will make things a little easier.

For countable ordinals α and β, let $\circledast(\alpha, \beta)$ denote the statement:

$\circledast(\alpha, \beta)$: Whenever $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$ is a suitable tower, $\xi < \zeta$ in $N_0 \cap (\kappa + 1)$, and $p \in N_0 \cap P_\zeta$, then there is a condition $r \in P_\zeta$ such that $\operatorname{dom} r \subseteq N_\alpha \cap [\xi, \zeta)$, and whenever $q \in P_\xi$ satisfies

$$\text{(5.2)} \qquad q \leq p \restriction \xi$$

and

$$\text{(5.3)} \qquad q \Vdash \operatorname{Spec}(\mathfrak{N}, \dot{G}_\xi) \text{ is of order–type } \beta + 1,$$

then $q \cup r$ satisfies

$$\text{(5.4)} \qquad q \cup r \leq p$$

and

$$\text{(5.5)} \qquad q \cup r \Vdash \operatorname{Spec}(\mathfrak{N}, \dot{G}_\zeta) \text{ is of order–type } \beta + 1.$$

We prove the statement $\circledast(\alpha, \beta)$ by induction on $\omega_1 \times \omega_1$ with the lexicographic order. Note that if $\beta > \alpha$, then $\circledast(\alpha, \beta)$ holds trivially as there are no conditions q satisfying (5.3)

Assume that the statement $\circledast(\alpha', \beta')$ holds for all pairs $(\alpha', \beta') <_{\text{lex}} (\alpha, \beta)$ and let $\mathfrak{N}$, ξ, ζ, and p be given.

The proof of the following claim is a standard "patching" argument. We include it for the convenience of readers who do not deal with iterated forcing on a daily basis.

Claim 5.1. *Suppose $E \subseteq P_\xi$ is dense and open in the set of conditions that satisfy (5.2) and (5.3). If for each $u \in E$ there is an $r_u \in P_\zeta$ such that $\operatorname{dom} r_u \subseteq N_\alpha \cap [\xi, \zeta)$, $u \cup r_u \leq p$, and*

$$\text{(5.6)} \qquad u \cup r_u \Vdash \operatorname{otp}(\operatorname{Spec}(\mathfrak{N}, \dot{G}_\zeta)) = \beta + 1,$$

then there is a single $r \in P_\zeta$ such that $\operatorname{dom} r \subseteq N_\alpha \cap [\xi, \zeta)$ and whenever $q \in P_\xi$ satisfies (5.2) and (5.3), $q \cup r$ satisfies (5.4) and (5.5).

Proof. Let D be the set of conditions in P_ξ that are either in E are that have no extension in E. The set D is dense and open in P_ξ, so we can find an antichain $\mathcal{A}$ maximal in P_ξ that consists of elements of D. Let $\mathcal{A}_0 = \mathcal{A} \cap E$. For each $u \in \mathcal{A}_0$, we let r_u be the object given to us by our hypothesis, and let s_u be r_u augmented in the canonical way (each partial order has a maximum element!) so that $\operatorname{dom} s_u = N_\alpha \cap [\xi, \zeta)$. For $u \in \mathcal{A} \setminus \mathcal{A}_0$, we let s_u be any condition in P_ζ with domain $N_\alpha \cap [\xi, \zeta)$.

What follows now is a typical argument — we patch together all of the s_u's into a single object r of the type we need. We define r by induction on $N_\alpha \cap [\xi, \zeta)$. Given $\eta \in N_\alpha \cap [\xi, \zeta)$, we define

$$\text{(5.7)} \qquad r(\eta) = \{\langle u \cup (r \restriction \eta), s_u(\eta) \rangle : u \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is a maximal antichain, it follows that $r(\eta)$ is a P_η–name for a condition in $\dot{Q}_\eta$. This defines r and it follows from the construction that $r \in P_\zeta$ and $\operatorname{dom} r = N_\alpha \cap [\xi, \zeta)$.

Now why does r work? Note that if $q \in P_\xi$ satisfies (5.2) and (5.3), then q forces the existence of a $u \in \mathcal{A}_0$ for which "$r = s_u$", and so (5.4) and (5.5) follow. $\square$

Case β is not closed under addition. . If β is not closed under addition, then we can find ordinals γ and γ' less than β so that $\beta = \gamma + 1 + \gamma'$. Note that we do not require that γ and γ' are non–zero.

Definition 5.2. We define E_i to be the set of all conditions $u \in P_\xi$ that extend $p \upharpoonright \xi$ and that force each of the following statements to hold:

$$(5.8) \qquad\qquad \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi)) = \beta + 1$$

and

$$(5.9) \qquad\qquad i \text{ is the } \gamma + 1^{\mathrm{st}} \text{ member of } \mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi).$$

We also define $E = \bigcup_{i < \alpha} E_i$.

Note that E is dense below any condition in P_ξ that satisfies (5.2) and (5.3) — we need only extend to decide a specific value i for the $\gamma + 1^{\mathrm{st}}$ element of $\mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi)$.

Claim 5.3. *For each $i < \alpha$, there is a condition $r_i \in P_\zeta$ such that if $u \in E_i$, then*

$$(5.10) \qquad\qquad u \cup r_i \leq p$$

and

$$(5.11) \qquad\qquad u \cup r_i \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}, \dot{G}_\zeta)) = \beta + 1.$$

Proof. We will obtain r_i by applying our induction hypothesis twice. Inside the model N_{i+1}, we can apply our induction hypothesis to $\mathfrak{N}_i = \langle N_j : j \leq i \rangle$ to get a condition $s_i \in N_{i+1} \cap P_\zeta$ such that whenever $q \in P_\xi$ extends $p \upharpoonright \xi$ and forces the order–type of $\mathrm{Spec}(\mathfrak{N}_i, \dot{G}_\zeta)$ to be $\gamma + 1$, then

$$(5.12) \qquad\qquad q \cup s_i \leq p$$

and

$$(5.13) \qquad\qquad q \cup s_i \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}_i, \dot{G}_\zeta)) = \gamma + 1.$$

Let $\mathfrak{N}^i = \langle N_j : i < j \leq \alpha \rangle$. This tower is of length $\leq \alpha$, and since $\gamma' < \beta$ we may apply our induction hypothesis to get a condition $r_i \in P_\zeta$ such that $\mathrm{dom}\, r_i \subseteq N_\alpha \cap [\xi, \zeta)$ and whenever $q \in P_\xi$ forces $\mathrm{Spec}(\mathfrak{N}^i, \dot{G}_\xi)$ to be of order–type $\gamma' + 1$, we have

$$(5.14) \qquad\qquad q \cup r_i \leq s_i$$

and

$$(5.15) \qquad\qquad q \cup r_i \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}^i, \dot{G}_\zeta)) = \gamma' + 1.$$

(Note that since $\xi \cap \mathrm{dom}\, s_i = \emptyset$, any condition in P_ξ extends $s_i \upharpoonright \xi$.)

Now suppose $u \in E_i$. Since u forces (5.8) and (5.9) to hold and $\beta = \gamma + 1 + \gamma'$, it must be the case that

$$(5.16) \qquad\qquad u \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}^i, \dot{G}_\xi)) = \gamma' + 1.$$

By our definition of r_i, we have that $u \cup r_i \leq s_i$ in P_ζ and

$$(5.17) \qquad\qquad u \cup r_i \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}^i, \dot{G}_\zeta)) = \gamma' + 1.$$

Taking this together with (5.9), we have that (5.11) holds as required. The proof is completed by a straightforward verification that $u \cup r_i \leq p$. $\qquad\square$

As commented earlier, the set E is dense below any condition that forces (5.2) and (5.3) to hold, and so we may apply Claim 5.1 to get an object r that will establish this particular instance of $\circledast(\alpha, \beta)$.

Case β is closed under addition. The proof of $\circledast(\alpha,\beta)$ in this case is by induction on the ordinal ζ. Let $\langle \zeta_n : n < \omega \rangle$ be increasing and cofinal in $N_\alpha \cap \zeta$ with $\zeta_0 = \xi$, and let $\langle \beta_n : n < \omega \rangle$ be increasing and cofinal in β.

Let us agree to call an element $s \in P_\xi$ *relevant* if $s \leq p \upharpoonright \xi$ and s forces the order–type of $\mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi)$ to be $\beta + 1$. If s is relevant, then we say that s *can be completed* if there is a $t \in P_\zeta$ such that $\mathrm{dom}\, t \subseteq N_\alpha \cap [\xi, \zeta)$, $s \cup t \leq p$, and

$$(5.18) \qquad s \cup t \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi)) = \beta + 1.$$

Notice that the set of relevant conditions that can be completed is closed under extension, so it suffices to prove that every relevant condition has an extension that can be completed — if we show this, then a patching argument (as in Claim 5.1) will give us the object r we need in order to finish.

Clearly every relevant condition can be extended to one that decides a particular value for the ordinal $\max \mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi)$, so what we will show is that any relevant condition that decides a value for $\max \mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi)$ can be completed.

Note that if s is relevant and $s \Vdash \max \mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi) = \gamma + 1 \leq \alpha$, then we are done by $\circledast(\gamma, \beta)$, so we need only worry about the other possibility.

Assume $s \in P_\xi$ satisfies

$$(5.19) \qquad s \leq p \upharpoonright \xi,$$

$$(5.20) \qquad s \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi)) = \beta + 1,$$

and

$$(5.21) \qquad s \Vdash \max \mathrm{Spec}(\mathfrak{N}, \dot{G}_\xi) = \alpha + 1.$$

Our goal is to prove that s can be completed; the first step in this task is a simple lemma about countable support iterations.

Lemma 5.4. *If we are given objects s_n and $\dot{p}_n$ (for $n < \omega$) such that*

- $s_n \in P_{\zeta_n}$,
- $p_0 \in P_\zeta$,
- $s_0 \leq p_0 \upharpoonright \xi$,
- $s_n \Vdash \dot{p}_{n+1} \in P_\zeta$,
- $s_n \Vdash \dot{p}_{n+1} \upharpoonright \zeta_n \in \dot{G}_{\zeta_n}$,
- $s_n \Vdash \dot{p}_{n+1} \leq \dot{p}_n$,
- $s_{n+1} \upharpoonright \zeta_n = s_n$,

then $\bigcup_{n<\omega} s_n$ (a condition in P_ζ) forces that each $\dot{p}_n$ is in $\dot{G}_\zeta$.

Proof. Let G be any generic subset of P_ζ that contains $\bigcup_{n<\omega} s_n$; we will work in the model $V[G]$. Remember that $G \cap P_{\zeta_n}$ is a generic subset of P_{ζ_n} that contains s_n. For each $i < \omega$, let p_{i+1} denote the interpretation of the name $\dot{p}_{i+1}$ using G (or $G \cap P_{\zeta_i}$ if you prefer).

We need to show that $p_i \in G$ for each i. To do this, it suffices to prove that $p_i \upharpoonright \zeta_n \in G$ for each $n < \omega$. Why is this sufficient? For any $w \in P_\zeta$, the set of conditions in P_ζ that either extend w or are incompatible with $w \upharpoonright \zeta_n$ for some n is dense, so the genericity of G comes to our rescue.

Since $s_n \in G$ for each n, we have that the sequence $\langle p_i : i < \omega \rangle$ is decreasing in P_ζ. Now fix $n < \omega$. Since $s_n \in G$, we have that $p_n \upharpoonright \zeta_n \in G$, and since the sequence of p_i's is decreasing, this means that $p_i \upharpoonright \zeta_n \in G$ for $i < n$ as well. For $i > n$, we note that since $p_i \upharpoonright \zeta_i \in G$ we have $p_i \upharpoonright \zeta_n \in G$ as well. $\square$

Now we return to the matter of showing that our given $s \in P_\xi$ can be completed. For $n \in \omega$, let D_n be the set of conditions in P_ζ that force the order–type of $\mathrm{Spec}(\mathfrak{N}, \dot{G}_\zeta)$ to be greater than β_n.

By induction on $n < \omega$, we will define objects $\dot{p}_n$, t_n, and s_n such that

(1) $\dot{p}_0 = p$, $t_0 = \emptyset$, $s_0 = s$,
(2) $s_n \in P_{\zeta_n}$,
(3) $s_n \Vdash \dot{p}_{n+1} \in N_\alpha \cap P_\zeta$,
(4) $s_n \Vdash \dot{p}_{n+1} \upharpoonright \zeta_n \in \dot{G}_{\zeta_n}$,
(5) $t_n \in P_\zeta$,
(6) $\mathrm{dom}\, t_{n+1} \subseteq N_\alpha \cap [\zeta_n, \zeta_{n+1})$,
(7) $s_{n+1} = s_n \cup t_{n+1}$,
(8) $s_n \Vdash \dot{p}_{n+1} \in D_n$,
(9) $s_n \Vdash \dot{p}_{n+1} \leq \dot{p}_n$.

Given s_n, t_n, and $\dot{p}_n$, we show how to obtain $\dot{p}_{n+1}$, t_{n+1}, and s_{n+1}. The next two lemmas will furnish us with $\dot{p}_{n+1}$.

Lemma 5.5. *The set of conditions $u \in P_{\zeta_n}$ for which there exist $\bar{p} \in P_\zeta$, γ_0, and γ_1 such that*

(1) $\gamma_0 < \gamma_1 < \alpha$,
(2) $\bar{p} \in N_{\gamma_0} \cap P_\zeta$,
(3) $u \Vdash \dot{p}_n = \bar{p}$,
(4) $u \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N} \upharpoonright [\gamma_0, \gamma_1], \dot{G}_\zeta)) = \beta_n + 1$

is dense below s_n.

Proof. Note that (5.21) together with our assumption that β is closed under addition implies that α is a limit ordinal. Given an extension of s, we can extend it further to decide a particular value $\bar{p}$ for $\dot{p}_n$. Since α is a limit ordinal, this means there is some least $\gamma_0 < \alpha$ for which $\bar{p} \in N_{\gamma_0}$. Since β is closed under addition, it must be the case that our condition forces that the order–type of $\mathrm{Spec}(\mathfrak{N} \upharpoonright [\gamma_0, \alpha], \dot{G}_\zeta)$ is still $\beta + 1$, and therefore we can extend again to decide a particular value γ_1 for the $\beta_n + 1^{\mathrm{st}}$ member of $\mathrm{Spec}(\mathfrak{N} \upharpoonright [\gamma_0, \alpha], \dot{G}_\zeta)$. $\square$

Lemma 5.6. *Given $\gamma_0 < \gamma_1 < \alpha$ and $\bar{p} \in N_{\gamma_0} \cap P_\zeta$, there exist $v \in N_{\gamma_1+1}$ such that whenever $w \in P_{\zeta_n}$ extends $\bar{p} \upharpoonright \zeta_n$ and w forces the order–type of $\mathrm{Spec}(\mathfrak{N} \upharpoonright [\gamma_0, \gamma_1], \dot{G}_{\zeta_n})$ to be $\beta_n + 1$, we have that $w \cup v \leq \bar{p}$ in P_ζ and $w \cup v$ forces the order–type of $\mathrm{Spec}(\mathfrak{N} \upharpoonright [\gamma_0, \gamma_1], \dot{G}_\zeta)$ to be $\beta_n + 1$.*

Proof. If we let $\alpha' = \mathrm{otp}([\gamma_0, \gamma_1])$, then the lemma follows from our induction hypothesis $\circledast(\alpha', \beta_n)$. $\square$

Let E_n be the set of all conditions $u \leq s_n$ in P_{ζ_n} that satisfy the assumptions of Lemma 5.5. For each $u \in E_n$, Lemma 5.6 provides us with an object v_u such that

- $v_u \in N_\alpha \cap P_\zeta$,
- $\mathrm{dom}\, v_u \subseteq N_\alpha \cap [\zeta_n, \zeta)$,
- $u \cup v_u \Vdash \dot{p}_n \in \dot{G}_\zeta$,
- $u \cup v_u \Vdash \mathrm{otp}(\mathrm{Spec}(\mathfrak{N}, \dot{G}_\zeta)) > \beta_n$.

Let $\mathcal{A}_n \subseteq E_n$ be an antichain pre–dense below s_n, and define

$$(5.22) \qquad\qquad \dot{p}_{n+1} = \{\langle u, u \cup v_u \rangle : u \in \mathcal{A}_n\}.$$

Now if G is any generic subset of P_{ζ_n} that contains s_n, then there is a unique $u \in G \cap \mathcal{A}_n$. Since $\dot{p}_{n+1}$ gets interpreted as $u \cup v_u$, it follows easily that s_n forces that $\dot{p}_{n+1}$ has all of the required properties.

To obtain t_{n+1}, we will do a similar argument in the poset $P_{\zeta_{n+1}}$. Note that if u extends s_n and there is some $\bar{p} \in P_\zeta$ such that $u \Vdash \dot{p}_{n+1} = \bar{p}$, then there is an object v_u such that

- $v_u \in P_{\zeta_{n+1}}$,
- $\operatorname{dom} v_u \subseteq N_\alpha \cap [\zeta_n, \zeta_{n+1})$,
- $u \cup v_u \Vdash \dot{p}_{n+1} \restriction \zeta_{n+1} \in \dot{G}_{\zeta_{n+1}}$,
- $u \cup v_u \Vdash \operatorname{otp}(\operatorname{Spec}(\mathfrak{N}, \dot{G}_{\zeta_{n+1}})) = \beta + 1$.

Such a v_u can be found because of our induction hypothesis for $\zeta_{n+1} < \zeta$. A patching argument analogous to that of Claim 5.1 provides us with an object t_{n+1} with the required properties. As s_{n+1} is determined by t_{n+1} and s_n, we have completed the construction for stage $n + 1$.

Now we let $t = \bigcup_{n<\omega} t_n$. We need to show that t completes s, i.e., that $\operatorname{dom} t \subseteq N_\alpha \cap [\xi, \zeta)$, $s \cup t \le p$, and

$$(5.23) \qquad s \cup t \Vdash \operatorname{otp}(\operatorname{Spec}(\mathfrak{N}, \dot{G}_\zeta)) = \beta + 1.$$

It is not hard to show that $\operatorname{dom} t \subseteq N_\alpha \cap [\xi, \zeta)$ and $s \cup t \le p$ — both of these follow from the way t was constructed. By Lemma 5.4, we have that $s \cup t$ forces that each $\dot{p}_n$ is in G_ζ, and since $s \cup t$ extends s_n, we have that $\dot{p}_{n+1} \in D_n$ for each n. Thus

$$(5.24) \qquad s \cup t \Vdash \operatorname{otp}(\operatorname{Spec}(\mathfrak{N}, \dot{G}_\zeta)) \ge \sup_{n<\omega} \beta_n = \beta.$$

Since $s \Vdash \operatorname{otp}(\operatorname{Spec}(\mathfrak{N}, \dot{G}_\xi)) = \beta + 1$, it follows that

$$(5.25) \qquad s \cup t \Vdash \operatorname{otp}(\operatorname{Spec}(\mathfrak{N}, \dot{G}_\zeta)) \le \beta + 1.$$

Since $\operatorname{Spec}(\mathfrak{N}, \dot{G}_\zeta)$ is always a closed subset of $\alpha + 1$ and β is a limit ordinal, it must be the case that (5.23) holds.

To recap, what we have shown is that if s is relevant and s decides a specific value for the ordinal $\max \operatorname{Spec}(\mathfrak{N}, \dot{G}_\xi)$, then s can be completed. Since the set of such s is dense in the set of all relevant conditions in P_ξ, we can use a patching argument to obtain a single r that uniformly completes every relevant s. This shows us that our induction continues past stage ζ, and from this we can deduce that $\circledast(\alpha, \beta)$ holds. This finishes the proof of Theorem 3. $\qquad\square$

6. On iterations of totally proper forcings

In this section, we show that weak $< \omega_1$–properness can serve as a replacement for $< \omega_1$–properness in the iteration theorems of [9] and Chapter V of [19] that deal with iterations that do not add reals. Shelah's paper [15] contains iteration theorems related to this one; he utilizes conditions even more technical than our requirement of weak $< \omega_1$–properness.

We will start with some notation germane to totally proper notions of forcing.

Definition 6.1. Let P be a notion of forcing, let N be a countable elementary submodel of some $H(\lambda)$ with $P \in N$, and let $p \in N \cap P$. We define

(1) $N^P = \{\dot{\tau} \in N : \dot{\tau} \text{ is a } P\text{–name}\}$,
(2) $\operatorname{Gen}(N, P) = \{G \subseteq N \cap P : G \text{ is an } N\text{–generic filter on } N \cap P\}$,
(3) $\operatorname{Gen}^+(N, P) = \{G \in \operatorname{Gen}(N, P) : G \text{ has a lower bound in } P\}$,

   (4) $\mathrm{Gen}(N, P, p) = \{G \in \mathrm{Gen}(N, P) : p \in G\}$,
   (5) $\mathrm{Gen}^+(N, P, p) = \mathrm{Gen}(N, P, p) \cap \mathrm{Gen}^+(N, P)$.

Lemma 6.2. *Assume P is totally proper and N is a countable elementary submodel of $H(\lambda)$ with $P \in N$.*

   (1) *Each $G \in \mathrm{Gen}(N, P)$ is a maximal filter of conditions in $N \cap P$.*
   (2) *If p is totally (N, P)–generic, then $\{q \in N \cap P : p \le q\}$ is in $\mathrm{Gen}(N, P)$.*
   (3) *If $G_0 \ne G_1$ in $\mathrm{Gen}(N, P)$, then there are $p_i \in G_i$ such that p_0 and p_1 are incompatible in P.*

Proof. Left to the reader. □

Definition 6.3. Let P be a notion of forcing, let $A \subseteq P$, and let θ be a sentence of the forcing language. We define

$$A \Vdash \theta$$

to mean that whenever $G \subseteq P$ is a generic filter such that $A \subseteq G$, then θ holds in the generic extension $V[G]$.

   If we adopt the convention that all notions of forcing are complete Boolean algebras, then "$A \Vdash \theta$" is equivalent to the condition $\wedge A$ (the infimum of A) forcing θ to be true.

Proposition 6.4. *Let P be a totally proper notion of forcing, and let N be a countable elementary submodel of $H(\lambda)$ that contains P. Let $\dot\tau_0, \ldots, \dot\tau_{n-1}$ be elements of N^P, and let $\theta(x_0, \ldots, x_{n-1})$ be a formula. Let $H \subseteq P$ be a filter of conditions that satisfies $N \cap H \in \mathrm{Gen}(N, P)$. Then either*

$$H \Vdash \theta(\dot\tau_0, \ldots, \dot\tau_{n-1}),$$

or

$$H \Vdash \neg\theta(\dot\tau_0, \ldots, \dot\tau_{n-1}).$$

Proof. This follows because elements of $\mathrm{Gen}(N, P)$ are maximal filters on the countable set $N \cap P$. □

Lemma 6.5. *Let P be totally proper, and let $N_0 \in N_1$ be countable elementary submodels of $H(\lambda)$. Suppose that $\bar G \in \mathrm{Gen}(N_0, P)$, $G \in \mathrm{Gen}^+(N_1, P)$, and $\bar G \subseteq G$. Then $\bar G \in N_1$.*

Proof. Let D be the set of conditions in P that are either totally (N_0, P)–generic, or that have no totally (N_0, P)–generic extension. Clearly D is dense in P, and $D \in N_1$. Thus there is a $p \in N_1 \cap G$. Let r be a lower bound for G. Then $r \le p$ and r is a lower bound for $\bar G$ as well, and therefore p has a totally (N_0, P)–generic extension. Since $p \in D$, it must be the case that p is totally (N_0, P)–generic. By part (2) of Lemma 6.2, $G' = \{q \in N_0 \cap P : p \le q\}$ is in $\mathrm{Gen}(N_0, P)$, and furthermore $G' \in N_1$ as it is definable from N_0, p, and P. Since G is a filter, every member of $\bar G$ is compatible with p, and hence by part (3) of Lemma 6.2, it must be the case that $G' = \bar G$, and thus $\bar G \in N_1$. □

   The proof of the next lemma is left to the reader — it follows easily from the definitions involved.

Lemma 6.6. *Let P be totally proper, and suppose $\dot{Q}$ is a P–name for a totally proper notion of forcing. Let $N \prec H(\lambda)$ be countable, with $\{P, \dot{Q}\} \in N$. Let $G \in \mathrm{Gen}(N, P)$, as suppose $\{\dot{q}_n : n \in \omega\}$ is an $(N[G], \dot{Q})$–generic sequence. Then $H := \{r \in N \cap P * \dot{Q} : (\exists p \in G)(\exists n \in \omega)[p * \dot{q}_n \le r]\}$ is an element of $\mathrm{Gen}(N, P * \dot{Q})$.*

The reader should contrast the result of the next lemma with the definition of weak $< \omega_1$–properness; here we find a condition q that explicitly determines the members of the spectrum of genericity.

Lemma 6.7. *Suppose P is totally proper and weakly $< \omega_1$–proper. Let $\mathfrak{N} = \langle N_i : i \le \alpha \rangle$ be a suitable tower of models. Given $p \in N \cap P$, there is a $q \le p$ such that $\{i \le \alpha : q$ is totally (N_i, P)–generic$\}$ has order–type $\alpha + 1$.*

Proof. Since P is weakly $< \omega_1$–proper, there is an $r \le p$ such that

(6.1) $r \Vdash \mathrm{Spec}(\mathfrak{N}, P)$ has order–type $\alpha + 1$.

Since P is totally proper and α is countable, there is a $q \le r$ and a set $I \subseteq \alpha + 1$ such that

(6.2) $q \Vdash I = \mathrm{Spec}(\mathfrak{N}, P)$.

Clearly I must have order–type $\alpha + 1$, and for $i \in I$,

$$q \Vdash N_i[\dot{G}] \cap \mathrm{Ord} = N_i \cap \mathrm{Ord},$$

hence q is (N_i, P)–generic. $\square$

Note that in the conclusion of the preceding lemma, the set of $i \le \alpha$ such that q is totally (N_i, P)–generic must be a suitable index set — this follows from Proposition 4.5.

The next definition appears implicitly in [19] and it is intimately related to $^\omega\omega$–bounding notions of forcing. The notation "$(N, P)^{\mathrm{fin}}$–generic" is new although the property so defined has played a prominent role in Shelah's work — our choice of notation seems like a convenient way of expressing the concepts involved.

Definition 6.8. Let P be a notion of forcing and let $N \prec H(\lambda)$ be countable with $P \in N$. A condition $q \in P$ is said to be $(N, P)^{\mathrm{fin}}$–generic if for every $D \in N$ dense in P, there is a finite subset of $N \cap D$ that is pre–dense below p.

It is instructive to compare the preceding definition to the definitions of "q is (N, P)–generic" and "q is totally (N, P)–generic". The best way to see what is going on is to look at maximal antichains $\mathcal{A} \subseteq P$ that are elements of N. If q is (N, P)–generic, then q forces $G \cap \mathcal{A} \in N$. If q is $(N, P)^{\mathrm{fin}}$–generic, then in addition there is a finite subset $\mathcal{A}_0$ of $N \cap \mathcal{A}$ such that $p \Vdash G \cap \mathcal{A} \in \mathcal{A}_0$. If q is totally (N, P)–generic, then there is some unique condition $q \in N \cap \mathcal{A}$ with $p \le q$.

We previously mentioned $^\omega\omega$–bounding notions of forcing, so we take a moment to recall the definition (from [19]) and pin down how it relates to $(N, P)^{\mathrm{fin}}$–genericity.

Definition 6.9. A forcing notion P is $^\omega\omega$–bounding if for every $f \in {^\omega\omega}^{V[G]}$ there is a $g \in {^\omega\omega}^V$ such that for all $n \in \omega$, $f(n) \le g(n)$.

The following result has a right to be called folklore — the authors are not aware if it has explicitly appeared in the literature, but it has certainly been known since very early on in the history of proper forcing.

Proposition 6.10. *A notion of forcing P is proper and $^{\omega}\omega$–bounding if and only if whenever $N \prec H(\lambda)$ is countable with $P \in N$, and $p \in N \cap P$, there is a $q \leq p$ such that q is $(N, P)^{\mathrm{fin}}$–generic.*

Proof. Suppose P is proper and $^{\omega}\omega$–bounding, and let $N \prec H(\lambda)$ be a countable model containing P. Given $p \in N \cap P$, let $r \leq q$ be (N, P)–generic. Next, choose $N_1 \prec H(\lambda)$ countable such that N and r are in N_1. Inside N_1, let $\langle \mathcal{A}_n : n \in \omega \rangle$ list all maximal antichains of P that are elements of N, and for each n, let $\langle a_n^i : i < \omega \rangle$ enumerate $N \cap \mathcal{A}_n$.

Since r is (N, P)–generic, there is a P–name $\dot{f}$ for a function in $^{\omega}\omega$ such that

$$(6.3) \qquad\qquad r \Vdash (\forall n < \omega)[N \cap \dot{G} \in \{a_n^i : i \leq \dot{f}(n)\}].$$

Since P is $^{\omega}\omega$–bounding, there is a $q \leq r$ and $g \in {}^{\omega}\omega$ such that for each n, q forces that $\dot{f}(n) \leq g(n)$. One can then easily check that q is $(N, P)^{\mathrm{fin}}$–generic as required.

For the converse, let $\dot{f}$ be a P–name for a function in $^{\omega}\omega$. Let N be a countable elementary submodel of $H(\lambda)$ that contains P and $\dot{f}$, and let $p \in N \cap P$ be arbitrary. There is a $q \leq p$ such that q is $(N, P)^{\mathrm{fin}}$–generic. For each n, the set of conditions that decide a value for $\dot{f}(n)$ is dense in P and an element of N. Since q is $(N, P)^{\mathrm{fin}}$–generic, for each n there is some $g(n) \leq \omega$ such that $q \Vdash \dot{f}(n) \in g(n)$, and we are done. □

If $P * \dot{Q}$ is a two step iteration, p is $(N, P)^{\mathrm{fin}}$–generic and

$$p \Vdash \text{``}\dot{q} \text{ is } (N[\dot{G}], \dot{Q})^{\mathrm{fin}}\text{–generic''},$$

then in general we cannot conclude that $p * \dot{q}$ is $(N, P * \dot{Q})^{\mathrm{fin}}$–generic. The following lemma, taken from Chapter XVIII of [19], tells us that in certain cases we can remedy this shortcoming; it will play a crucial role in the proof of Theorem 4.

Lemma 6.11 (Claim XVIII.2.6 of [19]). *Let $\mathbb{P} = \langle P_i, \dot{Q}_i : i < \kappa \rangle$ be a countable support iteration of $^{\omega}\omega$–bounding proper forcing notions. Given objects i, j, N_0, N_1, p, and q such that*

(1) *$i < j \leq \kappa$,*
(2) *$N_0 \in N_1$ are countable elementary submodels of $H(\lambda)$ such that $\{\mathbb{P}, i, j\} \in N_0$,*
(3) *$p \in N_0 \cap P_j$,*
(4) *$q \in P_i$,*
(5) *$q \leq p \restriction i$,*
(6) *q is $(N_0, P_i)^{\mathrm{fin}}$–generic,*
(7) *q is (N_1, P_i)–generic,*

there is an object r such that

(8) *$r \in P_j$,*
(9) *$r \restriction i = q$,*
(10) *$r \leq p$,*
(11) *r is $(N_0, P_j)^{\mathrm{fin}}$–generic,*
(12) *r is (N_1, P_j)–generic.*

The preceding lemma is an essential ingredient in the proof of the upcoming iteration theorem. Although it is not explicitly stated in the conclusion of the lemma, one can obtain that $\mathrm{dom}\, r \subseteq \mathrm{dom}\, q \cup N_1 \cap [i, j)$.

The next theorem is modelled on the proofs found in Chapter VIII of [19]. Conceptually, our proof is no more difficult than those in Shelah's book — the arguments are at the heart the same — but things are a lot more complex at the technical level in our proof because weak $< \omega_1$–properness is not as nicely behaved as honest–to–goodness $< \omega_1$–properness.

Theorem 4. *Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \epsilon \rangle$ be a countable support iteration such that:*

1. *$\Vdash_{P_\alpha} \dot{Q}_\alpha$ is totally proper.*
2. *$\Vdash_{P_\alpha} \dot{Q}_\alpha$ is weakly $< \omega_1$–proper.*
3. *For each α, if N_0, N_1, $\dot{q}$, $\bar{G}$, and $\langle G^\ell : \ell < k \rangle$ satisfy*
   - *N_0 and N_1 are countable elementary submodels of $H(\lambda)$,*
   - *$N_0 \in N_1$,*
   - *$\{\mathbb{P}, \alpha, \dot{q}\} \in N_0$,*
   - *$\Vdash_{P_\alpha} \dot{q} \in \dot{Q}_\alpha$,*
   - *$\bar{G} \in \mathrm{Gen}^+(N_0, P_\alpha, p) \cap N_1$,*
   - *for $\ell < k$, $G^\ell \in \mathrm{Gen}(N_1, P_\alpha)$,*
   - *for $\ell < k$, $\bar{G} \subseteq G^\ell$,*

   then there is a sequence $\langle \dot{q}_n : n \in \omega \rangle$ in $N_1 \cap \mathrm{Gen}(N_0[\bar{G}], \dot{Q}, \dot{q})$ such that for all $\ell < k$,

   $$G^\ell \Vdash \langle \dot{q}_n : n \in \omega \rangle \text{ has a lower bound in } \dot{Q}.$$

Then P_ϵ is totally proper.

Before we get into a proof, let us comment that condition (3) of the statement of the theorem is a weakened version of the condition used in the iteration theorem presented in [9], and all iteration theorems in the literature that involve not adding reals require something along these lines — in Shelah's terminology, this is "medicine against the weak diamond" (see [15]).

Let $N_0 \prec H(\lambda)$ be a countable model such that $\mathbb{P} \in N_0$.

Definition 6.12. For $\xi < \zeta \in N_0 \cap (\epsilon + 1)$, we define an ordinal $\alpha(\xi, \zeta)$ as follows: Given ζ, we let

$$(6.4) \qquad \alpha = \sup\{\alpha(\xi', \zeta') : (\xi', \zeta') \in N_0 \text{ and either } \zeta' < \zeta \text{ or } \zeta' = \zeta \wedge \xi' < \xi\},$$

and let $\alpha(\xi, \zeta) = \alpha_0 + 3$, where α_0 is the least ordinal greater than α that is closed under ordinal addition.

At this point, the previous definition may seem to be a bit mysterious. What's going on is that $\alpha(\xi, \zeta)$ is going to be the length of a tower of models that is "long enough" to allow us to use an induction hypothesis to advance from P_ξ to P_ζ in the proof of the iteration theorem.

Proposition 6.13. *Given $\xi < \zeta \in N_0 \cap (\epsilon + 1)$, p, $\mathfrak{N}$, $\bar{G}$, $\langle G^\ell : \ell < k \rangle$, and $\langle r^\ell : \ell < k \rangle$ such that*

1. *$p \in N_0 \cap P_\zeta$,*
2. *$\mathfrak{N} = \langle N_\beta : \beta \leq \alpha(\xi, \zeta) \rangle$ satisfies*
   - *$\mathfrak{N}$ is a continuous increasing $\in$–chain of countable elementary submodels of $H(\lambda)$,*
   - *$\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$,*
3. *$\bar{G} \in \mathrm{Gen}^+(N_0, P_\xi) \cap N_1$,*
4. *$p \upharpoonright \xi \in \bar{G}$,*

(5) $\langle G^\ell : \ell < k \rangle$ *satisfies*
- $G^\ell \in \mathrm{Gen}^+(N_{\alpha(\xi,\zeta)}, P_\xi)$,
- $\bar{G} \subseteq G^\ell$,
- $\{\beta \leq \alpha(\xi,\zeta) : (\forall \ell < k)[N_\beta \cap G^\ell \in \mathrm{Gen}(N_\beta, P_\xi)]\}$ *has order–type* $\alpha(\xi,\zeta)$,

(6) r^ℓ *is a lower bound for* G^ℓ,

then we can find objects G^* *and* s *such that*

(1) $G^* \in \mathrm{Gen}(N_0, P_\zeta)$,
(2) $p \in G^*$,
(3) $s \in P_\zeta$ *with* $\mathrm{dom}\, s \subseteq N_{\alpha(\xi,\zeta)} \cap [\xi, \zeta)$,
(4) *for each* $\ell < k$, $r^\ell \cup s$ *is a lower bound for* G^* *in* P_ζ.

Proof. The proof is by induction on $\zeta \in N_0 \cap (\epsilon + 1)$.

CASE 1: $\zeta = \zeta_0 + 1$.

Here, the way is clear — we will use our induction hypothesis to move things to P_{ζ_0}, and then use condition (3) of Theorem 4 to advance one more step.

Let $\xi < \zeta$ in $N_0 \cap (\epsilon + 1)$. The case where $\xi = \zeta_0$ follows immediately from our assumptions, so assume that $\xi < \zeta_0$.

Let p, $\mathfrak{N}$, $\bar{G}$, $\langle G^\ell : \ell < k \rangle$, and $\langle r^\ell : \ell < k \rangle$ be given. Let $\alpha = \alpha(\xi, \zeta_0)$. By Lemma 6.5, we know that for each $\ell < k$, $N_\alpha \cap G^\ell \in N_{\alpha+1}$ and so for each $\ell < k$ we can find a condition $r_\ell \in N_{\alpha+1} \cap G^\ell$ that is a lower bound for $N_\alpha \cap G^\ell$.

Now inside $N_{\alpha+1}$ we apply our induction hypothesis to the objects $p \restriction \zeta_0$, $\bar{G}$, $\langle N_\alpha \cap G^\ell : \ell < k \rangle$, and $\langle r_\ell : \ell < k \rangle$.

This gives us objects H^* and t such that

- $H^* \in \mathrm{Gen}(N_0, P_{\zeta_0}, p \restriction \zeta_0) \cap N_{\alpha+1}$,
- $r_\ell \cup t$ is a lower bound for H^* for each $\ell < k$.

Now in the model $N_{\alpha+2}$, we can find conditions $t^\ell \in P_{\zeta_0}$ such that $t^\ell \restriction \xi \in G^\ell$, $t^\ell \leq r_\ell \cup t$, and t^ℓ is totally $(N_{\alpha+1}, P_{\zeta_0})$–generic, say

(6.5) $$t^\ell \Vdash N_{\alpha+1} \cap \dot{G}_{\zeta_0} = H^\ell.$$

Now we apply assumption (3) of Theorem 4 to the objects $N_0 \in N_{\alpha+1}$, $p(\zeta)$, H^*, $\langle H^\ell : \ell < k \rangle$ to get an $(N_0[H^*], \dot{Q}_\zeta, p(\zeta))$–generic sequence $\langle \dot{q}_n : n < \omega \rangle$ in $N_{\alpha+1}$ that each H^ℓ forces to have a lower bound. Now that we have done this, Lemma 6.6 shows us how to obtain $G^* \in \mathrm{Gen}(N_0, P_\zeta)$ from H^* and the sequence $\{\dot{q}_n : n \in \omega\}$.

CASE 2: ζ a limit.

Let $\{D_n : n \in \omega\}$ list the dense open subsets of P_ζ that are elements of N_0, and let $\langle \zeta_n : n \in \omega \rangle$ be cofinal in $N_0 \cap \zeta$ with $\zeta_0 = \xi$. Also, let α_0 be the unique ordinal such that $\alpha(\xi, \zeta) = \alpha_0 + 3$. Recall that α_0 is closed under ordinal addition.

By induction, we will define objects p_n, G_n^*, I_n, k_n, $\langle G_n^{\bar{\ell}} : \bar{\ell} < k_n \rangle$, and s_n such that

(1) $p_0 = p$, $G_0^* = \bar{G}$, $I_0 = \alpha_0 + 1$, $k_0 = k$, $G_0^\ell = G^\ell \cap N_{\alpha_0+1}$, $s_0 = \emptyset$,
(2) $p_{n+1} \leq p_n$,
(3) $p_{n+1} \in N_0 \cap D_n$,
(4) $p_{n+1} \restriction \zeta_n \in G_n^*$,
(5) $G_n^* \in \mathrm{Gen}^+(N_0, P_{\zeta_n}) \cap N_{\min(I_n \setminus \{0\})}$,
(6) $G_{n+1}^* \cap P_{\zeta_n} = G_n^*$,
(7) $G_n^{\bar{\ell}} \in \mathrm{Gen}^+(N_{\alpha_0+1}, P_{\zeta_n}) \cap N_{\alpha_0+2}$,
(8) $G_n^* \subseteq G_n^{\bar{\ell}}$ for all $\bar{\ell} < k_n$,

(9) $I_n = \{\beta \leq \alpha_0 : N_\beta \cap G_n^{\bar{\ell}} \in \text{Gen}(N_\beta, P_{\zeta_n}) \text{ for all } \bar{\ell} < k_n\}$,

(10) the order–type of I_n is $\alpha_0 + 1$,

(11) $s_n \in P_{\zeta_n}$ with domain a subset of $[\xi, \zeta_n)$,

(12) for each $\ell < k$, we have

- $r^\ell \cup s_n \in P_{\zeta_n}$,
- $r^\ell \cup s_n \Vdash N_{\alpha_0+1} \cap \dot{G}_{\zeta_n} \in \{G_n^{\bar{\ell}} : \bar{\ell} < k_n\}$,
- $r^\ell \cup s$ is $(N_{\alpha_0+2}, P_{\zeta_n})^{\text{fin}}$–generic,
- $r^\ell \cup s_n$ is $(N_{\alpha_0+3}, P_{\zeta_n})$–generic.

Given the objects for stage n, we must show how to get the objects required for stage $n + 1$. Throughout the upcoming argument, we will be exchanging "vertical information" for "horizontal information". To understand what this means, the reader should look at how well the conditions $r^\ell \cup s_n$ "pin down" the identity of the generic object $\dot{G}_{\zeta_n}$.

For example, all of the conditions $r^\ell \cup s_n$ are $(N_{\alpha_0+3}, P_{\zeta_n})$–generic. This means that given a maximal antichain $\mathcal{A}$ of P_{ζ_n} that lies in N_{α_0+3}, $r^\ell \cup s_n$ will guarantee that the generic object hits the countable set $\mathcal{A} \cap N_{\alpha_0+3}$. If $\mathcal{A}$ happens to be in N_{α_0+2}, then $r^\ell \cup s_n$ can see a finite subset of $N_{\alpha_0+2} \cap \mathcal{A}$ such that it, i.e., $r^\ell \cup s_n$, can guarantee the generic object will meet that finite set.

As far as N_{α_0+1} is concerned, we are much better off — we actually have a finite list of candidates $\langle G_n^{\bar{\ell}} : \bar{\ell} < k_n \rangle$ such that for each $\bar{\ell} < k$,

$$r^\ell \cup s_n \Vdash (\exists \bar{\ell} < k_n)[N_{\alpha_0+1} \cap \dot{G}_{\zeta_n}] = G_n^{\bar{\ell}}.$$

Next, consider how things improve when we move to the model N_{α_0}. Not only does $r^\ell \cup s_n$ guarantee that the generic object looks like a restriction of one of the $G_n^{\bar{\ell}}$'s to N_{α_0}, we also have that the sets $G_n^{\bar{\ell}}$ for $\bar{\ell} < k_n$ are simultaneously generic for lots of the models N_β, $\beta < \alpha_0$.

Finally, each $r^\ell \cup s_n$ is totally (N_0, P_{ζ_n})–generic because they all agree that the generic object, when restricted to the model N_0, looks like G_n^*. One should think of this as "the degree of genericity of $r^\ell \cup s_n$ decreases as we move up through the tower".

Now we get back to the task of building the objects needed for stage $n + 1$.

Goal: Find p_{n+1}.

Let us define

$$D = \{q \in P_{\zeta_n} : (\exists p \in D_n)[\, p \in D_n \text{ extends } p_n \text{ and } p \upharpoonright \zeta_n = q\,]\}.$$

It is clear that D is dense below $p_n \upharpoonright \zeta_n$ in P_{ζ_n}. Since $G_n^* \in \text{Gen}(N_0, P_{\zeta_n})$ and $p_n \upharpoonright \zeta_n \in G_n^*$, we can find the required p_{n+1} using the genericity of G_n^*.

Goal: Find G_{n+1}^*.

Let $\beta_0 \in I_n$ be such that $\text{otp}(I_n \cap \beta_0) = \alpha(\zeta_n, \zeta_{n+1})$, and we let β_1 equal the next element of I_n past β_0. Inside N_{β_1}, for each $\bar{\ell} < k_n$ we choose $r_{\beta_1}^{\bar{\ell}} \in G_n^{\bar{\ell}}$ that is a lower bound for $N_{\beta_0} \cap G_n^{\bar{\ell}}$, and let $H^{\bar{\ell}} \in \text{Gen}(N_{\beta_0}, P_{\zeta_n})$ be such that $r_{\beta_1}^{\bar{\ell}} \Vdash N_{\beta_0} \cap G_n^{\bar{\ell}} = H^{\bar{\ell}}$.

We apply our induction hypothesis for $P_{\zeta_{n+1}}$ inside the model N_{β_1} to the objects

- $p \upharpoonright \zeta_{n+1}$,
- $\langle N_\beta : \beta \in I_n \cap (\beta_0 + 1) \rangle$,
- G_n^*,
- $\langle H^{\bar{\ell}} : \bar{\ell} < k_n \rangle$,
- $\langle r_{\beta_1}^{\bar{\ell}} : \bar{\ell} < k_n \rangle$,

and this gives us $G_{n+1}^* \in N_{\beta_1} \cap \mathrm{Gen}(N_0, P_{\zeta_{n+1}})$ and $t \in N_{\beta_1}$ such that $p_{n+1} \restriction \zeta_{n+1} \in G_{n+1}^*$, and for all $\bar{\ell} < k_n$, $r_{\beta_1}^{\bar{\ell}} \cup t$ is a lower bound for G_{n+1}^*. Note that Proposition 4.5 tells us that $\langle N_\beta : \beta \in I_n \cap \beta_0 + 1 \rangle$ is as needed in order to apply our induction hypothesis.

Goal: Find I_{n+1}.

We now work inside the model N_{α_0+1}. For $\bar{\ell} < k_n$, choose $r_{\alpha_0}^{\bar{\ell}} \in N_{\alpha_0+1} \cap G_n^{\bar{\ell}}$ that is a lower bound for $N_{\alpha_0} \cap G_n^{\bar{\ell}}$. Let $J_0 = I_n \setminus (\beta_1 + 1)$.

By induction on $\bar{\ell} < k_n$ we will define objects $t_{\bar{\ell}}$ and $J_{\bar{\ell}+1}$ such that

- $t_{\bar{\ell}} \leq r_{\alpha_0}^{\bar{\ell}} \cup t$ in $P_{\zeta_{n+1}}$,
- $t_{\bar{\ell}}$ is totally $(N_{\alpha_0}, P_{\zeta_{n+1}})$–generic,
- $t_{\bar{\ell}} \restriction \zeta_n \in N_{\alpha_0+1} \cap G_n^{\bar{\ell}}$,
- $t_{\bar{\ell}}$ is weakly $(\mathfrak{N} \restriction J_{\bar{\ell}}, P_{\zeta_{n+1}})$–generic,
- $J_{\bar{\ell}+1} := \{\beta \in J_{\bar{\ell}} : t_{\bar{\ell}}$ is totally $(N_\beta, P_{\zeta_{n+1}})$–generic$\}$ has order–type α_0+1.

Note that J_0 is a suitable index set and $J_0 \in N_{\alpha_0+1}$. Given $J_{\bar{\ell}}$, we show how to obtain $t_{\bar{\ell}}$ and $J_{\bar{\ell}+1}$.

Our assumptions on $J_{\bar{\ell}}$ imply that $\mathfrak{N} \restriction J_{\bar{\ell}}$ is a suitable tower of models and $\mathfrak{N} \restriction J_{\bar{\ell}} \in N_{\alpha_0+1}$. Also, since $J_{\bar{\ell}}$ is a subset of I_n, we know that $r_{\alpha_0}^{\bar{\ell}}$ is weakly $(\mathfrak{N} \restriction J_{\bar{\ell}}, P_{\zeta_n})$–generic.

We now apply Theorem 3 inside the model N_{α_0+1} for the partial order $P_{\zeta_{n+1}}$ with $\mathfrak{N} \restriction J_{\bar{\ell}}$, $r_{\alpha_0}^{\bar{\ell}}$, and $r_{\beta_1}^{\bar{\ell}} \cup t$ in place of the $\mathfrak{N}$, q, and p appearing there. This gives us an object $s^{\bar{\ell}}$ that "lengthens" $r_{\alpha_0}^{\bar{\ell}}$ into a weakly $(\mathfrak{N} \restriction J_{\bar{\ell}}, P_{\zeta_{n+1}})$–generic condition.

Claim 6.14. *There is a totally $(N_{\alpha_0}, P_{\zeta_{n+1}})$–generic condition $t^{\bar{\ell}}$ in N_{α_0+1} that extends $r_{\alpha_0}^{\bar{\ell}} \cup s^{\bar{\ell}}$ and satisfies $t_{\bar{\ell}} \restriction \zeta_n \in G_n^{\bar{\ell}}$.*

Proof. Since $P_{\zeta_{n+1}}$ is totally proper, the (countable) set $N_{\alpha_0} \cap \dot{G}_{\zeta_{n+1}}$ is forced to be an element of the ground model V. Thus the set of conditions that decide a particular value for $N_{\alpha_0} \cap \dot{G}_{\zeta_{n+1}}$ is a dense open subset of $P_{\zeta_{n+1}}$. In particular, the set

$$(6.6) \qquad D := \{u \leq r_{\alpha_0}^{\bar{\ell}} \cup s^{\bar{\ell}} : u \text{ decides a value for } N_{\alpha_0} \cap \dot{G}_{\zeta_{n+1}}\}$$

is dense below $r_{\alpha_0}^{\bar{\ell}}$ in $P_{\zeta_{n+1}}$. Note that $D \in N_{\alpha_0+1}$ as all parameters needed to define D are elements of N_{α_0+1}.

It is easily seen that the set

$$(6.7) \qquad\qquad D \restriction \zeta_n := \{u \restriction \zeta_n : u \in D\}$$

is dense below $r_{\alpha_0}^{\bar{\ell}}$ in P_{ζ_n}. Since $r_{\alpha_0}^{\bar{\ell}} \in G_n^{\bar{\ell}}$ and $D \restriction \zeta_n \in N_{\alpha_0+1}$, it must be the case that the intersection $N_{\alpha_0+1} \cap \dot{G}_n^{\bar{\ell}} \cap D \restriction \zeta_n$ is non–empty. Thus there is a condition $t_{\bar{\ell}} \in N_{\alpha_0+1} \cap D$ such that $t_{\bar{\ell}} \restriction \zeta_n \in G_n^{\bar{\ell}}$.

Now why is $t_{\bar{\ell}}$ totally $(N_{\alpha_0}, P_{\zeta_{n+1}})$–generic? We can find

$$H \in \mathrm{Gen}(N_{\alpha_0}, P_{\zeta_{n+1}}) \cap N_{\alpha_0+1}$$

such that

$$t_{\bar{\ell}} \Vdash N_{\alpha_0} \cap \dot{G}_{\zeta_n} = H.$$

If $u \in H$, then $t_{\bar{\ell}}$ forces that u is a member of $\dot{G}_{\zeta_n}$. Since without loss of generality all of our notions of forcing are separative, this means that $t_{\bar{\ell}}$ must be an extension of u. Thus $t_{\bar{\ell}}$ is a lower bound for H, and hence totally $(N_{\alpha_0}, P_{\zeta_{n+1}})$–generic. $\square$

Now we claim that $t_{\bar{\ell}}$ has all the required properties.

We have already ensured that $t_{\bar{\ell}} \le r_{\alpha_0}^{\bar{\ell}} \cup t$, $t_{\bar{\ell}}$ is totally $(N_{\alpha_0}, P_{\zeta_{n+1}})$–generic, and $t_{\bar{\ell}} \restriction \zeta_n \in N_{\alpha_0+1} \cap G_n^{\bar{\ell}}$. Since $t_{\bar{\ell}} \le r_{\alpha_0}^{\bar{\ell}} \cup s^{\bar{\ell}}$, we have that $t_{\bar{\ell}}$ is weakly $(\mathfrak{N} \restriction J_{\bar{\ell}}, P_{\zeta_{n+1}})$–generic. Since $t_{\bar{\ell}}$ is totally $(N_{\alpha_0}, P_{\zeta_{n+1}})$–generic, this implies

(6.8) $$J_{\bar{\ell}+1} := \{\beta \in J_{\bar{\ell}} : t_{\bar{\ell}} \text{ is totally } (N_\beta, P_{\zeta_{n+1}})\text{–generic}\}$$

has order–type $\alpha_0 + 1$. The fact that $J_{\bar{\ell}+1}$ is a suitable index set follows from the comments after the proof of Lemma 6.7.

Once we have defined $t_{\bar{\ell}}$ and $J_{\bar{\ell}+1}$ for $\bar{\ell} < k_n$, we let $I_{n+1} = J_{k_n}$.

Goal: Find s_{n+1}.

Let us take a moment to assess our current situation. For each $\bar{\ell} < k_n$, we have a condition $t_{\bar{\ell}} \in N_{\alpha_0+1} \cap P_{\zeta_{n+1}}$ such that

- $t_{\bar{\ell}}$ is totally $(N_{\alpha_0}, P_{\zeta_{n+1}})$–generic,
- $t_{\bar{\ell}}$ is weakly $(\mathfrak{N} \restriction I_{n+1}, P_{\zeta_{n+1}})$–generic,
- $t_{\bar{\ell}} \restriction \zeta_n \in G_n^{\bar{\ell}}$.

We have also arranged that I_{n+1} satisfies

- $\mathrm{otp}(I_{n+1}) = \alpha_0 + 1$,
- for each $\bar{\ell} < k_n$, $I_{n+1} \subseteq \{\beta \in I_n : t_{\bar{\ell}} \text{ is totally } (N_\beta, P_{\zeta_{n+1}})\text{–generic}\}$.

Since $\{t_{\bar{\ell}} : \bar{\ell} < k_n\} \subseteq N_{\alpha_0+1}$, we can patch these conditions together inside N_{α_0+1} to obtain a condition t (essentially the disjunction of the conditions t_ℓ) such that

- $t \in N_{\alpha_0+1} \cap P_{\zeta_{n+1}}$,
- for all $\ell < k$, $r^\ell \cup s_n \le t \restriction \zeta_n$,
- $t \Vdash (\exists \bar{\ell} < k_n)[t_{\bar{\ell}} \in \dot{G}_{\zeta_{n+1}}]$.

Given t, for each $\ell < k$ we apply Lemma 6.11 with ζ_n, ζ_{n+1}, N_{α_0+2}, N_{α_0+3}, t, and $r^\ell \cup s_n$ in place of the objects i, j, N_0, N_1, p, and q appearing there.

Thus for each $\ell < k$, we get an object u_ℓ such that

- $r^\ell \cup s_n \cup u_\ell \in P_{\zeta_{n+1}}$,
- $r^\ell \cup s_n \cup u_\ell \le t$,
- $r^\ell \cup s_n \cup u_\ell$ is $(N_{\alpha_0+2}, P_{\zeta_{n+1}})^{\mathrm{fin}}$–generic,
- $r^\ell \cup s_n \cup u_\ell$ is $(N_{\alpha_0+3}, P_{\zeta_{n+1}})$–generic.

Now another patching argument amalgamates the (finitely many) u_ℓ's to give us an s_{n+1} such that for all $\ell < k$,

- $r^\ell \cup s_{n+1}$ and $r^\ell \cup s_n \cup u_\ell$ are equivalent as far as forcing with $P_{\zeta_{n+1}}$ is concerned,
- $r^\ell \cup s_{n+1}$ is $(N_{\alpha_0+2}, P_{\zeta_{n+1}})^{\mathrm{fin}}$–generic,
- $r^\ell \cup s_{n+1}$ is $(N_{\alpha_0+3}, P_{\zeta_{n+1}})$–generic.

Note that we get $(N_{\alpha_0+2}, P_{\zeta_{n+1}})^{\mathrm{fin}}$–genericity because we have only finitely many u_ℓ's.

Goal: Find k_{n+1} and $\langle G_{n+1}^{\ell'} : \ell' < k_{n+1} \rangle$.

In the model N_{α_0+2}, we have a $P_{\zeta_{n+1}}$–name $\dot{\tau}$ for $N_{\alpha_0+1} \cap \dot{G}_{\zeta_{n+1}}$. Since $P_{\zeta_{n+1}}$ is totally proper,

(6.9) $$\Vdash_{P_{\zeta_{n+1}}} \dot{\tau} \in V.$$

Since each $r^\ell \cup s_{n+1}$ is $(N_{\alpha_0+2}, P_{\zeta_{n+1}})^{\mathrm{fin}}$–generic, there is a finite set

$$\mathcal{G} \subseteq \mathrm{Gen}(N_{\alpha_0+1}, P_{\zeta_{n+1}}) \cap N_{\alpha_0+2}$$

such that

(6.10) $(\forall \ell < k)[r^\ell \cup s_{n+1} \Vdash \dot\tau \in \mathcal{G}].$

Without loss of generality, $|\mathcal{G}|$ is as small as possible and now we define $k_{n+1} = |\mathcal{G}|$ and list $\mathcal{G}$ as $\langle G_{n+1}^{\ell'} : \ell' < k_{n+1} \rangle$.

We still need to see that the sequence $\langle G_{n+1}^{\ell'} : \ell' < k_{n+1} \rangle$ has the required attributes — in particular, we need properties (8), (9), and (10) of our induction hypothesis. Since $|\mathcal{G}|$ was taken to be as small as possible, our choice of t implies that for all $\ell' < k_{n+1}$, there exists $\bar\ell < k_n$ such that $t_{\bar\ell} \in G_{n+1}^{\ell'}$, and this suffices to obtain the required properties.

Thus our induction can be carried out, and suitable objects found for each $n < \omega$. To finish the proof of Proposition 6.13 for this case, note that the sequence $\{p_n : n \in \omega\}$ we constructed will generate a member G^* of $\mathrm{Gen}(N_0, P_\zeta)$, which by choice of p_0 satisfies $p \in G^*$. If we define $s = \bigcup_{n < \omega} s_n$, then for each $\ell < k$, $r^\ell \cup s$ is a lower bound for G^* in P_ζ — this is proved by showing that $r^\ell \cup s$ extends p_n for each $n < \omega$. The requirement on the domain of s causes no trouble either by the remark made after the statement of Lemma 6.11.

This completes the proof that Proposition 6.13 continues to hold for ζ a limit ordinal, and completes the proof by induction that Proposition 6.13 holds in all cases.   □

Now that Proposition 6.13 is done, it is quite easy to obtain the conclusion of Theorem 4 — we simply apply Proposition 6.13 to the case $\xi = 0$ and $\zeta = \epsilon$. Proposition 6.13 is more than strong enough to allow us to prove that P_ϵ is totally proper.

7. Constructing the model

In this section, we finally build a model of ZFC in which CH holds and every first countable, countably compact space is either compact, or contains a copy of ω_1. Our construction has a fairly standard form — we do a countable support iteration of length ω_2 over a model of GCH in which we kill every potential counterexample. We then appeal to the iteration theorem of the previous section to conclude that the Continuum Hypothesis remains true in the final generic extension.

The first thing we do is to sketch the proof of a lemma that has a right to be called folklore.

Lemma 7.1 (Folklore). *Let X be countably compact and countably tight. Then X contains an uncountable free sequence if and only if X contains a closed subset that is a closed pre–image of ω_1.*

Proof. Suppose first that X contains an uncountable free sequence, i.e., a subset $F = \{x_\alpha : \alpha < \omega_1\}$ such that for each $\alpha < \omega_1$,

(7.1) $\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset.$

For $\alpha < \omega_1$, we define $Y_\alpha = \overline{\{x_\beta : \beta < \alpha\}}$. Let $Y = \bigcup_{\alpha < \omega_1} Y_\alpha$, and we define a map $\pi : Y \to \omega_1$ by letting $\pi(x)$ be the least α such that $x \in Y_\alpha$.

Since X is countably tight, it is clear that Y is the closure of F in X. It is also straightforward to verify that the map π is continuous. The fact that π is closed follows from the countable compactness of X.

The converse has an even easier proof — let x_α be selected so that $\pi(x_\alpha) = \alpha+1$. Then the collection $\{x_\alpha : \alpha < \omega_1\}$ is a free sequence in X. $\square$

Proposition 7.2 ([6]). *Assume that X is a first countable, countably compact, non–compact space that does not contain an uncountable free sequence. Then there is a totally proper, $< \omega_1$–proper notion of forcing such that in the generic extension, X acquires an uncountable free sequence.*

Note that in the above proposition, since the notion of forcing is totally proper, X will remain a first countable, countably compact non–compact space in the generic extension.

Proposition 7.3. *Assume that CH holds, and X is a first countable, countably compact, non–compact Hausdorff space. Then there is a totally proper weakly $< \omega_1$– proper notion of forcing P_X such that in the P_X–generic extension, X contains a closed subspace homeomorphic to ω_1.*

Proof. There are two cases.

CASE 1: X contains a closed pre–image of ω_1.

Since X is first countable and the Continuum Hypothesis holds, it is straightforward to prove that X contains a closed pre–image Y of ω_1 that is both closed in X and of size $\aleph_1$. The notion of forcing detailed in section 3 of this paper (with Y in place of the X used there) then does the job — in the generic extension, Y will still be a closed subset of X, and the forcing adjoins a closed subset $Z \subseteq Y$ homeomorphic to ω_1.

CASE 2: X does not contain a closed pre–image of ω_1.

In this case, P_X will be the composition $P * \dot{Q}$ of two totally proper notions of forcing. The first stage, P, will be the notion of forcing from [6] mentioned in Proposition 7.2. Since P is totally proper, X remains a first countable, countably compact non–compact space in the generic extension, and furthermore X contains a closed pre–image of ω_1. Since totally proper forcing adds no new reals, the Continuum Hypothesis remains true as well, so the next step is to proceed as in the first case to get a notion of forcing that adjoins a closed copy of ω_1 to X. The composition of these two forcing notions is the P_X that is required. (Note that the composition of a $< \omega_1$–proper notion of forcing and a weakly $< \omega_1$–proper notion of forcing is weakly $< \omega_1$–proper.)

Construction: We will define our iteration $\mathbb{P} = \langle P_\xi, \dot{Q}_\xi : \xi < \omega_2 \rangle$ by induction on ξ. For this, it suffices to specify the identity of $\dot{Q}_\xi$ given P_ξ as the fact that we are constructing a countable support iteration determines what happens at limit stages.

We assume that we have fixed ahead of time some reasonable bookkeeping procedure that will, at each stage ξ of our construction, furnish us with (a P_ξ–name for) a first countable, countably compact, non–compact space X of size $\aleph_1$ that must be taken care of. We also assume that the Continuum Hypothesis remains true after forcing with P_ξ. Given this, Proposition 7.3 tells us that there is a P_ξ–name $\dot{Q}_\xi$ for a totally proper notion of forcing that will give X a closed subset homeomorphic to ω_1.

Claim 7.4. *Any countable support iteration $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ defined by the preceding recipe will satisfy the hypotheses of Theorem 4.*

Proof. Our earlier work (and work done in [6]) tells us that for each $\alpha < \omega_1$,

$$\Vdash_{P_\alpha} \dot{Q}_\alpha \text{ is totally proper}$$

and

$$\Vdash_{P_\alpha} \dot{Q}_\alpha \text{ is weakly } < \omega_1\text{–proper.}$$

Thus we need only worry about the third hypothesis of Theorem 4. Toward this goal, let us fix α, and suppose we have objects N_0, N_1, $\bar{G}$, and $\langle G^\ell : \ell < k \rangle$ such that

- N_0 and N_1 are countable elementary submodels of $H(\lambda)$,
- $N_0 \in N_1$,
- $\{\mathbb{P}, \alpha, \dot{q}\} \in N_0$,
- $\Vdash_{P_\alpha} \dot{q} \in \dot{Q}_\alpha$,
- $\bar{G} \in \text{Gen}^+(N_0, P_\alpha, p) \cap N_1$,
- for $\ell < k$, $G^\ell \in \text{Gen}(N_1, P_\alpha)$,
- for $\ell < k$, $\bar{G} \subseteq G^\ell$.

We will assume that our bookkeeping at stage α hands us a first countable closed pre–image of ω_1 of size $\aleph_1$, so that $\dot{Q}_\alpha$ will be the notion of forcing detailed earlier in the paper. The other alternative for $\dot{Q}_\alpha$ can be handled by a similar argument — in fact, in [6] the notion of forcing used is shown to satisfy an even more stringent requirement.

Let $\{\dot{D}_n : n \in \omega\} \in N_1$ be a list of all P_α–names for dense subsets of $\dot{Q}_\xi$ that are elements of N_0. Our goal is to construct a sequence $\{\dot{p}_n : n \in \omega\}$ of P_ξ–names from N_0 such that

- $\bar{G} \Vdash \dot{p}_{n+1} \leq \dot{p}_n$,
- $\bar{G} \Vdash \dot{p}_{n+1} \in \dot{D}_n$,
- for each $\ell < k$, $G^\ell \Vdash \{\dot{p}_n : n \in \omega\}$ has a lower bound.

Before we do this, we reflect for a moment on the proof of Theorem 1. We constructed a decreasing sequence of conditions $\{p_n : n \in \omega\}$ that hit every dense subset of P that is a member of N, and we had to take action to ensure that the sequence had a lower bound. The action we took was, essentially, to make sure that the sequence of sets $[p_{n+1}] \setminus [p_n]$ converged "fast enough" to a point in $\text{Tr}(N)$ — rapid enough convergence makes sure that the sequence has a lower bound. In our current situation, we will have to make sure that each G^ℓ thinks that the sequence $[\dot{p}_{n+1}] \setminus [\dot{p}_n]$ converges "fast enough" so that we will end up with a lower bound no matter which G^ℓ turns out to be $N_1 \cap \dot{G}$.

So how do we do this? In N_0, there are P_α–names $\dot{X}$ and $\dot{\mathcal{U}}$ for the topological space and filter of closed subsets under consideration. Let $\{\dot{D}_n : n \in \omega\} \in N_1$ be as above.

Claim 7.5. *In N_1 there are objects $\dot{z}$ and $\langle \dot{x}_n : n \in \omega \rangle$ such that*

(1) *$\dot{z} \in N_1^{P_\alpha}$ is a name for a point in $\dot{X}$,*
(2) *each $\dot{x}_n$ is a name in N_0 for a point in $\dot{X}$,*
(3) *if $\dot{A} \in N_0^{P_\alpha}$ is a name for an element of $\dot{\mathcal{U}}$, then for all sufficiently large n, $\bar{G} \Vdash \dot{x}_n \in \dot{A}$,*
(4) *for all $\ell < k$, $G^\ell \Vdash \langle \dot{x}_n : n \in \omega \rangle$ converges to $\dot{z}$.*

Proof. We work inside the model N_1. Let $\{\dot{A}_n : n \in \omega\}$ list all names in $N_0^{P_\alpha}$ for elements of $\dot{\mathcal{U}}$. We know that for each $n < \omega$,

$$\bar{G} \Vdash \bigcap_{i<n} \dot{A}_i \neq \emptyset,$$

and so for each n we can find a name $\dot{x}_n \in N_0^{P_\alpha}$ such that

$$\bar{G} \Vdash \dot{x}_n \in \dot{A}_0 \cap \cdots \cap \dot{A}_{n-1}.$$

This produces a sequence $\langle \dot{x}_n : n \in \omega \rangle$ satisfying requirement (3) of the claim, and since we were working in N_1, without loss of generality $\langle \dot{x}_n : n \in \omega \rangle \in N_1$.

$\dot{X}$ is (forced to be) a closed pre–image of ω_1, and so by our choice of $\langle \dot{x}_n : n \in \omega \rangle$ we know there are names $\dot{I}_0$ and $\dot{z}_0$ in N_1 such that

(7.2) $G^0 \Vdash \langle \dot{x}_n : n \in \dot{I}_0 \rangle$ converges to $\dot{z}_0$.

Now $\dot{I}_0$ is a name for a subset of ω and P_α is totally proper, so there is an honest–to–goodness $I_0 \subseteq \omega$ such that

(7.3) $G^0 \Vdash \dot{I}_0 = I_0$.

Note that $I_0 \in N_1$, hence the sequence $\langle \dot{x}_n : n \in I_0 \rangle$ is in N_1 as well. By re-indexing, we achieve that there is a name $\dot{z}_0 \in N_1$ such that

(7.4) $G^0 \Vdash \langle \dot{x}_n : n \in \omega \rangle$ converges to $\dot{z}_0$.

Now we repeat the same argument, running through the rest of the G^ℓ's. Since there are only finitely many G^ℓ's, we are left with a sequence $\langle \dot{x}_n : n \in \omega \rangle$ and names $\langle \dot{z}_\ell : \ell < k \rangle$ such that

(7.5) $G^\ell \Vdash \langle \dot{x}_n : n \in \omega \rangle$ converges to $\dot{z}_\ell$.

A standard argument allows us to replace the names $\dot{z}_0, \ldots, \dot{z}_{k-1}$ with a single name $\dot{z}$ — again, since we have only finitely many names to patch together, we can assume $\dot{z} \in N_1$. We finish by noting that requirement (3) is preserved by passing to infinite subsequences.   □

Note that for each $\ell < k$, we have

(7.6) $G^\ell \Vdash \dot{z} \in \text{Tr}(N_0[\dot{G}_\alpha])$;

this follows by our choice of $\langle \dot{x}_n : n \in \omega \rangle$. Let $\{\dot{U}_n : n \in \omega\}$ be a sequence of P_α–names such that every condition forces that the interpretations of the $\dot{U}_n$'s form a decreasing base of open neighborhoods for $\dot{z}$.

We will define by induction on n a sequence $\{\dot{p}_n : n \in \omega\}$ and function $h \in {}^\omega\omega$ such that

(1) $\dot{p}_0 = \dot{q}$, $h(0) = 0$,
(2) $\dot{p}_n \in N_0^P \ (\subseteq N_1)$,
(3) $\bar{G} \Vdash \dot{p}_{n+1} \leq \dot{p}_n$,
(4) $\bar{G} \Vdash \dot{p}_{n+1} \in \dot{D}_n$,
(5) $h(n) < h(n+1)$,
(6) for each $\ell < k$, $G^\ell \Vdash [\dot{p}_{n+1}] \setminus [\dot{p}_n] \subseteq \dot{U}_{h(n+1)}$,
(7) if $\dot{f}$ is a name in N_0^P such that $\bar{G} \Vdash \dot{f} \in \Phi_{\dot{p}_i}$ for some i, then there is a stage $n \geq i$ such that for each $\ell < k$,

(7.7) $G^\ell \Vdash \{x \in Y(\dot{f}, \dot{p}_n, \dot{p}_i) : \dot{U}_{h(n+1)} \subseteq \dot{f}(x)\}$ is large.

Just as in the proof of Theorem 1, we assume that some bookkeeping procedure has been fixed at the start so that at each stage we will be handed a (name for a) promise $\dot{f}$ that must be taken care of. Assume we are given p_n and $h \upharpoonright n + 1$ and that our bookkeeping hands us a $\dot{f}$ such that for some $i \leq n$,

$$(7.8) \qquad\qquad \bar{G} \Vdash \dot{f} \in \Phi_{\dot{p}_i}.$$

In N_0^P, we can find a P_α–name $\dot{f}'$ such that

$$\bar{G} \Vdash \dot{f}' = \dot{f} \upharpoonright Y(\dot{f}, \dot{p}_n, \dot{p}_i).$$

Fix $\ell < k$. We know

$$G^\ell \Vdash \dot{z} \notin \operatorname{Ban} \dot{f}',$$

and so there is a natural number m_ℓ such that

$$G^\ell \Vdash \{x \in Y(\dot{f}, \dot{p}_n, \dot{p}_i) : \dot{U}_{m_\ell} \subseteq \dot{f}(x)\} \text{ is large.}$$

We define $h(n+1)$ to be the least number greater than $h(n)$ and all m_ℓ's for $\ell < k$. With this choice of $h(n+1)$, we have that for each $\ell < k$,

$$(7.9) \qquad G^\ell \Vdash \{x \in Y(\dot{f}, \dot{p}_n, \dot{p}_i) : \dot{U}_{h(n+1)} \subseteq \dot{f}(x)\} \text{ is large.}$$

Note that we have taken care of this particular instance of requirement (7.7). Next, we choose $i_n < \omega$ large enough so that for all $\ell < k$,

$$(7.10) \qquad\qquad G^\ell \Vdash \dot{x}_{i_n} \in \dot{U}_{h(n+1)}.$$

This is possible because there are only finitely many G^ℓ's to worry about.

The model N_0 will contain names for all the members of a countable neighborhood base of $\dot{x}_{i_n}$, so we can find $\dot{V} \in N_0^{P_\alpha}$ such that

$$(7.11) \qquad\qquad \bar{G} \Vdash \dot{V} \text{ is a neighborhood of } \dot{x}_{i_n}$$

and

$$(7.12) \qquad\qquad (\forall \ell < k) G^\ell \Vdash \dot{V} \subseteq \dot{U}_{h(n+1)}.$$

Now choose $\dot{p}_{n+1} \in N_0^{P_\alpha}$ such that $\bar{G}$ forces $\dot{p}_{n+1} \leq \dot{p}_n$, $\dot{p}_{n+1} \in \dot{D}_n$, and $[\dot{p}_{n+1}] \setminus [\dot{p}_n] \subseteq \dot{V}$. All this is possible by our work earlier in the paper, and just as in the proof of Theorem 1, we have

$$(7.13) \qquad\qquad (\forall \ell < k) G^\ell \Vdash \{\dot{p}_n : n \in \omega\} \text{ has a lower bound,}$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now that we have established that our construction satisfies the assumptions of our iteration theorem, the rest of the proof follows standard lines. $\qquad\square$

Theorem 5. *It is consistent that the Continuum Hypothesis holds and that first countable, countably compact regular spaces are either compact or contain a closed copy of ω_1.*

Proof. We start with a model of ZFC+CH, and force with an iteration $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ as described earlier in this section. We have shown that any such iteration satisfies the hypotheses of our iteration theorem, and so the limit forcing P_{ω_2} is totally proper. It is also easy to see that all iterands of $\mathbb{P}$ are ω_2–centered, and so P_{ω_2} will preserve all cardinals and cofinalities. A standard argument lets us construct $\mathbb{P}$ so that every first countable, countably compact, non–compact space of size $\aleph_1$ is "taken care of" at some stage of the iteration, i.e., every such space contains a

closed copy of ω_1 in $V[G_{\omega_2}]$. Assuming CH holds, every first countable, countably compact, non–compact space contains a closed non–compact subset of size $\aleph_1$. Since CH holds in $V[G_{\omega_2}]$, we see that in our model, every first countable, countably compact, non–compact T_3 space contains a (closed) subset homeomorphic to ω_1. $\square$

8. What's special about ω_1?

In this section, we prove in ZFC that there is a two–to–one closed pre–image of ω_2 that does not contain a copy of ω_2. We use Shelah's method of club–guessing — the references for this are [18] and [17] of [16].

We start by fixing some notation.

Definition 8.1. If λ and μ are regular cardinals, then we define

(8.1) $$S_\mu^\lambda = \{\delta < \lambda : \mathrm{cf}(\delta) = \mu\}.$$

By Claim III.2.3 of [16], there is a family

(8.2) $$\bar{C} = \langle C_\delta : \delta \in S_{\aleph_0}^{\aleph_2} \rangle$$

such that each C_δ is cofinal in δ, C_δ has order–type ω, and for every closed unbounded $C \subseteq \omega_2$, there is a stationary set of δ's such that $C_\delta \setminus C$ is finite.

For $\delta \in S_{\aleph_0}^{\aleph_2}$, let $\eta_\delta : \omega \to \delta$ be the increasing enumeration of C_δ.

We will obtain the underlying set of our space X by taking ω_2 and "doubling" the points of cofinality ω, i.e., the underlying set of X is $(\omega_2 \setminus S_{\aleph_0}^{\aleph_2}) \cup (S_{\aleph_0}^{\aleph_2} \times \{0,1\})$. We will topologize X in such a way that the natural projection $\pi : X \to \omega_2$ becomes a closed map.

For $\alpha < \omega_2$, by "level α" we mean the set $\pi^{-1}(\{\alpha\})$. Each level of our space has cardinality at most 2. We will define the topology by induction on the level of the points in X.

To start, the first level of our space (i.e., level 0) is an isolated point.

Our induction hypothesis for stage α is that we have topologized $\pi^{-1}(\alpha)$ in such a way that the restriction of π to this initial segment of X is a closed mapping.

If α is a successor ordinal, the level α consists of a single point, and we declare that this point is isolated in X.

If α is a limit ordinal of cofinality ω_1, then level α consists of a single point, and basic open neighborhoods of this point will consist of sets of the form $\pi^{-1}([\beta+1, \alpha])$, where $\beta < \alpha$.

The work comes in when α is a limit ordinal of countable cofinality. Let x_0 and x_1 denote the two points in X at level α.

We will arrange that x_0 and x_1 each have a countable base in X. A basic open neighborhood of x_0 is of the form

(8.3) $$\bigcup_{n \geq m} \pi^{-1}([\eta_\alpha(2n) + 1, \eta_\alpha(2n+1)]),$$

for $m < \omega$, while a basic open neighborhood of x_1 is of the form

(8.4) $$\bigcup_{n \geq m} \pi^{-1}([\eta_\alpha(2n+1) + 1, \eta_\alpha(2n+2)]).$$

In plain language, we use C_α to partition α into an ω–sequence of intervals. A basic open neighborhood of x_0 looks like the pre–image (under π) of a tail of the "even intervals", while a basic open neighborhood of x_1 is similar with even replaced by odd.

It is not hard to verify that this defines a topology on X with the advertised bases at each point, and also that the mapping π is continuous and closed.

Proposition 8.2. *X does not contain a closed set homeomorphic to ω_2.*

Proof. We work by contradiction. Assume $Y \subseteq X$ is homeomorphic to ω_2, witnessed by the homeomorphism f. Since π is a closed mapping, we know the set $C := \pi[Y]$ is closed and unbounded in ω_2.

Let M be an elementary submodel of $H(\lambda)$ for some large λ satisfying

- $\{X, Y, \bar{C}, f\} \in M$,
- $M \cap \omega_2$ is an ordinal of cofinality ω,
- if $\delta = M \cap \omega_2$, then $C_\delta \setminus C$ is finite.

Such an M can be found as the set of such δ's is stationary in ω_2.

Note that $M \cap Y = f^{-1}(\delta)$, and this means that the $\overline{M \cap Y} \setminus (M \cap Y)$ consists of the singleton $f^{-1}(\{\delta\})$. We will get a contradiction by showing that $\pi^{-1}(\{\delta\}) \subseteq \overline{M \cap Y}$.

We show that $x_0 \in \overline{M \cap Y}$; the proof that $x_1 \in \overline{M \cap Y}$ is entirely analogous.

Let U be a basic open neighborhood of x_0. We need only show that $U \cap Y \neq \emptyset$. By definition, there is an m such that

$$(8.5) \qquad U = \bigcup_{n \geq m} \pi^{-1}([\eta_\delta(2n) + 1, \eta_\delta(2n+1)]).$$

In particular,

$$(8.6) \qquad \bigcup_{n \geq m} \pi^{-1}(\{\eta_\delta(2n+1)\}) \subseteq U.$$

For all large enough k, we know

$$(8.7) \qquad Y \cap \pi^{-1}(\{\eta_\delta(k)\}) \neq \emptyset,$$

as $\eta_\delta(k) \in C = \pi[Y]$. This means that $U \cap Y \neq \emptyset$ as desired. We therefore have a contradiction, and the proposition is established. $\square$

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