TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 357, Number 12, Pages 4771–4804
S 0002-9947(05)03784-0
Article electronically published on July 20, 2005

A HYPERBOLIC FREE BOUNDARY PROBLEM MODELING TUMOR GROWTH: ASYMPTOTIC BEHAVIOR

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Abstract. In this paper we study a free boundary problem modeling the growth of radially symmetric tumors with two populations of cells: proliferating cells and quiescent cells. The densities of these cells satisfy a system of nonlinear first order hyperbolic equations in the tumor, and the tumor's surface is a free boundary $r = R(t)$. The nutrient concentration satisfies a diffusion equation, and $R(t)$ satisfies an integro-differential equation. It is known that this problem has a unique stationary solution with $R(t) \equiv R_s$. We prove that (i) if $\lim_{T\to\infty} \int_T^{T+1} |\dot{R}(t)|\, dt = 0$, then $\lim_{t\to\infty} R(t) = R_s$, and (ii) the stationary solution is linearly asymptotically stable.

1. The model

A variety of PDE models for tumor growth have been developed in the last three decades. These models are based on mass conservation laws and on reaction-diffusion processes for cell densities and nutrient concentrations within the tumor. The surface of the tumor is a free boundary, and one seeks to determine both the tumor's region and the solutions of the differential equations within the tumor. Some models assume that all cells are in proliferating state, while other models include cells in a quiescent and/or in a necrotic state. In some of the latter models, the cells in different states are assumed to be mixed together, while in other models the cells in different states are assumed to occupy separate regions in the tumor.

We refer to [1], [5]–[9], [18], [19], [23] and references therein for models which are based on reaction-diffusion equations, and to [4], [20]–[22], [24] for models which include hyperbolic equations; the hyperbolic equations arise from mass conservation laws of densities of cells. Some of these articles include numerical results. Rigorous mathematical analysis including existence, uniqueness, and stability theorems, as well as properties of the free boundaries, have been obtained in [2], [3], [10]–[17].

In this paper we consider a model which includes densities P, Q of proliferating and quiescent cells, respectively, and concentration C of nutrients. The cells, in different states, are assumed to be mixed within the tumor, and to have the same size and mass. We also assume that the tumor is uniformly packed with cells, so that

$$P + Q = const. \equiv N. \tag{1.1}$$

Received by the editors September 24, 2002.
2000 *Mathematics Subject Classification.* Primary 34B15; Secondary 35C10, 35Q80, 92C15.
Key words and phrases. Tumor growth, free boundary problem, stationary solution, asymptotic behavior.

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Due to proliferation of cells and to removal of necrotic cells, there is a continuous movement of cells within the tumor. We represent this movement by a velocity field $\vec{v}$. We treat the tumor tissue as a porous medium so that, by Darcy's law,

$$\vec{v} = \nabla\sigma, \qquad \sigma \text{ pressure.} \tag{1.2}$$

Next we assume that living cells can change from proliferating state to quiescent state at a rate $\overline{K}_Q(C)$, and from quiescent state to proliferating state at a rate $\overline{K}_P(C)$. Clearly,

- $\overline{K}_P(C)$ is monotone increasing in C, since the tumor grows (i.e., proliferation increases) if the supply of nutrients increases, and, similarly,
- $\overline{K}_Q(C)$ is monotone decreasing in C.

We also assume that quiescent cells become necrotic (primarily because of insufficient nutrition) at a rate $\overline{K}_D(C)$, where

- $\overline{K}_D(C)$ is monotone decreasing in C,

i.e., the death rate increases as the supply of nutrients decreases.

The proliferating cells undergo proliferation as well as apoptosis (natural death). For simplicity we neglect apoptosis. We denote the proliferation rate by $\overline{K}_B(C)$. Then,

- $\overline{K}_B(C)$ is monotone increasing in C.

In a previous paper [14], Cui and Friedman considered a tumor model, which includes necrotic cells, and proved, in the radially symmetric case, that there exists a unique solution with tumor volume $\{r < R(t)\}$, where $R(t)$ remains uniformly positive and uniformly bounded for all $t > 0$. However, the asymptotic behavior of $R(t)$ as $t \to \infty$ remained unresolved. In the present paper we address this latter problem in the special case where the presence of necrotic cells is neglected. This situation occurs if we assume that $\overline{K}_D(C) = 0$ (this assumption does not affect our results), or if we assume that the necrotic cells are cleared from the tumor on a fast time scale.

We assume that C satisfies a diffusion equation which, for simplicity, we take to be

$$\nabla^2 C - \lambda C = 0 \quad \text{ in } \Omega(t) \ (\lambda > 0), \tag{1.3}$$

and

$$C = C_0 \quad \text{on } \partial\Omega(t), \tag{1.4}$$

where $\Omega(t)$ is the tumor region at time t. The mass conservation laws for the densities of proliferating cells and quiescent cells in $\Omega(t)$ take the following form:

$$\frac{\partial P}{\partial t} + \operatorname{div}(P\vec{v}) = [\overline{K}_B(C) - \overline{K}_Q(C)]P + \overline{K}_P(C)Q, \tag{1.5}$$

$$\frac{\partial Q}{\partial t} + \operatorname{div}(Q\vec{v}) = \overline{K}_Q(C)P - [\overline{K}_P(C) + \overline{K}_D(C)]Q. \tag{1.6}$$

If we add Eqs. (1.5), (1.6) and use (1.1), (1.2), we obtain an equation for the pressure σ:

$$N\nabla^2\sigma = \overline{K}_B(C)P - \overline{K}_D(C)Q. \tag{1.7}$$

Clearly, Eq. (1.6) may be replaced by Eq. (1.7). If we replace Q by $N - P$ and set

$$\overline{c} = \frac{C}{C_0}, \qquad \overline{p} = \frac{P}{N},$$

we arrive at the following system of equations:

$$\nabla^2\overline{c} - \lambda\overline{c} = 0 \quad \text{in } \Omega(t), \tag{1.8}$$

$$\overline{c} = 1 \quad \text{on } \partial\Omega(t), \tag{1.9}$$

$$\frac{\partial\overline{p}}{\partial t} + \operatorname{div}(\overline{p}\nabla\sigma) = K_P(\overline{c}) + [K_B(\overline{c}) - K_N(\overline{c})]\overline{p} \quad \text{in } \Omega(t), \tag{1.10}$$

$$\nabla^2\sigma = -K_D(\overline{c}) + K_M(\overline{c})\overline{p} \quad \text{in } \Omega(t), \tag{1.11}$$

where

$$K_M(\overline{c}) = K_B(\overline{c}) + K_D(\overline{c}), \qquad K_N(\overline{c}) = K_P(\overline{c}) + K_Q(\overline{c}) \tag{1.12}$$

and

$$K_i(\overline{c}) = \overline{K}_i(C_0\overline{c}) \quad \text{for } i = B, D, P, Q.$$

Since we shall deal with the radially symmetric case, we can take

$$\vec{v} = \overline{u}(r)\frac{x}{r} \qquad (r = |x|),$$

and rewrite Eqs. (1.8)–(1.11) in the form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\overline{c}}{\partial r}\right) = \lambda\overline{c} \qquad (0 < r < R(t),\ t \geq 0), \tag{1.13}$$

$$\frac{\partial\overline{c}}{\partial r}(0,t) = 0, \qquad \overline{c}(R(t),t) = 1 \quad (t \geq 0), \tag{1.14}$$

$$\frac{\partial\overline{p}}{\partial t} + u\frac{\partial\overline{p}}{\partial r} = K_P(\overline{c}) + [K_M(\overline{c}) - K_N(\overline{c})]\overline{p} - K_M(\overline{c})\overline{p}^2 \qquad (0 \leq r \leq R(t),\ t > 0), \tag{1.15}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{u}) = -K_D(\overline{c}) + K_M(\overline{c})\overline{p} \qquad (0 < r \leq R(t),\ t \geq 0), \tag{1.16}$$

$$\overline{u}(0,t) = 0 \qquad (t \geq 0). \tag{1.17}$$

The motion of the free boundary is given by the continuity equation:

$$\frac{dR(t)}{dt} = R(t)\overline{u}(R(t),t) \qquad (t > 0). \tag{1.18}$$

Finally, we prescribe initial conditions:

$$\overline{p}(r,0) = \overline{p}_0(r) \qquad (0 \leq r \leq R_0), \tag{1.19}$$

$$R(0) = R_0, \tag{1.20}$$

where R_0 is a positive constant and $\overline{p}_0(r)$ is a continuously differentiable function satisfying $0 \leq \overline{p}_0(r) \leq 1$ $(0 \leq r \leq R_0)$.

Recently, the last two authors have proved [13] that there exists a unique stationary solution $(\overline{c}_s(r), \overline{p}_s(r), \overline{u}_s(r), R_s)$ of the problem (1.13)–(1.18). They also proved [14] that there exists a unique global solution of the time-dependent problem (1.13)–(1.20), and

$$\delta_0 \leq R(t) \leq M \quad \text{for all } t \geq 0, \tag{1.21}$$

where δ_0, M are positive constants. Based on (1.21) we expect that the stationary solution is globally asymptotically stable. In this paper we prove the following two partial results in this direction.

(i) If $R(t)$ is non-oscillating for large t or, more generally, if

$$\int_T^{T+1} |\dot{R}(t)|\, dt \to 0 \quad \text{as } T \to \infty,$$

then

$$R(t) \to R_s, \quad \overline{p}(r,t) \to \overline{p}_s(r), \quad \overline{u}(r,t) \to \overline{u}_s(r), \quad \overline{c}(r,t) \to \overline{c}_s(r) \tag{1.22}$$

as $t \to \infty$;

(ii) the stationary solution is linearly asymptotically stable.

The proof of (i) is given in §§3, 4. It is based on comparison theorems and on the following estimates:

$$\overline{p}_r(r,t) > 0 \quad \text{if } r > c_0 e^{-\nu_0 t},\ t \geq t_0, \tag{1.23}$$

$$-c_1 \leq \frac{R(t)\overline{u}(r,t) - r\overline{u}(R(t),t)}{r(R(t)-r)} \leq -c_2, \qquad t \geq t_0, \tag{1.24}$$

$$r|\overline{p}_r(r,t)| \leq c_3, \tag{1.25}$$

where ν_0, t_0 and the c_i's are positive constants. In §5 we state more precisely assertion (ii). The proof of this assertion is based on reducing the linearized evolution system by means of a solution to a singular integro-differential equation to the study of two simpler, partially decoupled, evolution equations. The integro-differential equation is solved in §§6–7, and the study of the two evolution equations is given in §8, where the proof of (ii) is completed.

It will be convenient to reduce the problem (1.13)–(1.20) to a problem in the fixed region $\{0 \leq r \leq 1,\ t > 0\}$; this is done in §2.

2. Reformulation of the problem

Set

$$p(r,t) = \overline{p}(rR(t),t), \quad u(r,t) = \frac{\overline{u}(rR(t),t)}{R(t)} \qquad (0 \leq r \leq 1),$$
$$z(t) = \log R(t),$$

and let $c(r,z)$ denote the solution of the problem

$$\begin{cases} -\Delta c = \lambda e^{2z} c & \text{in } B_1 = \{r < 1\},\ z \in \mathbf{R}, \\ c = 1 & \text{on } \partial B_1, \end{cases}$$

namely,

$$c(r,z) = \frac{\sinh(\sqrt{\lambda}e^z r)}{r\sinh(\sqrt{\lambda}e^z)} \quad (0 < r \leq 1), \qquad c(0,z) = \frac{\sqrt{\lambda}e^z}{\sinh(\sqrt{\lambda}e^z)}. \tag{2.1}$$

It can be easily verified that, for a given $R(t)$, the solution of (1.13), (1.14) is given by

$$\overline{c}(r,t) = c\left(\frac{r}{R(t)}, \log R(t)\right) = c(re^{-z(t)}, z(t)).$$

Hence, in the future we shall only consider Eqs. (1.15)–(1.20), where $\overline{c}$ is replaced with the above expression. We introduce the functions

$$f(c,p) = K_P(c) + [K_M(c) - K_N(c)]p - K_M(c)p^2,$$
$$g(c,p) = -K_D(c) + K_M(c)p,$$
$$v(r,t) = u(r,t) - ru(1,t).$$

Then (1.1)–(1.20) can be rewritten in the form

$$p_t + vp_r = f(c(r,z(t)), p(r,t)) \qquad (0 \le r \le 1,\ t > 0), \tag{2.2}$$

$$\dot{z} = \int_0^1 r^2 g(c(r,z(t)), p(r,t))\,dr \qquad (t > 0), \tag{2.3}$$

$$r^2 v = (1 - r^3)\dot{z} - \int_r^1 \rho^2 g(c(\rho,z(t)), p(\rho,t))\,d\rho \qquad (0 < r \le 1,\ t > 0), \tag{2.4}$$

$$p(r,0) = p_0(r) \equiv \overline{p}_0(rR_0) \quad (0 \le r \le 1), \qquad z(0) = z_0 \equiv \log R_0. \tag{2.5}$$

We make the following assumptions:

(a) $K_i(c)$ $(i = B, D, P, Q)$ are continuously differentiable for $0 \le c \le 1$, and

$$K_B'(c) > 0, \quad K_P'(c) > 0 \qquad (0 \le c \le 1),$$
$$K_B(0) = K_P(0) = 0,$$
$$K_D'(c) < 0, \quad K_Q'(c) < 0 \qquad (0 \le c \le 1),$$
$$K_D(1) = K_Q(1) = 0,$$
$$K_M'(c) = K_B'(c) + K_D'(c) > 0 \qquad (0 \le c \le 1);$$

(b) $p_0(r)$ is continuously differentiable for $0 \le r \le 1$, and

$$0 \le p_0(r) \le 1 \quad \text{for } 0 \le r \le 1.$$

The last assumption in (a) is based on experimental observations (see [20]).

It can be easily verified that $0 \le p(r,t) \le 1$ for $0 \le r \le 1$, $t \ge 0$ and

$$\frac{1}{r}\frac{\partial c}{\partial r} > 0, \quad \frac{1}{1-r}\frac{\partial c}{\partial z} < 0, \quad \frac{\partial f}{\partial c} > 0, \quad \frac{\partial g}{\partial c} > 0, \quad \frac{\partial g}{\partial p} > 0. \tag{2.6}$$

We also have

$$\frac{\partial f}{\partial p}(c(r,z_s), p_s(r)) < 0, \tag{2.7}$$

where $p_s(r)$, z_s are the corresponding components of the stationary solution (so that $p_s(r) = \overline{p}_s(rR_s)$, $z_s = \log R_s$). Indeed, setting $c_s = c(r,z_s)$ and letting α be the positive root of the equation $f(c_s,\alpha) = 0$, we have $p_s > \alpha$ in $(0,1)$ (see [13, Remark 7.1(2)]), so that

$$\begin{aligned} f_p(c_s,p_s) &= K_M(c_s) - K_N(c_s) - 2K_M(c_s)p_s \\ &< K_M(c_s) - K_N(c_s) - 2K_M(c_s)\alpha \\ &= -\sqrt{[K_M(c_s) - K_N(c_s)]^2 + 4K_M(c_s)K_P(c_s)} < 0. \end{aligned}$$

Note also that

$$u(1,t) = \frac{\dot{R}(t)}{R(t)} = \dot{z},$$

so that

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v) = g - 3u(1,t) = g - 3\dot{z}.$$

Hence

$$r^2 v = \int_0^r \rho^2 g(c(\rho, z(t)), p(\rho, t))\, d\rho - r^3 \dot{z}; \tag{2.8}$$

this formula for v as well as formula (2.4) will both be used later on.

3. Properties of the solution

In this section we establish (1.23)–(1.25) and some other estimates for the solution $(z(t), v(r,t), p(r,t))$.

Lemma 3.1. *There holds*

$$|\dot{z}(t)|,\ |\ddot{z}(t)|,\ |z^{(3)}(t)| \le C \quad \text{for } t > 0, \tag{3.1}$$

where C is a constant.

Proof. From (2.3) we have

$$|\dot{z}| \le \int_0^1 r^2 |g|\, dr \le \frac{1}{3}[K_D(0) + K_B(1)].$$

Next,

$$\ddot{z} = \int_0^1 r^2 [g_c c_z \dot{z} + g_p p_t]\, dr = \int_0^1 r^2 [g_c c_z \dot{z} + g_p(-v p_r + f)]\, dr,$$

by (2.2). Since

$$\begin{aligned} -r^2 v g_p p_r &= -r^2 v (g(c,p))_r + r^2 v g_c c_r \\ &= (-r^2 c g)_r + (r^2 v)_r g + r^2 v g_c c_r, \end{aligned}$$

we see that

$$\begin{aligned} \ddot{z} &= \int_0^1 [r^2 g_c c_z \dot{z} + (-r^2 v g)_r + (r^2 v)_r g + r^2 v g_c c_r + r^2 g_p f]\, dr \\ &= \int_0^1 \left[r^2 g_c c_z \dot{z} - 3r^2 \dot{z} g + r^2 g^2 + g_c c_r \left((1 - r^3)\dot{z} - \int_r^1 \rho^2 g\, d\rho \right) + r^2 g_p f \right] dr \\ &\qquad\qquad (\text{by (2.4) and } (-r^2 v g)|_{r=0}^1 = 0) \\ &= -3\dot{z}^2 + \dot{z} \int_0^1 g_c [r^2 c_z + (1 - r^3) c_r]\, dr + \int_0^1 r^2 \left[g^2 + g_p f - g_c c_r \int_r^1 \rho^2 g\, d\rho \right] dr. \end{aligned}$$

Clearly, the last two integrals are bounded for all $t \ge 0$. Hence $|\ddot{z}| \le C$. Similarly we can prove that $|z^{(3)}| \le C$. □

We introduce the characteristic curves $r = r(\xi, t)$ of (2.2) for $0 \le \xi \le 1$:

$$\begin{cases} \dot{r} = v(r,t), & t > 0, \\ r(\xi, 0) = \xi. \end{cases} \tag{3.2}$$

Since $v(0,t) = v(1,t) = 0$, these curves remain in $0 < r < 1$ if $0 < \xi < 1$.

Lemma 3.2. *For any $0 < \xi < 1$ and $t > 0$,*

$$\frac{\partial r(\xi, t)}{\partial \xi} = \exp\left(\int_0^t \frac{\partial v}{\partial r}(r(\xi, \tau), \tau)\, d\tau \right) > 0.$$

The proof is immediate since, by (3.2),

$$\frac{d}{dt}\left(\frac{\partial r}{\partial \xi}\right) = \frac{\partial v}{\partial r}(r(\xi,t),t)\cdot\frac{\partial r}{\partial \xi}$$

and $(\partial r/\partial\xi)|_{t=0} = 1$. □

Set

$$P(\xi,t) = p(r(\xi,t),t).$$

Lemma 3.3. *There exists a positive constant c_0 such that*

$$c_0 \le p(r,t) \le 1 \qquad (0 \le r \le 1,\ t \ge 1). \tag{3.3}$$

Proof. Let

$$c_* = \inf_{t>0} c(0,z(t)).$$

Since $\sup|z(t)| < \infty$ (by (1.21)), we have $c_* > 0$. It follows that

$$P_t = p_t + vp_r = f(c,p) \ge f(c_*,P) \qquad (t>0),$$
$$P(\xi,0) = p_0(\xi) \ge 0.$$

Let $y(t)$ be the solution of the equation $\dot y = f(c_*,y)$ with initial value $y(0)=0$. Then by comparison we have $P(\xi,t) \ge y(t)$ for all $0 \le \xi \le 1$, $t \ge 0$. From the form of $f(c_*,y)$ we deduce that $y(t) > 0$ for $t>0$ and $\lim_{t\to\infty} y(t) = \alpha(c_*)$, where $\alpha(c_*)$ is the unique positive root of the quadratic equation $f(c_*,\alpha)=0$. Hence (3.3) follows. □

Lemma 3.4. *There exist positive constants c_1, ν_1 such that*

$$P_\xi(\xi,t) \ge -c_1 e^{-\nu_1 t} \qquad (0<\xi<1,\ t\ge 1), \tag{3.4}$$

so that, in particular,

$$-\int_0^1 \min\{P_\xi(\xi,t),0\}\,d\xi \le c_1 e^{-\nu_1 t} \qquad (t \ge 1).$$

Proof. Since $\dot P = f(c,P)$, we have

$$\dot P_\xi = f_p P_\xi + f_c c_r r_\xi \ge f_p P_\xi.$$

Hence

$$P_\xi(\xi,t) \ge P_\xi(\xi,0)\exp\left(\int_0^t f_p(c(r(\xi,\tau),z(\tau)),P(\xi,\tau))\,d\tau\right). \tag{3.5}$$

Observe that

$$f_p = \frac{f}{P} - \frac{K_P(c)}{P} - K_M(c)P = \frac{f}{P} - \nu(r,t),$$

where $\nu(r,t) = \frac{K_P(c)}{P} + K_M(c)P$. Since, by (3.3), $c_0 \le P \le 1$, we have $\nu_0 \le \nu(r,t) \le \nu_1$ $(0\le r\le 1,\ t\ge 1)$, where ν_0, ν_1 are positive constants. Hence

$$-\nu_1 \le f_p - \frac{\dot P}{P} \le -\nu_0 \qquad (t \ge 1). \tag{3.6}$$

Using the first inequality and (3.3) in (3.5), assertion (3.4) follows. □

Lemma 3.5. *There exist positive constants c_2, T_0 such that*

$$p_r(r,t) > 0 \quad \textit{if}\ c_2 e^{-\frac{1}{2}\nu_0 t} \le r \le 1,\ t \ge T_0. \tag{3.7}$$

Proof. Set $W(\xi,t) = p_r(r(\xi,t),t)$. Then

$$\dot{W} = (f_p - v_r)W + f_c c_r.$$

Setting

$$\Gamma(\xi,t) = \int_0^t (f_p - v_r)|_{r=r(\xi,\tau)}\,d\tau,$$

we can write

$$W(\xi,t) = e^{\Gamma(\xi,t)}\left[W(\xi,0) + \int_0^t e^{-\Gamma(\xi,\tau)} f_c c_r\,d\tau\right] \tag{3.8}$$

and

$$f_c c_r \geq \eta_0 r \qquad (\eta_0 > 0 \text{ a constant}) \tag{3.9}$$

since f_c is uniformly positive and $c_r \geq \eta_0' r$ for some positive constant η_0'.

To estimate the v_r in Γ we write

$$\begin{aligned} v_r &= g - \frac{2}{r^3}\int_0^r \rho^2 g\,d\rho - \dot{z} \\ &= \left(g - \frac{3}{r^3}\int_0^r \rho^2 g\,d\rho\right) + \frac{1}{r}\left(\frac{1}{r^2}\int_0^r \rho^2 g\,d\rho - r\dot{z}\right) \equiv J_1(p) + J_2(p). \end{aligned} \tag{3.10}$$

Since

$$p_r(r,t) = \min\{p_r(r,t),0\} + \max\{p_r(r,t),0\},$$

we can write

$$p(r,t) = \int_0^r p_r(\rho,t)\,d\rho + p(0,t) = p_1(r,t) + p_r(r,t),$$

where

$$p_1(r,t) = \int_0^r \min\{p_r(\rho,t),0\}\,d\rho,$$
$$p_2(r,t) = \int_0^r \max\{p_r(\rho,t),0\}\,d\rho + p(0,t).$$

Notice that

$$\begin{aligned} |p_1| &\leq -\int_0^1 \min\{p_r(r,t),0\}\,dr \\ &= -\int_0^1 \min\{P_\xi(\xi,t),0\}\,d\xi \quad \text{(by Lemma 3.2)} \\ &\leq C_1 e^{-\nu_1 t}, \qquad t \geq 1 \text{ (by Lemma 3.4)}. \end{aligned} \tag{3.11}$$

Hence

$$|J_1(p) - J_1(p_2)| \leq C\|g_p\|_\infty \|p - p_2\|_\infty \leq C\|p_1\|_\infty \leq C_2 e^{-\nu_1 t}, \qquad t \geq 1,$$

so that

$$J_1(p) = J_1(p_2) + [J_1(p) - J_1(p_2)] \geq J_1(p) - J_1(p_2) \geq -C_2 e^{-\nu_1 t}, \qquad t \geq 1. \tag{3.12}$$

Here we have used the fact that $J_1(p_2) \geq 0$ because, by (2.6) and the fact that

$$\frac{\partial p_2}{\partial r} = \max\{p_r, 0\} \geq 0,$$

$g(c(r,z(t)),p_2(r,t))$ is monotone increasing in r. Since $\dot{r}=v$ by (3.2), we also have

$$J_2(p)=\frac{v}{r}=\frac{\dot{r}}{r}.$$

Substituting this and (3.12) into (3.10) and using (3.6), we get

$$\begin{aligned}\Gamma(\xi,t) &\le \int_0^t\left(\frac{\dot{P}}{P}-\nu_0+c_2e^{-\nu_1 t}-\frac{\dot{r}}{r}\right)d\tau\\ &\le -\nu_0 t+\frac{c_2}{\nu_1}+\log\left(\frac{P(\xi,t)}{P(\xi,0)}\right)+\log\left(\frac{\xi}{r(\xi,t)}\right)\\ &\le C_3-\nu_0 t+\log\left(\frac{\xi}{r(\xi,t)}\right)\ \text{(by (3.3))}.\end{aligned}\tag{3.13}$$

Substituting this and (3.9) into (3.8) we find that

$$\begin{aligned}W(\xi,t)e^{-\Gamma(\xi,t)} &\ge W(\xi,0)+C_4\int_0^t e^{\nu_0\tau}\cdot\frac{r}{\xi}\cdot\eta_0 r\,d\tau\\ &\ge W(\xi,0)+C_5e^{\nu_0 t}\frac{r^2}{\xi}\qquad (t\ge 1).\end{aligned}$$

Hence if

$$r^2\ge C_6e^{-\nu_0 t}\sup_{0<\xi<1}|\xi P_\xi(\xi,0)|,$$

then $W(\xi,t)\ge 0$. □

Lemma 3.6. *There exist positive constants c_3, c_4, T_0 such that*

$$-c_3\le\frac{v(r,t)}{r(1-r)}\le -c_4\quad \textit{if } 0<r<1,\ t\ge T_0.\tag{3.14}$$

Proof. The first inequality follows from

$$\begin{aligned}v(r,t) &= \frac{1}{r^2}\int_0^r\rho^2 g\,d\rho-r\int_0^1\rho^2 g\,d\rho\\ &=\frac{1-r^3}{r^2}\int_0^r\rho^2 g\,d\rho-r\int_r^1\rho^2 g\,d\rho\ge -c_1r(1-r).\end{aligned}$$

To prove the second inequality we write $v=v_1+v_2$, where

$$v_1=\frac{1-r^3}{r^2}\int_0^r\rho^2[g(c,p)-g(c,p_2)]\,d\rho-r\int_r^1\rho^2[g(c,p)-g(c,p_2)]\,d\rho,$$
$$v_2=\frac{1-r^3}{r^2}\int_0^r\rho^2 g(c,p_2)\,d\rho-r\int_r^1\rho^2 g(c,p_2)\,d\rho,$$

and p_2 is as in Lemma 3.5. By (3.11) we have

$$\begin{aligned}|v_1| &\le\frac{1-r^3}{r^2}\int_0^r\rho^2|g(c,p)-g(c,p_2)|\,d\rho+r\int_r^1\rho^2|g(c,p)-g(c,p_2)|\,d\rho\\ &\le\frac{2}{3}r(1-r^3)\|g_p\|_\infty\|p_1\|_\infty\le C_1e^{-\nu_0 t}(1-r)\end{aligned}\tag{3.15}$$

provided $t\ge T_0$. Since $\partial p_2/\partial r=\max\{p_r,0\}\ge 0$,

$$\frac{d}{dr}g(c(r,z(t)),p_2(r,t))=g_c c_r+g_p\frac{\partial p_2}{\partial r}\ge g_c c_r\ge C_2 r,$$

so that

$$
\begin{aligned}
v_2 &= \frac{1}{r^2}\int_0^r \rho^2 g(c,p_2)\,d\rho - r\int_0^1 \rho^2 g(c,p_2)\,d\rho \\
&= -r\int_0^1 \rho^2\,d\rho \int_{r\rho}^{\rho} \frac{d}{ds} g(c(s,z(t)),p_2(s,t))\,ds \\
&\le -r\int_0^1 \rho^2 \int_{r\rho}^{\rho} C_2 s\,ds \le -C_3 r(1-r).
\end{aligned}
\tag{3.16}
$$

Combining (3.15) with (3.16), we obtain

$$
v \le -C_3 r(1-r) + C_1 e^{-\nu_0 t} r(1-r) \le -C_4 r(1-r)
$$

for $t \ge T_0$, if T_0 is sufficiently large. □

Lemma 3.7. *There exists a positive constant c_5 such that*

$$
|rp_r(r,t)| \le c_5, \tag{3.17}
$$

$$
|p_t| \le c_5 \tag{3.18}
$$

for $0 \le r \le 1$, $t \ge 0$.

Proof. The inequality (3.18) follows from (3.17) and (2.2):

$$
|p_t| \le |vp_r| + |f| \le Cr|p_r| + C.
$$

To prove (3.17) we recall from (3.13) that

$$
\begin{aligned}
|e^{\Gamma(\xi,t)} W(\xi,0)| &\le Ce^{-\nu_0 t}|p_r(\xi,0)|\frac{\xi}{r(\xi,t)} \\
&\le \frac{C}{r(\xi,t)} e^{-\nu_0 t} \sup_{0\le\rho\le 1} |\rho p_r(\rho,0)|.
\end{aligned}
$$

Similarly as before, for $0 \le \tau < t$,

$$
\begin{aligned}
\Gamma(\xi,t) - \Gamma(\xi,\tau) &= \int_\tau^t (f_p - v_r)\,dt' \\
&\le \int_\tau^t \left[-\nu_0 + \frac{\dot P}{P} + Ce^{-\nu_0 t} - \frac{\dot r}{r}\right] dt' \\
&\le -\nu_0(t-\tau) + \log\left(\frac{P(\xi,t)}{P(\xi,\tau)}\right) - \log\left(\frac{r(\xi,t)}{r(\xi,\tau)}\right).
\end{aligned}
\tag{3.19}
$$

Hence

$$
\int_0^t e^{\Gamma(\xi,t)-\Gamma(\xi,\tau)} f_r\,d\tau \le C\int_0^t e^{-\nu_0(t-\tau)}\frac{r^2(\xi,\tau)}{r(\xi,t)}\,d\tau \le \frac{c_0(1-e^{-\nu_0 t})}{r(\xi,t)};
$$

here we used the inequality $f_r \le const.r(\xi,\tau)$. Therefore, from (3.8) we get

$$
r|p_r(r,t)| = r(\xi,t)|W(\xi,t)| \le c_0\left[\sup_{0\le\rho\le 1}|\rho p_r(\rho,0)|e^{-\nu_0 t} + 1 - e^{-\nu_0 t}\right],
$$

and (3.18) follows. □

Remark. Similarly we can prove that

$$
r^2|p_{rr}| \le C,
$$

but this inequality will not be needed in the sequel.

4. Non-oscillation theorem

We denote the stationary solution of (2.2)–(2.4) by z_s, $v_s(r)$, $p_s(r)$.

Theorem 4.1. *If*

$$\lim_{T\to\infty}\int_T^{T+1}|\dot z(t)|\,dt=0, \tag{4.1}$$

then

$$z(t)\to z_s,\quad v(r,t)\to v_s(r),\quad p(r,t)\to p_s(r)$$

uniformly, as $t\to\infty$.

Remark. The condition (4.1) is satisfied if, for instance,

$$\dot z(t)\text{ changes sign only a finite number of times.}$$

Indeed, in this case either $\dot z(t)>0$ or $\dot z(t)<0$ for all t sufficiently large, so that $z(t)$ has a limit as $t\to\infty$, and, consequently,

$$\int_T^{T+1}|\dot z(t)|\,dt=\left|\int_T^{T+1}\dot z(t)\,dt\right|=|z(T+1)-z(T)|\to 0$$

as $T\to\infty$.

The proof of Theorem 4.1 depends on several lemmas.

Lemma 4.2. *Under the assumption* (4.1)*, the following hold*:

$$\dot z(t)\to 0\quad\text{as } t\to\infty; \tag{4.2}$$

$$\ddot z(t)\to 0\quad\text{as } t\to\infty. \tag{4.3}$$

Proof. Assertion (4.2) follows from Lemma 3.1 and the estimate, for $h=\dot z$,

$$(h(t))^2=\int_t^{t+1}\left\{h^2(\tau)-2\int_t^{\tau}h(s)\dot h(s)\,ds\right\}d\tau\le 2\sup\{|h|+|\dot h|\}\int_t^{t+1}|h(\tau)|\,d\tau. \tag{4.4}$$

Assertion (4.3) follows from (4.4) with $h=\ddot z$ and

$$\int_T^{T+1}\ddot z^2(t)\;dt=\dot z\ddot z|_T^{T+1}-\int_T^{T+1}\dot z(t)z^{(3)}z(t)\,dt\to 0\quad\text{as } T\to\infty. \qquad \square$$

Lemma 4.3. *Under the assumption* (4.1)*, the following hold:*

$$\sup_{0\le r\le 1} r|p_t(r,t)|\to 0\quad\text{as } t\to\infty, \tag{4.5}$$

$$\sup_{0\le r\le 1}|u_t(r,t)|\to 0\quad\text{as } t\to\infty. \tag{4.6}$$

Proof. Set $\theta=p_t$. Differentiating (2.2) and (2.4) in t, we get

$$\theta_t+v\theta_r=f_p\theta+f_cc_z\dot z-p_rv_t, \tag{4.7}$$

$$v_t=\left(\frac{1}{r^2}-r\right)\ddot z-\frac{\dot z}{r^2}\int_r^1\rho^2g_cc_z\,d\rho-\frac{1}{r^2}\int_r^1\rho^2g_p\theta\,d\rho. \tag{4.8}$$

It follows that

$$\theta_t+v\theta_r=f_p\theta+\frac{p_r}{r^2}\int_r^1\rho^2g_p\theta\,d\rho+\left(f_cc_z+\frac{p_r}{r^2}\int_r^1\rho^2g_cc_z\,d\rho\right)\dot z+\left(r-\frac{1}{r^2}\right)p_r\ddot z. \tag{4.9}$$

Set

$$\widetilde{\Gamma}(\xi,t)=\int_0^t f_p|_{r=r(\xi,\tau)}\,d\tau.$$

Integrating (4.9) along the characteristics $r=r(\xi,t)$ and using (3.17), we get

$$|\theta(r(\xi,t),t)|\le e^{\widetilde{\Gamma}(\xi,t)}|\theta(\xi,0)|+\int_0^t\frac{e^{\widetilde{\Gamma}(\xi,t)-\widetilde{\Gamma}(\xi,\tau)}}{r^3(\xi,\tau)}\,d\tau\int_{r(\xi,\tau)}^1\rho^2|\theta|\,d\rho$$
$$+C\int_0^t\frac{e^{\widetilde{\Gamma}(\xi,t)-\widetilde{\Gamma}(\xi,\tau)}}{r^3(\xi,\tau)}(|\dot z|+|\ddot z|)\,d\tau.$$

Define

$$M(r,t)=\sup_{r\le\rho\le 1}|\theta(\rho,t)|=\sup_{r\le\rho\le 1}|p_t(\rho,t)|.$$

By Lemma 3.6, we may without loss of generality assume that $v<0$ in $(0,1)\times[0,\infty)$, so that $r(\xi,t)\le r(\xi,\tau)\le\xi$ for $0\le\tau\le t$. Similarly as before we have (cf. (3.19) with $v_r=0$)

$$\widetilde{\Gamma}(\xi,t)-\widetilde{\Gamma}(\xi,\tau)\le\log\left(\frac{P(\xi,t)}{P(\xi,\tau)}\right)-\nu_0(t-\tau).$$

Hence

$$r^3M(r,t)\le c_0e^{-\nu_0 t}+c_0\int_0^t e^{-\nu_0(t-\tau)}\int_r^1\rho^2M(\rho,\tau)\,d\rho\,d\tau$$
$$+c_0\int_0^t e^{-\nu_0(t-\tau)}(|\dot z(\tau)|+|\ddot z(\tau)|)\,d\tau.$$

Let k be a positive number larger than 3. Multiplying the last inequality by r^{k-3} and integrating over $r\in(0,1)$, we get

$$\int_0^1 r^kM(r,t)\,dr$$
$$\le c_0ke^{-\nu_0 t}\left\{1+\int_0^t e^{\nu_0\tau}(|\dot z|+|\ddot z|)\,d\tau+\frac{1}{k}\int_0^t e^{\nu_0\tau}\,d\tau\int_0^1 r^kM(r,\tau)\,dr\right\};$$

here we have used the identity

$$\int_0^1 r^{k-3}\,dr\int_r^1\rho^2M(\rho,\tau)\,d\rho=\frac{1}{k-2}\int_0^1 r^kM(r,\tau)\,dr.$$

It follows from the Gronwall inequality that

$$\int_0^1 r^kM(r,t)\,dr\le Ce^{-(\nu_0-\frac{c_0}{k})t}+\int_0^t e^{-(\nu_0-\frac{c_0}{k})(t-\tau)}(|\dot z(\tau)|+|\ddot z(\tau)|)\,d\tau.$$

Taking $k>c_0/\nu_0$ and using Lemma 4.2, we obtain

$$\lim_{t\to\infty}\int_0^1 r^kM(r,t)\,dr=0.$$

Recalling (3.18), we conclude that (4.5) holds. Finally, (4.6) follows from (4.2), (4.5) and

$$|u_t|=\frac{1}{r^2}\left|\int_0^r\rho^2(g_cc_z\dot z+g_pp_t)\,d\rho\right|\le C(|\dot z|+\sup_{0<r<1}|rp_t|).\qquad\square$$

Lemma 4.4. *Let $f_n(r,t)$ be a sequence of uniformly bounded functions monotone increasing in r and monotone increasing (resp. decreasing) in t, $0 \le r \le 1$, $0 \le t < \infty$. Then there exists a subsequence $f_{n_k}(r,t)$ and a bounded function $f(r,t)$ which is monotone increasing in r and monotone increasing (resp. decreasing) in t, such that $f_{n_k}(r,t) \to f(r,t)$ a.e.*

The proof of this well-known result is omitted. □

Lemma 4.5. *Let (z,v,p) be the solution of (2.2)–(2.5). Assume that*

$$\overline{z}(t) \le z(t) \le \hat{z}(t), \qquad \overline{v}(r,t) \le v(r,t) \le \hat{v}(r,t) \tag{4.10}$$

for $0 \le r \le 1$, $0 \le t \le T$, where $\overline{z}(t)$, $\hat{z}(t)$ are continuous in t, and $\overline{v}(r,t)$, $\hat{v}(r,t)$ are continuous in (r,t) and continuously differentiable in r. Then

$$\overline{p}(r,t) \le p(r,t) \le \hat{p}(r,t) \tag{4.11}$$

for $0 \le r \le 1$, $0 \le t \le T$, where $\overline{p}$ and $\hat{p}$ are respectively the solutions of the problems:

$$\begin{cases} \overline{p}_t + \hat{v}\overline{p}_r = f(c(r,\hat{z}(t)), \overline{p}(r,t)), \ 0 < t \le T, \\ \overline{p}(r,0) = 0 \end{cases} \tag{4.12}$$

and

$$\begin{cases} \hat{p}_t + \overline{v}\hat{p}_r = f(c(r,\overline{z}(t)), \hat{p}(r,t)), \ 0 < t \le T, \\ \hat{p}(r,0) = 1. \end{cases} \tag{4.13}$$

Proof. Let $r = \hat{r}(\xi,t)$ be the characteristic curves of (4.12) and let $w(\xi,t) = \overline{p}_r(\hat{r}(\xi,t),t)$. By differentiating (4.12) in r and setting $r = \hat{r}(\xi,t)$, we get

$$\begin{cases} w_t + (\hat{v}_r - f_p)w = f_c c_r, \ 0 < t \le T, \\ w(r,0) = 0. \end{cases}$$

Since $f_c > 0$, $c_r > 0$, we conclude that $w(\xi,t) > 0$ for $0 < \xi < 1$, $0 < t \le T$, so that $\overline{p}_r(r,t) > 0$ for $0 < r < 1$, $0 < t \le T$. Similarly $\hat{p}_r(r,t) > 0$ for $0 < r < 1$, $0 < t \le T$. It then follows from (4.10) that

$$\begin{cases} \overline{p}_t + v\overline{p}_r \le f(c(r,z(t)), \overline{p}(r,t)), \ 0 < t \le T, \\ \overline{p}(r,0) = 0. \end{cases} \tag{4.14}$$

Rewriting (2.2) and (4.14) as ODEs along characteristics and using a comparison argument, we obtain the first inequality in (4.11). The proof of the second inequality is similar. □

Lemma 4.6. *Assume that*

$$\lim_{t\to\infty} u(r,t) = u_\infty(r) \quad \text{uniformly for } 0 \le r \le 1. \tag{4.15}$$

Then $\lim_{t\to\infty}(z(t), u(r,t), p(r,t)) \equiv (z_\infty, u_\infty(r), p_\infty(r))$ exists, and

$$z_\infty = z_s, \quad u_\infty(r) = u_s(r), \quad p_\infty(r) = p_s(r). \tag{4.16}$$

The convergence of $p(r,t)$ to $p_s(r)$ is uniform.

Proof. Since $u(1,t) = \dot{z}(t)$, if $u_\infty(1) = \lim_{t\to\infty} u(1,t) \neq 0$, then $\lim_{t\to\infty} |z(t)| = \infty$, which contradicts (1.21). It follows that $u_\infty(1) = 0$, i.e.,

$$\lim_{t\to\infty} \dot{z}(t) = 0. \tag{4.17}$$

Letting $t \to \infty$ in (3.14), we get

$$-c_3 \le \frac{u_\infty(r)}{r(1-r)} \le -c_4, \qquad 0 < r < 1. \tag{4.18}$$

Next, we prove that

$$\lim_{t\to\infty} z(t) = z_s \quad \text{and} \quad u_\infty(r) = u_s(r). \tag{4.19}$$

To this end, it suffices to prove that if

$$t_n \uparrow \infty \quad \text{and} \quad \lim_{n\to\infty} z(t_n) = a, \tag{4.20}$$

then $a = z_s$ and $u_\infty(r) = u_s(r)$.

In view of (4.17), we can find two sequences of numbers, $\{T_n\}$ and $\{\varepsilon_n\}$, where $T_n \uparrow \infty$ and $\varepsilon_n \downarrow 0$, such that

$$a - \varepsilon_n \le z(t) \le a + \varepsilon_n \quad \text{for } t_n \le t \le t_n + T_n. \tag{4.21}$$

Since $\lim_{t\to\infty} u(r,t) = u_\infty(r)$ uniformly, we may further assume that also

$$u_\infty(r) - \varepsilon_n \le u(r,t) \le u_\infty + \varepsilon_n \quad \text{for } 0 \le r \le 1,\ t_n \le t \le t_n + T_n. \tag{4.22}$$

Now let

$$z_n(t) = z(t+t_n), \quad u_n(r,t) = u(r,t+t_n), \quad p_n(r,t) = p(r,t+t_n).$$

Then, by (4.21) and (4.22),

$$a - \varepsilon_n \le z_n(t) \le a + \varepsilon_n, \qquad 0 \le t \le T_n, \tag{4.23}$$

$$u_\infty(r) - \varepsilon_n \le u_n(r,t) \le u_\infty(r) + \varepsilon_n, \qquad 0 \le r \le 1,\ 0 \le t \le T_n. \tag{4.24}$$

Let

$$\hat{v}_n(r) = (u_\infty(r) + \varepsilon_n) - r(u_\infty(r) - \varepsilon_n), \qquad \overline{v}_n(r) = (u_\infty(r) - \varepsilon_n) - r(u_\infty(r) + \varepsilon_n).$$

Then, by (4.24), the function $v_n(r,t) \equiv v(r,t+t_n) = u_n(r,t) - ru_n(1,t)$ satisfies

$$\overline{v}_n(r) \le v_n(r,t) \le \hat{v}_n(r), \qquad 0 \le r \le 1,\ 0 \le t \le T_n. \tag{4.25}$$

Let $\overline{p}_n$, $\hat{p}_n$ be solutions of the problems

$$\begin{cases} \overline{p}_{n,t} + \hat{v}_n \overline{p}_{n,r} = f(c(r, a+\varepsilon_n), \overline{p}_n), & 0 \le r \le 1,\ t > 0, \\ \overline{p}_n(r,0) = 0, & 0 \le r \le 1, \end{cases} \tag{4.26}$$

$$\begin{cases} \hat{p}_{n,t} + \overline{v}_n \hat{p}_{n,r} = f(c(r, a-\varepsilon_n), \hat{p}_n), & 0 \le r \le 1,\ t > 0, \\ \hat{p}_n(r,0) = 1, & 0 \le r \le 1. \end{cases} \tag{4.27}$$

Then by Lemma 4.5,

$$\overline{p}_n(r,t) \le p_n(r,t) \le \hat{p}_n(r,t), \qquad 0 \le r \le 1,\ 0 \le t \le T_n. \tag{4.28}$$

Differentiating (4.26) in t and integrating along the characteristics, we find that $\partial\overline{p}_n/\partial t \ge 0$; similarly $\partial\hat{p}_n/\partial t \le 0$. Recalling that $\overline{p}_n(r,t)$ and $\hat{p}_n(r,t)$ are also monotone increasing in r (cf. the proof of Lemma 4.5), we may use Lemma 4.4

to conclude that, without loss of generality, the sequences $\{\overline{p}_n\}$ and $\{\hat{p}_n\}$ are convergent for almost all (r,t). Let $\overline{p}$ and $\hat{p}$ denote their limits, respectively. Taking $n \to \infty$ in (4.26) and (4.27), we get

$$\overline{p}_t + u_\infty \overline{p}_r = f(c(r,a), \overline{p}), \tag{4.29}$$

$$\hat{p}_t + u_\infty \hat{p}_r = f(c(r,a), \hat{p}) \tag{4.30}$$

in a weak sense. Since each $\overline{p}_n(r,t)$ is monotone increasing in t, $\overline{p}(r,t)$ is also monotone increasing in t. Similarly, $\hat{p}(r,t)$ is monotone decreasing in t. It follows that the pointwise limits

$$\overline{p}_\infty(r) = \lim_{t\to\infty} \overline{p}(r,t) \quad \text{and} \quad \hat{p}_\infty(r) = \lim_{t\to\infty} \hat{p}(r,t)$$

exist. Taking $t \to \infty$ in (4.29) and (4.30), we find that both $\overline{p}_\infty(r)$ and $\hat{p}_\infty(r)$ satisfy the equation

$$u_\infty(r) p'(r) = f(c(r,a), p(r)) \qquad (0 < r < 1) \tag{4.31}$$

in a weak sense. Since $u_\infty(r) < 0$ for $0 < r < 1$, we can use similar arguments as [13, §8] to show that this equation is satisfied in the classical sense. We claim that

$$\overline{p}_\infty(r) = \hat{p}_\infty(r).$$

Indeed, from (4.18) and the boundedness of $\overline{p}_\infty$ it follows that $f(c(r,a), \overline{p}_\infty(r))$ must vanish at $r=0$ and $r=1$, which determines uniquely the values of $\overline{p}_\infty(r)$ at $r=0$ and $r=1$ (in particular, $\overline{p}_\infty(1) = 1$). Since the same holds for $\hat{p}_\infty$, the function $\pi(r) \equiv \hat{p}_\infty(r) - \overline{p}_\infty(r)$ vanishes at $r=0$ and $r=1$. By direct computation we get

$$\pi'(r) + \alpha(r)\pi(r) = 0, \qquad 0 < r < 1,$$

where $\alpha(r) < 0$ for $0 < r < 1$. It follows that $\pi \equiv 0$ (otherwise we get a contradiction at a point where $\pi(r)$ takes its maximum or minimum).

Now let $\tilde{p}(r,t)$ be any weak limit of a subsequence of $\{p_n(r,t)\}$; for simplicity we may take it to be the entire sequence $\{p_n\}$. Then by (4.28) we have, for any finite $T > 0$,

$$\overline{p}(r,t) \le \tilde{p}(r,t) \le \hat{p}(r,t), \qquad 0 \le r \le 1,$$

for $0 \le t \le T$, so that this holds also for all $t \ge 0$. It follows that

$$\lim_{t\to\infty} \tilde{p}(r,t) = \overline{p}_\infty(r) = \hat{p}_\infty(r) \equiv p_\infty(r). \tag{4.32}$$

Letting $n \to \infty$ in the equation

$$u_n(r,t) = \frac{1}{r^2}\int_0^r \rho^2 g(c(\rho, z_n(t)), p_n(\rho,t))\, d\rho$$

and using (4.23), (4.24), we obtain the relation

$$u_\infty(r) = \frac{1}{r^2}\int_0^r \rho^2 g(c(\rho,a), \tilde{p}(\rho,t))\, d\rho.$$

If we take $t \to \infty$ and use (4.32), we get

$$u_\infty(r) = \frac{1}{r^2}\int_0^r \rho^2 g(c(\rho,a), p_\infty(\rho))\, d\rho. \tag{4.33}$$

Next, integrating the equation

$$\dot{z}_n(t) = \int_0^1 r^2 g(c(r, z_n(t)), p_n(r,t))\, dr$$

over $[0, T]$ for any $T > 0$, we get

$$z_n(T) - z_n(0) = \int_0^T \int_0^1 r^2 g(c(r, z_n(t)), p_n(r,t))\, dr.$$

Again letting $n \to \infty$ and using (4.23), (4.24), we obtain

$$0 = \int_0^T \int_0^1 r^2 g(c(r, a), \tilde{p}(r,t))\, dr.$$

Since T is arbitrary, this implies that

$$\int_0^1 r^2 g(c(r, a), \tilde{p}(r,t))\, dr = 0$$

for all $t > 0$, and, as $t \to \infty$, we arrive at the relation

$$\int_0^1 r^2 g(c(r, a), p_\infty(r))\, dr = 0. \tag{4.34}$$

From (4.31), (4.33), (4.34) we see that $(z, u(r), p(r)) = (a, u_\infty(r), p_\infty(r))$ is a weak solution of the stationary problem. Since weak solutions are classical solutions [13, §8], by the uniqueness of classical solutions ([13, §11]) we conclude that

$$a = z_s, \quad u_\infty(r) = u_s(r), \quad p_\infty(r) = p_s(r).$$

This completes the proof of the lemma, except for the assumption that $p(r,t) \to p_s(r)$ uniformly as $t \to \infty$. This statement follows from Lemma 4.7.

Lemma 4.7. *Let (z, v, p) be the solution of (2.2)–(2.5). Assume that*

$$\lim_{t\to\infty} z(t) = z_s \quad \textit{and} \quad \lim_{t\to\infty} v(r,t) = v_s(r) \tag{4.35}$$

uniformly for $0 \le r \le 1$. Then also

$$\lim_{t\to\infty} p(r,t) = p_s(r) \tag{4.36}$$

uniformly for $0 \le r \le 1$.

Proof. Set

$$\zeta(t) = z(t) - z_s, \quad \eta(r,t) = v(r,t) - v_s(r), \quad \phi(r,t) = p(r,t) - p_s(r).$$

Then

$$\phi_t + v\phi_r = -p_s'(r)\eta + a(r,t)\zeta - b(r,t)\phi, \tag{4.37}$$

where $a(r,t)$ is a bounded continuous function and

$$\begin{aligned} b(r,t) &= -[K_M(c_s) - K_N(c_s)] + K_M(c_s)[p(r,t) + p_s(r)] \\ &\ge -[K_M(c_s) - K_N(c_s)] + K_M(c_s)p_s(r) \\ &= [K_P(c_s) - v_s(r)p_s'(r)]/p_s(r) \ge \textit{const.} > 0. \end{aligned}$$

Since $\zeta(t) \to 0$, $\eta(r,t) \to 0$ as $t \to \infty$, writing (4.37) as an ODE along characteristics we conclude that $\phi(r,t) \to 0$ uniformly as $t \to \infty$. □

Lemma 4.8. *Assume that*

$$\lim_{t\to\infty} u_t(r,t) = 0 \quad \textit{uniformly for } 0 \le r \le 1. \tag{4.38}$$

Then

$$\lim_{t\to\infty} z(t) = z_s, \quad \lim_{t\to\infty} u(r,t) = u_s(r), \quad \lim_{t\to\infty} p(r,t) = p_s(r) \tag{4.39}$$

uniformly for $0 \le r \le 1$.

Proof. Since $u(r,t)/r$ and $u_r(r,t) + (2/r)u(r,t)$ are bounded for $0 < r \le 1$, $t \ge 0$, for any sequence $t_n \uparrow \infty$ we can find a subsequence, which we still denote by t_n, and a continuous function $u_\infty(r)$ such that

$$\lim_{n\to\infty} u(r,t_n) = u_\infty(r) \quad \text{uniformly for } 0 \le r \le 1. \tag{4.40}$$

We want to prove that $u_\infty(r) = u_s(r)$.

Using a similar argument as in the proof of Lemma 4.5, we can deduce from (4.38) and (4.40) that there exist positive sequences $\varepsilon_n \downarrow 0$ and $T_n \uparrow \infty$ such that

$$u_\infty(r) - \varepsilon_n \le u(r,t) \le u_\infty(r) + \varepsilon_n \quad \text{for } 0 \le r \le 1,\ t_n \le t \le t_n + T_n. \tag{4.41}$$

We assert that $u_\infty(1) = 0$. Indeed, if $u_\infty(1) > 0$, then from the bounds in (4.41) and the fact $\varepsilon_n \to 0$ we can find an n_0 such that $\dot z(t) = u(1,t) \ge (1/2)u_\infty(1)$ for $t_n \le t \le t_n + T_n$, $n \ge n_0$. It follows that

$$z(t_n + T_n) \ge z(t_n) + \frac{T_n}{2}u_\infty(1) \ge -C + \frac{T_n}{2}u_\infty(1) \to \infty \quad \text{as } n \to \infty,$$

contradicting the fact that $z(t)$ is a bounded function. Hence $u_\infty(1) \le 0$. Similarly $u_\infty(1) \ge 0$ and, hence, $u_\infty(1) = 0$.

From (4.41) (for $r = 1$) and the equation $\dot z(t) = u(1,t)$ we get

$$-\varepsilon_n \le \dot z(t) \le \varepsilon_n \quad \text{for } t_n \le t \le t_n + T_n. \tag{4.42}$$

Since $z(t_n)$ is a bounded sequence, by replacing $\{t_n\}$ by a subsequence we may assume that $z(t_n)$ is convergent. Let $a = \lim_{n\to\infty} z(t_n)$. Using (4.42) and a similar argument as in the proof of Lemma 4.5, we can show that, after replacing $\{\varepsilon_n\}$ and $\{T_n\}$ by other sequences $\varepsilon_n' \downarrow 0$, $T_n' \uparrow \infty$ (which for simplicity we again denote by ε_n, T_n),

$$a - \varepsilon_n \le z(t) \le a + \varepsilon_n \quad \text{for } t_n \le t \le t_n + T_n. \tag{4.43}$$

Using (4.41) and (4.43), we can apply similar arguments as in the proof of Lemma 4.5 to conclude that $a = z_s$ and, in particular, $u_\infty(r) = u_s(r)$.

The above argument shows that (4.15) holds, so that, by Lemma 4.6, the desired assertion follows. □

Proof of Theorem 4.1. Theorem 4.1 follows from Lemmas 4.3 and 4.8. □

5. Linear asymptotic stability

Setting in (2.2)–(2.4):

$$z(t) = z_s + \varepsilon\zeta(t) + O(\varepsilon^2),$$
$$v(r,t) = u_s(r) + \varepsilon\eta(r,t) + O(\varepsilon^2),$$
$$p(r,t) = p_s(r) + \varepsilon\phi(r,t) + O(\varepsilon^2),$$

and letting $\varepsilon \to 0$, we get the linearized problem of (2.2)–(2.4) about the stationary point $(z_s, v_s(r), p_s(r))$:

$$\phi_t + \mathcal{L}\phi = -A(r)\zeta - B(r)\dot{\zeta} \qquad (0 < r < 1,\ t > 0), \tag{5.1}$$

$$\dot{\zeta} = -\mu\zeta + J(\phi) \qquad (t > 0), \tag{5.2}$$

where

$$\mathcal{L}\phi(r) = \overline{v}(r)\phi_r(r) - \overline{f}_p(r)\phi(r) - \frac{\overline{p}_r(r)}{r^2}\int_r^1 \rho^2 \overline{g}_p(\rho)\phi(\rho)\,d\rho, \tag{5.3}$$

$$A(r) = -\overline{f}_c(r)\overline{c}_z(r) - \frac{\overline{p}_r(r)}{r^2}\int_r^1 \rho^2 \overline{g}_c(\rho)\overline{c}_z(\rho)\,d\rho > 0, \tag{5.4}$$

$$B(r) = \left(\frac{1}{r^2} - r\right)\overline{p}_r(r) > 0, \tag{5.5}$$

$$\mu = -\int_0^1 \rho^2 \overline{g}_c(\rho)\overline{c}_z(\rho)\,d\rho > 0, \tag{5.6}$$

$$J(\phi) = \int_0^1 \rho^2 \overline{g}_p(\rho)\phi(\rho)\,d\rho; \tag{5.7}$$

here $\overline{v}(r) = v_s(r)$, $\overline{p}_r(r) = p_s'(r)$,

$$\overline{c}_z(r) = \frac{\partial c}{\partial z}(r, z_s), \quad \overline{f}_p(r) = \frac{\partial f}{\partial p}(c(r, z_s), p_s(r)), \quad \overline{f}_c(r) = \frac{\partial f}{\partial c}(c(r, z_s), p_s(r)),$$
$$\overline{g}_p(r) = \frac{\partial g}{\partial p}(c(r, z_s), p_s(r)), \qquad \overline{g}_c(r) = \frac{\partial g}{\partial c}(c(r, z_s), p_s(r)).$$

The inequalities $A(r) > 0$, $B(r) > 0$, $\mu > 0$ follow from the inequalities (2.6) and (2.7). Notice that by using Eq. (2.4), we have eliminated η from the linearized system.

Since $\overline{v} \in C^1[0,1]$ and $\overline{v}(0) = \overline{v}(1) = 0$, the characteristic curves of (5.1) given by $dr/dt = \overline{v}(r)$ and initiating from ξ, $0 < \xi < 1$, remain at this interval for all $t > 0$. Using this fact, one can easily prove that the initial value problem for the system (5.1)–(5.2) is well-posed, namely, for any $\zeta_0 \in \mathbf{R}$ and $\phi_0 \in C[0,1] \cap C^1(0,1]$, (5.1)–(5.2) has a unique solution $\zeta \in C^1[0,\infty)$, $\phi \in C([0,1]\times[0,\infty)) \cap C^1((0,1]\times[0,\infty))$ satisfying

$$\zeta(0) = \zeta_0, \qquad \phi(r, 0) = \phi_0(r) \quad (0 \le r \le 1).$$

In the following sections we shall prove that the trivial stationary solution $(\phi, \zeta) = (0, 0)$ of the system (5.1)–(5.2) is asymptotically stable. Note that if we assume that $\phi_0(1) = 0$, then from (5.1) we get $\phi(1, t) = 0$ for all $t \ge 0$. For simplicity we shall first consider this case, and defer the general case to the remark at the end of §8.

Theorem 5.1. *There exist a positive constant σ^* and a function $\phi^*(r)$ satisfying $\phi^*(r) > 0$ $(0 < r < 1)$, $\phi^*(r) \to \infty$ as $r \to 0$, such that for any solution (ϕ, ζ) of (5.1)–(5.2) with $\phi_0(1) = 0$ the following holds*:

$$|\zeta(t)| + \sup_{0<r<1}\left|\frac{\phi(r,t)}{\phi^*(r)}\right| \le C(1+t)^2 e^{-\sigma^* t} \quad \textit{for all } t \ge 0, \tag{5.8}$$

where C is a constant which may depend on the initial value of (ϕ, ζ).

The following lemma will be needed later on.

Lemma 5.2. *Set* $\kappa = -\overline{f}_p(0)/\overline{v}'(0)$. *Then* $\kappa > 0$ *and, for some* $c_0 > 0$,

$$\overline{p}_r(r) = (c_0 + o(1)) \begin{cases} r & \text{if } \kappa > 2, \\ r \log\left(\frac{1}{r}\right) & \text{if } \kappa = 2, \\ r^{\kappa-1} & \text{if } \kappa < 2, \end{cases} \tag{5.9}$$

as $r \to 0$.

Proof. Indeed, in the notation of [13], we have $\kappa = \sigma(\lambda) + 1,\ \lambda = c(0, z)$ (see [13, (5.9), (4.4), (3.8)] for the definition of the function $\sigma(\lambda)$). Hence (5.9) follows from [13, Theorems 5.3, 5.4]. □

In view of the above lemma, the functions $A(r)$, $B(r)$ are singular at the point $r = 0$, and there exists a constant $C > 0$ such that

$$0 < A(r) \le Cr^{-2}(1-r)\overline{p}_r(r), \quad 0 < B(r) \le Cr^{-2}(1-r)\overline{p}_r(r) \qquad (0 < r < 1); \tag{5.10}$$

here we used the facts that $\overline{c}_z(1) = 0$ and $\overline{p}_r(1) > 0$.

6. Steady-state problems with a parameter z

For any $z \in \mathbf{R}$, consider the problem

$$v(r)p'(r) = f(c(r,z), p(r)), \qquad 0 \le p(r) \le 1 \ (0 < r < 1), \tag{6.1}$$

$$r^2 v(r) = -\int_r^1 \rho^2 g(c(\rho, z), p(\rho))\, d\rho \qquad (0 \le r \le 1). \tag{6.2}$$

As in the proof of [13, Theorem 5.3(1)], this system has a unique smooth solution in an interval $(1-\delta, 1]$ for some $0 < \delta < 1$, with $v(1) = 0$, $p(1) = 1$ (which follows from $f(c(1,z), p(1)) = 0$) and $v'(1) = g(c(1,z), p(1)) = K_B(1) > 0$, so that $v(r) < 0$ if δ is small enough. By standard ODE theory, we can uniquely extend the solution either to a maximal interval $(r^*(z), 1]$ with $0 \le r^*(z) < 1$ such that either $v(r) < 0$ in $(r^*(z), 1)$ and

$$v(r^*(z)) \equiv \lim_{r \downarrow r^*(z)} v(r) = 0, \tag{6.3}$$

or $r^*(z) = 0$ and $v(r) < 0$ for all $0 < r < 1$ whereas $v(0+) = -\infty$. Furthermore, using a similar (but simpler) argument as in the proof of [13, Theorem 6.1] we can show that both $r^*(z)$ and the solution are continuously differentiable in the parameter z. We shall denote the solution by $(P(r,z), V(r,z))$ to emphasize its dependence on z.

Lemma 6.1. (1) *There hold, for* $r \in (r^*(z), 1)$,

$$V(r,z) < 0, \qquad P_r(r,z) > 0, \tag{6.4}$$

$$P_z(r,z) < 0, \qquad V_z(r,z) > 0. \tag{6.5}$$

(2) *There exists a unique* $z^* \in \mathbf{R}$ *such that* $r^*(z^*) = 0$, $r^*(z) > 0$ *for* $z > z^*$, $r^*(z) = 0$ *for* $z < z^*$, $V(0, z^*) \equiv \lim_{r\downarrow 0} V(r, z^*) = 0$, *and*

$$\lim_{r \downarrow 0} r^2 V(r,z) = -\int_0^1 \rho^2 g(c(\rho,z), P(\rho,z))\, d\rho < 0 \quad \text{for } z < z^*, \tag{6.6}$$

so that, in particular, $(z^*, V(r,z^*), P(r,z^*)) = (z_s, v_s(r), p_s(r))$.

Proof. The proof that $P_r > 0$ follows as in [13, §7]. Differentiating (6.1) in z we get

$$V_z P_r + V P_{rz} = f_c c_z + f_p P_z. \tag{6.7}$$

Differentiating this equation in r and taking $r = 1$, we obtain

$$(V_r - f_p)\frac{\partial P_z}{\partial r} = f_c c_{zr}, \qquad r = 1;$$

here we have used the facts that $V_z = 0$, $V_{rz} = 0$, $P_z = 0$ and $c_z = 0$ at $r = 1$. Noting that $V_r > 0$, $f_p < 0$, $c_{zr} > 0$ at $r = 1$ and $f_c > 0$, we see that $\partial P_z/\partial r > 0$ at $r = 1$. Since $P_z = 0$ at $r = 1$, it follows that $P_z(r, z) < 0$ if $1 - \delta < r < 1$, for some $0 < \delta < 1$. Let $(r_0, 1)$ be the maximal interval such that $P_z(r, z) < 0$ for $r_0 < r < 1$. Then from (6.2) we get

$$V_z(r, z) = -\frac{1}{r^2}\int_r^1 \rho^2[g_c c_z + g_p P_z]\,d\rho > 0 \quad \text{if } r_0 < r < 1. \tag{6.8}$$

We claim that $r_0 = r^*(z)$. Indeed, otherwise $P_z(r_0, z) = 0$, $P_{rz}(r_0, z) \le 0$ and, by (6.8), $V_z(r_0, z) > 0$. It follows that, at $r = r_0$, the left-hand side of (6.7) is positive whereas the right-hand side is negative, a contradiction. Hence assertion (1) holds.

Assertion (2) can be proved by using similar arguments as in [13, §§9, 10]. We omit the details. □

Remark. The above lemma gives a simplified proof of the existence of a stationary solution for the system (2.2)–(2.4).

Lemma 6.2. *There exist positive functions $c_1(z)$, $c_2(z)$ such that*

$$c_1(z) r^{-1}(1 - r) \le \left(\frac{1}{r^2} - r\right)\frac{\partial c}{\partial r} + \frac{\partial c}{\partial z} \le c_2(z) r^{-1}(1 - r) \tag{6.9}$$

for $0 < r < 1$, $z \in \mathbf{R}$.

Proof. Set $F(c) = \lambda c$, so that

$$\Delta c = e^{2z} F(c).$$

By direct calculation we find that

$$\Delta\left\{\left(\frac{1}{r^2} - r\right)\frac{\partial c}{\partial r}\right\} - e^{2z}F_c \cdot \left(\frac{1}{r^2} - r\right)\frac{\partial c}{\partial r} = -2e^{2z}F(c) - \frac{4}{r^2}\frac{\partial}{\partial r}\left(\frac{c_r}{r}\right),$$

$$\Delta c_z - e^{2z}F_c \cdot c_z = 2e^{2z}F(c).$$

Hence

$$-\Delta\left\{\left(\frac{1}{r^2} - r\right)\frac{\partial c}{\partial r} + \frac{\partial c}{\partial z}\right\} + e^{2z}F_c \cdot \left\{\left(\frac{1}{r^2} - r\right)\frac{\partial c}{\partial r} + \frac{\partial c}{\partial z}\right\} = \frac{4}{r^2}\frac{\partial}{\partial r}\left(\frac{c_r}{r}\right).$$

Let $w(r, z) = \left(\frac{1}{r^2} - r\right)\frac{\partial c}{\partial r} + \frac{\partial c}{\partial z}$. Then

$$-\Delta(rw) + \frac{2}{r}\frac{\partial}{\partial r}(rw) + e^{2z}F_c \cdot (rw) = r(-\Delta w + e^{2z}F_c \cdot w) = \frac{4}{r}\frac{\partial}{\partial r}\left(\frac{c_r}{r}\right);$$

the right-hand side is positive since

$$r^2 c_r = e^{2z}\int_0^r \rho^2 F(c(\rho, z))\,d\rho = r^3 e^{2z}\int_0^1 \theta^2 F(c(r\theta, z))\,d\theta,$$

so that

$$\frac{\partial}{\partial r}\left(\frac{c_r}{r}\right) = e^{2z}\int_0^1 \theta^3 F_c(c(r\theta, z)) c_r(r\theta, z)\,d\theta > 0.$$

The function rw also satisfies the boundary conditions

$$rw|_{r=0} = \lim_{r\to 0} \frac{1}{r}\frac{\partial c}{\partial r} = \frac{1}{3}e^{2z}F(c(0,z)) > 0,$$
$$rw|_{r=1} = \left\{ \left(\frac{1}{r} - r^2\right)\frac{\partial c}{\partial r} + r\frac{\partial c}{\partial z}\right\}\bigg|_{r=1} = 0.$$

Hence, by the maximum principle and the continuity of rw,

$$0 < rw(r,z) \le c_0(z) \quad \text{for } 0 < r < 1.$$

The inequalities in (6.9) now easily follow from this estimate on rw and from the Taylor expansions of $rw(r,z)$, $w(r,z)$ at $r=0$ and $r=1$, respectively. □

Lemma 6.3. *Let $\mathcal{L}$, $A(r)$ and $B(r)$ be as in* (5.3)–(5.5). *Then*

$$A = -\mathcal{L}\left(\frac{\partial P}{\partial z}\bigg|_{z=z^*}\right), \tag{6.10}$$

and there exist positive constants c_1, c_2 and c_3 such that

$$c_1 \le \frac{r^2(\mathcal{L}B - A)}{\overline{p}_r(r)(1-r)} \le c_2, \qquad 0 < r < 1; \tag{6.11}$$
$$r^2(\mathcal{L}B - A)(r)/\overline{p}_r(r) = c_3(1+o(1)) \quad \textit{as } r \to 0.$$

Proof. By differentiating (6.1), (6.2) in z and taking $z = z^* = z_s$ we get, respectively,

$$\overline{v}(r)\frac{d}{dr}\left(\frac{\partial P}{\partial z}\bigg|_{z=z^*}\right) - \overline{f}_p(r)\,\frac{\partial P}{\partial z}\bigg|_{z=z^*} = \overline{f}_c(r)\overline{c}_z(r) - \overline{p}_r(r)\,\frac{\partial V}{\partial z}\bigg|_{z=z^*},$$
$$\frac{\partial V}{\partial z}\bigg|_{z=z^*} = -\frac{1}{r^2}\int_r^1 \rho^2\overline{g}_c(\rho)\overline{c}_z(\rho)\,d\rho - \frac{1}{r^2}\int_r^1 \rho^2\overline{g}_p(\rho)\,\frac{\partial P}{\partial z}\bigg|_{z=z^*}\,d\rho.$$

Hence

$$\begin{aligned}\mathcal{L}\left(\frac{\partial P}{\partial z}\bigg|_{z=z^*}\right) &= \overline{v}(r)\frac{d}{dr}\left(\frac{\partial P}{\partial z}\bigg|_{z=z^*}\right) - \overline{f}_p(r)\,\frac{\partial P}{\partial z}\bigg|_{z=z^*} \\ &\quad - \frac{\overline{p}_r(r)}{r^2}\int_r^1 \rho^2\overline{g}_p(\rho)\,\frac{\partial P}{\partial z}\bigg|_{z=z^*}\,d\rho \\ &= \overline{f}_c(r)\overline{c}_z(r) + \frac{\overline{p}_r(r)}{r^2}\int_r^1 \rho^2\overline{g}_c(\rho)\overline{c}_z(\rho)\,d\rho = -A,\end{aligned}$$

and assertion (6.10) follows. Next, by taking $z = z^*$ in Eq. (6.1) and differentiating it in r, we get

$$\overline{v}(r)\frac{d}{dr}(\overline{p}_r(r)) = \overline{f}_c(r)\overline{c}_r(r) + \overline{f}_p(r)\overline{p}_r(r) - \overline{v}'(r)\overline{p}_r(r),$$

so that

$$\overline{v}\frac{d}{dr}\left\{\left(\frac{1}{r^2} - r\right)\overline{p}_r\right\} - \overline{f}_p\cdot\left(\frac{1}{r^2} - r\right)\overline{p}_r = \left(\frac{1}{r^2} - r\right)\{\overline{f}_c\overline{c}_r - \overline{v}'\overline{p}_r\} - \left(\frac{2}{r^3} + 1\right)\overline{v}\,\overline{p}_r.$$

By integration by parts (using the relation $\partial \overline{g}/\partial\rho = \overline{g}_p\overline{p}_r + \overline{g}_c\overline{c}_r$) we also have

$$\begin{aligned}
&-\frac{1}{r^2}\int_r^1 \rho^2\overline{g}_p(\rho)\cdot\left(\frac{1}{\rho^2}-\rho\right)\overline{p}_r(\rho)\,d\rho \\
&\quad= -\frac{1}{r^2}\int_r^1\left\{\frac{d}{d\rho}[(1-\rho^3)\overline{g}(\rho)] - (1-\rho^3)\overline{g}_c(\rho)\overline{c}_r(\rho) + 3\rho^2\overline{g}(\rho)\right\}d\rho \\
&\quad= \frac{1}{r^2}(1-r^3)\overline{g}(r) + \frac{1}{r^2}\int_r^1 (1-\rho^3)\overline{g}_c(\rho)\overline{c}_r(\rho)\,d\rho + 3\overline{v}(r).
\end{aligned}$$

Using these relations we obtain

$$\mathcal{L}B = \left(\frac{1}{r^2}-r\right)\overline{f}_c\overline{c}_r + \frac{\overline{p}_r}{r^2}\int_r^1 \rho^2\left(\frac{1}{\rho^2}-\rho\right)\overline{g}_c\overline{c}_r\,d\rho.$$

Hence

$$\begin{aligned}
\mathcal{L}B - A &= \left(\frac{1}{r^2}-r\right)\overline{f}_c\overline{c}_r + \frac{\overline{p}_r}{r^2}\int_r^1 \rho^2\left(\frac{1}{\rho^2}-\rho\right)\overline{g}_c\overline{c}_r\,d\rho + \overline{f}_c\overline{c}_z + \frac{\overline{p}_r}{r^2}\int_r^1 \rho^2\overline{g}_c\overline{c}_z\,d\rho \\
&= \overline{f}_c(r)\left\{\left(\frac{1}{r^2}-r\right)\overline{c}_r(r) + \overline{c}_z(r)\right\} \\
&\quad + \frac{\overline{p}_r(r)}{r^2}\int_r^1 \rho^2\overline{g}_c(\rho)\left\{\left(\frac{1}{\rho^2}-\rho\right)\overline{c}_r(\rho) + \overline{c}_z(\rho)\right\}d\rho.
\end{aligned}$$

Since $c_r(r,z) = (r/3)e^{2z}F(c(0,z)) + o(r)$ as $r \to 0$, (6.11) as well as the last assertion of the lemma follow immediately by Lemmas 6.2 and 5.2. □

7. A singular integro-differential equation

In this section we study the following singular integral-differential equation for ϕ^*:

$$\mathcal{L}\phi^* - (\mu + J(B))\phi^* + J(\phi^*)\phi^* = \mathcal{L}B - A,$$

or, equivalently, the system for (ϕ^*, σ^*):

$$\begin{cases} (\mathcal{L}-\sigma^*)\phi^* = \mathcal{L}B - A, \\ \sigma^* = \mu + J(B) - J(\phi^*), \end{cases} \tag{7.1}$$

where A, B, μ, $\mathcal{L}$ and J are as in (5.3)–(5.7). To prove that this equation has a unique solution in $C(0,1]\cap C^1(0,1)$ we need the following lemma.

Lemma 7.1. *Let* $\sigma_1 = -\overline{f}_p(1) = K_B(1) + K_P(1) > 0$, *and assume that* $\psi \in C(0,1]$. *Then the following assertions hold*:

(1) *For any* $\sigma < \sigma_1$ *the equation*

$$(\mathcal{L}-\sigma)\phi = \psi \quad \text{in } (0,1) \tag{7.2}$$

has a unique solution $\phi = \phi(r,\sigma) \in C(0,1]\cap C^1(0,1)$. *Moreover,* $\phi(1,\sigma) = \psi(1)/(\sigma_1-\sigma)$, *and if* $\psi > 0$ *in* $(0,1)$, *then also*

$$\phi > 0, \quad \frac{\partial\phi}{\partial\sigma} > 0 \quad \text{in } (0,1). \tag{7.3}$$

(2) *If also* $\psi(1) = 0$, *then Eq.* (7.2) *with* $\sigma = \sigma_1$ *has a 1-parameter family of solutions, given by*

$$\phi = \phi_{sp} + c\phi_{ge}, \qquad c \text{ is a real parameter}, \tag{7.4}$$

where ϕ_{sp}, ϕ_{ge} are respectively the unique solutions of the equations:

$$\begin{cases}(\mathcal{L}-\sigma_1)\phi_{sp}=\psi \text{ in } (0,1)\\ \phi_{sp}(1)=0\end{cases} \quad \text{and} \quad \begin{cases}(\mathcal{L}-\sigma_1)\phi_{ge}=0 \text{ in } (0,1)\\ \phi_{ge}(1)=1.\end{cases} \tag{7.5}$$

Furthermore, $\phi_{sp}(r)=\lim_{\sigma\nearrow\sigma_1}\phi(r,\sigma)$, $\phi_{sp}>0$ *in* $(0,1)$, *and* $\phi_{ge}>0$ *in* $(0,1]$.

Proof. We first consider the case $\sigma<\sigma_1$. Eq. (7.2) can be explicitly written as

$$\overline{v}(r)\phi_r(r)-(\overline{f}_p(r)+\sigma)\phi(r)-\frac{\overline{p}_r(r)}{r^2}\int_r^1\rho^2\overline{g}_p(\rho)\phi(\rho)\,d\rho=\psi(r). \tag{7.6}$$

Take any $r_0\in(0,1)$ and set

$$W(r)=\exp\left(-\int_{r_0}^r\frac{\overline{f}_p(\rho)+\sigma}{\overline{v}(\rho)}\,d\rho\right) \quad \text{and} \quad \alpha=\frac{\sigma_1-\sigma}{\overline{v}'(1)}.$$

Since $\overline{v}'(1)=K_B(1)>0$, we have $\alpha>0$. Also, since $\overline{v}(1)=0$ and $\overline{f}_p(1)=-\sigma_1$,

$$\frac{\overline{f}_p(r)+\sigma}{\overline{v}(r)}=\frac{\alpha+o(1)}{1-r} \quad \text{as } r\to1,$$

so that $W(r)=C(r_0)(1-r)^\alpha(1+o(1))$ as $r\to1$, where $C(r_0)$ is a positive constant. Using these facts and rewriting (7.6) in the form

$$(W(r)\phi(r))'=\frac{W(r)}{\overline{v}(r)}\left[\psi(r)+\frac{\overline{p}_r(r)}{r^2}\int_r^1\rho^2\overline{g}_p(\rho)\phi(\rho)\,d\rho\right],$$

we can apply an argument as in the proof of [13, Theorem 5.3(1)] to show that (7.2) is equivalent to the integral equation

$$\phi(r)=\frac{1}{W(r)}\int_r^1\frac{W(\rho)}{\overline{v}(\rho)}\left[\psi(\rho)+\frac{\overline{p}_r(\rho)}{\rho^2}\int_\rho^1 s^2\overline{g}_p(s)\phi(s)\,ds\right]d\rho.$$

It then follows from the contraction mapping argument that there exists a $0<\delta<1$ such that (7.6) has a unique solution in the interval $(1-\delta,1)$, satisfying $\phi\in C(1-\delta,1]\cap C^1(1-\delta,1)$, and $\phi(1,\sigma)=\psi(1)/(\sigma_1-\sigma)$. Since $\overline{v}(r)<0$ for $0<r<1$, by standard ODE theory we can uniquely extend the solution to the whole interval $(0,1)$. To prove that $\phi>0$ in $(0,1)$ we first assume that $\psi(1)>0$. Then also $\phi(1,\sigma)>0$, so that $\phi(r,\sigma)>0$ in an interval $r_0<r\le1$, for some $0\le r_0<1$. If $0<r_0<1$ and $\phi(r_0,\sigma)=0$, then, at the point $r=r_0$, the left-hand side of (7.6) is negative while the right-hand side is positive, which is a contradiction. Hence $\phi(r,\sigma)>0$ for all $0<r\le1$ provided $\psi(1)>0$. If $\psi(1)=0$, then we can approximate ψ with $\psi+\varepsilon$ and, letting $\varepsilon\to0$, deduce that $\phi\ge0$ in $(0,1]$. The function ϕ cannot vanish at any point in $0<r<1$, for otherwise a similar argument as above will lead to a contradiction. Hence $\phi>0$ in $(0,1)$. Differentiating (7.2) in σ we get

$$(\mathcal{L}-\sigma)\left(\frac{\partial\phi}{\partial\sigma}\right)=\phi>0 \quad \text{in } (0,1),$$

so that $\partial\phi/\partial\sigma>0$ in $(0,1)$. This proves assertion (1).

Next we consider the case $\sigma=\sigma_1$. In this case we have $W(r)=C(r_0)(1+o(1))$ as $r\to1$. Using this fact and the condition that $\psi(1)=0$, we can apply a similar argument as in the proof of [13, Theorem 5.4(1)] to show that (7.6) is equivalent to the following integral equation:

$$\phi(r) = \frac{1}{W(r)}\left\{c + \int_r^1 \frac{W(\rho)}{\overline{v}(\rho)}\left[\psi(\rho) + \frac{\overline{p}_r(\rho)}{\rho^2}\int_\rho^1 s^2\overline{g}_p(s)\phi(s)\,ds\right]d\rho\right\},$$

where c is an arbitrary real number. Hence, by using the contraction mapping theorem we conclude that Eq. (7.6) has a 1-parameter family of solutions: for each $c \in \mathbf{R}$ there is a unique solution such that $\phi(1) = c$. From the above equation we also see that ϕ can be expressed in the form (7.4), where ϕ_{sp} and ϕ_{ge} are respectively the unique solutions of the equations:

$$\phi_{sp}(r) = \frac{1}{W(r)}\int_r^1 \frac{W(\rho)}{\overline{v}(\rho)}\left[\psi(\rho) + \frac{\overline{p}_r(\rho)}{\rho^2}\int_\rho^1 s^2\overline{g}_p(s)\phi_{sp}(s)\,ds\right]d\rho$$

and

$$\phi_{ge}(r) = \frac{1}{W(r)}\left\{1 + \int_r^1 \frac{W(\rho)}{\overline{v}(\rho)}\left[\frac{\overline{p}_r(\rho)}{\rho^2}\int_\rho^1 s^2\overline{g}_p(s)\phi_{ge}(s)\,ds\right]d\rho\right\},$$

and they satisfy the corresponding systems in (7.5). By a similar argument as for the case $\sigma < \sigma_1$ one can prove that $\phi_{sp}(r) > 0$ for $0 < r < 1$ and $\phi_{ge}(r) > 0$ for $0 < r \le 1$. Finally, the assertion $\lim_{\sigma\nearrow\sigma_1}\phi(r,\sigma) = \phi_{sp}(r)$ follows from similar arguments as in the proof of [13, Theorem 6.1]. ☐

Lemma 7.2. *Assume that* $\psi \in C^1(0,1]$ *and the following conditions are satisfied*:

$$\psi > 0 \quad \text{in } (0,1), \qquad \psi(1) = 0 \qquad \text{and} \qquad \sup_{0<r<1}\frac{r^2\psi(r)}{\overline{p}_r(r)} < \infty.$$

Let $\sigma \le \sigma_1$, *and let* ϕ *be the solution of Eq.* (7.2), *satisfying* $\phi(1) \ge 0$ *in case* $\sigma = \sigma_1$. *Then the following assertions hold*:

(1) $\phi \in C^1(0,1]$.

(2) *If* $\sigma \ge \sigma_0$, *where*

$$\sigma_0 = -3\overline{v}'(0) - \overline{f}_p(0), \tag{7.7}$$

then $\phi(r) \ge c_0 r^{-3}(1-r)$ $(c_0 > 0)$, *so that* $\int_0^1 \rho^2\phi(\rho)\,d\rho = \infty$.

(3) *If* $\sigma < \sigma_0$, *then* $\int_0^1 \rho^2\phi(\rho)\,d\rho < \infty$, *and there exist constants* c_1, c_2 *such that*

$$\phi(r) = (c_1 + o(1))r^{-3+\frac{\sigma_0-\sigma}{|\overline{v}'(0)|}} + (c_2 + o(1))r^{-2}\overline{p}_r(r) \quad \text{as } r \to 0. \tag{7.8}$$

Proof. Assertion (1) follows from a similar argument as in the proof of [13, (8.10)]. To prove (2) and (3) we shall use the same function $W(r)$ as in the previous lemma. Then $W(r) = c_0 r^{-\frac{\sigma+\overline{f}_p(0)}{\overline{v}'(0)}}(1+o(1))$ as $r \to 0$, for some positive constant c_0, and

$$\phi(r) = \frac{1}{W(r)}\left\{\phi(r_0) + \int_r^{r_0} \frac{W(\rho)}{|\overline{v}(\rho)|}\left[\psi(\rho) + \frac{\overline{p}_r(\rho)}{\rho^2}\int_\rho^1 s^2\overline{g}_p(s)\phi(s)\,ds\right]d\rho\right\}. \tag{7.9}$$

If $\sigma \ge \sigma_0$, then

$$\frac{\sigma + \overline{f}_p(0)}{|\overline{v}'(0)|} = \frac{\sigma - \sigma_0}{|\overline{v}'(0)|} + 3 \ge 3,$$

so that

$$\phi(r) \ge \frac{\phi(r_0)}{W(r)} \ge c_0 r^{-3} \quad \text{for } 0 < r < r_0,$$

and assertion (2) follows. If $\sigma < \sigma_0$, then, setting $\theta = \frac{\sigma + \overline{f}_p(0)}{|\overline{v}'(0)|}$, we have $\theta < 3$, so that

$$\begin{aligned}\phi(r) &\le \frac{1}{W(r)}\left\{\phi(r_0) + C\int_r^{r_0} \frac{\rho^\theta}{\rho}\cdot\frac{\overline{p}_r(\rho)}{\rho^2}\left[C + \int_r^1 s^2\phi(s)\,ds\right]d\rho\right\}\\ &\le \frac{1}{W(r)}\left\{\phi(r_0) + Cr^{\theta-2}\overline{p}_r(r)\left[C + \int_r^1 s^2\phi(s)\,ds\right]d\rho\right\} \quad \text{(using (5.9))}\\ &\le Cr^{-\theta} + Cr^{-2}\overline{p}_r(r)\left[C + \int_r^1 s^2\phi(s)\,ds\right].\end{aligned}$$

This implies that, for $0 < r < \delta$,

$$\int_r^\delta \rho^2\phi(\rho)\,d\rho \le C + o(\delta)\left[1 + \int_r^1 s^2\phi(s)\,ds\right] \quad \text{as } \delta \to 0.$$

Hence $\int_0^1 r^2\phi(r)\,dr < \infty$. Using this fact and recalling that $W(r) = c_0 r^\theta(1 + o(1))$ as $r \to 0$, we get (7.8) from (7.9). □

Lemma 7.3. *Equation* (7.1) *has a unique solution* (ϕ^*, σ^*) *which satisfies*: $\phi^* \in C^1(0,1]$,

$$c_1 r^{-2}(1-r)\overline{p}_r(r) \le \phi^*(r) \le c_\theta r^{-\theta} \qquad (c_1 > 0,\ c_\theta > 0,\ 1 \le \theta < 3), \tag{7.10}$$

and

$$0 < \sigma^* \le \sigma_1, \qquad 0 < \sigma^* < \sigma_0. \tag{7.11}$$

Proof. For each $\sigma \in [0, \sigma_1)$, let $\phi(r,\sigma)$ be the unique solution in $C(0,1] \cap C^1(0,1)$ of the equation

$$(\mathcal{L} - \sigma)\phi = \mathcal{L}B - A$$

and define, for $0 \le \sigma < \hat{\sigma}$, where $\hat{\sigma} = \min\{\sigma_0, \sigma_1\}$,

$$H(\sigma) = \sigma + J(\phi(\cdot,\sigma)) - \mu - J(B).$$

By Lemma 7.2(3) we see that H is well-defined, and, by (7.3), $H \in C^1[0,\hat{\sigma})$ and

$$H'(\sigma) = 1 + J\left(\frac{\partial\phi}{\partial\sigma}(\cdot,\sigma)\right) > 1. \tag{7.12}$$

Strictly speaking $J(\frac{\partial\phi}{\partial\sigma}(\cdot,\sigma))$ may not exist, but working with finite differences with respect to σ we obtain

$$\frac{H(\sigma+\delta) - H(\sigma)}{\delta} = 1 + J\left(\frac{\phi(\cdot,\sigma+\delta) - \phi(\cdot,\sigma)}{\delta}\right) > 1,$$

so that $H'(\delta) > 1$ holds in a weak sense and, in particular, $H(\sigma)$ is strictly monotone increasing. Since, by (6.10), $\phi(r,0) = B(r) + \frac{\partial P}{\partial z}\big|_{z=z^*}$, we have

$$H(0) = J\left(\frac{\partial P}{\partial z}\bigg|_{z=z^*}\right) - \mu = \int_0^1 \rho^2\left[\overline{g}_c\overline{c}_z + \overline{g}_p\,\frac{\partial P}{\partial z}\bigg|_{z=z^*}\right]d\rho < 0.$$

Next, if $\sigma_0 \le \sigma_1$ so that $\hat{\sigma} = \sigma_0$, then, by (6.2) and Lemma 7.2,

$$\lim_{\sigma\uparrow\hat{\sigma}} H(\sigma) = \sigma_0 + \lim_{\sigma\uparrow\sigma_0} J(\sigma) - \mu - J(B) = \infty.$$

It follows that there exists a unique $0 < \sigma^* < \hat{\sigma}$ such that $H(\sigma^*) = 0$, which implies that $(\phi^*, \sigma^*) = (\phi(\cdot,\sigma^*), \sigma^*)$ is the unique solution of (7.1) which, by Lemmas 6.3

and 7.2, satisfies (7.10). Consider next the case $\sigma_0 > \sigma_1$ so that $\hat{\sigma} = \sigma_1$. Since $\lim_{\sigma\to\sigma_1} \phi(r,\sigma) = \phi_{sp}(r)$, we have

$$H(\sigma_1 - 0) = \sigma_1 + J(\phi_{sp}) - \mu - J(B).$$

It follows that if $J(\phi_{sp}) > \mu + J(B) - \sigma_1$, then $H(\sigma_1 - 0) > 0$, and, consequently, there exists a unique $\sigma^* \in (0, \sigma_1)$ such that $H(\sigma^*) = 0$, so that (7.1) has a unique solution satisfying (7.10). It remains to consider the case where $J(\phi_{sp}) \leq \mu + J(B) - \sigma_1$. In this case, since $J(\phi_{ge}) > 0$, we can find a unique number $c^* \geq 0$ such that

$$\phi^*(r) = \phi_{sp}(r) + c^*\phi_{ge}(r) \quad \text{and} \quad \sigma^* = \sigma_1$$

form the unique solution of (7.1). This completes the proof of the lemma with σ^* satisfying (7.11). □

Remark. Note that $(\phi, \zeta) = (\phi^* - B, 1)e^{-\sigma^* t}$ solves (5.1), (5.2); namely, $-\sigma^*$ is an eigenvalue of the linearized operator associated with the steady state.

8. The proof of Theorem 5.1

Introducing the function

$$\phi(r,t) = \theta(r,t) + \zeta(t)(\phi^*(r) - B(r)), \tag{8.1}$$

the system (5.1)–(5.2) becomes:

$$\theta_t + \mathcal{L}\theta = -\phi^* J(\theta), \tag{8.2}$$

$$\dot{\zeta} = -\sigma^*\zeta + J(\theta). \tag{8.3}$$

Note that (8.2) can be decoupled from (8.3). We shall first estimate the asymptotic behavior of θ and then use it to analyse the behavior of ζ and ϕ as $t \to \infty$.

From §5 we know that $\phi \in C[0,1] \cap C^1(0,1]$, $\phi(1,t) = 0$. Using (7.10) and (5.10) we see that the function $\theta(r,t)$ introduced by (8.1) may be unbounded at $r = 0$, but nevertheless it satisfies:

$$\sup_{0<r<1} \left| \frac{\theta(r,t)}{\phi^*(r)} \right| < \infty; \tag{8.4}$$

here we used the fact that if $\phi^*(1) = 0$, then by (7.10) ϕ^* only vanishes linearly and by (8.1) $\theta(1,t) = 0$, so that θ/ϕ^* remains bounded also at $r = 1$. It follows that $J(\theta)$ is finite.

We introduce the set of functions

$$\mathcal{M} = \left\{ \theta_0 \in C(0,1] \cap C^1(0,1) : \sup_{0<r<1} \left| \frac{\theta_0(r)}{\phi^*(r)} \right| < \infty \right\}$$

and prove that if the initial value of θ belongs to $\mathcal{M}$, then, as $t \to \infty$,

$$\sup_{0<r<1} \left| \frac{\theta(r,t)}{\phi^*(r)} \right| \to 0, \qquad J(\theta(\cdot,t)) \to 0 \tag{8.5}$$

and

$$\zeta(t) \to 0 \quad \text{as } t \to \infty. \tag{8.6}$$

We begin with some lemmas.

Lemma 8.1. *Let* $\theta_0 \in \mathcal{M}$*,* $\sigma \in \mathbf{R}$*,* $\Psi \in C([0,1]\times[0,\infty))$*, and* $\Psi(\cdot,t) \in \mathcal{M}$ *for each* $t \geq 0$*. Then the initial value problem*

$$\begin{cases} \Phi_t + (\mathcal{L}-\sigma)\Phi = \Psi(r,t), \quad 0<r<1,\ t>0, \\ \Phi(r,t) = \theta_0(r), \quad 0<r<1, \end{cases} \tag{8.7}$$

has a unique solution $\Phi(r,t)$ *which is continuous for* $0 \leq r \leq 1$*,* $t \geq 0$*, continuously differentiable for* $0<r<1$*,* $t>0$*, and belongs to* $\mathcal{M}$ *for each fixed* $t \geq 0$*. Furthermore, if* $\theta_0 \geq 0$ *and* $\Psi \geq 0$*, then also* $\Phi \geq 0$*.*

Proof. Let $r = r(\xi,t)$ $(0 \leq \xi \leq 1,\ t \geq 0)$ be the characteristic curves of (8.7) and denote $\widetilde{\Phi}(\xi,t) = \Phi(r(\xi,t),t)$. Then (8.7) is equivalent to the problem:

$$\begin{cases} \widetilde{\Phi}_t = (\overline{f}_p(r(\xi,t))+\sigma)\widetilde{\Phi} \\ \qquad +\frac{\overline{p}_r(r(\xi,t))}{r^2(\xi,t)}\int_\xi^1 r^2(\eta,t)a(\eta,t)\overline{g}_p(r(\eta,t))\widetilde{\Phi}(\eta,t)\,d\eta + \widetilde{\Psi}(\xi,t),\ t>0, \\ \widetilde{\Phi}|_{t=0} = \theta_0, \end{cases}$$

where $a(\eta,t) = \exp(\int_0^t \overline{v}(r(\eta,\tau))\,d\tau)$ and $\widetilde{\Psi}(\xi,t) = \Psi(r(\xi,t),t)$. One can easily transform this problem into an integral equation and solve it locally in t by using the contraction mapping theorem. Global existence follows by step-by-step continuation, since, for each $T>0$, Φ is *a priori* bounded in $0<t<T$. The proof that $\Phi(r,t)$ belongs to $\mathcal{M}$ for fixed t follows from similar arguments as in the preceding section. Finally, to prove that $\theta_0 \geq 0$ implies $\Phi \geq 0$ we first assume that $\theta_0 \geq$ *const.* >0 and prove that $\Phi(r,t)>0$ for all $0 \leq r \leq 1$, $t \geq 0$. If the assertion is false, then we can find a $t_0>0$ and a $0 \leq r_0 \leq 1$ such that

$$\Phi(r,t)>0 \quad \text{for } 0 \leq r \leq 1,\ 0 \leq t < t_0; \qquad \Phi(r,t_0) \geq 0 \quad \text{for } 0 \leq r \leq 1,$$

and $\Phi(r_0,t_0)=0$. It follows that

$$\Phi_t(r_0,t_0) \leq 0 \quad \text{and} \quad \int_r^1 \rho^2 \overline{g}_p(\rho)\Phi(\rho,t_0)\,d\rho \geq 0 \qquad (0 \leq r \leq 1).$$

Since $\Phi(1,t)$ is the solution of the equation

$$\frac{d}{dt}(\Phi(1,t)) = (\overline{f}_p(1)+\sigma)\Phi(1,t) + \Psi(1,t),$$

by the assumptions $\phi_0(1)>0$ and $\Psi(1,t) \geq 0$ we infer that $\Phi(1,t)>0$ for all $t>0$. Hence $r_0<1$ and

$$\int_{r_0}^1 \rho^2 \overline{g}_p(\rho)\Phi(\rho,t_0)\,d\rho > 0.$$

Further, if $0<r_0<1$, then $\Phi_r(r_0,t_0)=0$ so that

$$\overline{v}(r_0,t_0)\Phi_r(r_0,t_0) = 0.$$

If $r_0=0$ the above relation also holds (see [13, §8]). It follows that

$$0 = \{\Phi_t+(\mathcal{L}-\sigma)\Phi-\Psi\}|_{(r,t)=(r_0,t_0)} = -\frac{\overline{p}_r(r_0)}{r_0^2}\int_{r_0}^1 \rho^2\overline{g}_p(\rho)\Phi(\rho,t_0)\,d\rho - \Psi(r_0,t_0) < 0,$$

which is a contradiction. For non-negative ϕ_0, we can first apply the above to $\phi_0+\varepsilon$ $(\varepsilon>0)$ and then take $\varepsilon \to 0$. □

Remark. From Lemma 8.1 it follows that for any $\theta_0 \in \mathcal{M}$ Eq. (8.2) has a unique solution satisfying $\theta(\cdot,0) = \theta_0$ and $\theta(\cdot,t) \in \mathcal{M}$.

Lemma 8.2. *Let $\theta(r,t)$ be the solution of Eq. (8.2) with initial value $\theta(r,0) = \theta_0(r) \in \mathcal{M}$. Let $\Phi^0(r,t)$ and $\Phi^*(r,t)$ be respectively the solutions of the initial value problems*

$$\begin{cases} \Phi^0_t + \mathcal{L}\Phi^0 = 0, & t > 0, \\ \Phi^0|_{t=0} = \theta_0, \end{cases} \tag{8.8}$$

and

$$\begin{cases} \Phi^*_t + \mathcal{L}\Phi^* = 0, & t > 0, \\ \Phi^*|_{t=0} = \phi^*. \end{cases} \tag{8.9}$$

Set $\Lambda(t) = J(\theta(\cdot,t))$. Then $\Lambda(t)$ satisfies the equation:

$$\Lambda(t) + \int_0^t \Lambda(\tau)K(t-\tau)\,d\tau = \Psi(t), \qquad t > 0, \tag{8.10}$$

where

$$K(t) = J(\Phi^*(\cdot,t)) = \int_0^1 r^2\overline{g}_p(r)\Phi^*(r,t)\,dr,$$
$$\Psi(t) = J(\Phi^0(\cdot,t)) = \int_0^1 r^2\overline{g}_p(r)\Phi^0(r,t)\,dr.$$

Proof. We first note that, by Lemma 8.1, the integrals $J(\theta)$, $J(\Phi^*)$ and $J(\Phi^0)$ are finite. We claim that

$$\theta(\cdot,t) - \Phi^0(\cdot,t) + \int_0^t \Lambda(\tau)\Phi^*(\cdot,t-\tau)\,d\tau = 0. \tag{8.11}$$

Indeed, denoting the left-hand side by $M(r,t)$, one can verify that

$$M_t + \mathcal{L}M = 0 \quad (t > 0) \qquad \text{and} \qquad M|_{t=0} = 0;$$

hence $M \equiv 0$. From (8.11) we get

$$J(\theta(\cdot,t)) = J(\Phi^0(\cdot,t)) - \int_0^t \Lambda(\tau)J(\Phi^*(\cdot,t-\tau))\,d\tau,$$

and (8.10) follows. □

Lemma 8.3. *Let $K \in C^1[0,\infty)$ and assume that*

$$K(t) \geq 0, \quad \frac{d}{dt}(e^{\sigma t}K(t)) \leq 0 \qquad \text{for } t \geq 0. \tag{8.12}$$

Then for any $\Psi \in C[0,\infty)$, Eq. (8.10) has a unique solution $\Lambda \in C[0,\infty)$, and the solution satisfies:

$$|\Lambda(t) - \Psi(t)| \leq K(0)\int_0^t e^{-\sigma(t-\tau)}|\Psi(\tau)|\,d\tau \quad \text{for } t \geq 0. \tag{8.13}$$

Proof. The existence of a unique solution of (8.10) follows by iteration. To prove (8.13) consider first the case $\sigma = 0$. Set $m(t) = \int_0^t \Lambda(\tau)\, d\tau$. We claim that, for any $\varepsilon > 0$,

$$m(t) < \int_0^t [\max\{\Psi(\tau), 0\} + \varepsilon]\, d\tau \quad \text{for all } t \geq 0. \tag{8.14}$$

Indeed, since $m(0) = 0$ and $m'(0) = \Lambda(0) = \Psi(0) < \max\{\Psi(0), 0\} + \varepsilon$, the above inequality holds for small positive t. It follows that if the assertion is not true, then we can find a $t_0 > 0$ such that

$$m(t) < \int_0^t [\max\{\Psi(\tau), 0\} + \varepsilon]\, d\tau \quad \text{for all } 0 \leq t < t_0 \tag{8.15}$$

and

$$m(t_0) = \int_0^{t_0} [\max\{\Psi(\tau), 0\} + \varepsilon]\, d\tau. \tag{8.16}$$

This implies that $\Lambda(t_0) = m'(t_0) \geq \max\{\Psi(t_0), 0\} + \varepsilon$ and

$$m(t) < \int_0^t [\max\{\Psi(\tau), 0\} + \varepsilon]\, d\tau < \int_0^{t_0} [\max\{\Psi(\tau), 0\} + \varepsilon]\, d\tau = m(t_0) \tag{8.17}$$

for $0 \leq t < t_0$. Note that

$$\begin{aligned}
\int_0^{t_0} \Lambda(\tau) K(t_0 - \tau)\, d\tau &= \int_0^{t_0} [m(\tau) - m(t_0)]' K(t_0 - \tau)\, d\tau \\
&= [m(\tau) - m(t_0)] K(t_0 - \tau)|_{\tau=0}^{t_0} \\
&\quad + \int_0^{t_0} [m(\tau) - m(t_0)] K'(t_0 - \tau)\, d\tau \\
&\geq [m(\tau) - m(t_0)] K(t_0 - \tau)|_{\tau=0}^{t_0} \quad \text{(by (8.11) and (8.16))} \\
&= m(t_0) K(t_0) \geq 0.
\end{aligned}$$

Hence

$$\Psi(t_0) = \Lambda(t_0) + \int_0^{t_0} \Lambda(\tau) K(t_0 - \tau)\, d\tau \geq \Lambda(t_0) \geq \max\{\Psi(t_0), 0\} + \varepsilon,$$

which is a contradiction. Therefore, (8.14) holds. Setting $\varepsilon \to 0$ in (8.14) we get

$$m(t) \leq \int_0^t \max\{\Psi(\tau), 0\}\, d\tau, \qquad t \geq 0.$$

Similarly one can prove that

$$m(t) \geq \int_0^t \min\{\Psi(\tau), 0\}\, d\tau, \qquad t \geq 0.$$

Combining these estimates we obtain

$$\max_{0 \leq \tau \leq t} m(\tau) - \min_{0 \leq \tau \leq t} m(\tau) \leq \int_0^t |\Psi(\tau)|\, d\tau, \qquad t \geq 0.$$

Since also

$$\int_0^t \Lambda(\tau)K(t-\tau)\,d\tau = \int_0^t [m(\tau)-m(t)]'K(t-\tau)\,d\tau$$
$$= m(t)K(t) + \int_0^t [m(\tau)-m(t)]K'(t-\tau)\,d\tau,$$

we get

$$\begin{aligned}|\Lambda(t)-\Psi(t)| &= \left|\int_0^t \Gamma(\tau)K(t-\tau)\,d\tau\right| \\ &\le [\max_{0\le\tau\le t} m(\tau) - \min_{0\le\tau\le t} m(\tau)][K(t)+K(0)-K(t)] \\ &\le K(0)\int_0^t |\Psi(\tau)|\,d\tau,\end{aligned}$$

so that the desired assertion follows for the case $\sigma = 0$.

In the case $\sigma \neq 0$ we have, by (8.10),

$$e^{\sigma t}\Lambda(t) + \int_0^t e^{\sigma\tau}\Lambda(\tau)\cdot e^{\sigma(t-\tau)}K(t-\tau)\,d\tau = e^{\sigma t}\Psi(t).$$

Hence, regarding $e^{\sigma t}\Lambda(t)$, $e^{\sigma t}K(t)$ and $e^{\sigma t}\Psi(t)$ respectively as new $\Lambda(t)$, $K(t)$ and $\Psi(t)$, we immediately deduce the assertion of the lemma from the special case $\sigma = 0$. □

Lemma 8.4. *For the solution* Φ^* *of* (8.9)*, the following hold*:

$$\Phi^*(r,t) > 0, \quad \frac{d}{dt}(e^{\sigma^* t}\Phi^*(r,t)) < 0 \quad \text{for } 0 < r < 1,\ t \ge 0, \tag{8.18}$$

and

$$J(\Phi^*(\cdot,t)) > 0, \quad \frac{d}{dt}(e^{\sigma^* t}J(\Phi^*(\cdot,t)))) < 0 \quad \text{for } t \ge 0. \tag{8.19}$$

Proof. Since $\phi^* > 0$, by Lemma 8.1 we have $\Phi^* > 0$. Now set $\psi = e^{\sigma^* t}\Phi^*$. Then clearly ψ satisfies

$$\begin{cases}\psi_t + (\mathcal{L}-\sigma^*)\psi = 0, & t > 0,\\ \psi|_{t=0} = \phi^*.\end{cases}$$

Differentiating this system in t we get

$$\begin{cases}(\psi_t)_t + (\mathcal{L}-\sigma^*)\psi_t = 0, & t > 0,\\ \psi_t|_{t=0} = -(\mathcal{L}-\sigma^*)\phi^* = -(\mathcal{L}B - A) < 0.\end{cases}$$

Hence, by Lemma 8.1, $\psi_t < 0$, or $\frac{\partial}{\partial t}(e^{\sigma^* t}\Phi^*) < 0$. This proves (8.18). The inequalities in (8.19) then follow immediately. □

Corollary 8.5. *Let* Φ^* *be as above. Then*

$$0 < \Phi^*(r,t) \le \phi^*(r)e^{-\sigma^* t} \quad \text{for } 0 < r < 1,\ t \ge 0, \tag{8.20}$$

$$0 < J(\Phi^*(\cdot,t)) \le J(\phi^*)e^{-\sigma^* t} \quad \text{for } t \ge 0. \tag{8.21}$$

The proof is immediate. □

Lemma 8.6. *Assume that* $\theta_0 \in \mathcal{M}$ *and let* Φ^0 *be the solution of* (8.8)*. Then there exists a constant* C *such that*

$$|\Phi^0(r,t)| \le C\phi^*(r)e^{-\sigma^* t} \quad \text{for } 0<r<1,\ t\ge 0, \tag{8.22}$$

and

$$|J(\Phi^0(\cdot,t))| \le Ce^{-\sigma^* t} \quad \text{for } t\ge 0. \tag{8.23}$$

Proof. Since $\theta_0 \in \mathcal{M}$, there exists a constant $C>0$ such that $|\theta_0| \le C\phi^*$. Applying Lemma 8.1 to $C\Phi^* \pm \Phi^0$ we get $|\Phi^0| \le C\Phi^*$. Hence from Corollary 8.5 we quickly get (8.22), and consequently also (8.23). □

Proof of Theorem 5.1. Let $\Lambda(t)$, $K(t)$ and $\Psi(t)$ be as in Lemma 8.2. By Corollary 8.5 and Lemmas 8.4, 8.6 we see that

$$0<K(t)<Ce^{-\sigma^* t}, \quad |\Psi(t)| \le e^{-\sigma^* t} \quad \text{for } t\ge 0$$

and $K(t)>0$, $\frac{d}{dt}(e^{\sigma^* t}K(t))<0$. It follows from Lemma 8.3 that

$$\begin{aligned}|\Lambda(t)| &\le |\Lambda(t)-\Psi(t)| + |\Psi(t)|\\ &\le K(0)\int_0^t e^{-\sigma^*(t-\tau)}|\Psi(\tau)|\,d\tau + |\Psi(t)|\\ &\le C\left(\int_0^t e^{-\sigma^*(t-\tau)}e^{-\sigma^*\tau}\,d\tau + e^{-\sigma^* t}\right)\\ &\le C(1+t)e^{-\sigma^* t}, \qquad t\ge 0.\end{aligned} \tag{8.24}$$

Using this and (8.20), (8.22), we quickly get from (8.11) that

$$|\theta(r,t)| \le C\phi^*(r)(1+t)^2e^{-\sigma^* t}, \qquad 0<r<1,\ t\ge 0. \tag{8.25}$$

Next, from (8.3) we get

$$\zeta(t) = \zeta_0 e^{-\sigma^* t} + \int_0^t e^{-\sigma^*(t-\tau)}\Lambda(\tau)\,d\tau, \qquad t\ge 0,$$

so that by (8.24),

$$|\zeta(t)| \le C(1+t)^2e^{-\sigma^* t} \quad \text{for } t\ge 0. \tag{8.26}$$

Using this and (8.25) in (8.1), we get

$$|\phi(r,t)| \le C\phi^*(r)(1+t)^2e^{-\sigma^* t} \quad \text{for } 0<r<1,\ t\ge 0, \tag{8.27}$$

and this completes the proof. □

Remark. Theorem 5.1 assumes that $\phi_0(1)=0$. If $\phi_0(1)\ne 0$, then, since $\phi^*(1)$ may vanish, the second term on the left-hand side of (5.8) may not be finite. In this case we can replace ϕ^* in this theorem by another function $\hat\phi$ which is positive in $(0,1]$ and $\hat\phi(0+)=\infty$, such that the assertion still holds. The function $\hat\phi$ can be obtained in the following way: If $\sigma^*<\sigma_1$, then take $\hat\phi$ to be the unique solution of the equation

$$(\mathcal{L}-\sigma^*)\hat\phi = 1+\mathcal{L}B - A \quad \text{in } (0,1);$$

if $\sigma^*=\sigma_1$, then take

$$\hat\phi = \phi_{sp} + \hat c\phi_{ge}, \qquad \hat c > c^*,$$

where c^* is as in the proof of Lemma 7.3. Clearly $\hat\phi(r)>0$ for all $0<r\le 1$ and $(\mathcal{L}-\sigma^*)\hat\phi \ge 0$ in $(0,1)$. Also from (7.8), $\hat\phi(0+)=\infty$ and $\int_0^1 \rho^2\hat\phi(\rho)\,d\rho<\infty$. To prove that the assertion of Theorem 5.1 still holds one needs to change the definition of the set $\mathcal{M}$ into

$$\widehat{\mathcal{M}} = \left\{ \theta_0 \in C(0,1] \cap C^1(0,1) : \sup_{0<r<1} \left| \frac{\theta_0(r)}{\hat{\phi}(r)} \right| < \infty \right\},$$

and assume that $\theta_0 \in \widehat{\mathcal{M}}$. By using a similar argument as in the proof of Lemma 8.4 one can prove that the function $\hat{\psi} = e^{\sigma^* t}\widehat{\Phi}$, where $\widehat{\Phi}$ is a function obtained by replacing ϕ^* in (8.9) with $\hat{\phi}$, satisfies similar inequalities as those in (8.18) and (8.19), so that

$$0 < \widehat{\Phi}(r,t) \leq \hat{\phi}(r) e^{-\sigma^* t} \quad \text{and} \quad 0 < J(\widehat{\Phi}(\cdot,t)) \leq J(\hat{\phi}) e^{-\sigma^* t}.$$

One can then deduce, as in the proof of Lemma 8.6, that $C\widehat{\Phi} \pm \Phi^0 \geq 0$ for some $C > 0$, so that

$$|\Phi^0(r,t)| \leq C\hat{\phi}(r) e^{-\sigma^* t} \quad \text{and} \quad |J(\Phi^0(\cdot,t))| \leq C e^{-\sigma^* t}.$$

These estimates imply the desired assertion.

Conclusion

In [14], Cui and Friedman considered a tumor model with three types of cells: proliferating, quiescent, and necrotic. Proliferating cells may become quiescent at a rate which depends on the nutrient concentration, and quiescent cells may become proliferating or necrotic at rates which also depend on the nutrient concentration. It was proved in [14] that, in the radially symmetric case, there exists a unique global solution with tumor volume $\{r < R(t)\}$ where

$$0 < \delta_0 \leq R(t) \leq M < \infty \quad \text{for all } t > 0.$$

The asymptotic behavior of $R(t)$ as $t \to \infty$ remained an open problem. In the present paper, we addressed this problem in the special case when necrotic cells are not present in the tumor; this may occur either because quiescent cells do not become necrotic, or if we assume that necrotic cells are cleared out very fast from the tumor. In this special case, it was proved by Cui and Friedman [13] that there exists a unique stationary solution with radius R_s. In the present paper we proved two results: (i) the stationary solution is linearly asymptotically stable and (ii) if $R(t)$ is monotone, or if just

$$\int_T^{T+1} |\dot{R}(t)|\, dt \to 0 \quad \text{as } T \to \infty,$$

then $R(t) \to R_s$ as $t \to \infty$, and the solution converges to the stationary solution.

The question whether the stationary solution is asymptotically stable is thus still not completely resolved. Furthermore, the existence of a stationary solution for the full model (i.e., including necrotic cells) remains another open problem. The present paper includes an alternate proof for the existence of the unique stationary solution, which is simpler than the proof in [13]; this new proof might possibly extend to the full model (with necrotic cells).

Acknowledgement

The first author was supported by National Science Foundation Grant DMS-0203991. The second author was supported by China National Natural Science Foundation Grant-10171112 and the Distinguished Visiting Scholar Program of the China Scholarship Council. The third author was supported by National Science Foundation Grant DMS-0098520. The second author wishes to thank the staff of the Department of Mathematics of The Ohio State University for their hospitality when he was a visiting scholar during October, 2001 to September, 2002.

References

1. J. Adam, *A simplified mathematical model of tumor growth*, Math. Biosci. **81** (1986), 224–229.
2. B. V. Bazaliy and A. Friedman, *A free boundary problem for an elliptic-parabolic system: application to a model of tumor growth*, Comm. P. D. E. **28** (2003), 627–675. MR1976462 (2004c:35420)
3. B. V. Bazaliy and A. Friedman, *Global existence and stability for an elliptic-parabolic free boundary problem: an application of a model of tumor growth*, Indiana Univ. Math. J. **52** (2003), 1265–1304. MR2010327 (2004j:35302)
4. N. Bellomo and L. Preziosi, *Modelling and mathematical problems related to tumor evolution and its interaction with the immune system*, Mathematical and Computer Modelling **32** (2000), 413–452. MR1775113 (2001i:92016)
5. N. Britton and M. Chaplain, *A qualitative analysis of some models of tissue growth*, Math. Biosci. **113** (1993), 77–89.
6. H. M. Byrne, *A weakly nonlinear analysis of a model of vascular solid tumor growth*, J. Math. Biol. **39** (1999), 59–89. MR1705626 (2000i:92011)
7. H. Byrne and M. Chaplain, *Growth of nonnecrotic tumors in the presence and absence of inhibitors*, Math. Biosci. **131** (1995), 130–151.
8. H. Byrne and M. Chaplain, *Growth of necrotic tumors in the presence and absence of inhibitors*, Math. Biosci. **135** (1996), 187–216.
9. H. Byrne and M. Chaplain, *Free boundary value problems associated with growth and development of multicellular spheroid*, Euro. J. Appl. Math. **8** (1997), 639–658. MR1608619 (99c:92020)
10. S. Cui, *Analysis of a mathematical model for the growth of tumors under the action of external inhibitors*, J. Math. Biol. **44** (2002), 395–426. MR1908130 (2003f:92019)
11. S. Cui and A. Friedman, *Analysis of a mathematical model of the effect of inhibitors on the growth of tumors*, Math. Biosci. **164** (2000), 103–137. MR1751267 (2001d:92006)
12. S. Cui and A. Friedman, *Analysis of a mathematical model of the growth of necrotic tumors*, J. Math. Anal. Appl. **255** (2001), 636–677. MR1815805 (2002a:35195)
13. S. Cui and A. Friedman, *A free boundary problem for a singular system of differential equations: an application to a model of tumor growth*, Trans. Amer. Math. Soc. **355** (2003), 3537–3590. MR1990162 (2004c:34051)
14. S. Cui and A. Friedman, *A hyperbolic free boundary problem modeling tumor growth*, Interfaces and Free Boundaries **5** (2003), 159–182. MR1980470 (2004b:35342)
15. A. Friedman and F. Reitich, *Analysis of a mathematical model for the growth of tumors*, J. Math. Biol. **38** (1999), 262–284. MR1684873 (2001f:92011)
16. A. Friedman and F. Reitich, *Symmetry-breaking bifurcation of analytic solutions to free boundary problems: an application to a model of tumor growth*, Trans. Amer. Math. Soc. **353** (2000), 1587–1634. MR1806728 (2002a:35208)
17. A. Friedman and F. Reitich, *On the existence of spatially patterned dormant malignancies in the model for the growth of non-necrotic vascular tumor*, Math. Models and Methods in Appl. Sci. **11** (2001), 601–625. MR1832995 (2002c:92012)
18. H. Greenspan, *Models for the growth of solid tumor by diffusion*, Stud. Appl. Math. **51** (1972), 317–340.
19. H. Greenspan, *On the growth and stability of cell cultures and solid tumors*, J. Theor. Biol. **56** (1976), 229–242. MR0429164 (55:2183)

20. D. McElwain and G. Pettet, *Cell migration in multicell spheroids*: *swimming against the tide*, Bull. Math. Biol. **55** (1993), 655–674.
21. G. Pettet, C. P. Please, M. J. Tindall and D. McElwain, *The migration of cells in multicell tumor spheroids*, Bull. Math. Biol. **63** (2001), 231–257.
22. J. Sherrat and M. Chaplain, *A new mathematical model for avascular tumor growth*, J. Math. Biol. **43** (2001), 291–312.
23. K. Thompson and H. Byrne, *Modelling the internalisation of labelled cells in tumor spheroids*, Bull. Math. Biol. **61** (1999), 601–623.
24. J. Ward and J. King, *Mathematical modelling of avascular-tumor growth* II: *Modelling growth saturation*, IMA J. Math. Appl. Med. Biol. **15** (1998), 1–42.

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