

A SIMPLE ALGORITHM FOR PRINCIPALIZATION OF MONOMIAL IDEALS

RUSSELL A. GOWARD, JR.

ABSTRACT. In this paper, we give a simple constructive proof of principalization of monomial ideals and the global analog. This also gives an algorithm for principalization.

1. INTRODUCTION

Let X be a scheme of finite type over a field k of arbitrary characteristic. Suppose that D_1, \dots, D_h are effective divisors such that $\sum D_i$ has simple normal crossings on X . Then, in characteristic zero, a simple case of Hironaka's famous theorem on principalization shows that there exists a sequence of blow-ups at non-singular centers principalizing the ideal $\mathcal{I}_{D_1} + \dots + \mathcal{I}_{D_h}$ locally generated by the defining equations of the D_i . That is, if we denote by $\pi : X' \rightarrow X$ the composite of these blow-ups, then the ideal $(\sum \mathcal{I}_{D_i})\mathcal{O}_{X'}$ is a principal ideal defining a divisor with simple normal crossings.

In this paper, we give a simple constructive proof of this result that is valid in any characteristic. We show furthermore that the non-singular centers can always be chosen to be the intersections of precisely two of the prime divisors in the support of the divisor $D_1 + \dots + D_h$ (or its pullback). In particular, one can principalize a monomial ideal in the polynomial ring $k[x_1, \dots, x_n]$ by performing "monomial" blow-ups.

A basic idea in resolution of singularities is to define invariants which behave well under blowing up. Then the locus of points, or a subvariety thereof, where the invariant is a maximum, is the center of a blow-up. Ideally, the invariants will decrease after blowing up until finally resolution has occurred.

We follow that idea here as well. We define a global invariant $(\sigma(X), \tau(X))$ of a pair of divisors D_1, D_2 such that $D_1 + D_2$ has simple normal crossings on X . We then show that after a single blow-up $X_1 \xrightarrow{\pi} X$ of a monomial center determined by $(\sigma(X), \tau(X))$, the invariant decreases $(\sigma(X), \tau(X)) > (\sigma(X_1), \tau(X_1))$ in the lexicographic order. Because $(\sigma, \tau) \in \mathbf{N}^2 \times \mathbf{N}$, this decreasing cannot continue indefinitely and eventually ends in a local principalization of $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$. Because the

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general case easily reduces to the case of two divisors, this produces a simple algorithm for principalization for any number of divisors D_i , which can be implemented on a computer.

In the proof, the following points will become clear: 1) the algorithm is constructive and independent of a characteristic of the ground field; 2) the centers of the blow-ups are monomial; and 3) only codimension two blow-ups are necessary for principalization. The latter two points do not appear explicitly in the literature as far as we know.

In section 2, we define the invariant $(\sigma(X), \tau(X))$ and explain the ordering on these pairs. We also give some examples to illustrate how $\sigma(X)$ determines the non-principal locus. It will be clear that $(\sigma(X), \tau(X))$ together with an arbitrary ordering of the prime divisors in the support of $D_1 + \dots + D_h$ uniquely determine the monomial centers of blowing-up in our algorithm.

In section 3, we prove the main results. In section 4, we present the algorithm in pseudocode. In section 5, we apply the algorithm in an example in dimension three.

For a general introduction to blowing-up we refer to [H1]. For a treatment of algorithms of resolution of singularities, we refer to [BM], [EV] and we refer to [H2] for Hironaka's results on resolution.

2. DEFINITION OF INVARIANT

Before we define the invariant used in the algorithm, we give an example to motivate the definition.

Example. Let k be a field and consider the monomial ideal $I = (x^a y^b, x^c y^d)$ of $k[x, y]$. When is I a principal ideal? For I to be principal, we must have one monomial dividing the other. This happens when $a \leq c$, $b \leq d$ or $a \geq c$, $b \geq d$. Equivalently, I fails to be principal if and only if $a - c$ and $b - d$ have opposite signs. The same idea permeates our definition of the invariant σ in the general case.

Suppose that X is a non-singular algebraic variety over a field k . We review some well-known terminology in order to fix notation.

Definition 1. An **effective divisor** D on X is a formal sum of the form $D = \sum_{i=1}^n a_i E_i$, where the E_i are irreducible codimension one subvarieties of X . The E_i are called **prime divisors**.

Definition 2. A locally principal ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ has **simple normal crossings** at $p \in X$ if there exist regular parameters (x_1, \dots, x_r) at p such that $\mathcal{I}_p = x_1^{a_1} \dots x_r^{a_r} \mathcal{O}_{X,p}$ for some natural numbers a_i .

An effective divisor $D = \sum_{i=1}^k a_i E_i$ on X has simple normal crossings if its ideal sheaf $\mathcal{I}_D = \mathcal{I}_{E_1}^{a_1} \dots \mathcal{I}_{E_k}^{a_k}$ has simple normal crossings at every point $p \in X$.

Definition 3. Let $D_1 = \sum_{i=1}^k a_i E_i$ and $D_2 = \sum_{i=1}^k b_i E_i$ be divisors with simple normal crossings on X . Then, for each pair of irreducible prime components with non-trivial intersection, define $\sigma(E_i \cap E_j)$ by

$$\sigma(E_i \cap E_j) = \begin{cases} \max\{(|a_i - b_i|, |a_j - b_j|), (|a_j - b_j|, |a_i - b_i|)\} & \text{if } a_i - b_i \text{ and } a_j - b_j \\ & \text{have opposite signs,} \\ (-\infty, -\infty) & \text{otherwise.} \end{cases}$$

The max is taken according to the lexicographic order on $\mathbf{N} \times \mathbf{N}$. For example, $(1, 3) < (2, 5)$ and $(3, 1) < (3, 2)$. Of course $(-\infty, -\infty) < (i, j)$ for all $i, j \in \mathbf{Z}$.

Definition 4. With the notation given in Definition 3, define

$$\begin{aligned}\sigma(X) &= \max\{ \sigma(E_i \cap E_j) \mid E_i \cap E_j \neq \emptyset, i \neq j \}, \\ \tau(X) &= \#\{ C \subset X \mid C = E_i \cap E_j, \sigma(C) = \sigma(X) \}.\end{aligned}$$

It is easy to see that the non-principal locus of $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is determined by σ : it is exactly the union of the $E_i \cap E_j$ such that $\sigma(E_i \cap E_j) \neq (-\infty, -\infty)$ (see Proposition 1). Our algorithm calls for us to blow-up the centers $E_i \cap E_j$ such that $\sigma(E_i \cap E_j)$ is maximal. If $\tau > 1$, then there are several possible choices of centers to blow-up. We make this choice unique, for example, by always choosing $E_a \cap E_b$ so that (a, b) is maximal over all such (i, j) in the lex order on $\mathbf{N} \times \mathbf{N}$.

Example 1. Let $X = \text{Spec}(k[x_1, x_2])$ and consider the effective divisors $D_1 = 3E_1 + 4E_2$ and $D_2 = E_1 + 5E_2$, where $E_i = V(x_i)$. Then $\sigma(X) = \sigma(E_1 \cap E_2) = (2, 1)$ and $\tau = 1$. We note that the order of the divisors E_i does not effect the value σ . If we change the order, then we still get that $\sigma(X) = (2, 1)$.

Note that at the origin $p = E_1 \cap E_2$, $\mathcal{I}_{D_1, p} + \mathcal{I}_{D_2, p} = (x_1^3 x_2^4, x_1 x_2^5)$ is not principal.

Example 2. Let $X = \text{Spec}(k[x_1, x_2, x_3])$. Let $D_1 = 5E_1 + 3E_2 + 2E_3$ and $D_2 = E_1 + 4E_2 + E_3$, where E_i is the divisor whose local equation is $x_i = 0$. Since $a_1 - b_1$ and $a_3 - b_3$ have the same sign, $\sigma(E_1 \cap E_3) = (-\infty, -\infty)$. On the other hand, $a_1 - b_1$ and $a_2 - b_2$ have opposite signs, so $\sigma(E_1 \cap E_2) = (4, 1)$. Also $a_2 - b_2$ and $a_3 - b_3$ have opposite signs and $\sigma(E_2 \cap E_3) = (1, 1)$. Then $\sigma(X) = \max\{(1, 1), (4, 1), (-\infty, -\infty)\} = (4, 1)$ and $\tau(X) = 1$.

In this case, our algorithm calls for blowing-up the line $E_1 \cap E_2$. Doing so, we introduce a new (exceptional) divisor E_4 and our divisors become

$$\begin{aligned}\pi_1^* D_1 &= 5E_1 + 3E_2 + 2E_3 + 8E_4, \\ \pi_1^* D_2 &= E_1 + 4E_2 + E_3 + 5E_4.\end{aligned}$$

We abuse notation and write E_i for their proper transforms. Then $\sigma(E_2 \cap E_3) = (1, 1)$, $\sigma(E_2 \cap E_4) = (3, 2)$, and all the other pairs of intersecting divisors have $\sigma(E_i \cap E_j) = (-\infty, -\infty)$. Hence our next center of blowing-up will be $E_2 \cap E_4$.

3. PROOF OF PRINCIPALIZATION

We begin with the case of two divisors.

Proposition 1. *Let X be a non-singular algebraic variety, and let D_1, D_2 be effective divisors on X such that $D_1 + D_2$ has simple normal crossings. Then $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is principal at $p \in X$ if and only if $\sigma(E_i \cap E_j) = (-\infty, -\infty)$ whenever $p \in E_i \cap E_j$. In particular, $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is locally principal if and only if $\sigma(X) = (-\infty, -\infty)$.*

Proof. Let $x_1, \dots, x_n \in \mathcal{O}_{X, p}$ be regular parameters at p . Let

$$\begin{aligned}f_1 &= \delta_1 x_1^{a_1} \dots x_n^{a_r}, \\ f_2 &= \delta_2 x_1^{b_1} \dots x_n^{b_r}\end{aligned}$$

be local equations for D_1 and D_2 with $\delta_i \in \mathcal{O}_{X, p}$ units. Suppose $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is not principal at p . Then (f_1, f_2) is not principal and it follows that, for some i, j with $i \neq j$, $a_i - b_i$ and $a_j - b_j$ have opposite signs. Otherwise, the $a_i - b_i$ all are ≤ 0 or ≥ 0 which implies $f_1 \mid f_2$ or $f_2 \mid f_1$. Let E_i be the component with local equation

$x_i = 0$, and let E_j be the component with local equation $x_j = 0$. Then $p \in E_i \cap E_j$ and $\sigma(E_i \cap E_j) > 0$. Conversely, suppose $p \in E_i \cap E_j$ and $\sigma(E_i \cap E_j) > 0$. Let $x_i = 0$ and $x_j = 0$ be local equations for E_i and E_j in a neighborhood of p . Then by definition of σ , $a_i - b_i$ and $a_j - b_j$ have opposite signs. Hence it follows that (f_1, f_2) is not principal, and so $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is not principal at p . \square

Remark. It follows from the proof of Proposition 1 that if $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is principal at p , then either $(\mathcal{I}_{D_1} + \mathcal{I}_{D_2})_p = (\mathcal{I}_{D_1})_p$ or $(\mathcal{I}_{D_1} + \mathcal{I}_{D_2})_p = (\mathcal{I}_{D_2})_p$. We will make use of this fact later in the proof of Lemma 1.

Theorem 1. *Let X be a non-singular algebraic variety. Suppose D_1 and D_2 are effective divisors on X such that $D_1 + D_2$ has simple normal crossings. Suppose that $C \subset X$ is the intersection of two components of $D_1 + D_2$ with $\sigma(C) = \sigma(X) > (-\infty, -\infty)$. Let $\pi : X' \rightarrow X$ be the blow-up of X centered at C . Then we have $(\sigma(X'), \tau(X')) > (\sigma(X_1), \tau(X_1))$ in the lexicographic order.*

Proof. Write $D_1 = \sum_{i=1}^k a_i E_i$ and $D_2 = \sum_{i=1}^k b_i E_i$, where the E_i are irreducible components. By relabeling the E_i if necessary we may assume that $C = E_1 \cap E_2$. Likewise we may also assume that $a_1 - b_1 < 0$ and $a_2 - b_2 > 0$.

Let $\pi : X' \rightarrow X$ be the blow-up of X with center C , and let E be the exceptional divisor of π . Denote by E'_i the proper transform of E_i . A simple local computation shows that

$$(1) \quad \pi^* E_i = \begin{cases} E'_i + E & \text{if } i = 1, 2, \\ E'_i & \text{if } i > 2. \end{cases}$$

Hence

$$(2) \quad \pi^* D_1 = (a_1 + a_2)E + \sum_{i=1}^k a_i E'_i,$$

$$(3) \quad \pi^* D_2 = (b_1 + b_2)E + \sum_{i=1}^k b_i E'_i.$$

Now we compute σ to show that it has not increased under this blow-up. For the divisors E'_i which are transforms of divisors on X , we have $\sigma(E'_i \cap E'_j) = \sigma(E_i \cap E_j)$ whenever E'_i and E'_j intersect. So it suffices to consider pairs where one of the divisors is exceptional. Denote $C_j = E \cap E'_j$. Because the argument is symmetric for $j = 1$ or 2 , and likewise for any $j \geq 3$, there are only really two cases to consider.

Case 1. $j = 1$. By our initial assumption, $a_1 - b_1 < 0$. If $a_1 + a_2 - (b_1 + b_2) \leq 0$, then $\sigma(C_1) = (-\infty, -\infty)$ and we are done. So suppose $a_1 + a_2 - (b_1 + b_2) > 0$. Then $b_1 - a_1 < a_2 - b_2$ and we have

$$\begin{aligned} \sigma &= (a_2 - b_2, b_1 - a_1) \text{ and} \\ \sigma(C_1) &= \max\{(b_1 - a_1, a_1 + a_2 - (b_1 + b_2)), (a_1 + a_2 - (b_1 + b_2), b_1 - a_1)\}. \end{aligned}$$

Now since $a_1 - b_1$ is negative, we see that

$$a_1 + a_2 - (b_1 + b_2) < a_2 - b_2.$$

Since also

$$a_2 - b_2 > a_1 - b_1,$$

we see directly that in either case $\sigma(C_1)$ is strictly less than σ .

Case 2. $j \geq 3$. Assume that $a_3 - b_3 < 0$. (The case when $a_3 - b_3 > 0$ has a similar argument.) If $a_1 + a_2 - (b_1 + b_2) \leq 0$, then $\sigma(E \cap E_3) = (-\infty, -\infty)$ and we are done. So assume that $a_1 + a_2 - (b_1 + b_2) > 0$. As before this implies that $\sigma = (a_2 - b_2, b_1 - a_1)$. Then there are two possibilities for $\sigma(C_3) = \sigma(E \cap E_3)$. One possibility is that $\sigma(C_3) = (a_1 + a_2 - (b_1 + b_2), b_3 - a_3)$. In this case, we have $a_2 - b_2 > (a_2 - b_2) + (a_1 - b_1)$ since $a_1 - b_1$ is negative. Hence $\sigma = (a_2 - b_2, b_1 - a_1) > (a_1 + a_2 - (b_1 + b_2), b_3 - a_3) = \sigma(C_3)$ and we are done in this case. The other possibility is that $\sigma(C_3) = (b_3 - a_3, a_1 + a_2 - (b_1 + b_2))$. Note that we can assume that $E_2 \cap E_3 \neq \emptyset$, otherwise E_3 does not intersect E at all. Thus by maximality of σ , $a_2 - b_2 \geq b_3 - a_3$. It follows that $\sigma > \sigma(C_3)$ except possibly when $a_2 - b_2 = b_3 - a_3$. However this cannot happen since this equality and the fact that $\sigma \geq \sigma(E_2 \cap E_3)$ would imply $b_1 - a_1 \geq b_3 - a_3 = a_2 - b_2$, which contradicts the computation above that $a_2 - b_2 > b_1 - a_1$.

Now we have shown that the invariant σ definitely decreases locally under blowing-up of our chosen center. Thus the global invariant (σ, τ) decreases as well, in the lexicographic order: under the blow-up $\pi : X' \rightarrow X$, either $\sigma(X') < \sigma(X)$ or $\sigma(X') = \sigma(X)$ and $\tau(X') = \tau(X) - 1$. \square

It is now easy to prove the theorem that was mentioned in the Introduction. First, we need the following lemma.

Lemma 1. *If D_1, \dots, D_h are divisors on a non-singular variety X such that $D_1 + \dots + D_h$ has simple normal crossings, then $\mathcal{I}_{D_1} + \dots + \mathcal{I}_{D_h}$ is locally principal if $\mathcal{I}_{D_i} + \mathcal{I}_{D_j}$ is locally principal for all pairs i, j .*

Proof. If $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is principal at p , then either $(\mathcal{I}_{D_1} + \mathcal{I}_{D_2})_p = (\mathcal{I}_{D_1})_p$ or $(\mathcal{I}_{D_1} + \mathcal{I}_{D_2})_p = (\mathcal{I}_{D_2})_p$, but then either

$$(\mathcal{I}_{D_1} + \dots + \mathcal{I}_{D_h})_p = (\mathcal{I}_{D_1} + \mathcal{I}_{D_3} + \dots + \mathcal{I}_{D_h})_p$$

or

$$(\mathcal{I}_{D_1} + \dots + \mathcal{I}_{D_h})_p = (\mathcal{I}_{D_2} + \mathcal{I}_{D_3} + \dots + \mathcal{I}_{D_h})_p$$

and we are done by induction on h . \square

Theorem 2. *Let D_1, \dots, D_h be effective divisors, such that $\sum D_i$ has simple normal crossings. Then there exists a sequence of blow-ups of non-singular centers $\pi : X' \rightarrow X$ such that $(\mathcal{I}_{D_1} + \dots + \mathcal{I}_{D_h})_{\mathcal{O}_{X'}}$ is locally principal and defines a simple normal crossing divisor.*

Proof. By Lemma 1, we reduce immediately to the case when $h = 2$. Fix an ordering of E_1, \dots, E_k of the divisors in the support of $D_1 + D_2$. Compute (σ, τ) . If $\sigma = (-\infty, -\infty)$, then the ideal $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is locally principal by Proposition 3.1 and we are done. Otherwise consider all the non-empty intersections $E_i \cap E_j$ where $\sigma(E_i \cap E_j)$ is maximal. From among these, let $C = E_a \cap E_b$, where (a, b) is maximal in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$. Let $\pi : X_1 \rightarrow X$ be the blow-up of X with center C . On X_1 , we have the divisors E_1, \dots, E_k, E_{k+1} still with normal crossings, where E_{k+1} is the exceptional divisor of π and by abuse of notation we have written E_1, \dots, E_k for their proper transforms. The divisors π^*D_1, π^*D_2 are made up of the prime divisors E_1, \dots, E_k, E_{k+1} . By Theorem 1, we have $(\sigma(X), \tau(X)) > (\sigma(X_1), \tau(X_1))$. We can then repeat this process on X_1 , using the divisors π^*D_1, π^*D_2 . Since $(\sigma, \tau) \in \mathbf{N}^2 \times \mathbf{N}$, this process must eventually stop.

When it stops, we have $(\sigma, \tau) = (-\infty, -\infty)$, and $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ has been principalized. The proof is complete. \square

Corollary 1. *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k . If $I \subset R$ is a monomial ideal, then there exists a monomial ideal $J \subset R$ such that $X = \mathbf{Proj} R[Jt]$ is non-singular and \mathcal{IO}_X is locally principal.*

Proof. Let $X = \mathbf{Spec} R$. Suppose $I = (m_1, \dots, m_j)$ with each m_t a monomial in the x_i . Then each monomial defines an effective divisor and their sum has normal crossing support on X . By the above theorem, there exists a sequence of blow-ups

$$X_l \xrightarrow{\pi_l} X_{l-1} \xrightarrow{\pi_{l-1}} \dots \longrightarrow X_1 \xrightarrow{\pi_1} X = \mathbf{A}_k^n$$

so that \mathcal{IO}_{X_l} is principal and defines a simple normal crossings divisor. Let $X_l = X$ and $\pi = \pi_l \circ \pi_{l-1} \circ \dots \circ \pi_1$ and define $J = \mathcal{IO}_X$. Then J is the desired monomial ideal. \square

In another paper [G], we show, given the monomial ideal $I \subset R$, how to explicitly construct a monomial ideal $J \subset I$ such that $X = \mathbf{Proj} R[Jt]$ is non-singular and \mathcal{IO}_X is locally principal.

4. ALGORITHM: PRINCIPALIZATION OF MONOMIAL IDEALS

INPUT: $(X, \mathcal{E}, \mathcal{D})$, where X is a non-singular algebraic variety, $\mathcal{E} = \{E_i\}$ is a collection of prime divisors in simple normal crossings, and $\mathcal{D} = \{D_1, \dots, D_h\}$ is a finite collection of effective divisors supported on \mathcal{E} .

OUTPUT: a sequence of blow-ups

$$X_l \xrightarrow{\pi_l} X_{l-1} \xrightarrow{\pi_{l-1}} \dots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

of non-singular centers such that \mathcal{IO}_{X_l} is principal where $\mathcal{I} = \mathcal{I}_{D_1} + \dots + \mathcal{I}_{D_h}$. Furthermore, each center will be of the form $E \cap F$ where E and F are either the proper transforms of the E_i or exceptional divisors (or their proper transforms) of the maps.

INITIALIZATION: $X_p := X_0 := X$, $\mathcal{E} := \{E_1, \dots, E_n\}$, $\mathcal{D} = \{D_1, \dots, D_h\}$.

STEP 1: For $(a, b) \in \{1, \dots, h\} \times \{1, \dots, h\}, a \neq b$ **DO**.

STEP 2: Consider divisors $D_a = \sum a_i E_i$ and $D_b = \sum b_i E_i$. If $E_i \cap E_j \neq \emptyset$, then compute

$$\sigma(E_i \cap E_j) = \begin{cases} \max\{(|a_i - b_i|, |a_j - b_j|), (|a_j - b_j|, |a_i - b_i|)\} & \text{if } a_i - b_i \text{ and } a_j - b_j \\ & \text{have opposite signs,} \\ (-\infty, -\infty) & \text{otherwise.} \end{cases}$$

Here the maximum is taken with respect to the lex order on $\mathbf{N} \times \mathbf{N}$. Compute $\sigma(X_p) = \max \sigma(E_i \cap E_j)$. Set $\tau(X_p) = \#E_i \cap E_j$ such that $\sigma(E_i \cap E_j) = \sigma(X_p)$.

WHILE $\sigma(X_p) > (-\infty, -\infty)$ **DO**

Set $C = E_c \cap E_d$ where (c, d) is uniquely determined as the max in the lex order among the indices i, j such that $\sigma(E_i \cap E_j) = \sigma(X_p)$.

Let $\pi_{p+1} : X_{p+1} \longrightarrow X_p$ be the blow-up of X_p with center C . Let $E_{n+1} = \pi_{p+1}^{-1}(C)$ be the exceptional divisor of this map.

For each $E_i \in \mathcal{E}$, set $E_i :=$ the proper transform of E_i on X_{p+1} . For each $D_i \in \mathcal{D}$, set $D_i := \pi_{p+1}^* D_i$. Set $\mathcal{E} := \mathcal{E} \cup \{E_{n+1}\}$, $n := n + 1$, and $p := p + 1$.

GO TO STEP 2

OTHERWISE: $\sigma(X_p) = (-\infty, -\infty)$ and composite map $X_p \longrightarrow X_0$ is a sequence of blow-ups which principalizes $\mathcal{I}_{D_a} + \mathcal{I}_{D_b}$.

GO TO STEP 1

□

5. EXAMPLE IN DIMENSION 3

Example. Let $X = \mathbf{A}_k^3$, k an arbitrary field. Let E_1, E_2 and E_3 be any prime divisors in \mathbf{A}_k^3 intersecting in simple normal crossings — for example, the coordinate planes. Let $D_1 = E_1 + 3E_2 + 2E_3$ and $D_2 = 2E_1 + E_2 + E_3$. Then

$$\begin{aligned}\sigma(E_1 \cap E_2) &= (2, 1), \\ \sigma(E_1 \cap E_3) &= (1, 1), \\ \sigma(E_2 \cap E_3) &= (-\infty, -\infty),\end{aligned}$$

and $\sigma(X) = (2, 1)$, $\tau(X) = 1$ so $C_1 = E_1 \cap E_2$ will be the center of the first blow-up.

Let $\pi_1 : X_1 \longrightarrow X_0 = X$ be the blow-up of X with center C_1 and let $E_{-1} = \pi_1^{-1}(C_1)$ be the exceptional divisor. Then

$$\begin{aligned}\pi_1^* D_1 &= 4E_{-1} + E'_1 + 3E'_2 + 2E'_3, \\ \pi_1^* D_2 &= 3E_{-1} + 2E'_1 + E'_2 + E'_3.\end{aligned}$$

Note that E'_1 and E'_2 do not intersect at all. We compute $\sigma(E_{-1} \cap E'_1) = (1, 1)$, $\sigma(E'_1 \cap E'_3) = (1, 1)$, and all other pairs have $\sigma = (-\infty, -\infty)$. Hence $(\sigma(X_1), \tau(X_1)) = ((1, 1), 2)$. To decide what center to blow-up, we use the lex order on the indices (i, j) to choose the center $C_2 = E'_1 \cap E'_3$.

Let $\pi_2 : X_2 \longrightarrow X_1$ be the blow-up of X_1 with center $C_2 = E'_1 \cap E'_3$. Let $E_{-2} = \pi_2^{-1}(C_2)$ be the exceptional divisor. Then

$$\begin{aligned}\pi_2^* D_1 &= 3E_{-2} + 4E'_{-1} + E''_1 + 3E''_2 + 2E''_3, \\ \pi_2^* D_2 &= 3E_{-2} + 3E'_{-1} + 2E''_1 + E''_2 + E''_3.\end{aligned}$$

Note that among the intersecting pairs of divisors, we have for all such pairs $\sigma = (-\infty, -\infty)$ except for $\sigma(E'_{-1} \cap E''_1) = (1, 1)$. Hence $\sigma(X_2) = (1, 1)$ and $\tau(X_2) = 1$. Again we see that

$$\begin{aligned}(\sigma(X_2), \tau(X_2)) &= ((1, 1), 1) \\ &< ((1, 1), 2) \\ &= (\sigma(X_1), \tau(X_1)).\end{aligned}$$

Our next center is thus $C_3 = E'_{-1} \cap E''_1$.

Let $\pi_3 : X_3 \longrightarrow X_2$ be the blow-up of X_2 with center $C_3 = E'_{-1} \cap E''_1$. Let $E_{-3} = \pi_3^{-1}(C_3)$ be the exceptional divisor. Then

$$\begin{aligned}\pi_3^* D_1 &= 5E_{-3} + 3E'_{-2} + 4E''_{-1} + E'''_1 + 3E'''_2 + 2E'''_3, \\ \pi_3^* D_2 &= 5E_{-3} + 3E'_{-2} + 3E''_{-1} + 2E'''_1 + E'''_2 + E'''_3.\end{aligned}$$

One now easily verifies that $\sigma = (-\infty, -\infty)$ for these two divisors because whenever two of the prime divisors in their support intersect, the differences of the corresponding coefficients have the same sign. In other words, the ideal $\mathcal{I}_{D_1} + \mathcal{I}_{D_2}$ is locally principal. Note however that this ideal is not globally principal.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN-ANN ARBOR, ANN ARBOR, MICHIGAN 48109-1109