

COMPLETE ANALYTIC EQUIVALENCE RELATIONS

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ABSTRACT. We prove that various concrete analytic equivalence relations arising in model theory or analysis are complete, i.e. maximum in the Borel reducibility ordering. The proofs use some general results concerning the wider class of analytic quasi-orders.

INTRODUCTION

This paper is part of the general theory of analytic equivalence relations, i.e. structures (X, E) , where X is a Polish space (or more generally a standard Borel space, for the notions only depend on the Borel structure of X) and E is an equivalence relation on X which is Σ_1^1 (analytic) as a subset of X^2 . This theory is organized by a quasi-order, \leq_B , called Borel reducibility: If E and F with respective domains X and Y are Σ_1^1 equivalence relations, a Borel reduction from E to F is a Borel map $f : X \rightarrow Y$ which satisfies, for all x, y in X ,

$$xEy \iff f(x)Ff(y)$$

and $E \leq_B F$ if there exists a Borel reduction from E to F . The associated equivalence relation \sim_B , defined by

$$E \sim_B F \iff E \leq_B F \text{ and } F \leq_B E,$$

is called Borel bi-reducibility.

The intuitive meaning of the order of Borel reducibility is the following: If we view X as a set of mathematical objects, which we are interested in up to E -equivalence, the classification problem for (X, E) consists of finding some (concrete, or nicely definable) set I of invariants, together with some (concrete, or nicely definable) assignment $\varphi : X \rightarrow I$, which completely classifies elements in X up to E -equivalence, i.e. satisfies $xEy \iff \varphi(x) = \varphi(y)$. As any solution to the classification problem for (Y, F) gives, by composing with a Borel reduction from E to F , a solution to the classification problem for (X, E) , Borel reducibility intuitively corresponds to a comparison of the complexity of classification problems — at least when the sets are Polish and the equivalence relations are Σ_1^1 , which is very often the case.

It is now well established that the ordering \leq_B is extremely complicated (see [1, 22]), and a good part of the recent developments of the theory has been to isolate some (\sim_B -degrees of) specific equivalence relations which can be used as milestones to get information about the complexity of various natural classification problems.

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It so happens that in most cases, these milestones are the \leq_B -maximum elements in various natural classes of Σ_1^1 equivalence relations (a fact which is very useful to get upper bounds to the complexity of natural problems).

The purpose of this paper is to study one of these milestones, namely the degree consisting of all \leq_B -maximum Σ_1^1 equivalence relations. Elements in this \sim_B -degree, i.e. Σ_1^1 equivalence relations which Borel reduce all other Σ_1^1 equivalence relations, will be called *complete* Σ_1^1 equivalence relations (they are sometimes called universal). Our main goal is to develop a “completeness” method for them, i.e. tools for proving that a given pair (X, E) is complete, and then to apply these tools to specific natural classification problems.

In order to make what we plan to do more precise, let us look first at a very similar — but much simpler — situation, in dimension 1. The objects there are pairs (X, A) , where A is now a Σ_1^1 subset of the Polish space X , and Borel reducibility of A to B now corresponds to the existence of a Borel map f between the ambient spaces with $A = f^{-1}(B)$. Again maximum elements are called complete Σ_1^1 sets, and their existence follows from the existence of universal Σ_1^1 sets (which are clearly necessarily complete).

In dimension 1, the ordering of Borel reducibility is not very interesting, for apart from the trivial cases of $\{\emptyset\}$ and its dual class (consisting of all (X, X) , X Polish), all Borel sets fall in one degree. Moreover the determinacy of Σ_1^1 games implies that every Σ_1^1 non-Borel set is complete. However, the so-called “completeness” method, which goes back to the beginning of Descriptive Set Theory, has proved to be very fruitful: in order to prove that a given Σ_1^1 set is not Borel, one just Borel reduces to it some other Σ_1^1 set, already known to be complete. The method has the advantage of being cumulative, even if one very often uses as test sets a few very specific complete sets, which are combinatorially easy to deal with, like the set *NWF* of ill-founded countable trees, or the set *NWO* of non-well-ordered linear orderings. Of course, one must first establish the completeness of these test sets, which is classically done by deriving it from a representation, or “normal form”, result for Σ_1^1 sets.

In this paper, we will follow the same patterns, but in dimension 2, and for Σ_1^1 equivalence relations (and, as we will see shortly below, for the larger class of Σ_1^1 quasi-orders).

As in dimension 1, the existence of complete objects follows easily from the existence of universal ones. Although the reducibility ordering is now very complicated, so that in particular there are lots of non-Borel but also non-complete Σ_1^1 equivalence relations, one can still start a completeness method, by first defining a “combinatorially simple” object, then proving a representation result which ensures its completeness, and in a third step using it to prove the completeness of various natural classification problems in Logic and Analysis.

This is the program we will follow, with one important change: For reasons which are not totally clear to us, but might be intrinsic, the preceding program is easier to develop for the class of Σ_1^1 quasi-orders (or partial pre-orders, i.e. binary relations which are reflexive and transitive, but not necessarily symmetric). By symmetrization, we will recover all desired results for the class of Σ_1^1 equivalence relations, which is our main concern. So in some sense we are performing an even broader task. However, from a different viewpoint, this can be viewed as a serious limitation to our method, for the “combinatorial simplicity” is lost in the process

of symmetrizing. In particular, we do not know how to prove that a given Σ_1^1 equivalence relation is complete, unless it naturally comes with a Σ_1^1 quasi-order which we can prove to be complete.

The paper is organized as follows: In section 1, we collect some facts about the reducibility order \leq_B , including the existence of \leq_B -maximum elements for various classes. We also very briefly describe the milestones among Σ_1^1 equivalence relations that will be useful in the paper.

In section 2, we define our basic complete Σ_1^1 quasi-order, \leq_{max} , and prove a representation result which implies its completeness. We also use it to briefly discuss the case of Borel quasi-orders and Borel equivalence relations.

The last three sections contain applications of our completeness method to various classification problems in Model Theory (section 3), in Analysis (section 4) and to Polish groups and monoids (section 5).

For example, we prove in section 3 that bi-embeddability between countable partial orders, and bi-embeddability between countable combinatorial trees, are complete Σ_1^1 equivalence relations. In section 4, we prove the same result for continuous bi-embeddability between compact metrizable spaces, isometric bi-embeddability between Polish metric spaces, and linear isometric bi-embeddability between separable Banach spaces. Finally we see in section 5 that there is a Polish semigroup (in fact a monoid) acting on a Polish space inducing a complete Σ_1^1 quasi-order.

The main results in this paper were announced in a Note aux Comptes Rendus de l'Académie des Sciences, [21].

1. THE BOREL REDUCIBILITY ORDERING

We first define the order \leq_B in a very general form.

Definition 1.1. Let A_1 and A_2 be two binary relations on Polish spaces X_1 and X_2 , respectively.

A map f from X_1 to X_2 is a reduction of A_1 to A_2 if for all x, y in X_1 ,

$$(x, y) \in A_1 \iff (f(x), f(y)) \in A_2.$$

We say that A_1 is Borel reducible to A_2 , or A_2 Borel reduces A_1 , in symbols $A_1 \leq_B A_2$, if there is a Borel reduction f from A_1 to A_2 . If moreover the reduction f can be found one-to-one, we say that A_1 Borel embeds into A_2 ($A_1 \sqsubseteq_B A_2$).

Finally, we denote by \sim_B the associated equivalence relation of Borel bireducibility, defined by

$$A_1 \sim_B A_2 \iff A_1 \leq_B A_2 \wedge A_2 \leq_B A_1.$$

As we said in the Introduction, Borel reducibility has been studied mainly in the context of Σ_1^1 equivalence relations. There, because of the reflexivity condition, the domain of the relation is important, and Polish spaces are often replaced by standard Borel spaces — a harmless change, as a standard Borel space is just a Polish space of which only the Borel structure is considered — and sometimes are replaced by arbitrary subsets of Polish spaces. We will not consider this second generalization here. Another possible generalization would be to consider more general structures with Polish domain, and relations and functions of arbitrary arity.

First, let us note some simple features of \leq_B . It is clearly reflexive and transitive, i.e. a quasi-order. Also, the Borelness condition imposed on the reductions

implies that for most descriptive classes Γ , and in particular for the class Σ_1^1 we are interested in, binary relations in Γ are downward closed under \leq_B , or equivalently relations in Γ can only reduce relations in Γ . But there are also other limitations, of a more algebraic type, due to the special “square” form of the reductions: The properties of reflexivity, antireflexivity, symmetry and transitivity are all preserved downwards by reductions. So quasi-orders can only reduce quasi-orders, and equivalence relations can only reduce equivalence relations.

One can push these remarks a bit further. Let R be a quasi-ordering. Denote by R^\sim its dual (or reverse) ordering, i.e. $xR^\sim y$ in the case yRx . Let $\equiv_R = R \cap R^\sim$ be the associated equivalence relation, and let $<_R = (R \setminus \equiv_R)$ be the associated strict order. Then if R_1 and R_2 are quasi-orders and f is a reduction of R_1 to R_2 , f also reduces R_1^\sim to R_2^\sim , \equiv_{R_1} to \equiv_{R_2} , and $<_{R_1}$ to $<_{R_2}$. This explains why we will be able to get results about Σ_1^1 equivalence relations from similar results about Σ_1^1 quasi-orders.

Another nice feature of \leq_B is that the particular ambient Polish space X is irrelevant in the general theory, as any two uncountable Polish spaces are Borel isomorphic, and hence any \sim_B -degree has members in any given uncountable Polish space X_0 .

Definition 1.2. Let \mathcal{C} be a class of binary relations on Polish spaces. A relation A is \mathcal{C} -complete, or complete in \mathcal{C} , if $A \in \mathcal{C}$ and A Borel reduces all elements of \mathcal{C} , i.e. has maximum \leq_B -degree among elements of \mathcal{C} .

When the class \mathcal{C} is clear from the context, we will just say that A is complete. Most classes \mathcal{C} that we will consider are downward closed under \leq_B , so that this abuse of terminology is harmless. E.g. a Σ_1^1 equivalence relation can only be complete among Σ_1^1 equivalence relations.

Proposition 1.3. *The following classes admit complete elements:*

- Σ_1^1 binary relations,
- Σ_1^1 reflexive (resp. irreflexive) relations,
- Σ_1^1 symmetric (resp. symmetric reflexive, symmetric irreflexive) relations,
- Σ_1^1 quasi-orders,
- Σ_1^1 equivalence relations.

Proof. We use the existence of universal sets. Let $W_0 \subseteq (2^\omega)^3$ be a Σ_1^1 set universal for Σ_1^1 subsets of $2^\omega \times 2^\omega$. Define W_1 on $2^\omega \times 2^\omega$ by

$$(\alpha_1, \alpha_2)W_1(\beta_1, \beta_2) \leftrightarrow \alpha_1 = \beta_1 \wedge (\alpha_1, \alpha_2, \beta_2) \in W_0.$$

W_1 is a Σ_1^1 binary relation on $2^\omega \times 2^\omega$. To check that it is complete, it is enough to Borel reduce to it any binary relation A on 2^ω , by the remark above. But if α is a W_0 -code of A , the map $\beta \mapsto (\alpha, \beta)$ reduces A to W_1 .

The other cases are very similar. If Δ is equality on $2^\omega \times 2^\omega$, then $W_1 \cup \Delta$ is complete for reflexive Σ_1^1 relations, $W_1 \setminus \Delta$ for irreflexive ones, and $W_1 \cap W_1^\sim$, $(W_1 \cap W_1^\sim) \cup \Delta$ and $(W_1 \cap W_1^\sim) \setminus \Delta$ are complete for symmetric, symmetric reflexive, and symmetric irreflexive Σ_1^1 relations, respectively.

The quasi-order W_2 generated by W_1 is also complete for Σ_1^1 quasi-orders. First it is easily Σ_1^1 (as Σ_1^1 is closed under projections). Also, if R is any Σ_1^1 quasi-order on 2^ω , with W_0 -code α , one still has

$$\beta_1 R \beta_2 \leftrightarrow (\alpha, \beta_1)W_2(\alpha, \beta_2),$$

so that W_2 is indeed “universal” for Σ_1^1 quasi-orders, and the map $\beta \mapsto (\alpha, \beta)$ again reduces R to W_2 . The case of equivalence relations is similar, working now with the equivalence relation W_3 generated by W_1 . \square

This very easy proof used only closure of Σ_1^1 under Borel preimages, countable unions and projections, and hence works as well for many descriptive classes Γ , e.g. Σ_n^1 . It can also be adapted to the class \mathbf{K}_σ (which is closed under countable unions and projections, but not Borel preimages).

Proposition 1.4. *The classes of \mathbf{K}_σ quasi-orders and \mathbf{K}_σ equivalence relations (on compact Polish spaces) admit complete elements.*

Proof. Starting from a \mathbf{K}_σ subset of $2^\omega \times (2^\omega \times 2^\omega)$ universal for \mathbf{K}_σ subsets of $2^\omega \times 2^\omega$, we get as before a \mathbf{K}_σ quasi-order on $2^\omega \times 2^\omega$ which is “universal” for \mathbf{K}_σ quasi-orders on 2^ω . So it is enough to check that any \mathbf{K}_σ quasi-order (X, R) , with X compact Polish, is Borel bi-reducible with one on 2^ω . But there is a continuous onto map $\pi : 2^\omega \rightarrow X$, with Borel selection $s : X \rightarrow 2^\omega$. Then $R' = (\pi \otimes \pi)^{-1}(R)$ is a \mathbf{K}_σ quasi-order on 2^ω , and π and s witness that $R \sim_B R'$. The proof is similar for \mathbf{K}_σ equivalence relations. \square

For classes with different closure properties, other techniques have to be used. In [11], Hjorth proves the existence of a complete (in fact “universal”) Π_1^1 equivalence relation, using the norm and boundedness properties of this class. His proof extends to Π_1^1 quasi-orders as well. In the next section, we will see a different approach for compact quasi-orders, which also extends to closed quasi-orders (on Polish spaces) with some more work; see Louveau [19]. This other approach works also for \mathbf{K}_σ and Σ_1^1 quasi-orders, and will be instrumental for the results of section 2.

In the opposite direction, there are classes with no complete element. H. Friedman proved this first for the class of Borel equivalence relations (see [29] and [17]), and it extends easily to Borel quasi-orders. The class of Π_2^0 quasi-orders has no complete element (Louveau [18]) (but equality on 2^ω is complete for Π_2^0 equivalence relations). Also, the class of Σ_1^1 strict orders has no complete element (Louveau [20]).

Proposition 1.5. *Let E be a Σ_1^1 equivalence relation on the Polish space X . Then E is complete if and only if $E = \equiv_R$, for some complete Σ_1^1 quasi-order R on X .*

Proof. Suppose first that R is a complete Σ_1^1 quasi-order on X . If F is a Σ_1^1 equivalence relation on the Polish space Y , F is in particular a quasi-order, hence Borel reduces to R . But then $F = \equiv_F$ Borel reduces to \equiv_R . This proves that \equiv_R is a complete Σ_1^1 equivalence relation.

Conversely, suppose E is complete, and let R_0 be a complete Σ_1^1 quasi-order, on say 2^ω , by Proposition 1.3. Then let $f : 2^\omega \rightarrow X$ be a Borel map reducing \equiv_{R_0} to E . Set

$$xRy \leftrightarrow xEy \vee \exists \alpha \in 2^\omega \exists \beta \in 2^\omega (xEf(\alpha) \wedge yEf(\beta) \wedge \alpha R_0 \beta).$$

We claim that R works. It is clearly Σ_1^1 , with $E \subseteq R$. Let

$$X_0 = \{x \in X \exists \alpha x = f(\alpha)\}$$

and $X_1 = X \setminus X_0$. Note that for $x, y \in X_0$ with $xR \setminus Ey$ and any α and β with $xEf(\alpha)$ and $yEf(\beta)$, one must have $\alpha R_0 \beta$, for f reduces \equiv_{R_0} to E . Also points in X_0 and X_1 are R -unrelated, and $R|_{X_1} = E|_{X_1}$. This easily implies that R is

indeed a quasi-order, that $\equiv_R = E$, and that f is a reduction of R_0 to R , so that R is complete. \square

In the applications, we will only use the “if” part, proving first that certain Σ_1^1 quasi-orders are complete, and derive that the associated equivalence relation is complete too. But the “only if” part indicates that complete Σ_1^1 equivalence relations are always of this form (although the quasi-ordering might not be as “natural” as the equivalence relation). This is some indication that the limitations of our method might be intrinsic.

To end this introductory section, let us briefly discuss some of the milestones discovered in the theory of Σ_1^1 equivalence relations that will be useful later to put the results of sections 3 and 4 in perspective. References can be found in Becker-Kechris [2], Hjorth [12] and Kechris [13, 14, 15].

Given a Polish group G and a Borel action a of G on some Polish space X , the orbit equivalence relation E_G^X is defined, on X , by

$$xE_G^X y \leftrightarrow \exists g \in G \ a(g, x) = y.$$

Call such an E_G^X a G -equivalence relation (it is always Σ_1^1). Then the class of all G -equivalence relations admits a complete element (see Becker-Kechris [2]), usually denoted by E_G^∞ , which we call the G -complete equivalence relation. Here are some examples:

(i) $(2^\omega, =)$ is G -complete for all compact groups G , and also for all Π_1^0 (and even Π_2^0) equivalence relations.

(ii) The relation E_0 on 2^ω , defined by

$$\alpha E_0 \beta \leftrightarrow \{n : \alpha(n) \neq \beta(n)\} \text{ is finite,}$$

is \mathbb{Z} -complete.

(iii) $E_{F_2}^\infty$, where F_2 is the free group with 2 generators, is complete for the class of all G -equivalences, G a Polish locally compact group.

(iv) Graph isomorphism (i.e. isomorphism between countable graphs with domain \mathbb{N}) is S_∞ -complete, where S_∞ is the symmetric group (of permutations of \mathbb{N}).

(v) If G_0 is either the group of homeomorphisms of the Hilbert cube, or the group of isometries of the Urysohn space (see section 3), $E_{G_0}^\infty$ is complete for the class of all Σ_1^1 equivalence relations which are G -equivalence relations for some Polish group G .

(vi) Not every Σ_1^1 equivalence relation is Borel reducible to a G -equivalence relation for some Polish group G . The simplest counterexample is E_1 , defined on $(2^\omega)^\omega$ by

$$(\alpha_n)E_1(\beta_n) \leftrightarrow \{n : \alpha_n \neq \beta_n\} \text{ is finite.}$$

(vii) From the previous discussion, we also have complete equivalence relations $E_{\Sigma_1^1}$ and $E_{\mathbf{K}_\sigma}$ in the classes of Σ_1^1 and \mathbf{K}_σ equivalence relations, respectively.

(viii) The relation E_{cntble} is defined on \mathbb{R}^ω by

$$(\alpha_n)E_{cntble}(\beta_n) \leftrightarrow \{\alpha_n : n \in \omega\} = \{\beta_n : n \in \omega\}.$$

The order \leq_B , between these milestones, is given by:

- a) $(\omega, =) <_B (2^\omega, =) <_B E_0 <_B E_1 <_B E_{\mathbf{K}_\sigma} <_B E_{\Sigma_1^1}$,
- b) $E_0 <_B E_{F_2}^\infty <_B E_{\mathbf{K}_\sigma}$,
- c) $E_{F_2}^\infty <_B E_{cntble} <_B E_{S_\infty}^\infty <_B E_{G_0}^\infty <_B E_{\Sigma_1^1}$,

d) the pairs $(E_1, E_{G_0}^\infty)$ and $(E_{cntble}, E_{\mathbf{K}_\sigma})$ are \leq_B -incomparable.

One should add that this picture does not reflect at all the complexity of \leq_B . For example, the interval $[E_0, E_{F_2}^\infty]$ is already immensely complicated. Also, there are Σ_1^1 equivalence relations “on the side”, even at the $(\omega, =)$ level, i.e. Σ_1^1 equivalence relations having \aleph_1 classes but no perfect set of inequivalent elements. For more details, we refer the reader to the above-mentioned books.

2. THE BASIC EXAMPLE \leq_{max}

Our aim is to define a “combinatorially simple” complete Σ_1^1 quasi-order. But let us first discuss two simpler cases, the compact and \mathbf{K}_σ quasi-orders.

Suppose first that R is a compact quasi-order, on some compact Polish space X . To each $x \in X$, we can associate the compact set $R_x = \{y \in X : yRx\}$, and clearly

$$xRy \leftrightarrow R_x \subseteq R_y.$$

So the map $x \mapsto R_x$ from X into the space $\mathcal{K}(X)$ of compact subsets of X is a reduction of R to \subseteq . As a consequence, we get:

Proposition 2.1. $(\mathcal{P}(\omega), \subseteq)$ is a complete compact quasi-order.

Proof. Given (X, R) as above, choose an open basis $(U_n)_{n \in \omega}$ of X , and set $f(x) = \{n : U_n \cap R_x \neq \emptyset\}$. By the previous discussion, f is a reduction of R to \subseteq , and f is Borel (this is the point that does not work for Π_1^0 quasi-orders. It is still true that $(\mathcal{P}(\omega), \subseteq)$ is complete for them, but some work has to be done). \square

If we want to do the same thing for a \mathbf{K}_σ quasi-order R , there is a problem. The reduction map $x \mapsto R_x$ is still Borel (in the codes), but inclusion between \mathbf{K}_σ sets is Π_1^1 in the codes, but not \mathbf{K}_σ (not even Σ_1^1). The idea is to replace inclusion by a stronger relation, taking advantage of the transitivity of R .

Let $\leq_{\mathbf{K}_\sigma}$ be the quasi-order on $\mathcal{P}(\omega)^\omega$ defined by

$$(A_n) \leq_{\mathbf{K}_\sigma} (B_n) \leftrightarrow \exists n \forall m A_m \subseteq B_{m+n}.$$

Proposition 2.2. $\leq_{\mathbf{K}_\sigma}$ is complete for \mathbf{K}_σ quasi-orders (on compact Polish spaces).

Proof. Let R be a \mathbf{K}_σ quasi-order on the compact Polish space X . Write $R = \bigcup_n R^n$, where the sets R^n are compact, increasing, and satisfy $R_0 = \Delta$, the diagonal of X , and for all n $R^n \circ R^n \subseteq R^{n+1}$, where $R^n \circ R^n = \{(x, z) : \exists y (x, y) \in R^n \wedge (y, z) \in R^n\}$. This is easy to get from an arbitrary sequence of compact sets with union R , as the operation \circ preserves compactness. Now fix an open basis (U_n) for X , and define $f : x \in X \mapsto (A_x^n) \in \mathcal{P}(\omega)^\omega$, by setting $A_x^n = \{m : U_m \cap R_x^n \neq \emptyset\}$. Clearly f is Borel, so we have to check that it reduces R to $\leq_{\mathbf{K}_\sigma}$. If $f(x) \leq_{\mathbf{K}_\sigma} f(y)$, then in particular $R_x \subseteq R_y$ and xRy . Conversely if xRy , then for some n $xR^n y$. But then for $m \geq n$, $z \in R_x^m$ implies $z \in R_y^{m+1}$. So for all m $R_x^m \subseteq R_y^{m+n+1}$, hence $n + 1$ witnesses that $f(x) \leq_{\mathbf{K}_\sigma} f(y)$. \square

By Proposition 1.5, the associated equivalence relation $\equiv_{\mathbf{K}_\sigma}$ is complete among \mathbf{K}_σ equivalence relations. Another very similar \mathbf{K}_σ equivalence relation was already known to be complete (see Kechris [13]).

For the general case of Σ_1^1 quasi-orders, our plan is similar. Inclusion between Σ_1^1 sets is now Π_2^1 in the codes, so we must replace it by a stronger quasi-order, again taking advantage of transitivity. And, as we did above, we will change it slightly to get a more combinatorial object.

First we need some notations and terminology.

Given a set X , $X^{<\omega}$ denotes the set of finite sequences from X . If $s \in X^{<\omega}$, $|s|$ is its length (its domain), and $s \subseteq t$ means s is a restriction of t . If $X = \omega$, we also define $s \leq t$ by $|s| = |t|$ and $\forall i < |s| \ s(i) \leq t(i)$. Similarly $s + t$, for s and t of the same length, is defined by pointwise addition. We view elements in $(X \times Y)^{<\omega}$ as pairs in $X^{<\omega} \times Y^{<\omega}$ of the same length.

A (set-theoretical) *tree* on X is a subset of $X^{<\omega}$ closed under restrictions. If T is a tree on $X \times \omega$, we say that T is *normal* if whenever $(u, s) \in T$ and $s \leq t$, $(u, t) \in T$, too. For $s \in \omega^{<\omega}$, set

$$T(s) = \{u \in X^{<\omega} : |u| = |s| \wedge (u, s) \in T\}.$$

So T is normal if $s \leq t$ implies $T(s) \subseteq T(t)$.

A map $f : \omega^{<\omega} \rightarrow \omega^{<\omega}$ is *Lipschitz* if f preserves both length and extension. This is equivalent to saying that there is a map $f^* : \omega^{<\omega} \times \omega \rightarrow \omega$ such that f is obtained by the recursion $f(\emptyset) = \emptyset$ and $f(s \frown n) = f(s) \frown f^*(s, n)$. It is also the same as saying that f corresponds to a strategy of Player II in the usual type of games on ω , where the two players alternate playing integers.

Definition 2.3. We let \mathcal{T} be the space of normal trees on $2 \times \omega$. Topologized as a subset of $2^{(2 \times \omega)^{<\omega}}$, it is Polish. We then define \leq_{max} on \mathcal{T} by

$$S \leq_{max} T \leftrightarrow \exists f : \omega^{<\omega} \rightarrow \omega^{<\omega} \ \forall s \in \omega^{<\omega} \ S(s) \subseteq T(f(s)).$$

(Going back to the discussion above, if we view a tree T as coding the Σ_1^1 subset of 2^ω ,

$$p([T]) = \{\alpha : \exists \beta \in \omega^\omega \ \forall n \ (\alpha|_n, \beta|_n) \in T\},$$

then $S \leq_{max} T$ is a strong way of saying that $p([S]) \subseteq p([T])$.)

\leq_{max} is clearly a Σ_1^1 quasi-order on \mathcal{T} . In order to show it is complete, we first prove a normal form result for Σ_1^1 quasi-orders on 2^ω .

Theorem 2.4. *Let R be a Σ_1^1 quasi-order on 2^ω . Then there exists a tree S on $2 \times 2 \times \omega$ satisfying:*

- (i) $R = p([S])$, i.e. $\alpha R \beta \leftrightarrow \exists \gamma \in \omega^\omega \ \forall n \ (\alpha|_n, \beta|_n, \gamma|_n) \in S$.
- (ii) S is normal, i.e. $(u, v, s) \in S$ and $s \leq t$ imply $(u, v, t) \in S$.
- (iii) If $u \in 2^{<\omega}$ and $s \in \omega^{<\omega}$ are of the same length, $(u, u, s) \in S$.
- (iv) If $(u, v, s) \in S$ and $(v, w, t) \in S$, then $(u, w, s + t) \in S$.

Proof. Start with any tree T_0 on $2 \times 2 \times \omega$ with $R = p([T_0])$. As is well known, if we set

$$T_1 = \{(u, v, t) : \exists s \leq t \ (u, v, s) \in T_0\},$$

T_1 is normal, and we still have $R = p([T_1])$. Also, if we let $T_2 = T_1 \cup \{(u, u, s) : |u| = |s|\}$, T_2 now satisfies (i), (ii) and (iii) (it satisfies (i) because R is reflexive).

Finally we define S by $(\emptyset, \emptyset, \emptyset) \in S$, and for all $k, n \in \omega, u, v \in 2^k, s \in \omega^k$ and $i, j \in 2$,

$$(2.1) \quad \begin{aligned} &(u \frown i, v \frown j, n \frown s) \in S \\ &\iff \exists u_0, u_1, \dots, u_n \in 2^k \ (u_0 = u \wedge u_n = v \wedge \forall l < n \ (u_l, u_{l+1}, s) \in T_2) \end{aligned}$$

(so if $n = 0$, $(u \frown i, v \frown j, 0 \frown s) \in S$ if and only if $u = v$).

We claim that S works. Clearly it is a tree. To check (i), note first that if (x, y, α) is a branch through T_2 , then $(x, y, 1 \frown \alpha)$ is a branch through S . So $R \subseteq p([S])$. Conversely, suppose $(x, y, n \frown \alpha)$ is a branch through S . If $n = 0$, $x = y$, and

$(x, y) \in R$. If $n > 0$, we get, for each k , sequences $(u_i^k)_{i \leq n}$ in 2^k , with $u_0^k = x|_k$, $u_n^k = y|_k$, and for $i < n$, $(u_i^k, u_{i+1}^k, \alpha|_k) \in T_2$. By compactness of 2^ω , we can find a subsequence (k_l) and for $i \leq n$ elements $z_i \in 2^\omega$ such that $u_i^{k_l} \rightarrow z_i$ as $l \rightarrow \infty$. But then we get that for $i < n$, (z_i, z_{i+1}, α) is a branch through T_2 , hence $z_i R z_{i+1}$. As $z_0 = x$ and $z_n = y$, we get by transitivity $x R y$, as desired.

To check (ii), let $(u, v, s) \in S$ and $t \geq s$. The case of $(\emptyset, \emptyset, \emptyset)$ is trivial. So suppose $u = u' \frown i$, $v = v' \frown j$, $s = n \frown s'$ and $t = m \frown t'$, with $n \leq m$ and $s' \leq t'$. As T_2 is normal, we also have $(u, v, n \frown t') \in S$, with the same witnesses $(u_i)_{i \leq n}$. Also, using property (iii) of T_2 , we can first repeat $m - n$ times the witness u_0 to get witnesses for $(u, v, m \frown t') \in S$, as desired.

(iii) follows from (ii) and the immediate remark that if $|u| = |s|$ and $s(0) = 0$, $(u, u, s) \in S$.

Finally, to check (iv), let $u = u' \frown i$, $v = v' \frown j$, $w = w' \frown k$, $s = n \frown s'$ and $t = m \frown t'$ satisfy $(u, v, s) \in S$ and $(v, w, t) \in S$. By (ii), we also have

$$(u, v, n \frown (s' + t')) \in S \quad \text{and} \quad (v, w, m \frown (s' + t')) \in S,$$

as witnessed by say $(u_i)_{i \leq n}$ and $(v_j)_{j \leq m}$, respectively. But then $(u_i)_{i < n} \widehat{\frown} (v_j)_{j \leq m}$ is a witness that $(u, w, (n + m) \frown (s' + t')) \in S$, as desired. \square

As a consequence, we get:

Theorem 2.5. *The quasi-order \leq_{max} is a complete Σ_1^1 quasi-order.*

Proof. We only need to Borel reduce to \leq_{max} any Σ_1^1 quasi-order R on 2^ω . Then let S be the tree associated to R by Theorem 2.4, and define $f : 2^\omega \rightarrow \mathcal{T}$ by

$$f(x) = S^x = \{(u, s) \in (2 \times \omega)^{<\omega} : (u, x|_{|u|}, s) \in S\}.$$

Indeed the tree S^x is normal, as S is. And f is Borel, in fact continuous. We check that it is a reduction. Suppose first that $S^x \leq_{max} S^y$, as witnessed by the Lipschitz map $\varphi : \omega^{<\omega} \rightarrow \omega^{<\omega}$. If 0^k is the sequence of length k with constant value 0, the sequences $\varphi(0^k)$, $k \in \omega$, extend each other, hence build $\alpha \in \omega^\omega$. By property (iii), for all k , $(x|_k, 0^k) \in S^x$, hence $(x|_k, \varphi(0^k)) \in S^y$. So (x, y, α) is a branch through S and by (i) $x R y$.

Conversely, suppose $x R y$, and let α be such that (x, y, α) is a branch through S . Define $\varphi : \omega^{<\omega} \rightarrow \omega^{<\omega}$ by $\varphi(s) = s + \alpha|_{|s|}$. The map φ is clearly Lipschitz. Also, if $s \in 2^k$, and u is such that $(u, s) \in S^x$, we get $(u, x|_k, s) \in S$, and $(x|_k, y|_k, \alpha|_k) \in S$; hence by property (iv) of S , $(u, y|_k, \varphi(s)) \in S$ and $(u, \varphi(s)) \in S^y$. So φ witnesses $S^x \leq_{max} S^y$, as desired. \square

Remark 2.6. 1. In the previous proof, we only used a weak consequence of the fact that $S^x \leq_{max} S^y$ in one direction, and proved a strong form of it in the other direction. So this leaves room for proving that many variants of \leq_{max} are also complete. The simplest to define is probably \leq_{max}^* , defined on \mathcal{T} by

$$S \leq_{max}^* T \leftrightarrow \exists \alpha \in \omega^\omega \forall (u, s) \in S (u, s + \alpha|_{|s|}) \in T,$$

which is also more in the spirit of the \mathbf{K}_σ case. But our goal is to get a complete Σ_1^1 quasi-order which is easy to reduce to other quasi-orders, and in this respect, \leq_{max} seems more useful in practice.

2. Some of the features of \leq_{max} may look superfluous, like e.g. insisting that we consider only normal trees (of course its natural extension to arbitrary trees is

complete, too). But note that because the trees are normal, Lipschitz witnesses to $S \leq_{max} T$ can always be chosen one-to-one, a fact we will use later on.

Also, one can view the trees T as labelings of the basic tree $\omega^{<\omega}$ by finite subsets of a countable set (associating to $s \in \omega^{<\omega}$ the labeling $T(s)$), and naturally extend \leq_{max} to more general labelings, by finite or even infinite subsets of say ω , and again obtain complete Σ_1^1 quasi-orders. But again, we will later use the specific labeling given by the elements of \mathcal{T} .

3. The ordering \leq_{max} also admits a game-theoretic interpretation. For S, T in \mathcal{T} , consider the game $G(S, T)$ where two players alternate playing integers, Player I building a sequence α and Player II a sequence β , and where Player II wins the run if for all n , $S(\alpha|_n) \subseteq T(\beta|_n)$. Then clearly $S \leq_{max} T$ if and only if Player II has a winning strategy in $G(S, T)$.

Now $G(S, T)$ is open, uniformly in S, T . So by the usual analysis of open games, we get that $\neg(S \leq_{max} T)$ is equivalent to the existence of an ordinal $\xi < \aleph_1$ such that Player I has a winning strategy in the following game $G^\xi(S, T)$: Alternately Player I plays integers and strictly decreasing ordinals below ξ and Player II answers with integers. The game finishes when the ordinal 0 is reached, the players having produced finite sequences s and t , and Player II wins if for $i \leq |s|$, $S(s|_i) \subseteq T(t|_i)$.

Moreover the least ξ for which Player I has a winning strategy in $G^\xi(S, T)$ provides a Π_1^1 -norm on the complement of \leq_{max} .

It follows that if we define $S \leq_{max}^\xi T$ if Player II has a winning strategy in $G^\xi(S, T)$, we get an ω_1 -sequence of Borel quasi-orders which is \leq_B -cofinal in the Borel quasi-orders. For if R is a Borel quasi-order, any Borel reduction f of R to \leq_{max} must, by a boundedness argument, also reduce R to some \leq_{max}^ξ . This line of research is pursued in a forthcoming paper by the second author (Rosendal [26]).

3. MODEL-THEORETIC EXAMPLES

3.1. Embeddability in model theory. Given two structures \mathcal{A} and \mathcal{B} in the same language L , we say that \mathcal{A} embeds into \mathcal{B} , in symbols $\mathcal{A} \sqsubseteq \mathcal{B}$, if there is a one-to-one map f which realizes an isomorphism between \mathcal{A} and the substructure $\mathcal{B}|_{Im(f)}$.

Below, we will consider only countable structures in a countable language, and will assume that the domain of the structures we consider is always \mathbb{N} (so we exclude finite structures). The set X_L of L -structures with domain \mathbb{N} is naturally topologized as a compact zero-dimensional Polish space, and when restricted to X_L , \sqsubseteq becomes a Σ_1^1 quasi-order, and the associated equivalence \equiv of bi-embeddability becomes a Σ_1^1 equivalence relation.

If \mathcal{C} is a class of countable L -structures, denote by $\sqsubseteq_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ the restrictions of \sqsubseteq and \equiv to the set $X_L \cap \mathcal{C}$. When \mathcal{C} is the set of models of a first-order theory, or of some $L_{\omega_1\omega}$ -sentence, $X_L \cap \mathcal{C}$ is Borel in X_L , and $\sqsubseteq_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ are Σ_1^1 and fall under our study.

Historically, the first case considered was \sqsubseteq_{LO} , where LO is the class of countable linear orders. Fraïssé [6] conjectured that \sqsubseteq_{LO} is a well-quasi-order (wqo), i.e. a quasi-ordering with no infinite antichains and no infinite descending chains (or, to rephrase it in a way closer to this paper, a quasi-ordering which (Borel) reduces neither $(\omega, =)$ nor (ω, \geq)). Fraïssé's conjecture was proved by Laver [16], who proved in fact that \sqsubseteq_{LO} is a better-quasi-ordering (bqo), a technical strengthening of wqo due to Nash-Williams and instrumental in the proof. We will not get into

bqo theory here (the reader may consult the nice introduction by Simpson in [23]), but just want to stress that Laver’s result implies that \sqsubseteq_{LO} is extremely far from complete. \equiv_{LO} is also very far from complete, as it has \aleph_1 classes, but does not Borel reduce equality on 2^ω . These two facts should be contrasted to the situation of the equivalence \cong_{LO} of isomorphism between countable linear orders: By a result of Friedman and Stanley (see Hjorth [12]), it is S_∞ -complete.

There have been other similar bqo results. For example, Nash-Williams [24] proves that \sqsubseteq_{TO} is bqo, where TO is the class of countable tree-orderings (partial orders in which the set of predecessors of a point is finite and totally ordered).

There is another notion of “tree”, which we call here a combinatorial tree (to distinguish it from the set-theoretic notion of section 2, and the tree-orderings above). A combinatorial tree is a graph on some set X (a symmetric antireflexive binary relation) which is connected and acyclic. A rooted combinatorial tree is a combinatorial tree together with a distinguished vertex called its root.

There are obvious relations between these different notions of “trees”: If (X, \leq) is a tree-ordering with a least element x_0 , one gets a combinatorial tree by connecting x to y if one is the predecessor of the other in \leq . Conversely, if (X, G, x_0) is a rooted combinatorial tree, one can define a tree-ordering on X with least element x_0 by saying that $x \leq y$ if the necessarily unique path from x_0 to y contains x . Moreover, any set-theoretical subtree T of $\omega^{<\omega}$ comes with a natural tree-ordering (extension), and a natural rooted combinatorial treeing, given by \emptyset and immediate extension. Up to isomorphism, we get in this way all countable tree-orderings with least element and all countable (rooted) combinatorial trees.

However, the notions of embedding for these structures are very different, as exemplified by Nash-Williams’ result above and the following result:

Theorem 3.1. *Let CT (respectively RCT) be the class of countable combinatorial trees (resp rooted ones). Then \sqsubseteq_{CT} and \sqsubseteq_{RCT} are complete Σ_1^1 quasi-orders, and hence \equiv_{CT} and \equiv_{RCT} are complete Σ_1^1 equivalence relations.*

Proof. We will define a Borel reduction $T \mapsto G_T$ of \leq_{max} to \sqsubseteq_{CT} and \sqsubseteq_{RCT} (simultaneously). First, fix some one-to-one enumeration θ of $2^{<\omega}$ such that $|s| \leq |t|$ implies $\theta(s) \leq \theta(t)$ (so $\theta(\emptyset) = 0$), e.g. by using the lexicographic ordering. We now describe the combinatorial tree G_T , for $T \in \mathcal{T}$. Its domain is not \mathbb{N} , but it is easy to find a Borel in T bijection between its domain and \mathbb{N} to get the reduction we want.

First, we “double” the set $\omega^{<\omega}$, i.e. add, for each $s \in \omega^{<\omega} \setminus \{\emptyset\}$, another vertex s^* , and put an edge between s^* and s , and between s^* and the predecessor s^- of s . This defines a graph G_0 (not depending on T), which is clearly a combinatorial tree. Then for each pair $(u, s) \in T$, we add vertices (u, s, x) , where x is either 0^k or $0^{2\theta(u)+2} \frown 1 \frown 0^k$, for $k \in \omega$. Also, we link each (u, s, x) to (u, s, x') , where x' is the predecessor of x (as a sequence), and link (u, s, \emptyset) to s . This completely describes G_T , which is obviously a combinatorial tree. Let us make easy observations about it. First, one can compute the valence v_T (the number of neighbors) of vertices in G_T : elements in $\omega^{<\omega}$ have valence ω , elements $(u, s, 0^{2\theta(u)+2})$, for $(u, s) \in T$, have valence 3, and all other vertices have valence 2. Next consider the distance d_T between vertices (the length of the unique path joining them). Then (because of the doubling) the distance between vertices in $\omega^{<\omega}$ is even. The distance between a vertex $(u, s, 0^{2\theta(u)+2})$ and points in $\omega^{<\omega}$ is odd, and at least $2\theta(u) + 3$ (obtained at s).

We now check that

$$S \leq_{max} T \rightarrow (G_S, \emptyset) \sqsubseteq (G_T, \emptyset)$$

and

$$G_S \sqsubseteq G_T \rightarrow S \leq_{max} T,$$

which will finish the proof.

First, suppose $S \leq_{max} T$. Then, by the remark following Theorem 2.5, there is a one-to-one Lipschitz map f from $\omega^{<\omega}$ into $\omega^{<\omega}$ with $s(s) \subseteq T(f(s))$ for $s \in \omega^{<\omega}$. Define an embedding of G_S into G_T as follows. Send $s \in \omega^{<\omega}$ to $f(s)$ (so in particular roots are preserved), and s^* to $f(s)^*$. This defines an embedding of G_0 into itself. Next if $(u, s) \in S$, we have $(u, f(s)) \in T$, hence we can send (u, s, x) to $(u, f(s), x)$. Clearly this map witnesses $(G_S, \emptyset) \sqsubseteq (G_T, \emptyset)$.

Suppose now $G_S \sqsubseteq G_T$, via some embedding g . Then we must have $v_T(g(y)) \geq v_S(y)$ and $d_T(g(y), g(z)) = d_S(y, z)$ for all vertices y, z in the domain of G_S . By the remarks above, it implies that g must send elements in $\omega^{<\omega}$ to elements in $\omega^{<\omega}$, i.e. defines a map $f : \omega^{<\omega} \rightarrow \omega^{<\omega}$. We claim that f is a witness to $S \leq_{max} T$.

First we show $f(\emptyset) = \emptyset$. Consider $x = (\emptyset, \emptyset, 0^2)$. It is a vertex in G_S (S is infinite, hence non-empty) of valence 3, and d_S -distance 3 from \emptyset . So it must be sent to some vertex of valence at least 3 in G_T , with d_T -distance 3 from $f(\emptyset)$. But by the remarks above, there is only one possible such vertex, namely $(\emptyset, \emptyset, 0^2)$, as points in $\omega^{<\omega}$ are at even distance of $f(\emptyset)$ and the other vertices of valence 3 are at a bigger distance. This implies in particular that $f(\emptyset) = \emptyset$.

Next we show that f is Lipschitz, by induction on the length of s . The first step was done above. As $s \frown n$ is within distance 2 from s in G_S , $f(s \frown n)$ must be within distance 2 of $f(s)$ in G_T ; it cannot be $f(s)^-$ which is $f(s^-)$ by the induction hypothesis. So it is $f(s) \frown k$ for some k , and f is Lipschitz.

Finally suppose $(u, s) \in S$, towards showing $(u, f(s)) \in T$. Consider the vertex $x = (u, s, 0^{2\theta(u)+2})$ in G_S . It must be sent by g to some vertex y in G_T of valence at least 3 and at distance $2\theta(u) + 3$ of $f(s)$. Again points in $\omega^{<\omega}$ are forbidden by parity, so $y = (v, t, 0^{2\theta(v)+2})$ for some $(v, t) \in T$. But as the path in G_S joining s to x does not contain s^- , the path in G_T joining $f(s)$ to y does not contain $f(s^-) = f(s)^-$, and t must extend $f(s)$. But if it extends it strictly, we get $|v| > |u|$ and $\theta(v) > \theta(u)$, so that the distance is too big. So $t = f(s)$ and $\theta(v) = \theta(u)$, hence $v = u$ and finally $(u, f(s)) \in T$, as desired. \square

The use of points of valence ω was clearly crucial in the previous proof, and one can ask whether there is a variant of the construction that would only use locally finite combinatorial trees, i.e. trees on which the valence is finite. The next result shows the answer is negative:

Proposition 3.2. *Let LFCT be the class of locally finite combinatorial trees, and let RLFCT be the class of rooted ones. Then:*

- (a) \sqsubseteq_{RLFCT} is complete for compact quasi-orders.
- (b) \sqsubseteq_{LFCT} is complete for \mathbf{K}_σ quasi-orders.

Proof. (a) First enumerate in a one-to-one way all isomorphism types of finite rooted trees, and let $t(H, x)$ be the number of the isomorphism type of the finite tree H with root x . Then associate to the locally finite rooted tree (G, x) the set

$$A_{G,x} = \{t(H, x) : H \text{ a finite subtree of } G \wedge x \in H\}.$$

This is clearly a Borel assignment, and we claim that it reduces \sqsubseteq_{RLFCT} to $(\mathcal{P}(\omega), \subseteq)$. If f is an embedding of (G_1, x_1) into (G_2, x_2) and $n \in A_{G_1, x_1}$, there is a finite subtree H of G_1 with $x_1 \in H$ and $n = t(H, x_1)$. But then $f(H)$ is a finite subtree of G_2 with $x_2 \in f(H)$, and $n = t(f(H), x_2) \in A_{G_2, x_2}$. Conversely, suppose $A_{G_1, x_1} \subseteq A_{G_2, x_2}$. Let $G_1^n = \{x \in G_1 : d_{G_1}(x_1, x) \leq n\}$. By local finiteness, G_1^n is finite and $t(G_1^n, x_1)$ is in A_{G_1, x_1} , hence also in A_{G_2, x_2} . So for each n there is an embedding from (G_1^n, x_1) into (G_2, x_2) . But there are only finitely many possible such embeddings, and by König's lemma there is an embedding of (G_1, x_1) into (G_2, x_2) .

We now prove the converse. For $A \subseteq \omega$, define a (rooted, locally finite) combinatorial tree G_A as follows. Its domain is $D_A = \omega \times \{0\} \cup A \times \{1\}$, its root $(0, 0)$ and its edges link $(i, 0)$ to $(i + 1, 0)$ for $i \in \omega$ and $(i, 0)$ to $(i, 1)$ for $i \in A$. Then trivially

$$A \subseteq B \leftrightarrow (G_A, (0, 0)) \sqsubseteq (G_B, (0, 0)).$$

As in G_A the valence is always ≤ 3 , and the vertices of valence 3 are the $(i, 0)$'s, $i \in A$. This finishes the proof of (a).

(b) First we show that \sqsubseteq_{LFCT} is Borel reducible to $\leq_{\mathbf{K}_\sigma}$. For this, associate with each locally finite combinatorial tree G with domain \mathbb{N} and with each integer n the set

$$A_G^n = \{t(H, k) : H \text{ is a finite subtree of } G \wedge k \in H \wedge d_G(0, k) \leq n\}.$$

We claim that $G \mapsto (A_G^n)_{n \in \omega}$ is the (clearly Borel) reduction we want.

First suppose $f : \mathbb{N} \rightarrow \mathbb{N}$ embeds G_1 into G_2 , and let $p = d_{G_2}(0, f(0))$. Let $m \in A_{G_1}^n$, towards showing that $m \in A_{G_2}^{n+p}$. Now $m = t(H, k)$ for some finite subtree H of G_1 and some $k \in H$ with $d_{G_1}(0, k) \leq n$. So $m = t(f(H), f(k))$ and $f(H)$ is a finite subtree of G_2 , $f(k) \in f(H)$ and $d_{G_2}(0, f(k)) \leq d_{G_2}(f(0), f(k)) + p \leq n + p$, as desired.

Conversely, suppose p is such that for all n , $A_{G_1}^n \subseteq A_{G_2}^{n+p}$. In particular (with the notations of part (a)) $A_{G_1}^0 = A_{G_1, 0}$ is contained in $A_{G_2}^p$, which is the finite union of the $A_{G_2, k}$'s for $k \in G_2^p$. But then the proof in (a) works as well to show that G_1 embeds in G_2 (with 0 sent to some $k \in G_2^p$).

It remains to find a Borel reduction of $\leq_{\mathbf{K}_\sigma}$ to \sqsubseteq_{LFCT} . For $\bar{A} = (A^n)$ a sequence of subsets of ω , define a locally finite combinatorial tree $G(\bar{A})$ as follows: With the notations of part (a), its domain consists of $\{0\} \times D_{\omega \setminus \{0\}}$, on which we copy $G_{\omega \setminus \{0\}}$ and the union of the $\{n + 1\} \times D_{A^n}$, on which we copy G_{A^n} ; and we also link $(n + 1, 0, 0)$ to $(0, n + 1, 0)$ for $n \in \omega$.

Suppose now $\bar{A} \leq_{\mathbf{K}_\sigma} \bar{B}$, so that for some p , $A^n \subseteq B^{n+p}$ for all n . Then let f send $(0, n, i)$ to $(0, n + p, i)$, and $(n + 1, x)$ for x in D_{A^n} to $(n + p + 1, x)$. Easily f embeds $G_{\bar{A}}$ into $G_{\bar{B}}$.

Conversely, suppose f embeds $G_{\bar{A}}$ into $G_{\bar{B}}$. In $G_{\bar{A}}$ and $G_{\bar{B}}$, all points have valence at most 4, and the points of valence 4 are the $(0, n + 1, 0)$'s, for $n \in \omega$. So in particular for some p , $f((0, 1, 0)) = (0, p + 1, 0)$, and then necessarily $f((0, n + 1, 0)) = (0, p + n + 1, 0)$ for all n (one cannot go backwards from $(0, p + 1, 0)$, as there are not enough points of valence 4 in that direction). But then for each n , f must embed the copy of $(G_{A^n}, (0, 0))$ into the copy of $(G_{B^{n+p}}, (0, 0))$, and hence $A^n \subseteq B^{n+p}$, as desired. \square

We now come back to orderings. We have seen a case, the class of countable tree-orderings, for which embeddability is very simple. Here is an opposite result.

Theorem 3.3. *Let PO be the class of countable partial orders. Then \sqsubseteq_{PO} is a complete Σ_1^1 quasi-order, and hence \equiv_{PO} is a complete Σ_1^1 equivalence relation.*

Proof. We will Borel reduce \sqsubseteq_{CT} to \sqsubseteq_{PO} . This is enough by Theorem 3.1.

For each combinatorial tree G on \mathbb{N} , we define a partial order \leq_G as follows: Its domain is $\mathbb{N} \cup G^*$, where $G^* = \{\{n, m\} : (n, m) \in G\}$ is the set of edges of G . Also, we set

$$x \leq_G y \leftrightarrow (x = y) \vee (x \in \mathbb{N} \wedge y \in G^* \wedge x \in y).$$

Clearly each \leq_G is a partial order (there is no transitivity to be checked), and it is easy to transform it into a partial order on \mathbb{N} , in a Borel way in G . So it remains to check that it gives a reduction.

First if f is an embedding of G_1 into G_2 , f extends to G_1^* by sending $\{n, m\}$ to $\{f(n), f(m)\}$, and this gives an embedding of \leq_{G_1} into \leq_{G_2} .

Conversely, if f is an embedding of \leq_{G_1} into \leq_{G_2} , as any $n \in \mathbb{N}$ is \leq_{G_1} -below at least some edge, f must send \mathbb{N} to \mathbb{N} , and similarly f must send G_1^* to G_2^* . As each edge $\{\{n, m\}$ in G_1^* is above n and m and nothing else, we must have $f(\{n, m\}) = \{f(n), f(m)\}$. This implies that $nG_1m \rightarrow f(n)G_2f(m)$, but also that adjacent edges in G_1 must go to adjacent edges in G_2 , so that finite paths are preserved, and hence distance, too. In particular if n and m are not G_1 -related, their images cannot be G_2 -related, and f defines an embedding of G_1 into G_2 . \square

Remark 3.4. 1. The partial orders above are very close to being lattices. Indeed if we add a maximum element and a minimum element to them, they become lattices. As the proof of Theorem 3.3 goes (essentially) unchanged, we get as a corollary of the proof that embeddability between countable lattices (viewed as partial orders), and embeddability between countable lattices (viewed as lattices) are complete Σ_1^1 quasi-orders.

2. Theorem 3.3 is one manifestation of a nice “reflection” phenomenon:

The quasi-ordering \leq_B should be considered (at least we hope) as one of the “natural” quasi-orderings occurring in mathematics. As it was defined in section 1, it is not of the correct type to fall within our scope, and be compared (using \leq_B !) to other quasi-orderings. But its restrictions to many classes are (usually only “in the codes”) of the correct type. This is what happens in Theorem 3.3: The restriction of \leq_B to quasi-orders with countable domain (and discrete topology) clearly corresponds to \sqsubseteq_{PO} . Also, Theorem 3.3 says that this restriction of \leq_B is already complete for Σ_1^1 quasi-orderings.

We do not know if a similar phenomenon holds at other levels. For example, \leq_B restricted to Σ_1^1 quasi-orders is Σ_3^1 (in the codes of Σ_1^1 quasi-orders). Is it a complete Σ_3^1 quasi-order?

The only known result in this direction is the following result of Adams and Kechris [1]): Consider the restriction of \leq_B to Borel equivalence relations with countable classes (i.e. below $E_{F_2}^\infty$). In the codes, it is a Σ_2^1 quasi-ordering, and as proved in [1], as a set, it is complete Σ_2^1 . But it is not known if it is a complete Σ_2^1 quasi-order, or even if it Borel reduces \leq_{max} . Adams-Kechris prove in [1] that it Borel reduces inclusion between Borel sets, hence also any Borel quasi-order.

We end this section with some open problems.

The results of this section give examples of classes \mathcal{C} of countable structures such that bi-embeddability $\equiv_{\mathcal{C}}$ and isomorphism $\cong_{\mathcal{C}}$ are extremely far apart. If \mathcal{C} is the

class of countable linear orders, \equiv_C has only \aleph_1 classes, whereas \cong_C is S_∞ -complete. Also, if \mathcal{C} is the class of countable partial orders, \cong_C is S_∞ -complete, whereas \equiv_C is a complete Σ_1^1 equivalence relation. The natural question here is whether one can increase these gaps:

Question 1: Is there an $L_{\omega_1\omega}$ -elementary class \mathcal{C} with \cong_C S_∞ -complete but \equiv_C with countably many classes?

Question 2: Is there an $L_{\omega_1\omega}$ -elementary class \mathcal{C} with \equiv_C a complete Σ_1^1 equivalence relation but \cong_C not S_∞ -complete?

In the same vein, here is another natural question:

Question 3: Is there an $L_{\omega_1\omega}$ -elementary class \mathcal{C} with \equiv_C a complete Σ_1^1 equivalence relation but \sqsubseteq_C not a complete Σ_1^1 quasi-order?

A negative answer to this last question would reinforce our heuristic view that the limitations to the method developed in this paper are indeed intrinsic (and a positive answer would need a different method).

3.2. Homomorphism in model theory. Here we will concentrate on a notion of model theory somewhat complementary to that of embeddability, namely homomorphism.

We will model a graph G on a set D as a symmetric irreflexive relation and say that $h : D \rightarrow E$ is a homomorphism between graphs G on D and H on E if for all $x, y \in D$, $xGy \rightarrow h(x)Hh(y)$. We write $G \preceq H$ in case there is such a homomorphism. The notion of homomorphism between graphs turns out to be very useful in combinatorics. For suppose that $G \preceq H$ and c is a vertex colouring of H ; then $c \circ h$ is a vertex colouring of G and the chromatic number of G will therefore be less than that of H .

It is known that \preceq when restricted to the class of finite graphs is a very complicated relation, in fact any countable partial order embeds into it, as shown by Z. Hedrlin (see [25]). Here we intend to extend this result to the class of countably infinite graphs by showing that the relation restricted to the Polish space of countably infinite graphs is a complete analytic quasi-order.

Theorem 3.5. *The quasi-order of homomorphism between countable graphs is Σ_1^1 complete.*

Proof. Let $\pi : 2^{<\omega} \hookrightarrow \omega \setminus 4$ be an enumeration such that $|u| < |v| \rightarrow \pi(u) < \pi(v)$.

Define for each $(u, s) \in (2 \times \omega)^{<\omega}$ a graph $C_{u,s} = (V_{u,s}, E_{u,s})$ by

$$V_{u,s} = \{s = a_0^{u,s}, a_1^{u,s}, \dots, a_{\pi(u)}^{u,s}, b_1^{u,s}, \dots, b_{2\pi(u)}^{u,s}\} \text{ and}$$

$$a_0^{u,s} E_{u,s} a_1^{u,s} E_{u,s} \dots E_{u,s} a_{\pi(u)}^{u,s} E_{u,s} b_1^{u,s} \text{ and } b_i^{u,s} E_{u,s} b_j^{u,s}, \forall i \neq j.$$

So $C_{u,s}$ contains a complete graph on $2\pi(u)$ points and therefore if $C_{u,s} \preceq C_{v,t}$, then $\pi(u) \leq \pi(v)$. Furthermore if such a homomorphism sends $s = a_0^{u,s}$ to $t = a_0^{v,t}$, then $\pi(u) = \pi(v)$, i.e. $u = v$.

For any $s \in \omega^{<\omega}$ let $K_s = (V_s, E_s)$ be a distinct copy of $K_{|s|+1}$ (the complete graph on $|s| + 1$ vertices) with one of the vertices being s . Finally for $T \in \mathcal{T}$ let $G_T = (V_T, E_T)$ be defined by

$$V_T = \bigcup_{s \in \omega^{<\omega}} V_s \cup \bigcup_{(u,s) \in T} V_{u,s} \text{ and}$$

E_T consists of the edges in E_s for $s \in \omega^{<\omega}$, in $E_{u,s}$ for $(u, s) \in T$ and $sE_T s \cap n$ for all $s \in \omega^{<\omega}$ and $n < \omega$.

For two vertices x, y in a connected graph H put $d_H(x, y) = n$ iff n is minimal such that $\exists x = x_0, x_1, \dots, x_n = y$ $x_0 H x_1 H \dots H x_n$. We note that for $\alpha \in \omega^\omega$ and $n < m$ we have $d_{G_T}(\alpha|_n, \alpha|_m) = m - n$ and in fact $\alpha|_n, \alpha|_{n+1}, \dots, \alpha|_m$ is the

unique shortest path from $\alpha|_n$ to $\alpha|_m$ in G_T . Also note that a homomorphism is necessarily distance decreasing.

Claim: If $h : G_S \preceq G_T$ for $S, T \in \mathcal{T}$, then there are some $\alpha \in \omega^\omega$ and $k \geq 0$ such that $\forall^\infty n \ h(0^n) = \alpha|_{k+n}$.

Since $s \in \omega^{<\omega}$ is a vertex in a complete subgraph of G_S on $|s| + 1$ vertices, so is $h(s)$. So $h(s)$ can either be a vertex in some K_t or be $b_i^{u,t}$ for some t and u (whenever $|s| \geq 2$). Suppose that $h(s) = b_i^{u,t}$ for $|s| \geq 2$ and $(u, t) \in T$. Then as $sE_S(s \frown 0)E_S(s \frown 0^2)E_S \dots$ is a path in G_S of vertices belonging to complete subgraphs of increasing cardinality, so must $h(s), h(s \frown 0), h(s \frown 0^2), \dots$. But the valence of $a_{\pi(u)}^{u,t}$ is 2, so $h(s \frown 0^n) \neq a_{\pi(u)}^{u,t}, \forall n$ and therefore $h(s \frown 0^n) \in \{b_1^{u,t}, \dots, b_{2\pi(u)}^{u,t}\}$ for all n . But this is impossible since the $b_i^{u,t}$ only belong to complete subgraphs of bounded cardinality $2\pi(u)$.

So $h(s) \in K_{p(s)}$ for all s with $|s| \geq 2$ and $p(s) \in \omega^{<\omega}$ such that $|p(s)| \geq |s|$. So then $d_{G_T}(\emptyset, h(s)) \geq |s|$ for $|s| \geq 2$.

Put $k = \inf_n \ d_{G_T}(\emptyset, h(0^n)) - n \geq 0$. Then as $d_{G_T}(0^n, 0^m) = m - n$ for $n < m$ and h is distance decreasing, we have that $d_{G_T}(h(0^n), h(0^m)) \leq m - n$. Therefore $\exists m_0 \ \forall n \geq m_0 \ d_{G_T}(\emptyset, h(0^n)) = n + k$ and $\forall m > n \geq m_0 \ d_{G_T}(h(0^n), h(0^m)) = m - n$.

Note that a path from K_s to K_t ($s \neq t$) in G_T necessarily passes through s and t , so take $n_0 \geq m_0$ such that $h(0^{n_0}) = t_0$ for some t_0 with $|t_0| \geq n_0$ and $n_1 > n_0$ such that $h(0^{n_1}) = t_1, |t_1| \geq n_1$. Then $d_{G_T}(t_0, t_1) = n_1 - n_0$ and $d_{G_T}(\emptyset, h(0_n))$ is strictly increasing for $n_0 \leq n \leq n_1$, so $t_1 = t_0 \widehat{\ } w$ for some w of length equal to $n_1 - n_0$. But $t_1|_{|t_0|}, t_1|_{|t_0|+1}, \dots, t_1$ is the unique shortest path from t_0 to t_1 , which $h(0^{n_0}), \dots, h(0^{n_1})$ has to follow, so $h(0^{n_0+n}) = t_1|_{|t_0|+n}$.

This shows that $h(0^{n_0}) \subseteq h(0^{n_0+1}) \subseteq \dots$ and that therefore $\alpha = \bigcup_n \ h(0^{n_0+n})$ will do. This proves the claim.

Now choose m_0 large enough so that $\forall n \geq m_0 \ h(0^n) = \alpha|_{k+n}$ and so that $\forall n \geq m_0 \ \forall u \in 2^n \ \pi(u) > k + n + 2$. Suppose $u \in 2^n, n \geq m_0$ and $(u, 0^n) \in S$. Then there is a vertex x in G_S of distance $\pi(u) + 1$ from 0^n which belongs to a complete subgraph on $2\pi(u)$ vertices. So there is a corresponding vertex y in G_T of distance less than $\pi(u) + 1$ from $\alpha|_{k+n} = h(0^n)$ which belongs to a complete subgraph of G_T on $2\pi(u)$ vertices. This y cannot belong to some K_t , for otherwise $|t| + 1 \geq 2\pi(u)$ and $|t| \leq k + n + \pi(u) + 1$, i.e. $\pi(u) \leq k + n + 2$, which is impossible. So this y has to be some $b_i^{v,t}$, with $\pi(u) \leq \pi(v)$, i.e. $|u| \leq |v|$. But for such $v \ d_{G_T}(\alpha|_{k+n}, b_i^{v,t}) \leq \pi(u) + 1$ iff $\pi(v) \leq \pi(u)$. So $\pi(u) = \pi(v)$ and $v = u$. But then $|t| = n$ and $\alpha|_{k+n} = t$, i.e. $k = 0$ and $(u, t) = (u, \alpha|_n) \in T$.

This shows that for $(u, 0^n) \in S$ with $n \geq m_0$ we have $(u, \alpha|_n) \in T$. But on the image of a canonical reduction of some Σ_1^1 quasi-order to \leq_{max} this is enough to insure that $S \leq_{max} T$ as is seen from the proof of the maximality of \leq_{max} .

On the other hand if $S \leq_{max} T$, then it is easily seen that G_S is in fact isomorphic to an induced subgraph of G_T , so in particular $G_S \preceq G_T$. This shows \leq_{max} reduces to the relation of homomorphism and finishes the proof. \square

Remark 3.6. Let us just mention that by a trivial variation of the proofs of Proposition 3.2 (b) and Theorem 3.5 one can show that the relation of homomorphism between countable locally finite connected graphs is Borel bi-reducible with \mathbf{K}_σ . Note also that the above proof of course produces connected graphs, as is easily seen.

4. EMBEDDABILITY IN ANALYSIS

In this section, we consider various “embeddability” notions that occur in analysis, i.e. that are defined between separable structures (as opposed to countable structures), like Polish metric spaces, compact metrizable spaces and separable Banach spaces.

4.1. Isometric embeddability. A Polish metric space (X, d) is a metric space in which the distance d is complete, and the induced topology is Polish (which just means separable here). The distance is very often understood.

A map $f : (X, d_X) \rightarrow (Y, d_Y)$ is an *isometric embedding* if it preserves the distances. It is an *isometry* between X and Y if moreover it is onto Y . This naturally leads to the notions of isometric embeddability \sqsubseteq^i , isometric bi-embeddability \equiv^i and isometry \cong^i , defined respectively by

$$X \sqsubseteq^i Y \leftrightarrow \exists f : X \rightarrow Y, f \text{ is an isometric embedding,}$$

$$X \equiv^i Y \leftrightarrow X \sqsubseteq^i Y \wedge Y \sqsubseteq^i X,$$

$$X \cong^i Y \leftrightarrow \exists f : X \rightarrow Y, f \text{ is an isometry.}$$

Recall also that a Polish metric space X is *homogeneous* if any point of X can be sent to any other point by an isometry of X , and *ultrahomogeneous* if any isometry between two finite subsets of X can be extended to an isometry of X . Also, X is *Heine-Borel* if any closed bounded subset of X is compact (so that in particular X is locally compact).

The complexity of \cong^i , for various classes of Polish metric spaces, has been studied in a very detailed way in a series of papers by Clemens, Gao and Kechris [4, 8, 5]. Our aim here is to study the complexity of the quasi-ordering \sqsubseteq^i and the equivalence relation \equiv^i for these classes of Polish metric spaces.

First, we have to put these objects into our frame. This is done in [8]: There is a \sqsubseteq^i -maximum space U among Polish metric spaces, called the Urysohn space. It is characterized, up to isometry, by this maximality property and the fact that it is ultrahomogeneous. So, up to isometry, we can view any Polish metric space as a (necessarily closed) subspace of U . And if we equip $F(U)$, the set of closed subspaces of U , with the Effros Borel structure (and any Polish topology that gives it), the relations \sqsubseteq^i , \equiv^i and \cong^i become Σ_1^1 on $F(U)$.

As we did in section 3, if \mathcal{C} is a class of Polish metric spaces, we use a subscript \mathcal{C} to indicate the restrictions of \sqsubseteq^i , \equiv^i and \cong^i to $\mathcal{C} \cap F(U)$.

Our first result is an immediate consequence of Theorem 3.1.

Proposition 4.1. \sqsubseteq^i is a complete Σ_1^1 quasi-order, and \equiv^i is a complete Σ_1^1 equivalence relation. In fact the same is true of \sqsubseteq_D^i and \equiv_D^i , where D is the class of discrete Polish metric spaces — and hence also for any intermediate class, like $\text{dim}0$ Polish metric spaces or locally compact Polish metric spaces.

Proof. Any countable combinatorial tree can be turned into a discrete Polish metric space by using the geodesic distance, and tree-embeddings then correspond exactly to isometric embeddings. So Theorem 3.1 gives the result. (To be less sloppy one should go, in a Borel way, from the trees to $F(U)$. But this is easy, e.g. by first embedding into U , as some space X , the tree $\omega^{<\omega}$ with its geodesic distance, and then redefine in a Borel way the reduction in Theorem 3.1 as taking values in $F(X)$, hence in $F(U)$.) □

This result should be contrasted with the results of Gao and Kechris concerning \cong^i for the same classes: The complexity varies from $E_{S_\infty}^\infty$ for discrete Polish metric spaces, to $E_{G_0}^\infty$ for the class of all Polish metric spaces. For some intermediate classes, like locally compact Polish spaces, the exact complexity is not known.

In [8], Gao and Kechris also study isometry for the subclasses UM of ultrametric Polish spaces, HUM of homogeneous ultrametric Polish spaces and $UHUM$ of ultrahomogeneous ultrametric Polish spaces. They prove that \cong_{UM}^i is Borel bi-reducible with $E_{S_\infty}^\infty$, and \cong_{HUM}^i and \cong_{UHUM}^i are Borel bi-reducible with E_{cntble} (see the end of section 1 for the definitions).

Here we have:

Proposition 4.2. (a) \sqsubseteq_{UM}^i is a complete Σ_1^1 quasi-order, and hence \equiv_{UM}^i is a complete Σ_1^1 equivalence relation.

(b) \sqsubseteq_{HUM}^i and \sqsubseteq_{UHUM}^i are Borel bi-reducible with the ordering \subseteq_{cntble} , defined on \mathbb{R}^ω by

$$(x_n) \subseteq_{cntble} (y_n) \leftrightarrow \{x_n : n \in \omega\} \subseteq \{y_n : n \in \omega\}.$$

So \equiv_{HUM}^i and \equiv_{UHUM}^i are bi-reducible with E_{cntble} , but in fact for homogeneous ultrametric spaces, isometric bi-embeddability and isometry coincide.

Proof. (a) If (G, x_0) is a rooted countable combinatorial tree, its body $[G, x_0]$ is the set of infinite paths through G starting at x_0 . We turn it into an ultrametric Polish space by defining the distance between distinct paths (x_0, x_1, \dots) and (x_0, y_1, \dots) as 2^{-i} , where i is least with $x_i \neq y_i$. It is easy to check that if the trees (G_1, x_1) and (G_2, x_2) are pruned (every point has valence at least 2), embeddings from (G_1, x_1) into (G_2, x_2) correspond exactly to isometric embeddings of their bodies.

But then note that in the proof of Theorem 3.1 we only used pruned trees, and working a bit to get the bodies as subspaces of U (by e.g. embedding first the “universal” body ω^ω), we get (a).

(b) The following argument is similar to the one in [8]. First recall that given a countable subset A of \mathbb{R}^+ , there is a Polish ultrametric space U_A which is ultrahomogeneous, has distances in $A \cup \{0\}$, and isometrically embeds all ultrametric Polish spaces with distances in $A \cup \{0\}$. So if θ is a homeomorphism between \mathbb{R} and $(0, 1)$ and we set $A^* = \{1 + \theta(r) : r \in A\}$ for A a countable subset of \mathbb{R} , then the map $A \mapsto U_{A^*}$ gives a Borel reduction of \subseteq_{cntble} to \sqsubseteq_{UHUM} (and even to its restriction to discrete spaces in $UHUM$).

For the other direction, we have to reduce \sqsubseteq_{HUM}^i to \subseteq_{cntble} . So suppose (X, d) is a homogeneous ultrametric Polish space. Then its set of distances $d[X^2]$ is countable, and for each $r \in \mathbb{R}^+$, the relation $d(x, y) < r$ defines on X an equivalence relation E_r . Fix $x \in X$, and set, for $n \geq 2$

$$A_n^X = \{r \in \mathbb{R}^+ : \{y : d(x, y) \leq r\} \text{ consists of at least } n \text{ } E_r\text{-equivalence classes}\}.$$

Note that by homogeneity, this does not depend on the choice of x . The sets A_n^X are decreasing in n , and $A_2^X = d[X^2] \setminus \{0\}$ is countable. Our reduction is $X \mapsto A^X = \bigcup_n (n + \theta[A_n^X])$. It is easy to see it can be enumerated in a Borel in X way, and trivially if $X \sqsubseteq^i Y$, $A^X \subseteq A^Y$. Conversely suppose $A^X \subseteq A^Y$, and let $D = \{x_n : n \in \omega\}$ be a countable dense subset of X . It is enough to build an isometric embedding f of D into Y , for it can then be extended isometrically to all of X . We build f by induction on n . First send x_0 to some $y_0 \in Y$. Now suppose f has been defined on $\{x_0, \dots, x_n\}$, with $f(x_i) = y_i$. Let $r = \inf\{d_X(x_{n+1}, x_i), i \leq n\}$, $D_r = \{x_i : i \leq n \wedge d_X(x_{n+1}, x_i) = r\}$, and let $k \geq 1$ be the number of E_r -classes

met by D_r inside $\{x : d_X(x_{n+1}, x) \leq r\}$. As x_{n+1} is E_r -inequivalent to all elements of D_r , $r \in A_{k+1}^X$, hence $r \in A_{k+1}^Y$. But this means that picking some i with x_i in D_r , there must be in Y an E_r^Y -class inside $\{y : d_Y(y_i, y) \leq r\}$ not met by $f(D_r)$, hence some point y_{n+1} with $d_Y(y_{n+1}, y) = r$ for $y \in f(D_r)$. Set $f(x_{n+1}) = y_{n+1}$. Defined in this way, f preserves distances on $D_r \cup \{x_{n+1}\}$ by construction, and on all of $\{x_0, \dots, x_{n+1}\}$ by the properties of ultrametrics. This finishes the proof.

For the last remark, note that the same proof, using now a back-and-forth argument, shows that for X, Y in HUM , $X \equiv^i Y$ is equivalent to $X \cong^i Y$ and to $A^X = A^Y$. □

In [8], Gao and Kechris also consider the classes C of connected locally compact Polish metric spaces, HB of Heine-Borel Polish metric spaces, and CHB of connected Heine-Borel Polish metric spaces. [8] contains a proof of Hjorth's result that isometry restricted to any of these classes is Borel-bireducible with $E_{F_2}^\infty$ (Hjorth proves in fact a stronger result, for the class of so-called pseudo-connected locally compact Polish metric spaces that contains both C and HB ; see [8]).

We do not know a good upper bound for the complexity of \sqsubseteq_C^i . The next result provides a lower bound, and solves the problem for HB and CHB .

Theorem 4.3. *The quasi-orders \sqsubseteq_{HB}^i and \sqsubseteq_{CHB}^i are Borel bi-reducible with $\leq_{\mathbf{K}_\sigma}$, and hence \equiv_{HB}^i and \equiv_{CHB}^i are Borel bi-reducible with $\equiv_{\mathbf{K}_\sigma}$.*

Proof. The lower bound is easy, and follows from Proposition 3.2: Given a locally finite combinatorial tree G , build X_G by replacing every edge in G by a copy of $[0, 1]$, and extending the geodesic distance in the natural way. The resulting space is clearly connected and Heine-Borel, and easily $G \mapsto X_G$ gives a Borel reduction of \sqsubseteq_{LFCT} to \sqsubseteq_{CHB}^i .

To get the upper bound, we must find a Borel reduction of \sqsubseteq_{HB}^i to some \mathbf{K}_σ quasi-order. This is done as follows.

We work with the Urysohn space U . First, for each sequence $\vec{x} = (x_i)_{i < n}$ in U^n , let its configuration $c(\vec{x})$ be the $n \times n$ matrix with coefficients $d(x_i, x_j)$. Let $D = \{(n, \vec{m}) : n \in \omega, \vec{m} \in \omega^n\}$. If $\vec{K} = (K_n)_{n \in \omega}$ is a sequence of compact subsets of U and $(n, \vec{m}) \in D$, set

$$C_{n, \vec{m}}(\vec{K}) = \{c(\vec{x}_0 \widehat{\ } \vec{x}_1 \widehat{\ } \dots \widehat{\ } \vec{x}_{n-1}) : \forall i < n \ \vec{x}_i \in K_i^{m_i}\}.$$

It is a compact subset of $\mathbb{R}^{l(\vec{m}) \times l(\vec{m})}$, where $l(\vec{m}) = \sum_{i < n} m_i$.

We first prove:

Lemma 4.4. *Suppose \vec{K} and \vec{L} are increasing sequences of compact subsets of U , with unions X and Y , respectively. Then the following are equivalent:*

(a) *There is an isometric embedding f from X into Y with $f(K_n) \subseteq L_n$ for all $n \in \omega$.*

(b) *For all $(n, \vec{m}) \in D$, $C_{n, \vec{m}}(\vec{K}) \subseteq C_{n, \vec{m}}(\vec{L})$.*

Proof. (a) implies (b) is obvious, as f preserves configurations. For (b) implies (a), fix a countable set $X_0 = \{x_n : n \in \omega\}$ such that $X_0 \cap K_n$ is dense in K_n for each n . For $N \in \omega$, reenumerate $(x_i)_{i < N}$ by first enumerating the points in K_0 , then in $K_1 \setminus K_0$, and so on. This gives some $(n, \vec{m}) \in D$ and some $\vec{z}_0, \dots, \vec{z}_{n-1}$ with $\vec{z}_i \in K_i^{m_i}$, such that the concatenation of the z_i 's reenumerate $(x_i)_{i < N}$. Then $c(\vec{z})$ is in $C_{n, \vec{m}}(\vec{K})$, hence in $C_{n, \vec{m}}(\vec{L})$, and by reenumerating again, this means that there is an isometric embedding f_N of $(x_i)_{i < N}$ into Y satisfying $x_i \in K_p \rightarrow f(x_i) \in L_p$

for all p and $i < N$. By compactness of the L_p 's, one can find a subsequence of N 's such that for all i , $f_N(x_i)$ converges to some $y_i \in L_{p(i)}$, where $p(i)$ is least with $x_i \in K_{p(i)}$. This defines an isometric embedding of X_0 to Y which can by continuity be extended to all of X , and satisfies $f(K_n) \subseteq L_n$ for all $n \in \omega$. \square

Using this, we now finish the proof of Theorem 4.3. Fix some $x_0 \in U$. For each Heine-Borel subspace X of U and $n \in \omega$, let $K_n(X) = \{x \in X : d(x_0, x) \leq n\}$, and $\vec{K}(X) = (K_n(X))_{n \in \omega}$. This is an increasing sequence of compact sets, with union X .

Now let

$$D' = \{(p, q, \vec{m}) : q \leq p \wedge (p - q, \vec{m}) \in D\},$$

and define, for $(p, q, \vec{m}) \in D'$,

$$C'_{p,q,\vec{m}}(X) = C_{p-q,\vec{m}}(T^q \vec{K}(X)),$$

where $T^q(\vec{K}(i)) = \vec{K}(i + q)$.

This defines a Borel map f from the (Polish) space HB to the compact Polish space

$$Z = \prod_{(p,q,\vec{m}) \in D'} \mathcal{K}(\overline{\mathbb{R}})^{l(\vec{m}) \times l(\vec{m})},$$

equipped with the product of the Hausdorff topologies on the factors.

On Z , define a \mathbf{K}_σ quasi-order \leq_Z by

$$(C_i^1)_{i \in D'} \leq_Z (C_i^2)_{i \in D'} \iff \exists n \forall (p, q, \vec{m}) \in D' C_{p,q,\vec{m}}^1 \subseteq C_{p+n,q+n,\vec{m}}^2.$$

To finish the proof, we show that f reduces \sqsubseteq_{HB}^i to \leq_Z . First, if n witnesses that $(C'_i(X))_{i \in D'} \leq_Z (C'_i(Y))_{i \in D'}$, we get in particular, for $q = 0$, that for any (p, \vec{m}) in D , $C_{p,\vec{m}}(\vec{K}(X)) \subseteq C_{p,\vec{m}}(T^n \vec{K}(Y))$, and by Lemma 4.4, X isometrically embeds into Y .

Conversely, if g is an isometric embedding of X into Y , pick x_1 in X , and n_0, n_1 such that $d(x_0, x_1) \leq n_0$ and $d(x_0, g(x_1)) \leq n_1$. Then for x in $K_p(X)$, we get $g(x)$ in $K_{p+n_0+n_1}(Y)$, and by lemma 4 again, $n_0 + n_1$ witnesses that $f(X) \leq_Z f(Y)$, as desired. \square

The preceding proof is an elaboration on the proof (see Gromov [10]) that isometry for compact Polish metric spaces is tame, i.e. Borel reducible to equality on 2^ω . Analogous proofs show that \sqsubseteq^i restricted to compact Polish metric spaces, and \sqsubseteq^i restricted to homogeneous locally compact spaces, are Borel bi-reducible with $(\mathcal{P}(\omega), \subseteq)$.

4.2. Continuous embeddability. In this subsection, we consider compact metrizable topological spaces, which we view, up to homeomorphism, as elements of the space $\mathcal{K}(I)$ of compact subsets of the Hilbert cube $I = [0, 1]^\omega$, with its Hausdorff topology. Taking as morphisms the one-to-one continuous maps, we get the Σ_1^1 relations of continuous embeddability \sqsubseteq^c , continuous bi-embeddability \equiv^c and homeomorphism \cong^c on $\mathcal{K}(I)$. As before, we indicate by a subscript \mathcal{C} the restrictions of these relations to a subclass \mathcal{C} of compact metrizable spaces.

The exact complexity of \cong^c is not known. Camerlo and Gao [3] prove it is at least $E_{S_\infty}^\infty$, and Hjorth [12] proves it is strictly above it. On the other end, an easy argument (see the next subsection) shows it is at most $\cong^i (\equiv_B E_{G_0}^\infty)$.

When restricted to 0-dimensional spaces (i.e. $\mathcal{K}(2^\omega)$), it becomes Borel bi-reducible with $E_{S_\infty}^\infty$ [3]. Using Cantor’s analysis of countable compact sets, one easily shows that $\sqsubseteq_{\mathcal{K}([0,1])}^c$ is a pre-well-ordering of type $\omega_1 + 2$.

For higher dimensions, one gets:

Theorem 4.5. *The quasi-order $\sqsubseteq_{\mathcal{K}([0,1]^2)}^c$ is a complete Σ_1^1 quasi-order, hence a fortiori \sqsubseteq^c , too. So, $\equiv_{\mathcal{K}([0,1]^2)}^c$ and \equiv^c are complete Σ_1^1 equivalence relations.*

Proof. First, consider the Borel reduction f of \leq_{max} to \sqsubseteq_{RCT} of Theorem 3.1, that we now view as taking values in the rooted set-theoretical subtrees of $(\omega^{<\omega}, \emptyset)$. Let X be the image of f . We will use the following properties of X : The elements of X are pruned trees (all vertices have valence at least 2) and have 2^{\aleph_0} infinite branches, and on X , \sqsubseteq_{CT} and \sqsubseteq_{RCT} coincide. Our plan is to define a Borel map $g : \mathcal{P}(\omega^{<\omega}) \rightarrow \mathcal{K}([0, 1]^2)$ which on X reduces \sqsubseteq_{CT} to \sqsubseteq^c . Composing the reductions f and g then gives the result.

For $A, B \in [0, 1]^2$, $m(A, B)$ denotes the middle point of the segment AB . Given a triangle (A, B, C) , we define sequences $(A_n)_{n < \omega}$, $(B_n)_{n < \omega}$, $(A'_n)_{n < \omega}$, $(A''_n)_{n < \omega}$ and compact sets $(K_n)_{n < \omega}$, all depending on (A, B, C) , as follows:

- (i) $A_\omega = m(A, C), A_0 = m(B, A_\omega), A_{n+1} = m(A_n, A_\omega)$.
- (ii) $B_0 = m(B, C), B_{n+1} = m(A_n, C)$.
- (iii) $A'_n = m(A, A_n), A''_n = m(A'_n, A'_{n+1})$.
- (iv) $K_n = AA_n \cup A_n A''_n \cup A''_n A$.

(So, except for the common origin A , the sets K_n are disjoint triangles which accumulate on the segment AA_ω .)

Then define inductively, for $s \in \omega^{<\omega}$, triangles (A_s, B_s, C_s) by

$$A_\emptyset = (0, 1), B_\emptyset = (0, 0), C = C_\emptyset = (1, 0), \text{ and}$$

$$A_{s \frown n} = A_n(A_s, B_s, C), B_{s \frown n} = B_n(A_s, B_s, C), C_{s \frown n} = C_s = C.$$

Also let $A'_{s \frown n} = A'_n(A_s, B_s, C), A''_{s \frown n} = A''_n(A_s, B_s, C)$, and set $K_\emptyset = \emptyset$ and $K_{s \frown n} = K_n(A_s, B_s, C)$.

Finally, for $T \subseteq \omega^{<\omega}$, set $g(T) = K_T = \overline{\bigcup_{s \in T} K_s}$. This is clearly a Borel map, and we claim it works.

Note first that for $T \in X$, the set K_T consists of the union of the segments $A_s A_{s \frown n}, A_s A''_{s \frown n}, A''_{s \frown n} A_{s \frown n}$ for $s \frown n \in T$, of the point C , and of the segments $A_s C$ for $s \in T$ with valence ω in T . This is because this last set is compact containing all K_s for $s \in T$. If $s \in T$ has infinite valence, one can pick for each n with $s \frown n \in T$ an infinite branch α_n through T starting by $s \frown n$, and the paths $P_n = \bigcup_{k \geq |s|} A_{\alpha_n|k} A_{\alpha_n|k+1}$ in K_T accumulate to $A_s C$.

We now check that g is a reduction. First suppose that $h : S \rightarrow T$ is a rooted embedding, with S, T in X . We then send C to C , A_s to $A_{h(s)}$, A''_s to $A''_{h(s)}$ for $s \in S$, and the other points in K_S accordingly, by linear interpolation on the corresponding segments — the only point to note here is that if s has valence ω in S , so has $h(s)$ in T . This clearly defines a continuous embedding of K_S into K_T .

Conversely, suppose $h : K_S \rightarrow K_T$ is a continuous embedding. A path in K is a continuous one-to-one map $p : [0, 1] \rightarrow K$. It starts at A if $p(0) = A$. Two paths p_1 and p_2 starting at A are essentially disjoint if for some $r \in (0, 1]$, $p_1((0, r))$ and $p_2((0, r))$ are disjoint. Given $A \in K$, let $v_K(A)$ be the supremum of the cardinals of all families of pairwise essentially disjoint paths in K starting at A . One immediately gets that under a continuous embedding, valence can only increase.

Now for $S \in X$, one checks that for $s \in S$, $v_{K_S}(A_s) = 2.v_S(s)$ (with the convention $2.\omega = \aleph_0$). Also $v_{K_S}(C) = 2^{\aleph_0}$, as branches through S naturally give continuous paths starting at C , which are essentially disjoint if the branches are distinct. Finally all other points in K_S have valence 2.

From this, we get that $h(C) = C$, and hence for $s \in S$, $h(A_s) = A_{k(s)}$ for some map $k : \omega^{<\omega} \rightarrow \omega^{<\omega}$. We finally check that k is a tree-embedding. So let s and $s \frown n$ be consecutive vertices in S , and consider the circuit $K_{s \frown n}$ in K_S . It must be sent by h to some circuit L containing $A_{h(s)}$ and $A_{h(s \frown n)}$ but not C , and easily there are no such circuits in K_T unless $h(s)$ and $h(s \frown n)$ are consecutive vertices in T , as desired. \square

4.3. Separable Banach spaces. We now consider the case of separable Banach spaces. Any such space is linearly isometric to a closed subspace of $\mathcal{C}([0, 1])$ equipped with the sup norm, so we can consider the Borel subset \mathcal{B} of the standard Borel space of closed subsets of $\mathcal{C}([0, 1])$ consisting of all closed linear subspaces of $\mathcal{C}([0, 1])$, as our basic space of separable Banach spaces (equipped with some Polish topology giving its Borel structure).

Taking as morphisms the linear isometric embeddings, i.e. the linear norm-preserving maps, we can define on \mathcal{B} linear isometric embeddability \sqsubseteq^{li} , linear isometric bi-embeddability \cong^{li} and linear isometry \cong^i , by

$$X \sqsubseteq^{li} Y \iff \exists f : X \rightarrow Y \text{ (} f \text{ is linear and norm-preserving),}$$

$$X \cong^{li} Y : \iff X \sqsubseteq^{li} Y \text{ and } Y \sqsubseteq^{li} X$$

and

$$X \cong^i Y \iff \exists f : X \rightarrow Y \text{ (} f \text{ is linear, norm-preserving and onto).}$$

They are Σ_1^1 relations on \mathcal{B} .

The exact complexity of the equivalence relation \cong^{li} is not known. But two classical results in Banach space theory give upper and lower bounds. First, by Banach-Stone, one has that for K, L compact metrizable spaces,

$$K \cong^c L \leftrightarrow (\mathcal{C}(K), \| \cdot \|_\infty) \cong^{li} (\mathcal{C}(L), \| \cdot \|_\infty),$$

and this gives a Borel reduction of \cong^c to \cong^{li} . Also, by a theorem of Mazur, any isometry between Banach spaces X and Y sending 0 to 0 is linear, hence on \mathcal{B} the relations \cong^i and \cong^{li} coincide (up to the codings, for one should first isometrically embed $\mathcal{C}([0, 1])$ into U). So we get $\cong^c \leq_B \cong^{li} \leq_B \cong^i$ (so that $\cong^c \leq_B \cong^i$ too, as stated in section 4.2). It is not known if any of these inequalities is strict.

Now consider the ordering \sqsubseteq^{li} . It is not true anymore that an isometric embedding from a Banach space X into a Banach space Y sending 0 to 0 is necessarily linear. However, Godefroy and Kalton have very recently proved in [9] that for *separable* Banach spaces, \sqsubseteq^{li} still coincides with \sqsubseteq^i . So the following result can also be viewed as computing the complexity of $\sqsubseteq^i_{\mathcal{B}}$ and $\cong^i_{\mathcal{B}}$. This result is one of a sequence of results about the complexity of various comparison notions in the theory of separable Banach spaces which are due to the second author and have appeared elsewhere [27]. Most of them need a good deal of Banach space theory, and we will not discuss them here. But we could not end this paper without giving the reduction of \sqsubseteq_{CT} to \sqsubseteq^{li} , which was obtained by Rosendal prior to all results in this paper, and indeed was the main motivation for both developing a completeness

method in dimension 2, and also for replacing equivalence relations by quasi-orders in this development.

Theorem 4.6. *The ordering \sqsubseteq_{CT} is Borel reducible to \sqsubseteq^{li} , and even to its restriction to the Banach spaces isomorphic to c_0 . In particular, \sqsubseteq^{li} is a complete Σ_1^1 quasi-order, and \equiv^{li} a complete Σ_1^1 equivalence relation.*

Proof. Let c_0 be the usual Banach space of sequences of reals converging to 0 at infinity, with the sup-norm $\| \cdot \|_\infty$. We denote by $(e_n)_{n \in \mathbb{N}}$ the usual basis of c_0 , and will write interchangeably elements of c_0 as $(\alpha_n)_{n \in \mathbb{N}}$ or $\sum_n \alpha_n e_n$.

Given a combinatorial tree G with domain \mathbb{N} and geodesic distance d_G , we define a new norm $\| \cdot \|_G$ on c_0 by

$$\|(\alpha_n)\|_G = \sup\{|\alpha_i| + \frac{|\alpha_j|}{1 + d_G(i, j)} : i, j \in \mathbb{N}, i \neq j\}.$$

It is easy to check that $\| \cdot \|_G$ is indeed a norm, and that for any $(\alpha_n) \in c_0$ one has

$$\|(\alpha_n)\|_\infty \leq \|(\alpha_n)\|_G \leq \frac{3}{2}\|(\alpha_n)\|_\infty,$$

so that $\| \cdot \|_G$ is an equivalent norm on c_0 .

Moreover, by suitably embedding isometrically $(c_0, \| \cdot \|_G)$ into $\mathcal{C}([0, 1])$, one can easily make the map $G \mapsto (c_0, \| \cdot \|_G)$ Borel. So the proof will be finished if we can prove that this map is a reduction of \sqsubseteq_{CT} to \sqsubseteq^{li} , i.e. that for G_1, G_2 two combinatorial trees, one has

$$G_1 \sqsubseteq G_2 \iff (c_0, \| \cdot \|_{G_1}) \sqsubseteq^{li} (c_0, \| \cdot \|_{G_2}).$$

One direction is easy: If $f : \mathbb{N} \rightarrow \mathbb{N}$ is an embedding of G_1 into G_2 , f must preserve the geodesic distances. So, if we define $g : c_0 \rightarrow c_0$ by $g(\sum_n \alpha_n e_n) = \sum_n \alpha_n e_{f(n)}$, we get the linear isometric embedding we wanted.

For the other direction, fix a linear isometric embedding

$$g : (c_0, \| \cdot \|_{G_1}) \rightarrow (c_0, \| \cdot \|_{G_2}).$$

Claim. For every $p \in \mathbb{N}$, there are $i \in \mathbb{N}$ and $\epsilon \in \{-1, 1\}$ such that $g(e_p) = \epsilon e_i$.

Granting this claim, the end of the proof is easy: Write $g(p) = \epsilon_p e_{f(p)}$, with $f : \mathbb{N} \rightarrow \mathbb{N}$. By computing norms, we get for $p \neq q$

$$\|\epsilon_p e_p + \epsilon_q e_q\|_{G_1} = \|e_{f(p)} + e_{f(q)}\|_{G_2}$$

$$\text{and } \|\epsilon_p e_p + \epsilon_q e_q\|_{G_1} = 1 + \frac{1}{1 + d_{G_1}(p, q)},$$

$$\text{whereas } \|e_{f(p)} + e_{f(q)}\|_{G_2} = 2 \text{ if } f(p) = f(q),$$

$$\text{and } \|e_{f(p)} + e_{f(q)}\|_{G_2} = 1 + \frac{1}{1 + d_{G_2}(f(p), f(q))} \text{ otherwise.}$$

From these equalities, we get $f(p) \neq f(q)$ and $d_{G_1}(p, q) = d_{G_2}(f(p), f(q))$, so that f embeds G_1 into G_2 , as desired.

So it remains to prove the above claim. For $p \in \mathbb{N}$, set $g(e_p) = \sum_i \alpha_i^p e_i$. As $\|e_p\|_{G_1} = 1$, we have $\frac{2}{3} \leq \|g(e_p)\|_\infty \leq 1$.

If $\|g(e_p)\|_\infty = 1$, we are done, for it implies that for some i , $|\alpha_i^p| = 1$, and as $\|g(e_p)\|_{G_2} = 1$ too, necessarily $\alpha_j^p = 0$ for $j \neq i$.

So, arguing by contradiction, suppose there is some p with $\|g(e_p)\|_\infty = 1 - a$, for some a with $0 < a \leq \frac{1}{3}$.

Consider, for $q \neq p$, the vector $e_p + \frac{a}{3}e_q$. Its G_1 -norm is $1 + \frac{a}{3} \cdot \frac{1}{1+d_{G_1}(p,q)} > 1$. Now notice that the supremum, in the definition of the G -norms, is always attained. Applying this to the G_2 -norm of $g(e_p + \frac{a}{3}e_q)$, we get that there are i and j (depending on q) with

$$\|g(e_p + \frac{a}{3}e_q)\|_{G_2} = \left| \alpha_i^p + \frac{a}{3}\alpha_i^q \right| + \left| \alpha_j^p + \frac{a}{3}\alpha_j^q \right| \cdot \frac{1}{1+d_{G_2}(i,j)} > 1.$$

From this inequality, majorizing $|\alpha_i^q|$ and $|\alpha_j^q|$ by 1, minorizing $d_{G_2}(i,j)$ by 1, and majorizing either $|\alpha_i^p|$ or $|\alpha_j^p|$ by $1 - a$, we get $|\alpha_i^p| > \frac{1}{2}$ and $|\alpha_j^p| > a$. This implies in particular that when q varies over $\mathbb{N} - \{p\}$, the set of corresponding pairs (i, j) is finite.

It also implies that $|\alpha_i^p| > \frac{a}{3}|\alpha_i^q|$ and $|\alpha_j^p| > \frac{a}{3}|\alpha_j^q|$. So we get

$$1 + \frac{a}{3} \cdot \frac{1}{1+d_{G_1}(p,q)} = A_q + B_q,$$

with

$$A_q = |\alpha_i^p| + |\alpha_j^p| \cdot \frac{1}{1+d_{G_2}(i,j)}$$

(which depends on q only via (i, j)) and

$$B_q = \epsilon \cdot \frac{a}{3} \cdot |\alpha_i^q| + \epsilon' \cdot \frac{a}{3} \cdot \frac{1}{1+d_{G_2}(i,j)} \cdot |\alpha_j^q|,$$

where ϵ is the sign of $\alpha_i^p \alpha_i^q$ and ϵ' the sign of $\alpha_j^p \alpha_j^q$.

Note also that $A_q \leq \|g(e_p)\|_{G_2} = 1$, hence $B_q > 0$.

Now by the remark above, we can find $q_1 \neq q_2$ in $\mathbb{N} - \{p\}$ with the same i, j, ϵ and ϵ' , and say $d_{G_1}(p, q_1) \leq d_{G_1}(p, q_2)$. In particular $A_{q_1} = A_{q_2} = A$ (say). But then consider the vector $x = e_p + \frac{a}{3}e_{q_1} + \frac{a}{3}e_{q_2}$. Its G_1 -norm is $A + B_{q_1}$, whereas we have, by the same computation as before,

$$\begin{aligned} \|g(x)\|_{G_2} &\geq \left| \alpha_i^p + \frac{a}{3}\alpha_i^{q_1} + \frac{a}{3}\alpha_i^{q_2} \right| + \frac{1}{1+d_{G_2}(i,j)} \cdot \left| \alpha_j^p + \frac{a}{3}\alpha_j^{q_1} + \frac{a}{3}\alpha_j^{q_2} \right| \\ &= A + B_{q_1} + B_{q_2}, \end{aligned}$$

a contradiction which proves the claim and the theorem. □

Remark 4.7. The construction in the previous proof also gives that for combinatorial trees G_1, G_2 ,

$$G_1 \cong G_2 \leftrightarrow (c_0, \| \cdot \|_{G_1}) \cong^{li} (c_0, \| \cdot \|_{G_2}),$$

hence gives an alternative proof of $E_{S_\infty}^\infty \leq_B \cong^{li}$, which is very different from the proof based on Mazur's theorem, and gives the reduction to spaces isomorphic to c_0 .

In Banach space theory, there is of course another set of natural Σ_1^1 quasi-orders and equivalence relations, associated to linear isomorphisms instead of linear isometries. Very few things are known about the complexity of these relations. It is not known in particular if isomorphism, or isomorphic bi-embeddability, between separable Banach spaces are complete Σ_1^1 equivalence relations. The best known result in this direction is due to Rosendal [27]: The relation E_1 Borel reduces to isomorphism, so that this last relation (unlike isometry) is not Borel reducible to any G -equivalence, G some Polish group.

5. QUASI-ORDERS INDUCED BY GROUPS

In this section we will look at some quasi-order relations induced by the actions of monoids and groups. Suppose that G is a Polish group acting in a Borel manner on a standard Borel space X . Then the induced orbit equivalence relation is analytic, but cannot be analytic complete, as even E_1 cannot reduce to it. Now if instead G was not a group but only a monoid (i.e. a semigroup with an identity), then the corresponding relation on X , defined by $xR_X^G y \leftrightarrow \exists g \in G \ g.x = y$, is only a quasi-order though still analytic.

Theorem 5.1. *There is a Polish monoid G acting continuously on a Polish space X , such that R_X^G is a complete Σ_1^1 quasi-order.*

Proof. Let $G = S_\infty \times \mathcal{P}(\omega \times \omega)$ be equipped with the following product:

$$(x, A).(y, B) = (x.y, A \cup (x \times x)[B]),$$

where

$$(n, m) \in (x \times x)[B] \leftrightarrow (x^{-1}(n), x^{-1}(m)) \in B.$$

It is easily checked that this is indeed an associative product and that the element $(1_{S_\infty}, \emptyset)$ is a both-sided identity. Furthermore the product is continuous, so the monoid is Polish.

We let G act on $\mathcal{P}(\omega \times \omega)$ by $(x, A).B = A \cup (x \times x)[B]$ and note that it is indeed associative and also clearly continuous. Moreover $(1_{S_\infty}, \emptyset).A = A$ for any $A \subset \omega \times \omega$. So this is a continuous action of the Polish monoid G on the space $\mathcal{P}(\omega \times \omega)$, whereby $R_{\mathcal{P}(\omega \times \omega)}^G$ is Σ_1^1 .

We will now reduce \sqsubseteq^{CT} to R_X^G . For $T \in \mathcal{T}$ we let $T' = \{(2n, 2m) \mid (n, m) \in T\}$. Then as combinatorial trees are connected and without cycles one has for $T, S \in \mathcal{T}$

$$T \sqsubseteq S \Leftrightarrow \exists x \in S_\infty \ nT'm \rightarrow x(n)S'x(m).$$

We now show that for $T, S \in \mathcal{T}$, we have

$$T \sqsubseteq S \Leftrightarrow \exists g \in G \ g.T' = S'.$$

So suppose $T \sqsubseteq S$ and take $x \in S_\infty$ such that $nT'm \rightarrow x(n)S'x(m)$; then $(x, S').T' = S'$. Conversely if $(x, A).T' = A \cup (x \times x)[T'] = S'$, then obviously $(x \times x)[T'] \subset S'$, so $nT'm \rightarrow x(n)S'x(m)$ and $T \sqsubseteq S$. \square

We will now turn to some relations induced by the translation and conjugation actions of a group on itself. For G a Polish group we denote by $F(G)$ and $Sg(G)$ the standard Borel spaces of the closed subsets, respectively Polish (and therefore closed) subgroups, of G with the Effros Borel structure. G acts on these sets by pointwise translation and conjugation, and the sets are naturally ordered by a potentially closed relation, namely that of set inclusion. But one set can also be included in another up to a translation or up to a conjugation, which are easily seen to be quasi-orders. So for $K, L \in F(G)$ let $K \subseteq^{G,t} L \Leftrightarrow \exists g \in G \ g.K \subseteq L$ and $K \subseteq^{G,c} L \Leftrightarrow \exists g \in G \ g.K.g^{-1} \subseteq L$. We write $\subseteq_{F(G)}^{G,t}$ etc. to indicate the space on which the relation is defined.

Note that if G is a closed subgroup of H Polish, then the inclusion mapping from $F(G)$ to $F(H)$ is a reduction of $\subseteq_{F(G)}^{G,t}$ to $\subseteq_{F(H)}^{H,t}$.

Also if G is a topological factor-group of H , i.e. there is a continuous homomorphism of H surjectively onto G , then $\subseteq_{F(G)}^{G,t} \leq_B \subseteq_{F(H)}^{H,t}$ and $\subseteq_{Sg(G)}^{G,c} \leq_B \subseteq_{Sg(H)}^{H,c}$. This is easily checked noting first that in this case the factor mapping will in fact

be open (see [2], page 6) and that therefore the function taking the inverse image by the factor map will be Borel from $F(G)$ to $F(H)$.

Note also that if G is countable, then $F(G) = \mathcal{P}(G)$.

Proposition 5.2. *The quasi-orders $\subseteq_{Sg(F_2)}^{F_2,c}$ and $\subseteq_{\mathcal{P}(\mathbb{Z}^2)}^{\mathbb{Z}^2,t}$ are \mathbf{K}_σ complete.*

Proof. For $(A_n) \in \mathcal{P}(\omega)^\omega$ let

$$f((A_n)) = \{(m, n) \mid n \in A_m\} \cup \mathbb{Z} \times \{-1\}.$$

Then as $\mathbb{Z} \times \{-1\}$ is the only bi-infinite line in $f((A_n))$, if

$$(m, n) + \mathbb{Z} \times \{-1\} \subseteq f((A_n)),$$

then $n = 0$. So it is easily seen that f is a reduction of $\leq_{\mathbf{K}_\sigma}$ to $\subseteq_{\mathcal{P}(\mathbb{Z}^2)}^{\mathbb{Z}^2,t}$. On the other hand, since \mathbb{Z}^2 is countable, it is easily seen that the relation $\subseteq_{\mathcal{P}(\mathbb{Z}^2)}^{\mathbb{Z}^2,t}$ is indeed \mathbf{K}_σ .

For the other relation, we will use a construction by S. Gao in [7] to reduce $\subseteq_{[F_2]^\omega}^{F_2,t}$ to $\subseteq_{Sg(F_2)}^{F_2,c}$. Note that this will suffice to show that $\subseteq_{Sg(F_2)}^{F_2,c}$ reduces $\leq_{\mathbf{K}_\sigma}$. \mathbb{Z}^2 is a topological factor-group of F_2 , so the above reduction of $\leq_{\mathbf{K}_\sigma}$ to $\subseteq_{\mathcal{P}(\mathbb{Z}^2)}^{\mathbb{Z}^2,t}$ composed with the canonical reduction of $\subseteq_{\mathcal{P}(\mathbb{Z}^2)}^{\mathbb{Z}^2,t}$ to $\subseteq_{\mathcal{P}(F_2)}^{F_2,t}$ will be a reduction of $\leq_{\mathbf{K}_\sigma}$ to $\subseteq_{[F_2]^\omega}^{F_2,t}$.

So let F_3 be the free group generated by a, b, c and let F_2 be the subgroup generated by a, b . For an infinite subset A of F_2 we let $K(A) = \langle xcx^{-1} \mid x \in A \rangle$ which is a subgroup of F_3 . Then $A \mapsto K(A)$ is Borel from $[F_2]^\omega$ to $Sg(F_3)$. Furthermore if $A, B \in [F_2]^\omega$ and $f \in F_2$ is such that $f.A \subseteq B$, then clearly $f.K(A).f^{-1} \subseteq K(B)$. Conversely it follows from the proof of lemma 2 in [7] that if $g.K(A).g^{-1} \subseteq K(B)$ for some $g \in F_3$, then there is an $f \in F_2$ with $f.A \subseteq B$. So this shows $\subseteq_{[F_2]^\omega}^{F_2,t}$ to reduce to $\subseteq_{Sg(F_3)}^{F_3,c}$. But it is noted by Thomas and Velickovic in [30] that F_3 embeds as a malnormal subgroup G_0 in F_2 (N is a malnormal subgroup of H iff $h.N.h^{-1} \cap N = \{1\}$ for all $h \in H \setminus N$). So in particular for $M, N \in Sg(F_3) = Sg(G_0)$ there is a $g \in F_3$ with $g.N.g^{-1} \subseteq M$ iff there is an $h \in F_2$ with $h.N.h^{-1} \subseteq M$. So this shows the inclusion mapping from $F_3 = G_0$ to F_2 to be a reduction from $\subseteq_{Sg(F_3)}^{F_3,c}$ to $\subseteq_{Sg(F_2)}^{F_2,c}$.

Again as F_2 is countable and $Sg(F_2)$ is compact, $\subseteq_{Sg(F_2)}^{F_2,c}$ is \mathbf{K}_σ . □

We note that what is really shown here is that $\subseteq_{\mathcal{P}(\mathbb{Z}^2)}^{\mathbb{Z},t}$ is \mathbf{K}_σ complete, where \mathbb{Z} is seen as the subgroup $\mathbb{Z} \times \{0\}$ of \mathbb{Z}^2 . This is also more or less clear from the definition of $\leq_{\mathbf{K}_\sigma}$. On the other hand $\subseteq_{\mathcal{P}(\mathbb{Z})}^{\mathbb{Z},t}$ is far from being \mathbf{K}_σ complete, because the induced equivalence relation turns out to be hyperfinite, and hence $\subseteq_{\mathcal{P}(\mathbb{Z})}^{\mathbb{Z},t}$ must be rather simple.

Proposition 5.3. *The quasi-orders $\subseteq_{F(S_\infty)}^{S_\infty,t}$ and $\subseteq_{F(S_\infty)}^{S_\infty,c}$ are Σ_1^1 complete.*

Proof. Define for $T \in \mathcal{T}$ a closed subset of S_∞ by $x \in B_T \leftrightarrow x(0)T'x(1)$, where T' is defined as in the proof of Theorem 5.1. Clearly $T \mapsto B_T$ is Borel. So suppose that $T, S \in \mathcal{T}$ with $T \sqsubseteq S$; then there is an $x \in S_\infty$ such that $nT'm \rightarrow x(n)S'x(m)$. So if $y \in B_T$, then $x.y \in B_S$, whereby $x.B_T \subseteq B_S$.

On the other hand if for some $x \in S_\infty$, $x.B_T \subseteq B_S$ and $nT'm$, then for some $y \in S_\infty$ with $y(0) = n$ and $y(1) = m$, $y \in B_T$ so $x.y \in B_S$ and therefore $x.y(0)S'y.y(1)$. Thus $x(n)S'x(m)$, which is enough to guarantee that $T \sqsubseteq S$. This shows \sqsubseteq^{CT} reduces to $\sqsubseteq_{F(S_\infty)}^{S_\infty, t}$, which finishes the proof of the first part of the proposition.

For the second part, for $n \neq m$ define $x_m^n \in S_\infty$ by $x_m^n(k) = k$ for $k \neq n, m$ and $x_m^n(n) = m$, $x_m^n(m) = n$. Then for $T \in \mathcal{T}$ let $A_T = \{x_m^n \mid nT'm\} \cup \{1_{S_\infty}\}$, which is a countable closed subset of S_∞ . We claim that for $T, S \in \mathcal{T}$ we have $T \sqsubseteq S$ iff $\exists y \in S_\infty y.A_T.y^{-1}$. To see this it is enough to note that $y.x_m^n.y^{-1} = x_{y(m)}^{y(n)}$, because then

$$y.A_T.y^{-1} = \{x_{y(m)}^{y(n)} \mid nT'm\} \cup \{1_{S_\infty}\} \subseteq A_S$$

iff $nT'm \rightarrow y(n)S'y(m)$. Again this shows \sqsubseteq^{CT} to reduce to $\sqsubseteq_{F(S_\infty)}^{S_\infty, c}$, which finishes the proof. \square

Apart from being a somewhat natural generalization of the orbit equivalence relation of a Polish group acting on its subsets by translation and conjugation, the quasi-orders above could hopefully prove useful in the classification of the relation of embeddability between groups. In the case of isomorphism, it is known by results of Thomas and Velickovic [30] that isomorphism between finitely generated groups is a complete essentially countable equivalence relation, and it would be natural to conjecture that embedding between finitely generated groups would be \mathbf{K}_σ complete as a quasi-order.

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