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AFFINE PSEUDO-PLANES AND CANCELLATION PROBLEM

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Abstract. We define affine pseudo-planes as one class of $\mathbb{Q}$-homology planes. It is shown that there exists an infinite-dimensional family of non-isomorphic affine pseudo-planes which become isomorphic to each other by taking products with the affine line $\mathbb{A}^1$. Moreover, we show that there exists an infinite-dimensional family of the universal coverings of affine pseudo-planes with a cyclic group acting as the Galois group, which have the equivariant non-cancellation property. Our family contains the surfaces without the cancellation property, due to Danielewski-Fieseler and tom Dieck.

1. Introduction

Let G be an algebraic group defined over the complex number field $\mathbb{C}$. We shall consider the following:

Equivariant Cancellation Problem. *Let X and Y be smooth affine varieties with algebraic G-actions. If $X \times W$ is G-isomorphic to $Y \times W$ for a G-module W, is X then G-isomorphic to Y?*

If we forget the actions, the problem is simply called the *Cancellation Problem.* When $Y \cong \mathbb{A}^2$, the cancellation holds by the results of Miyanishi-Sugie [18] and Fujita [7]. However, the Cancellation Problem for $Y \cong \mathbb{A}^n$ remains open if $n \geq 3$.

In the Equivariant Cancellation Problem, the intriguing case is when Y is isomorphic to a G-module, i.e., an affine space with a linear G-action. In this case, it is known that the answer is negative. In fact, for a reductive algebraic group G, there exist affine spaces with non-linearizable G-actions which are realized as the total spaces of non-trivial algebraic G-vector bundles over G-modules (Schwarz [19], see also references in [10]). By Bass-Haboush [2], every G-vector bundle over a G-module is stably trivial, namely, it becomes isomorphic to a trivial G-vector bundle by adding a certain trivial G-vector bundle. Hence non-trivial G-vector bundles over G-modules, whose total spaces have non-linearizable G-actions, give rise to counterexamples to the Equivariant Cancellation Problem with G-modules Y (cf. Masuda-Miyanishi [12]). All counterexamples to the Equivariant Cancellation Problem that we have so far for reductive algebraic groups G and G-modules Y are derived from non-trivial G-vector bundles over G-modules.

Next, consider the case where Y is not isomorphic to a G-module nor an affine space without G-action. Then there are well-known counterexamples due to Daniel-

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ewski [4], Fieseler [6], tom Dieck [5], and Wilkens [22]. In tom Dieck [5], it is shown that the smooth affine surfaces $\widetilde{V}(d,r)$ in $\mathbb{A}^3$ defined by $x^r z + y^d = 1$ for $r \geq 1$ and $d \geq 2$ have non-cancellation property; namely, $\widetilde{V}(d,r) \not\cong \widetilde{V}(d,s)$ for $r \neq s$, but $\widetilde{V}(d,r) \times \mathbb{A}^1 \cong \widetilde{V}(d,s) \times \mathbb{A}^1$ for any r and s. The surface $\widetilde{V}(d,r)$ has an action of the multiplicative group $G_m = \mathbb{C}^*$ and tom Dieck showed that the surfaces $\widetilde{V}(d,r)$ have in fact equivariant non-cancellation property. There exists an action of the additive group $G_a = \mathbb{C}^+$ on $\widetilde{V}(d,r)$ as well, and hence there is an $\mathbb{A}^1$-fibration $\widetilde{\rho} : \widetilde{V}(d,r) \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$, which has a unique reducible fiber $\widetilde{\rho}^{-1}(0)$. We call a smooth affine surface X an *affine pseudo-plane* if X has an $\mathbb{A}^1$-fibration $\rho : X \to \mathbb{A}^1$ such that every fiber of ρ is irreducible and there is only one multiple fiber. Then the surfaces $\widetilde{V}(d,r)$ are the universal coverings of affine pseudo-planes $V(d,r)$ with Galois group isomorphic to $\mathbb{Z}/d\mathbb{Z}$. The surfaces $V(d,r)$ constructed by tom Dieck for $r \geq 2$ are characterized as affine pseudo-planes with non-trivial $\mathbb{C}^*$-actions. See section 2 for definitions and relevant results.

In the present article, we observe various properties of affine pseudo-planes and their universal coverings. We shall show that affine pseudo-planes can be constructed from the Hirzebruch surfaces, as tom Dieck's surfaces $V(d,r)$ are constructed from the Hirzebruch surfaces. We also show that the universal coverings of affine pseudo-planes are isomorphic to the hypersurfaces in $\mathbb{A}^3$, and give their defining equations explicitly. Using the results on the properties of affine pseudo-planes and their universal coverings, we show that if $d \geq 2$, there exists an infinite-dimensional family of non-isomorphic smooth affine surfaces with actions of $G_a \times \mathbb{Z}/d\mathbb{Z}$, whose members are mutually equivariantly non-cancellative. The family consists of the universal coverings of affine pseudo-planes and includes the examples due to Danielewski-Fieseler and tom Dieck. By taking their quotients by $\mathbb{Z}/d\mathbb{Z}$, we also obtain an infinite-dimensional family of non-isomorphic affine pseudo-planes without cancellation property. In the last section, we show that there exist families of non-isomorphic affine G-varieties without equivariant cancellation property. The families are of infinite dimension and are derived from G-equivariant $\mathbb{A}^1$-fibrations and not G-vector bundles.

2. Affine pseudo-planes with unique $\mathbb{A}^1$-fibrations

An algebro-geometric characterization of the affine plane $\mathbb{A}^2$ is stated as follows: the affine plane $\mathbb{A}^2$ is an affine surface such that its coordinate ring R is factorial, $R^* = \mathbb{C}^*$, and there exists an $\mathbb{A}^1$-fibration with base curve isomorphic to $\mathbb{A}^1$. There are many related results on smooth affine surfaces with $\mathbb{A}^1$-fibrations (cf. Miyanishi [14]). Here we recall the following.

Lemma 2.1. *Let X be a smooth affine surface with an $\mathbb{A}^1$-fibration $\rho : X \to C \cong \mathbb{A}^1$. Suppose that every fiber of ρ is irreducible. Then* $\operatorname{Pic}(X) \cong \prod_{P \in C} \mathbb{Z}/d_P\mathbb{Z}$, *where d_P is the multiplicity of the fiber $\rho^{-1}(P)$. In particular, if there is only one multiple fiber dF with the multiplicity d and $F \cong \mathbb{A}^1$, then* $\operatorname{Pic}(X) \cong \mathbb{Z}/d\mathbb{Z}$.

Affine surfaces satisfying the assumptions in Lemma 2.1 are $\mathbb{Q}$-homology planes, and there are many such surfaces. We define *affine pseudo-plane* as one class of such affine surfaces.

Definition 2.1. A smooth affine surface X is an *affine pseudo-plane* if X satisfies the following conditions.

(1) X has an $\mathbb{A}^1$-fibration $\rho : X \to C$, where $C \cong \mathbb{A}^1$.
(2) The $\mathbb{A}^1$-fibration ρ has a unique multiple fiber dF with multiplicity $d \geq 2$ and $F \cong \mathbb{A}^1$, and every other fiber is isomorphic to $\mathbb{A}^1$.

We say that X has *type* (d, n, r) if X further satisfies the next condition:

(3) X has a smooth compactification (V, D) such that the boundary divisor $D = V - X$ has simple normal crossings and the dual graph of D is as given in Figure 1 below, where $n \geq 1$ and $r \geq 1$. Furthermore, $\overline{F}$ is the closure of F in V, and S' is the unique cross-section contained in D.

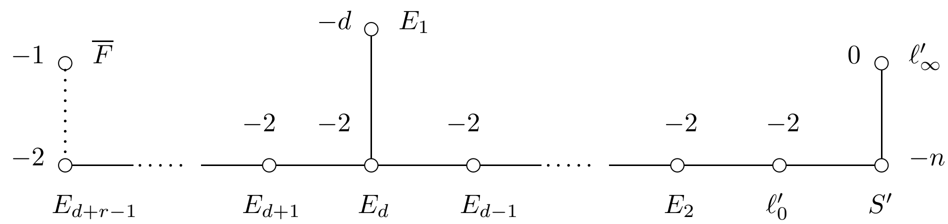

FIGURE 1

If X has a smooth compactification (V, D) with the dual graph as in Figure 1 and $({S'}^2) = -n$ for $n > 1$, then we can make $({S'}^2) = -1$. In fact, choose a point P on the fiber ℓ'_∞ and blow up the point P to obtain a (-1) curve E. Then the proper transform L of ℓ'_∞ is a (-1) curve. Contract L to obtain the same figure as before with ℓ'_∞ replaced by the image of E and with $({S'}^2) = -n+1$ if $P \neq S' \cap \ell'_\infty$, and $-n-1$ if $P = S' \cap \ell'_\infty$. This operation is called the *elementary transformation* with center P. After several elementary transformations, we obtain $({S'}^2) = -1$. Meanwhile, we have to consider the case $({S'}^2) < -1$ as well, e.g., in the proof of Theorem 2.3. We call an affine pseudo-plane of type $(d, 1, r)$ simply an affine pseudo-plane of type (d, r).

Lemma 2.2. *Let X be an affine pseudo-plane of type (d, r). Then X is isomorphic to the complement of $M_0 \cup C_d$ if $r < d$, and $M_1 \cup C_d$ if $r \geq d$ in the Hirzebruch surface Σ_n with $n = |r - d|$, where M_0 is the minimal section and where C_d and M_1 are specified as follows. In the case $r < d$, C_d is an irreducible member of the linear system $|M_0 + d\ell_0|$ which meets M_0 in the point $M_0 \cap \ell_0$ with multiplicity r, where ℓ_0 is a fiber of the $\mathbb{P}^1$-fibration of Σ_n. In the case $r \geq d$, M_1 is a section of Σ_n with $({M_1}^2) = n$, and C_d is an irreducible member of the linear system $|M_1 + d\ell_0|$ which meets M_1 in the point $M_1 \cap \ell_0$ with multiplicity r. In both cases, $\ell_0 \cap X = \overline{F} \cap X$.*

Proof. Contract $S', \ell'_0, E_2, \ldots, E_d, E_{d+1}, \ldots, E_{d+r-1}$ in this order. Then the resulting surface is the Hirzebruch surface Σ_n with $n = |r - d|$ and the image of ℓ'_∞ provides C_d. The image of E_1 provides M_0 or M_1 according to whether $r - d < 0$ or $r - d \geq 0$, while the image of $\overline{F}$ is the fiber ℓ_0. □

An affine pseudo-plane X of type (d, n, r) with $r \geq 2$ has the distinguished property as stated in the following theorem. An $\mathbb{A}^1$-fibration $\rho : X \to C \cong \mathbb{A}^1$ is called unique; if there is another $\mathbb{A}^1$-fibration $\sigma : X \to B \cong \mathbb{A}^1$, then $\sigma = \tau \circ \rho$ for an automorphism τ of $\mathbb{A}^1$. The next theorem follows from a theorem of Bertin [3], but we prefer to give a direct proof.

Theorem 2.3. *Let X be an affine pseudo-plane of type (d, n, r) with $r \geq 2$. Then ρ is a unique $\mathbb{A}^1$-fibration on X.*

Proof. Suppose that there exists another $\mathbb{A}^1$-fibration $\sigma : X \to B$ which is different from the fixed $\mathbb{A}^1$-fibration $\rho : X \to C$. Then $B \cong \mathbb{A}^1$ and every fiber of σ is isomorphic to $\mathbb{A}^1$ if taken with the reduced structure. Let M be a linear pencil on V spanned by the closures of general fibers of σ, where the notations V, D, etc. are the same as in Definition 2.1. Then a general member of M meets the curve ℓ'_∞, for otherwise the $\mathbb{A}^1$-fibrations ρ and σ coincide with each other. Suppose that M has no base points. Then the curve ℓ'_∞ is a cross-section of M and $S' + \ell'_0 + E_1 + \cdots + E_{d+r-1}$ supports a reducible fiber of M. Then $r = d = 1$. Since $d \geq 2$ by the hypothesis, this case does not take place. Hence M has a base point, say P, on ℓ'_∞. Let $Q := \ell'_\infty \cap S'$. We consider two cases separately.

Case $P \neq Q$. Then $\ell'_\infty + S' + \ell'_0 + E_1 + E_2 + \cdots + E_{d+r-1}$ will support a reducible member G_0 of the pencil M. Let $s = (\overline{F} \cdot G)$, where G is a general member of M. By comparing the intersection numbers of G with two fibers of ρ, ℓ'_∞ and the one containing $d\overline{F}$, it follows that $(\ell'_\infty \cdot G) = ds$. Let μ be the multiplicity of G at P, where P is a one-place point of G. We have $ds \geq \mu$. Consider first the case $n = 1$. The contraction of $S', \ell'_0, E_2, \ldots, E_{d-1}$ makes E_d a (-1) curve meeting three components $\ell'_\infty, E_1, E_{d+1}$, and this is impossible. So, suppose $n \geq 2$. The elimination of the base points of M will be achieved by blowing up the point P and its infinitely near points. After the elimination of the base points of M, the proper transform $\widetilde{M}$ gives rise to a $\mathbb{P}^1$-fibration, and the proper transform of ℓ'_∞ is a unique (-1) component. If $gs > \mu$, then the point P and its infinitely near point of the first order lying on ℓ'_∞ are blown up. Hence the proper transform of ℓ'_∞ is not a (-1) curve. This implies that $ds = \mu$. Let E be the exceptional curve arising from the blowing-up of P and let M' be the proper transform of M. Then E is contained in the member G'_0 of M' corresponding to G_0 of M. In fact, we otherwise have $ds = 1$, which is impossible because $d \geq 2$. Now contract ℓ'_∞ and take the image of E instead of ℓ'_∞. Then we have the same dual graph as Figure 1 with $({S'}^2) = -(n-1)$. By repeating this argument, we reach a contradiction.

Case $P = Q$. As above, let G_0 be a reducible member of M containing $S' + \ell'_0 + E_1 + E_2 + \cdots + E_{d+r-1}$. If ℓ'_∞ is not contained in G_0, the elimination of the base points of M, which is achieved by blowing up the point $P = Q$ and its infinitely near points, yields a $\mathbb{P}^1$-fibration in which the fiber corresponding to G_0 is a reducible fiber not containing any (-1) curve. This is a contradiction. Hence ℓ'_∞ is contained in G_0. So, G_0 is supported by $S' + \ell'_0 + E_1 + E_2 + \cdots + E_{d+r-1} + \ell'_\infty$. Now apply the elementary transformation with center P. Then we obtain the same dual graph as Figure 1, where $({S'}^2) = -(n+1)$ and ℓ'_∞ is replaced by the image of E. After repeating the elementary transformations several times, we are reduced to the case where $P \neq Q$. So, we reach a contradiction in the present case as well. □

Since the existence of an $\mathbb{A}^1$-fibration with affine base is equivalent to the existence of an action of the additive group G_a, it follows from Theorem 2.3 that there is an essentially unique G_a-action on an affine pseudo-plane of type (d, n, r) for $r \geq 2$. On the other hand, an affine pseudo-plane of type $(d, n, 1)$ has two algebraically independent G_a-actions, namely, it has trivial Makar-Limanov invariant (cf. [11]). This is a consequence of a more general result in Gurjar-Miyanishi [9, Theorem 3.1] which is stated below. We only note that the boundary divisor D in the case of type $(d, n, 1)$ is a linear chain for the normal compactification in Definition 2.1 and that $\pi_{1,\infty}(X)$ is then a finite cyclic group of order d^2.

Theorem 2.4. *Let X be a smooth affine surface. Then the Makar-Linamov invariant $\mathrm{ML}(X)$ is trivial if and only if X has a minimal normal compactification V such that the dual graph of $D := V - X$ is a linear chain of rational curves and $\pi_{1,\infty}(X)$ is a finite group.*

Lemma 2.2 gives rise to a construction of affine pseudo-planes from the Hirzebruch surfaces. We denote by $X(d, r)$ an affine pseudo-plane of type (d, r) constructed from the Hirzebruch surface as in Lemma 2.2. Some partial cases of affine pseudo-planes were observed in tom Dieck [5] as examples of affine surfaces without cancellation property. We shall recall and generalize a little bit his construction. Write $\Sigma_n = \mathrm{Proj}\,(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1})$ as the quotient of $(\mathbb{A}^2 \setminus \{0\}) \times \mathbb{P}^1$ under the relation

$$(z_0, z_1), [w_0, w_1] \sim (\nu z_0, \nu z_1), [\nu^n w_0, w_1]$$

for $\nu \in G_m = \mathbb{C}^*$. The projection $\{(z_0, z_1), [w_0, w_1]\} \mapsto [z_0, z_1]$ induces a $\mathbb{P}^1$-fibration $p_n : \Sigma_n \to \mathbb{P}^1$. In the above definition by quotient and in what follows, the integer n could be negative. If $n \geq 0$, the curve $w_0 = 0$ (resp. $w_1 = 0$) is a section M_1 of p_n with $({M_1}^2) = n$ (resp. the minimal section M_0 with $({M_0}^2) = -n$). Meanwhile, if $n < 0$, then the curve $w_0 = 0$ (resp. $w_1 = 0$) is the minimal section M_0 (resp. a section M_1 with $({M_1}^2) = |n|$) of $\Sigma_{|n|}$. Let $d \geq 2$ and $r = d + n \geq 1$. With the notations of Lemma 2.2, we assume that the fiber ℓ_0 is defined by $z_0 = 0$. Let $w = w_0/w_1$. Then $\{z_0/z_1, w/z_1^n\}$ is a system of local coordinates at the point $M_1 \cap \ell_0$ (resp. $M_0 \cap \ell_0$) if $n \geq 0$ (resp. $n < 0$). Let Λ be a linear subsystem of $|M_1 + d\ell_0|$ if $n \geq 0$ (resp. $|M_0 + d\ell_0|$ if $n < 0$), consisting of members which meet the curve M_1 (resp. M_0) at the point $M_1 \cap \ell_0$ (resp. $M_0 \cap \ell_0$) with multiplicity r if $n \geq 0$ (resp. $n < 0$). Then any member of Λ is defined by an equation

$$\frac{w}{z_1^n}\left\{a_0 + a_1\left(\frac{z_0}{z_1}\right) + \cdots + a_{d-1}\left(\frac{z_0}{z_1}\right)^{d-1} + a_d\left(\frac{z_0}{z_1}\right)^{d}\right\} + a_{d+1}\left(\frac{z_0}{z_1}\right)^{r} = 0$$

or equivalently by

$$(1) \qquad w_0\left(a_0 z_1^d + a_1 z_0 z_1^{d-1} + \cdots + a_{d-1} z_0^{d-1} z_1 + a_d z_0^d\right) + a_{d+1} z_0^r w_1 = 0$$

for $(a_0, a_1, \ldots, a_{d+1}) \in \mathbb{P}^{d+1}$. In fact, it is readily computed that $\dim \Lambda = d + 1$. So, the curve C_d is defined by such an equation with $a_0 \neq 0$ and $a_{d+1} \neq 0$. Hence it follows that

$$X(d, r) = \Sigma_{|n|} \setminus (\{w_0 = 0\} \cup C_d)$$

where $n = r - d$ and C_d is the curve defined by (1) with $a_0 \neq 0$ and $a_{d+1} \neq 0$. We shall verify the following result.

Lemma 2.5. *Let $r \geq 2$ and let $X = X(d, r)$ be an affine pseudo-plane defined as above. Let $\sigma : G_m \times X \to X$ be a non-trivial action of the algebraic torus $G_m = \mathbb{C}^*$. Then the following assertions hold true:*

(1) *The action σ induces an action $\sigma : G_m \times \Sigma_{|n|} \to \Sigma_{|n|}$ such that ${}^{\sigma(\mu)}M_i \subseteq M_i$ for $i = 0, 1$, ${}^{\sigma(\mu)}C_d \subseteq C_d$ and ${}^{\sigma(\mu)}\ell_0 \sim \ell_0$, where ${}^{\sigma(\mu)}M_i$ denotes the image of M_i under the action of $\mu \in \mathbb{C}^*$, etc.*

(2) *The curve C_d is defined by an equation*

$$z_1^d w_0 + a z_0^r w_1 = 0 \quad \textit{for} \quad a \in \mathbb{C}^*.$$

Proof. (1) We prove only the case $n = r-d \geq 0$. The proof of the case $n < 0$ is done in the same manner. Let $\rho : X \to C \cong \mathbb{A}^1$ be the unique $\mathbb{A}^1$-fibration (cf. Theorem 2.3). Then the fibers of ρ are permuted by the action σ. Hence σ extends to the cross-section S' and sends S' into itself. Let W be a G_m-equivariant smooth normal compactification of X whose existence is guaranteed by [21]. We may assume that $W \setminus X$ contains the cross-section S'. Let F_0 and F_∞ be two fibers of the $\mathbb{P}^1$-fibration $p : W \to \mathbb{P}^1$ whose supports partly or totally lie outside of X, where F_0 contains the multiple fiber of ρ. We may assume that all (-1) components of F_0 and F_∞ are fixed componentwise under the action σ. Then we may assume that F_∞ is irreducible and F_0 minus the component $\overline{F}$ contains no (-1) components, where $\overline{F} \cap X$ gives rise to the multiple fiber of ρ. Then we may assume that $W \setminus X$ has the dual graph as in Definition 2.1. So, the action σ induces a G_m-action on Σ_n such that ${}^{\sigma(\mu)}(M_1) \subseteq M_1, {}^{\sigma(\mu)}(C_d) \subseteq C_d$ and ${}^{\sigma(\mu)}(\ell_0) \subseteq \ell_0$ because M_1, C_d, ℓ_0 are the images on Σ_n of the components $E_1, \ell'_\infty, \overline{F}$, respectively. The minimal section M_0 is stable under the σ-action because the minimal section is unique on Σ_n.

(2) The G_m-action σ on $\Sigma_{|n|}$ is given as follows in terms of the coordinates:

$$\mu \cdot ((z_0, z_1), [w_0, w_1]) = \big((\mu^\alpha z_0, \mu^\beta z_1), [\mu^\gamma w_0, \mu^\delta w_1]\big)$$

for $\mu \in \mathbb{C}^*$. Since C_d is stable under the σ-action, the defining equation (1) must be semi-invariant. Note that $a_0 a_{d+1} \neq 0$ because C_d is irreducible. Hence we obtain $\alpha r + \delta = \beta d + \gamma$. Suppose that $(a_1, \ldots, a_d) \neq (0, \ldots, 0)$. Then we have an additional relation $\alpha i + \beta(d-i) + \gamma = \beta d + \gamma$ for some $1 \leq i \leq d$. The last relation implies $\alpha = \beta$. So, the first relation gives $\gamma = \alpha n + \delta$. Then we have

$$\begin{aligned} \mu \cdot ((z_0, z_1), [w_0, w_1]) &= \big((\mu^\alpha z_0, \mu^\beta z_1), [\mu^\gamma w_0, \mu^\delta w_1]\big) \\ &= \big((\mu^\alpha z_0, \mu^\beta z_1), [\mu^{\alpha n+\delta} w_0, \mu^\delta w_1]\big) \\ &\sim \big((z_0, z_1), [\mu^\delta w_0, \mu^\delta w_1]\big) \\ &= ((z_0, z_1), [w_0, w_1]) . \end{aligned}$$

Hence the σ-action is trivial. This proves the second assertion. □

Let $V(d, r)$ be the affine pseudo-plane defined by

$$V(d, r) = \Sigma_{|n|} \setminus (\{w_0 = 0\} \cup \{z_1^d w_0 + z_0^r w_1 = 0\})$$

where $n = r - d$. Then there exists a G_m-action on $V(d, r)$ defined by

$$\mu \cdot ((z_0, z_1), [w_0, w_1]) = \big((\mu z_0, z_1), [w_0, \mu^{-r} w_1]\big)$$

for $\mu \in \mathbb{C}^*$. For $r \geq 2$, one can show that any G_m-action on $V(d, r)$ is reduced to the G_m-action specified as above. In fact, with the notation in the proof of Lemma 2.5(2)

$$\begin{aligned} \mu \cdot ((z_0, z_1), [w_0, w_1]) &= \big((\mu^\alpha z_0, \mu^\beta z_1), [\mu^\gamma w_0, \mu^\delta w_1]\big) \\ &= \big((\mu^\alpha z_0, \mu^\beta z_1), [\mu^{\alpha r - \beta d + \delta} w_0, \mu^\delta w_1]\big) \\ &= \Big((\mu^\alpha z_0, \mu^\beta z_1), [\mu^{\beta n + (\alpha-\beta) r + \delta} w_0, \mu^\delta w_1]\Big) \\ &\sim \Big((\mu^{\alpha-\beta} z_0, z_1), [\mu^{r(\alpha-\beta)} w_0, w_1]\Big) \\ &= \Big((\mu^{\alpha-\beta} z_0, z_1), [w_0, \mu^{-r(\alpha-\beta)} w_1]\Big) . \end{aligned}$$

We shall consider the universal covering $\widetilde{X}(d, r)$ of an affine pseudo-plane $X(d, r)$.

Lemma 2.6. *The following assertions hold true:*

(1) *The universal covering* $\widetilde{X}(d,r)$ *is isomorphic to an affine hypersurface in* $\mathbb{A}^3 = \operatorname{Spec}\mathbb{C}[x,y,z]$ *defined by an equation*

$$x^r z + (y^d + a_1 x y^{d-1} + \cdots + a_{d-1}x^{d-1}y + a_d x^d) = 1. \tag{2}$$

The Galois group of the covering $\widetilde{X}(d,r) \to X(d,r)$ *is a cyclic group* $H(d) := \mathbb{Z}/d\mathbb{Z}$ *of order* d *and acts as*

$$\lambda \cdot (x,y,z) = (\lambda x, \lambda y, \lambda^{-r} z)$$

for $\lambda \in H(d)$.

(2) *The projection* $(x,y,z) \mapsto x$ *induces an* $\mathbb{A}^1$*-fibration* $\widetilde{\rho} : \widetilde{X}(d,r) \to \mathbb{A}^1$ *such that every fiber except for* $\widetilde{\rho}^{-1}(0)$ *is smooth and the fiber* $\widetilde{\rho}^{-1}(0)$ *consists of* d *copies of* $\mathbb{A}^1$ *which are reduced.*

(3) *There is a* G_a*-action on* $\widetilde{X}(d,r)$ *defined by*

$$\begin{aligned} &c \cdot (x,y,z) \\ &= (x, y + cx^r, z - x^{-r}\{((y+cx^r)^d + a_1 x(y+cx^r)^{d-1} + \cdots + a_d x^d) \\ &\qquad\qquad -(y^d + a_1 x y^{d-1} + \cdots + a_{d-1}x^{d-1}y + a_d x^d)\}), \end{aligned}$$

where $c \in G_a = \mathbb{C}$.

(4) *The* $\mathbb{A}^1$*-fibration* $\widetilde{\rho} : \widetilde{X}(d,r) \to \mathbb{A}^1$ *is unique for* $r \geq 2$.

(5) *Let* ω *be a* d*-th root of unity. Then there exist uniquely determined polynomials* $p_\omega(x), q_\omega(x) \in \mathbb{C}[x]$ *satisfying the following conditions:*
   (i) $\deg p_\omega(x) \leq r-1$.
   (ii) $p_\omega(0) = \omega$.
   (iii) $x^r q_\omega(x) + p_\omega(x)^d + a_1 x p_\omega(x)^{d-1} + \cdots + a_{d-1}x^{d-1}p_\omega(x) + a_d x^d = 1$.
   (iv) $p_{\lambda\omega}(\lambda x) = \lambda p_\omega(x)$, $q_{\lambda\omega}(\lambda x) = \lambda^{-r} q_\omega(x)$ *for any* d*-th root* λ *of unity.*

   By making use of these polynomials, we define the morphism

$$\varphi_\omega : \mathbb{A}^2 \cong \mathbb{A}^1 \times G_a \to \widetilde{X}(d,r), \quad (x,c) \mapsto c \cdot (x, p_\omega(x), q_\omega(x))$$

which is an open immersion onto an open set U_ω *which is the complement of* $\coprod_{\lambda \neq \omega} G_a \cdot (0,\lambda,0)$. *The inverse morphism* φ_ω^{-1} *on* U_ω *is defined by*

$$(x,y,z) \mapsto \begin{cases} (x, \dfrac{y - p_\omega(x)}{x^r}) & \textit{if } x \neq 0, \\[2ex] (0, \dfrac{-z + q_\omega(0)}{d\omega^{-1}}) & \textit{if } x = 0. \end{cases}$$

(6) $\widetilde{X}(d,r)$ *is obtained by glueing together the* d*-copies of the affine plane* $\mathbb{A}^2$ *by the transition functions*

$$\begin{aligned} g_{\lambda\omega} := \varphi_\lambda^{-1} \circ \varphi_\omega : \mathbb{A}^1_* \times \mathbb{A}^1 &\to \mathbb{A}^1_* \times \mathbb{A}^1 \\ (x,c) &\mapsto (x, c + \frac{p_\omega(x) - p_\lambda(x)}{x^r}), \end{aligned}$$

where $\omega, \lambda \in H(d)$ *and* $\mathbb{A}^1_* = \mathbb{A}^1 - \{0\}$.

(7) *The Galois group* $H(d)$ *acts as*

$$\lambda \cdot \varphi_\omega(x,c) = \varphi_{\lambda\omega}(\lambda x, \lambda^{1-r} c).$$

Proof. (1) Recall that $X(d,r)$ is the complement in $\Sigma_{|n|}$ of the curves C_d defined by the equation (1) with $a_0 \neq 0$ and $a_{d+1} \neq 0$, and the curve $w_0 = 0$ which is M_1 if $r-d \geq 0$ (resp. M_0 if $r-d<0$), where $n = r-d$. Since $w_0 \neq 0$, we can normalize to $w_0 = 1$. We can then normalize

$$w_0\left(a_0z_1^d + a_1z_0z_1^{d-1} + \cdots + a_{d-1}z_0^{d-1}z_1 + a_dz_0^d\right) + a_{d+1}z_0^rw_1 \neq 0$$

to the relation

$$z_0^rw_1 + (a_0z_1^d + a_1z_0z_1^{d-1} + \cdots + a_{d-1}z_0^{d-1}z_1 + a_dz_0^d) = 1,$$

where $a_0 \neq 0$. This normalization comes from the defining equivalence relation

$$(z_0, z_1), [w_0, w_1] \sim (\nu z_0, \nu z_1), [\nu^n w_0, w_1].$$

We may assume that $a_0 = 1$. The equivalence relation requires that the points $(\lambda z_0, \lambda z_1, \lambda^{-n}w_1)$ for $\lambda \in H(d)$ should be identified together, where we note that $n = r-d$. Hence the assertion follows. Note that $\widetilde{X}(d,r)$ is simply connected.

(3) Let δ be a derivation on the coordinate ring $\Gamma(\widetilde{X}(d,r))$ defined by

$$\begin{aligned}\delta(x) &= 0, \quad \delta(y) = x^r,\\ \delta(z) &= -(dy^{d-1} + (d-1)a_1xy^{d-2} + \cdots + a_{d-1}x^{d-1}).\end{aligned}$$

Then δ is locally nilpotent. Hence it defines a G_a-action on $\widetilde{X}(d,r)$ by

$$c \cdot a = \sum_{n=0}^{\infty} \frac{c^n}{n!}\delta^n(a), \qquad \text{for } c \in G_a, a \in \Gamma(\widetilde{X}(d,r)),$$

which is as specified as in the assertion. One easily verifies that $\operatorname{Ker}\delta = \mathbb{C}[x]$ and the inclusion $\operatorname{Ker}\delta \hookrightarrow \Gamma(\widetilde{X}(d,r))$ induces an $\mathbb{A}^1$-fibration $\tilde{\rho} : \widetilde{X}(d,r) \to \mathbb{A}^1$.

(4) We consider the smooth compactification (V, D) as given in Definition 2.1. Then we have a linear equivalence

$$\ell'_\infty \sim \ell'_0 + E_1 + 2E_2 + \cdots + (d-1)E_{d-1} + dE_d + d(E_{d+1} + \cdots + E_{d+r-1} + \overline{F}),$$

which is written as follows:

$$\begin{aligned}&\ell'_\infty + (d-1)(\ell'_0 + E_1) + (d-2)E_2 + \cdots + E_{d-1}\\ &\quad \sim d\left\{\ell'_0 + E_1 + E_2 + \cdots + E_{d-1} + E_d + E_{d+1} + \cdots + E_{d+r-1} + \overline{F}\right\}.\end{aligned}$$

Let $q : \widetilde{V} \to V$ be a d-ple cyclic covering which ramifies totally over the branch locus $\ell'_\infty + \ell'_0 + E_1 + E_2 + \cdots + E_{d-1}$. Then $\widetilde{V}$ has cyclic quoteint singularities over the intersection points $\ell'_0 \cap E_2, E_2 \cap E_3, \ldots, E_{d-2} \cap E_{d-1}$. The minimal resolution of these singularities will only insert linear chains of exceptional curves in between the proper transforms of the intersecting curves. Meanwhile, the inverse image $\widetilde{E}_d := q^{-1}(E_d)$ of the curve E_d remains irreducible on $\widetilde{V}$ and the linear branch $E_{d+1} + \cdots + E_{d+r-1} + \overline{F}$ splits into a disjoint union of d linear branches $\widetilde{E}_{d+1}^{(j)} + \cdots + \widetilde{E}_{d+r-1}^{(j)} + \widetilde{\overline{F}}^{(j)}$ $(1 \leq j \leq d)$. Each of the curves $\widetilde{E}_{d+1}^{(j)}$ $(1 \leq j \leq d)$ meets the curve $\widetilde{E}_d$ transversally in one point. Furthermore, $q^{-1}(X(d,r))$ is the universal covering space $\widetilde{X}(d,r)$. By the above observations, one knows that the boundary divisor of $\widetilde{X}(d,r)$ embedded minimally in a smooth projective surface, which is obtained from $\widetilde{V}$ by resolving minimally the above cyclic quotient singularities and contracting (-1) curves and consecutively contractible curves resulting from the inverse image of $q^{-1}(\ell'_\infty + S' + \ell'_0 + E_1 + \cdots + E_{d-1})$, is not a linear chain, for

$\widetilde{E}_d + \sum_{j=1}^{d}(\widetilde{E}^{(j)}_{d+1} + \cdots + \widetilde{E}^{(j)}_{d+r-1})$ cannot become a part of a linear chain if $d \geq 2$ and $r \geq 2$. So, we are done by a theorem of Bertin [3].

(5) Write $p_\omega(x) = \omega + c_1(\omega)x + \cdots + c_{r-1}(\omega)x^{r-1}$, where the coefficients are to be determined by the relation

$$x^r q_\omega(x) + p_\omega(x)^d + a_1 x p_\omega(x)^{d-1} + \cdots + a_{d-1}x^{d-1}p_\omega(x) + a_d x^d = 1 \tag{3}$$

which is obtained from equation (2) above by substituting $p_\omega(x), q_\omega(x)$ for y, z. By condition (i), it is easy to see that $p_\omega(x)$ is uniquely determined. Namely the coefficients $c_1(\omega), \ldots, c_{r-1}(\omega)$ are uniquely determined by putting the coefficients of the terms x^i $(1 \leq i \leq r-1)$ to be zero in the left-hand side of equation (3) above. Then $q_\omega(x)$ is uniquely determined as well. By multiplying $\lambda^d = 1$ to the relation (3), we obtain

$$\begin{aligned}&(\lambda x)^r \lambda^{-r} q_\omega(\lambda^{-1}(\lambda x))\\ &\quad + (\lambda p_\omega(\lambda^{-1}(\lambda x)))^d + a_1(\lambda x)(\lambda p_\omega(\lambda^{-1}(\lambda x)))^{d-1} + \cdots + a_d(\lambda x)^d = 1.\end{aligned}$$

Replace λx by x in the above relation. Then the uniqueness of the polynomials $p_{\lambda\omega}(x), q_{\lambda\omega}(x)$ imply that $p_{\lambda\omega}(x) = \lambda p_\omega(\lambda^{-1}x)$ and $q_{\lambda\omega}(x) = \lambda^{-r}q_\omega(\lambda^{-1}x)$. Now replace x by λx. Then we obtain the relation (iv). Note that $\varphi_\omega : \mathbb{A}^2 \to U_\omega$ is injective and $U_\omega \cong \mathbb{A}^2$. Hence φ_ω is an isomorphism by [1]. The φ_ω $(\omega \in H(d))$ are bundle charts of $\widetilde{X}(d,r)$ and $\widetilde{X}(d,r)$ is obtained by glueing d-copies $\{\omega\} \times \mathbb{A}^2$ $(\omega \in H(d))$ under the identification

$$(\omega, x, c) \sim (\lambda, x, c + \frac{p_\omega(x) - p_\lambda(x)}{x^r})$$

for $\omega, \lambda \in H(d)$ and $(x, c) \in \mathbb{A}^1_* \times \mathbb{A}^1$. The other assertions are verified in a straightforward manner. □

We say that a homogeneous polynomial $y^d + a_1 x y^{d-1} + \cdots + a_d x^d$ with the coefficient of the y^d-term equal to 1 is *monic*. Let $\widetilde{X}(d,r)$ and $\widetilde{X'}(d,s)$ be the affine hypersurfaces in $\mathbb{A}^3$ defined by $x^r z + f(x,y) = 1$ and $x^s z + h(x,y) = 1$, respectively, where $f(x,y)$ and $h(x,y)$ are monic homogeneous polynomials of degree d. If $r \neq s$, then $\widetilde{X}(d,r) \not\cong \widetilde{X'}(d,s)$ since $\pi_{1,\infty}(\widetilde{X}(d,r)) \neq \pi_{1,\infty}(\widetilde{X'}(d,s))$. Concerning the isomorphism classes of the hypersurfaces $\widetilde{X}(d,r)$, we have the following result. Note that we may assume that $f(x,y)$ and $h(x,y)$ are of the form $y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d$ by changing the coordinates.

Lemma 2.7. *Let $d \geq 2$ and $r > d$. Let $\widetilde{X}_1(d,r)$ and $\widetilde{X}_2(d,r)$ be the hypersurfaces defined by the equations $x^r z + y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d = 1$ and $x^r z + y^d + b_2 x^2 y^{d-2} + \cdots + b_d x^d = 1$, respectively. Then there is an isomorphism $\widetilde{X}_1(d,r) \cong \widetilde{X}_2(d,r)$ if and only if*

$$a_i = \mu^i b_i, \qquad \text{for} \quad \mu \in \mathbb{C}^*, \quad 2 \leq i \leq d.$$

Proof. Let $f : \widetilde{X}_1(d,r) \to \widetilde{X}_2(d,r)$ be an isomorphism and let φ be the induced isomorphism of the coordinate rings. Note that f preserves the unique $\mathbb{A}^1$-fibrations of $\widetilde{X}_1(d,r)$ and $\widetilde{X}_2(d,r)$ as well as the reduced reducible fibers. Hence f induces an automorphism of $\mathbb{C}[x]$ and $\varphi(x) = \mu x$ with $\mu \in \mathbb{C}^*$. Then φ induces an automorphism of the polynomial ring $\mathbb{C}[x, x^{-1}][y]$. So, one can write $\varphi(y) = u x^e y + F(x)$ with $u \in \mathbb{C}^*, e \in \mathbb{Z}$ and $F(x) \in \mathbb{C}[x, x^{-1}]$. Since φ is an isomorphism of the coordinate rings, it follows that $e \geq 0$ and $F \in \mathbb{C}[x]$. Furthermore, since f maps isomorphically the unique reducible fiber of $\widetilde{X}_1(d,r)$ to the unique reducible fiber of $\widetilde{X}_2(d,r)$, it follows that $e=0$, $F(0)=0$ and $u^d=1$. Since

$$\begin{aligned} z &= x^{-r}\{1-(y^d+b_2x^2y^{d-2}+\cdots+b_dx^d)\} \\ &\in \Gamma(\widetilde{X}_2(d,r))\otimes\mathbb{C}[x,x^{-1}]=\mathbb{C}[x,x^{-1}][y], \end{aligned}$$

it follows that $\varphi(z)=\mu^{-r}x^{-r}\{1-(\varphi(y)^d+b_2(\mu x)^2\varphi(y)^{d-2}+\cdots+b_d(\mu x)^d)\}$ in $\Gamma(\widetilde{X}_1(d,r))\otimes\mathbb{C}[x,x^{-1}]=\mathbb{C}[x,x^{-1}][y]$. While, φ is an isomorphism from $\Gamma(\widetilde{X}_2(d,r))$ to $\Gamma(\widetilde{X}_1(d,r))$, $\varphi(z)$ is written in a form $\sum_{k\geq 0}^N \varphi_k(x,y)z^k$, where $\varphi_k(x,y)=\sum_{0\leq j<d}\phi_{kj}(x)y^j$ for $\phi_{kj}(x)\in\mathbb{C}[x]$. Hence we have $\varphi(z)=\mu^{-r}z+\varphi_0(x,y)$, and

$$(\varphi(y))^d+b_2\mu^2x^2(\varphi(y))^{d-2}+\cdots+b_d\mu^dx^d-1$$

coincides with

$$y^d+a_2x^2y^{d-2}+\cdots+a_dx^d-1$$

modulo x^r. The comparison of the coefficients of the terms x^iy^{d-i} for $1\leq i\leq d$ implies that F is a multiple of x^r and $a_i=\mu^iu^{d-i}b_i$ for $2\leq i\leq d$. Replacing μu^{-1} by a new μ, we obtain $a_i=\mu^ib_i$ for $\mu\in\mathbb{C}^*$, $2\leq i\leq d$.

Conversely, if $a_i=\mu^ib_i$ ($2\leq i\leq d$) for $\mu\in\mathbb{C}^*$, then we can determine an isomorphism φ by

$$\begin{aligned} \varphi(x) &= \mu x, \\ \varphi(y) &= y+x^rG(x), \\ \varphi(z) &= \mu^{-r}\left[z-x^{-r}\left\{\left(\varphi(y)^d+b_2(\mu x)^2\varphi(y)^{d-2}+\cdots+b_d(\mu x)^d\right)\right.\right. \\ &\qquad \left.\left.-\left(y^d+a_2x^2y^{d-2}+\cdots+a_dx^d\right)\right\}\right], \end{aligned}$$

where $G(x)\in\mathbb{C}[x]$. □

Let $\widetilde{X}(d,r)$ be the affine hypersurface in $\mathbb{A}^3$ defined by equation (2) in Lemma 2.6 which has the transition functions given in assertion (6) of the same lemma. Let $\widetilde{X'}(d,s)$ be a similar affine hypersurface in $\mathbb{A}^3$ with the equation

$$x^sz+(y^d+a'_1xy^{d-1}+\cdots+a'_{d-1}x^{d-1}y+a'_dx^d)=1$$

and the transition functions

$$g'_{\lambda\omega}:={\varphi'}_\lambda^{-1}\circ{\varphi'}_\omega:\mathbb{A}^1_*\times\mathbb{A}^1\to\mathbb{A}^1_*\times\mathbb{A}^1,\quad (x,c)\mapsto(x,c+\frac{p'_\omega(x)-p'_\lambda(x)}{x^s}).$$

As in [5], we define a 3-dimensional affine variety $\widetilde{X}(d,r,s)$ by glueing together d-copies of the affine 3-space $\{\omega\}\times\mathbb{A}^3$ ($\omega\in H(d)$) by the following identification:

$$(\omega,x,c_1,c_2)\sim(\lambda,x,c_1+\frac{p_\omega(x)-p_\lambda(x)}{x^r},c_2+\frac{p'_\omega(x)-p'_\lambda(x)}{x^s}),\quad x\neq 0.$$

The projection $(\omega,x,c_1,c_2)\mapsto(\omega,x,c_1)$ yields a morphism $\pi_1:\widetilde{X}(d,r,s)\to\widetilde{X}(d,r)$ which is a principal G_a-bundle over $\widetilde{X}(d,r)$ with G_a acting naturally on the coordinate c_2. Similarly, the projection $(\omega,x,c_1,c_2)\mapsto(\omega,x,c_2)$ gives rise to a principal G_a-bundle $\pi_2:\widetilde{X}(d,r,s)\to\widetilde{X'}(d,s)$ with G_a acting naturally on the coordinate c_1. Since every principal G_a-bundle over an affine variety is trivial [20], it follows that

$$\widetilde{X}(d,r)\times\mathbb{A}^1\cong\widetilde{X}(d,r,s)\cong\widetilde{X'}(d,s)\times\mathbb{A}^1.$$

Hence the surfaces $\widetilde{X}(d,r)$ have the non-cancellation property.

Theorem 2.8. *Let $d \geq 2$ and let $r, s > d$. Let $\widetilde{X}(d,r)$ and $\widetilde{X'}(d,s)$ be the affine hypersurfaces defined by the equations $x^r z + (y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d) = 1$ and $x^s z + (y^d + a'_2 x^2 y^{d-2} + \cdots + a'_d x^d) = 1$, respectively. Then the following assertions hold:*

(1) *For any r and s,*

$$\widetilde{X}(d,r) \times \mathbb{A}^1 \cong \widetilde{X'}(d,s) \times \mathbb{A}^1.$$

(2) *The isomorphism $\widetilde{X}(d,r) \cong \widetilde{X'}(d,s)$ holds if and only if $r = s$ and $a'_i = \mu^i a_i$ for $\mu \in \mathbb{C}^*$ and $2 \leq i \leq d$.*

At this point, we do not know whether the isomorphism $\widetilde{X}(d,r) \times \mathbb{A}^1 \cong \widetilde{X'}(d,s) \times \mathbb{A}^1$ in Theorem 2.8 is $H(d)$-equivariant or not. We shall show that the isomorphism in Theorem 2.8(1) is in fact $H(d)$-equivariant in some cases. The $H(d)$-action specified in assertion (1) of Lemma 2.6 is lifted to $\widetilde{X}(d,r,s)$ on ω-charts as follows so that π_1 and π_2 are $H(d)$-equivariant:

$$\lambda \cdot (\omega, x, c_1, c_2) = (\lambda\omega, \lambda x, \lambda^{1-r} c_1, \lambda^{1-s} c_2) \quad \text{for } \lambda \in H(d).$$

We look for $H(d)$-equivariant sections of π_1 and π_2. A section of $\pi_1 : \widetilde{X}(d,r,s) \to \widetilde{X}(d,r)$ is expressed on ω-chart as

$$\{\omega\} \times \mathbb{A}^2 \to \{\omega\} \times \mathbb{A}^3, \qquad (x,c) \mapsto (x, c, \sigma_\omega(x,c)).$$

Hence an $H(d)$-equivariant section of π_1 is a family of $\sigma_\omega \in \mathbb{C}[x,c]$ ($\omega \in H(d)$), which is compatible with glueing maps and $H(d)$-actions.

Lemma 2.9. *For the principal G_a-bundle $\pi_1 : \widetilde{X}(d,r,s) \to \widetilde{X}(d,r)$, an $H(d)$-equivariant section is given by a family of polynomials $\sigma_\omega \in \mathbb{C}[x,c], \omega \in H(d)$, satisfying the following conditions:*

(1) *For all $\omega, \lambda \in H(d)$ and $(x,c) \in \mathbb{A}^1_* \times \mathbb{A}^1$,*

$$\sigma_\omega(x,c) + \frac{p'_\omega(x) - p'_\lambda(x)}{x^s} = \sigma_\lambda(x, c + \frac{p_\omega(x) - p_\lambda(x)}{x^r}).$$

(2) *For any $\omega, \lambda \in H(d)$,*

$$\lambda^{1-s}\sigma_\omega(x,c) = \sigma_{\lambda\omega}(\lambda x, \lambda^{1-r} c).$$

We can use relation (2) in the above lemma to compute σ_λ from σ_1:

$$\sigma_\lambda(x,c) = \lambda^{1-s}\sigma_1(\lambda^{-1}x, \lambda^{r-1}c). \tag{4}$$

In terms of the function σ_1, conditions (1) and (2) in Lemma 2.9 are reformulated as in the following result. The proof is essentially the same as in [5] if one takes into account relation (5)(iv) of Lemma 2.6.

Lemma 2.10. *Given a polynomial $\sigma = \sigma_1 \in \mathbb{C}[x,c]$, define polynomials $\{\sigma_\lambda \mid \lambda \in H(d)\}$ by equation (4) above. Then conditions (1) and (2) in Lemma 2.9 are satisfied if and only if σ satisfies*

$$\lambda^{1-s} x^s \sigma(\lambda^{-1}x, \lambda^{r-1}(c + \frac{p_1(x) - p_\lambda(x)}{x^r})) = x^s\sigma(x,c) + p'_1(x) - p'_\lambda(x) \tag{5}$$

for all $\lambda \in H(d), (x,c) \in \mathbb{A}^1_* \times \mathbb{A}^1$.

If there exists a polynomial σ satisfying condition (5) in Lemma 2.10, then there is an $H(d)$-equivariant isomorphism

$$\widetilde{X}(d,r,s) \cong \widetilde{X}(d,r) \times \mathbb{A}^1(1-s),$$

where $\mathbb{A}^1(a)$ denotes the one-dimensional $H(d)$-module of weight a. In fact, an $H(d)$-equivariant isomorphism $\widetilde{X}(d,r) \times \mathbb{A}^1(1-s) \cong \widetilde{X}(d,r,s)$ is defined as follows on the ω-chart for $\omega \in H(d)$:

$$\begin{array}{rcl} (\omega \times \mathbb{A}^2) \times \mathbb{A}^1 & \to & \omega \times \mathbb{A}^3, \\ ((\omega, x, c_1), c_2) & \mapsto & (\omega, x, c_1, \sigma_\omega(x, c_1) + c_2). \end{array}$$

Let $\widetilde{V}(d,r)$ be the affine surface defined by $x^r z + y^d = 1$. Then $\widetilde{V}(d,r)$ is the universal covering of the affine pseudo-plane $V(d,r)$. The polynomial $p_\omega(x)$ corresponding to $\widetilde{V}(d,r)$ is $p_\omega(x) = \omega$. Let $\widetilde{V}(d,r,s)$ be the affine variety glueing $\{\omega\} \times \mathbb{A}^3$ for $\omega \in H(d)$ with transition functions of $\widetilde{V}(d,r)$ and $\widetilde{V}(d,s)$ just as we constructed $\widetilde{X}(d,r,s)$. In [5], it is shown that there exist $H(d)$-equivariant sections of $\widetilde{V}(d,r,s) \to \widetilde{V}(d,r)$ for any r and s. The next result is a key fact in finding $H(d)$-equivariant sections, which is due to tom Dieck in the case $u = 1$.

Proposition 2.11. *Let u and t be positive integers and let $1 \le u \le d-1$.*

(1) *There exists a unique polynomial $Q_{u,t}(x) \in \mathbb{C}[x]$ satisfying the following properties:*
   (i) *$Q_{u,t}(\lambda x) = \lambda^u Q_{u,t}(x)$ for any $\lambda \in H(d)$.*
   (ii) *$\deg Q_{u,t}(x) = u + (t-1)d$.*
   (iii) *$Q_{u,t}(1+x) - 1$ is divisible by x^t.*

(2) *Let $P_{u,t}(x)$ be the polynomial defined by the equation $Q_{u,t}(1+x) - 1 = x^t P_{u,t}(x)$. Then for any $\lambda \in H(d)$,*

$$\lambda^{-t}(x+1-\lambda)^t P_{u,t}(\lambda^{-1}(x+1-\lambda)) = \lambda^{-u}(x^t P_{u,t}(x) + 1 - \lambda^u).$$

Proof. (1) By the property (i) and (ii), $Q_{u,t}(x)$ is written as

$$Q_{u,t}(x) = \sum_{j=0}^{t-1} a_j x^{u+jd}.$$

By property (iii), the coefficients a_j must satisfy linear equations. The determinant of the coefficient matrix of the system of the linear equations in $a_0, \ldots, a_{t-1}$ is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \binom{u}{1} & \binom{u+d}{1} & \cdots & \binom{u+(t-1)d}{1} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{u}{t-1} & \binom{u+d}{t-1} & \cdots & \binom{u+(t-1)d}{t-1} \end{vmatrix},$$

where the binomial coefficient $\binom{a}{b}$ for $a < b$ is defined to be zero. Note that $\binom{u+jd}{i}$ is a polynomial in jd of degree i. By adding a linear combination of the first i rows to the $(i+1)$-th row, the $(i+1)$-th row becomes $1/i!$ times of $(0, d^i, (2d)^i, \cdots, (t-1)^i d^i)$. Hence the determinant reduces to a non-zero multiple of the Vandermonde determinant and its value is non-zero. Thus we can determine the coefficients a_j, and the polynomial $Q_{u,t}(x)$ is uniquely determined.

(2) By the definition of $P_{u,t}(x)$, $Q_{u,t}(x)$ is written as

$$Q_{u,t}(x) = 1 + (x-1)^t P_{u,t}(x-1).$$

Then the required relation follows from property (i) of $Q_{u,t}(x)$. □

Theorem 2.12 (cf. tom Dieck [5]). *Let $d \geq 2$ and $r, s \geq 1$. Then for any r and s, there exists an $H(d)$-equivariant isomorphism*

$$\widetilde{V}(d,r) \times \mathbb{A}^1(1-s) \cong \widetilde{V}(d,s) \times \mathbb{A}^1(1-r).$$

Proof. It suffices to find $\sigma(x,c)$ satisfying

$$\lambda^{1-s} x^s \sigma(\lambda^{-1}x, \lambda^{r-1}(c + \frac{1-\lambda}{x^r})) = x^s \sigma(x,c) + 1 - \lambda$$

for all $\lambda \in H(d), (x,c) \in \mathbb{A}^1_* \times \mathbb{A}^1$. Let a and t be integers such that $a = -s + rt$, $t > 0$ and $a \geq 0$. Set $\sigma(x,c) = x^a c^t P_{1,t}(x^r c)$, where $P_{1,t}(x)$ is the polynomial defined in Proposition 2.11(2). Then one easily verifies that $\sigma(x,c)$ satisfies the above condition, and the assertion follows. □

Remark. There is a $\mathbb{C}^*$-action on $\widetilde{V}(d,r)$ defined by

$$\mu \cdot (x,y,z) = (\mu x, y, \mu^{-r} z) \qquad \text{for} \quad \mu \in \mathbb{C}^*,$$

which is the lift-up of the $\mathbb{C}^*$-action on the affine pseudo-plane $V(d,r)$ (cf. Lemma 2.5 and the statement below it). One verifies that the isomorphism in Theorem 2.12 is in fact an $H(d) \times \mathbb{C}^*$-equivariant isomorphism

$$\widetilde{V}(d,r) \times \mathbb{A}^1(1-s,-s) \cong \widetilde{V}(d,s) \times \mathbb{A}^1(1-r,-r),$$

where $\mathbb{A}^1(a,b)$ denotes the one-dimensional $H(d) \times \mathbb{C}^*$-module with weight a for $H(d)$ and with weight b for $\mathbb{C}^*$.

In some cases, we can find a polynomial σ satisfying the condition in Lemma 2.10 and write down σ explicitly. First, we consider the case $r = s = 2$.

Lemma 2.13. *Let $\widetilde{X}(d,2)$ and $\widetilde{X'}(d,2)$ be the affine surfaces defined by $x^2 z + f(x,y) = 1$ and $x^2 z + h(x,y) = 1$, respectively, where $f(x,y)$ and $h(x,y)$ are monic homogeneous polynomials of degree d. Then for any $d \geq 2$ there is an $H(d)$-equivariant isomorphism*

$$\widetilde{X}(d,2) \times \mathbb{A}^1(-1) \cong \widetilde{X'}(d,2) \times \mathbb{A}^1(-1).$$

Proof. Let $\widetilde{X}(d,2,2)$ be the affine variety obtained by $\widetilde{X}(d,2)$ and $\widetilde{X'}(d,2)$. For the principal G_a-bundle $\widetilde{X}(d,2,2) \to \widetilde{X}(d,2)$, $p_1(x)$ and $p'_1(x)$ in Lemma 2.9 are both of the form $1 + ax$ for $a \in \mathbb{C}$. Since $p_\lambda(x) = \lambda p_1(\lambda^{-1}x)$ and $p'_\lambda(x) = \lambda p'_1(\lambda^{-1}x)$, condition (5) in Lemma 2.10 is reduced to

$$\lambda^{-1} x^2 \sigma(\lambda^{-1}x, \lambda(c + \frac{1-\lambda}{x^2})) = x^2 \sigma(x,c) + 1 - \lambda$$

for all $\lambda \in H(d), (x,c) \in \mathbb{A}^1_* \times \mathbb{A}^1$. Set $\sigma(x,c) = x^{d-1} + c$. Then σ satisfies the above condition and it follows that $\widetilde{X}(d,2,2) \cong \widetilde{X}(d,2) \times \mathbb{A}^1(-1)$. Since $\sigma(x,c)$ gives rise to an $H(d)$-equivariant section of the principal G_a-bundle $\widetilde{X}(d,2,2) \to \widetilde{X'}(d,2)$, we have the assertion. □

Next, we consider the case where $\widetilde{X}(d,r)$ is defined by the equation $x^r z + y^d + ax^d = 1$ with $a \in \mathbb{C}$.

Lemma 2.14. *Let $d < r \leq 2d$. Suppose that $\widetilde{X}(d,r)$ is defined by $x^r z + y^d + ax^d = 1$ with $a \in \mathbb{C}$. Then there exists an $H(d)$-equivariant isomorphism*

$$\widetilde{X}(d,r) \times \mathbb{A}^1(1-r) \cong \widetilde{V}(d,r) \times \mathbb{A}^1(1-r).$$

Proof. Let $\widetilde{X}(d,r,r)$ be the affine variety obtained by $\widetilde{X}(d,r)$ and $\widetilde{V}(d,r)$. The polynomial σ which gives rise to an $H(d)$-equivariant section of the principal G_a-bundle $\widetilde{X}(d,r,r) \to \widetilde{X}(d,r)$ must satisfy condition (5) in Lemma 2.10 with $r = s$, $p_\lambda(x) = \lambda(1 - (a/d)x^d)$ and $p'_\lambda(x) = \lambda$. Define $\sigma(x,c) \in \mathbb{C}[x,c]$ by

$$\sigma(x,c) = -\frac{a^2}{d^2}x^{2d-r} + \left(1 + \frac{a}{d}x^d\right)c.$$

Then σ satisfies the condition and we obtain an $H(d)$-equivariant isomorphism $\widetilde{X}(d,r,r) \cong \widetilde{X}(d,r) \times \mathbb{A}^1(1-r)$. Similarly, one easily verifies that

$$\tau(x,c) = \left(1 - \frac{a}{d}x^d\right)c$$

satisfies

$$\lambda^{1-r}x^r\tau(\lambda^{-1}x, \lambda^{r-1}(c + \frac{1-\lambda}{x^r})) = x^r\tau(x,c) + p_1(x) - p_\lambda(x)$$

and gives rise to an $H(d)$-equivariant section of $\widetilde{X}(d,r,r) \to \widetilde{V}(d,r)$. Hence $\widetilde{X}(d,r,r) \cong \widetilde{V}(d,r) \times \mathbb{A}^1(1-r)$, and the assertion follows. □

Combining Lemma 2.14 and Theorem 2.12, we obtain the following.

Lemma 2.15. *Let $d < r \leq 2d$ and $d < s \leq 2d$. Suppose that $\widetilde{X}(d,r)$ and $\widetilde{X'}(d,s)$ are defined by $x^r z + y^d + ax^d = 1$ and $x^s z + y^d + a'x^d = 1$ with $a, a' \in \mathbb{C}$, respectively. Then there exists an $H(d)$-equivariant isomorphism*

$$\widetilde{X}(d,r) \times \mathbb{A}^1(1-r) \times \mathbb{A}^1(1-s) \cong \widetilde{X'}(d,s) \times \mathbb{A}^1(1-r) \times \mathbb{A}^1(1-s).$$

Now, resume the set-up in Lemmas 2.9 and 2.10, and suppose that $r \equiv s \equiv 1 \pmod d$. Then we can find a polynomial σ as satisfying the condition in Lemma 2.10 so that there exists an $H(d)$-equivariant isomorphism $\widetilde{X}(d,r) \times \mathbb{A}^1 \cong \widetilde{X'}(d,s) \times \mathbb{A}^1$, where $\mathbb{A}^1$ is the trivial $H(d)$-module.

Theorem 2.16. *Let $d \geq 2$ and $r \equiv s \equiv 1 \pmod d$. Then there is an $H(d)$-isomorphism*

$$\widetilde{X}(d,r) \times \mathbb{A}^1 \cong \widetilde{X'}(d,s) \times \mathbb{A}^1.$$

Proof. We first show that $\widetilde{X}(d,r) \times \mathbb{A}^1 \cong \widetilde{V}(d,1) \times \mathbb{A}^1$ for any $r \equiv 1 \pmod d$. Consider the affine variety $\widetilde{X}(d,r,1)$ obtained by $\widetilde{X}(d,r)$ and $\widetilde{V}(d,1)$. Then, the polynomial σ which gives rise to an $H(d)$-equivariant section of the principal G_a-bundle $\widetilde{X}(d,r,1) \to \widetilde{X}(d,r)$ must satisfy

$$x\sigma(\lambda^{-1}x, c + \frac{p_1(x) - p_\lambda(x)}{x^r}) = x\sigma(x,c) + 1 - \lambda$$

for all $\lambda \in H(d), (x,c) \in \mathbb{A}^1_* \times \mathbb{A}^1$, where $p_1(x) = 1 + a_1 x + \cdots + a_{r-1}x^{r-1}$ with $a_i \in \mathbb{C}$. It follows from $p_\lambda(x) = \lambda p_1(\lambda^{-1}x)$ that

$$\begin{aligned} p_1(x) - p_\lambda(x) &= p_1(x) - \lambda p_1(\lambda^{-1}x) \\ &= (1-\lambda) + \sum_{i=2}^{kd} a_i(1-\lambda^{1-i})x^i, \end{aligned}$$

where k is a non-negative integer such that $r = 1 + kd$. Set

$$\sigma(x, c) = f_0(x) + f_1(x)c,$$

where

$$f_0(x) = a_2 x + \cdots + a_{kd} x^{kd-1}, \qquad f_1(x) = x^{kd}.$$

Then σ satisfies the above condition, and it follows that $\widetilde{X}(d, r, 1) \cong \widetilde{X}(d, r) \times \mathbb{A}^1$ as $H(d)$-varieties. Next, we find a polynomial τ which gives rise to an $H(d)$-equivariant section of the principal G_a-bundle $\widetilde{X}(d, r, 1) \to \widetilde{V}(d, 1)$. The polynomial τ must satisfy

$$x^r \tau(\lambda^{-1} x, c + \frac{1-\lambda}{x}) = x^r \tau(x, c) + p_1(x) - p_\lambda(x)$$

for all $\lambda \in H(d), (x, c) \in \mathbb{A}^1_* \times \mathbb{A}^1$. Note that

$$p_1(x) - p_\lambda(x) = (1 - \lambda) + \sum_{\substack{2 \le i \le d \\ 0 \le j \le k-1}} a_{i+jd}(1 - \lambda^{1-i}) x^{i+jd}.$$

Set

$$\tau(x, c) = c^r P_{1,r}(xc) + \sum_{\substack{2 \le i \le d \\ 0 \le j \le k-1}} a_{i+jd} c^{r-i-jd} P_{d-i+1, r-i-jd}(xc),$$

where $P_{i,j}(x)$ is the polynomial defined in Proposition 2.11(2). Then τ satisfies the above equation, and an $H(d)$-equivariant isomorphism $\widetilde{X}(d, r, 1) \cong \widetilde{V}(d, 1) \times \mathbb{A}^1$ holds. Hence we obtain an $H(d)$-equivariant isomorphism $\widetilde{X}(d, r) \times \mathbb{A}^1 \cong \widetilde{V}(d, 1) \times \mathbb{A}^1$ for any $r \equiv 1 \pmod{d}$. Since we have an $H(d)$-isomorphim $\widetilde{X'}(d, s) \times \mathbb{A}^1 \cong \widetilde{V}(d, 1) \times \mathbb{A}^1$ for $s \equiv 1 \pmod{d}$, there exists an $H(d)$-isomorphism $\widetilde{X}(d, r) \times \mathbb{A}^1 \cong \widetilde{X'}(d, s) \times \mathbb{A}^1$. □

By Lemma 2.7 and Theorem 2.16, we obtain families of non-isomorphic affine surfaces $\widetilde{X}(d, r)$ with equivariant non-cancellation property. By taking the quotients by $H(d)$, we obtain families of affine pseudo-planes with non-cancellation property.

Theorem 2.17. *Let $d \ge 2$ and let $r, s > 1$ and $r \equiv s \equiv 1 \pmod{d}$. Let $\widetilde{X}(d, r; f)$ be the affine hypersurface defined by $x^r z + f(x, y) = 1$, where $f(x, y)$ is of the form $y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d$ with $a_j \in \mathbb{C}$. Then the quotient of $\widetilde{X}(d, r; f)$ by the Galois group $H(d)$ is an affine pseudo-plane $X(d, r; f)$ of type (d, r), and the following assertions hold:*

(1) *For any r and s,*

$$X(d, r; f_1) \times \mathbb{A}^1 \cong X(d, s; f_2) \times \mathbb{A}^1,$$

where f_1 and f_2 are monic homogeneous polynomials of the form stated above.

(2) *The isomorphism $X(d, r; f_1) \cong X(d, s; f_2)$ holds if and only if $r = s$ and $f_1(x, y) = f_2(\mu x, y)$ for $\mu \in \mathbb{C}^*$.*

3. An application

Let G be a reductive algebraic group. As an application of the results in the previous section, we construct the examples of families of affine G-varieties without equivariant cancellation property.

Let $Y = \operatorname{Spec} R$ be an affine G-variety such that the invariant subring R^G is a polynomial ring $\mathbb{C}[x]$ for $x \in R^G$. Let $\widetilde{Y}(d, r; f)$ be a hypersurface in $Y \times \mathbb{A}^2 = \operatorname{Spec} R[y, z]$ defined by $x^r z + f(x, y) = 1$, where $f(x, y)$ is a monic homogeneous polynomial of degree d. Then $\widetilde{Y}(d, r; f)$ has a G_a-action induced by a locally nilpotent R-derivation $D = x^r \partial_y - f_y \partial_z$. Since the G_a-action commutes with the G-action, the inclusion $\operatorname{Ker} D = R \hookrightarrow R[y, z]$ induces a G-equivariant $\mathbb{A}^1$-fibration $\widetilde{Y}(d, r; f) \to Y$. Let $\tilde{\pi} : \widetilde{Y}(d, r; f) \to \widetilde{X}(d, r; f)$ and $\pi : Y \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$ be the algebraic quotients by G, where $\widetilde{X}(d, r; f)$ is the affine hypersurface in $\mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$ defined by $x^r z + f(x, y) = 1$. Then it follows that $\widetilde{Y}(d, r; f) = \widetilde{X}(d, r; f) \times_{\operatorname{Spec} \mathbb{C}[x]} Y$.

$$
\begin{array}{ccc}
\widetilde{Y}(d, r; f) & \xrightarrow{\tilde{\pi}} & \widetilde{X}(d, r; f) \\
\downarrow & & \downarrow \tilde{\rho} \\
Y & \xrightarrow[\pi]{} & \mathbb{A}^1
\end{array}
$$

Theorem 3.1. *Let $d \geq 2$ and let $r, s > d$. Let $\widetilde{Y}(d, r; f_1)$ and $\widetilde{Y}(d, s; f_2)$ be affine G-varieties defined by $f_1(x, y)$ and $f_2(x, y)$ as above, respectively, where f_1 and f_2 are homogeneous polynomials of the form $y^d + a_2 x^2 y^{d-2} + \cdots + a_d x^d$ for $a_i \in \mathbb{C}$ $(2 \leq i \leq d)$. Then the following assertions hold:*

(1) *For any r and s, there is a G-equivariant isomorphism*

$$\widetilde{Y}(d, r; f_1) \times \mathbb{A}^1 \cong \widetilde{Y}(d, s; f_2) \times \mathbb{A}^1.$$

(2) *The isomorphism of G-varieties $\widetilde{Y}(d, r; f_1) \cong \widetilde{Y}(d, s; f_2)$ holds if and only if $r = s$ and $f_1(x, y) = f_2(\mu x, y)$ for $\mu \in \mathbb{C}^*$.*

Proof. The assertions follows from Theorem 2.8. □

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