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NORMS AND ESSENTIAL NORMS OF LINEAR COMBINATIONS OF ENDOMORPHISMS

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ABSTRACT. We compute norms and essential norms of linear combinations of endomorphisms on uniform algebras.

1. The introduction

Let X denote a compact Hausdorff space. Recall that a *uniform algebra* is a closed subalgebra of $C(X)$ that contains the constants and separates the points of X. We will denote the maximal ideal space of A (also known as the spectrum of A) by $M(A)$. When working in this setting, we may consider f as a function in $C(M(A))$ by identifying f with its Gelfand transform.

Now consider a uniform algebra A and an endomorphism $T : A \to A$. All our endomorphisms will satisfy $T(1) = 1$, so that $T \neq 0$ and $\|T\| = 1$. (See [12, Proposition 7] and [10] for more details on the information presented here.) If we consider the adjoint map $T^* : A^* \to A^*$ and restrict T^* to $M(A)$, we get a mapping $\phi : M(A) \to M(A)$ and more is true: T is given by the composition $T(f) = f \circ \phi$ for all $f \in A$. Since T has the special symbol ϕ associated with it, we will call ϕ the symbol associated to T and we will write $T = T_\phi$. Our unital endomorphisms are in one-to-one correspondence with continuous maps $\phi : M(A) \to M(A)$ satisfying $f \circ \phi \in A$ for all $f \in A$. For this reason, T is often denoted C_ϕ and called a composition operator.

Now that we have defined our "composition operators," we will study them, their norm, and their essential norm. We refer the reader to [10] or [6] for information about compactness, weak compactness, and complete continuity of a single operator. We begin our investigation by answering two basic questions about the algebras generated by these endomorphisms: we show that the algebra is, in general, neither closed nor dense. From there we move to more difficult questions as we compute the norm and essential norm of a linear combination of our endomorphisms. This part of the paper is closely related to work in [5] on interpolation in the double dual of A. Using interpolation results for general uniform algebras,

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for example, for symbols $\phi_1, \phi_2, \ldots, \phi_n$ and scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ we will show that if the pseudohyperbolic distance between distinct symbols tends in an appropriate way to 1, then $\| \sum_{j=1}^{N} \lambda_j T_{\phi_j} \|_e = \sum_{j=1}^{N} |\lambda_j|$, where $\|T\|_e$ denotes the essential norm of the operator T; that is, the distance of T to the compact operators. We will also determine conditions under which a linear combination of such operators is compact, and recover and extend results appearing in [1], [4], [7], [8], [13], and [14].

We note that it is well known that if A is a uniform algebra with connected spectrum $M(A)$ and T_ϕ is a compact (unital) endomorphism, then the range of ϕ is contained in a single Gleason part. In particular, as shown in [9], if X is a compact connected Hausdorff space, then every nonzero compact endomorphism T of $C(X)$ has the form $Tf = f(x_0)1$ for some $x_0 \in X$. In the final section of this paper, we will study conditions under which a linear combination of unital endomorphisms is compact. We are motivated, of course, by the case of a single operator. Our results are presented for uniform algebras, and then for the specific case of endomorphisms of $C(X)$. Related work can be found in [5], [9] and [10].

2. Investigating algebras of endomorphisms: Two examples

We begin our investigation by presenting two examples. Our first example will show that the algebra generated by the endomorphisms of a uniform algebra A is not, in general, closed. Our second example will show that it is not, in general, dense in the space of all bounded linear operators on A with respect to the operator norm topology.

We consider two examples in the setting of the disk algebra $A(\mathbf{D})$; that is, the algebra of functions continuous on $\overline{\mathbf{D}}$ and analytic on $\mathbf{D}$. Recall that the maximal ideal space of $A(\mathbf{D})$ is $\overline{\mathbf{D}}$. The algebra generated by $\mathrm{End}(A(\mathbf{D}))$ is denoted by $\langle \mathrm{End}(A(\mathbf{D})) \rangle$.

Example 1. In this first example we show that the algebra generated by $\mathrm{End}(A(\mathbf{D}))$ is not closed in the operator norm topology.

We consider the disk algebra, $A(\mathbf{D})$, and the self-maps of $\overline{\mathbf{D}}$ defined by

$$\phi_j(z) = \left(\frac{z+1}{2} \right)^{1/m_j},$$

where m_j is a positive integer chosen so that $(1/2)^{1/m_j} > 1 - 1/j^2$. We note that each map, ϕ_j, is indeed in the algebra $A(\mathbf{D})$.

Let $\{\lambda_j\}$ be an absolutely summable sequence of nonzero complex numbers and consider the operator $\sum_{j=1}^{\infty} \lambda_j T_{\phi_j}$. Now this operator is clearly in the closure of the algebra generated by $\mathrm{End}(A(\mathbf{D}))$. If it is also in the algebra $\langle \mathrm{End}(A(\mathbf{D})) \rangle$ (as opposed to the closure), then there exist complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and self-maps of $\overline{\mathbf{D}}$, denoted $\psi_1, \psi_2, \ldots, \psi_n$, in $A(\mathbf{D})$ with

$$\sum_{j=1}^{\infty} \lambda_j T_{\phi_j} = \sum_{j=1}^{n} \alpha_j T_{\psi_j}.$$

We may assume that all the self-maps are distinct, for otherwise we could move like terms to one side and begin again.

We now focus on the nonzero number, λ_1. Because all the maps are analytic, there must exist a real number, x, with $0 < x < 1$ for which

$$\phi_1(x) \neq \phi_k(x) \text{ for } k \neq 1 \text{ and } \phi_1(x) \neq \psi_k(x) \text{ for all } k = 1, \dots, n.$$

Now $\phi_k(x) > \phi_k(0) > 1 - 1/k^2$, so $\{\phi_k(x)\}_{k=1}^{\infty}$ is a Blaschke sequence. Since $\{\psi_j(x)\}_{j=1}^{n}$ is a finite set of points (which may lie in the disk or on the unit circle), we can find a Blaschke product B and a function $f \in A(\mathbf{D})$ satisfying

$$B(\phi_1(x)) \neq 0 \text{ and } B(\phi_k(x)) = 0 \text{ for } k \neq 1,$$

and $f(\phi_1(x)) \neq 0$ while

$$f(\psi_1(x)) = \cdots = f(\psi_n(x)) = 0.$$

Now B is an infinite Blaschke product. Since the zeros of B can be chosen to cluster precisely at the point 1 on the circle, the function $(z-1)Bf$ is in the disk algebra and vanishes in $\mathbf{D}$ precisely where Bf does. Thus

$$\begin{aligned}\lambda_1\big((z-1)Bf\big)(\phi_1(x)) &= \sum_{k=1}^{\infty} \big((\lambda_k T_{\phi_k})((z-1)Bf)\big)(x)\\ &= \sum_{l=1}^{n} \big((\alpha_l T_{\psi_l})((z-1)Bf)\big)(x) = 0,\end{aligned}$$

a contradiction.

Example 2. In this second example we show that the closure of the algebra generated by $\mathrm{End}(A(\mathbf{D}))$ is not the set of all bounded operators on $A(\mathbf{D})$.

Let $M_z : A(\mathbf{D}) \to A(\mathbf{D})$ denote the operator given by multiplication by z. We claim that this operator is not in the closure of $\langle \mathrm{End}(A(\mathbf{D}))\rangle$. So, suppose to the contrary that there exist complex numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ and functions, denoted $\phi_1, \phi_2, \dots, \phi_n$, in $A(\mathbf{D})$ that are self-maps of $\overline{\mathbf{D}}$, such that

$$\|M_z - \sum_{j=1}^{n} \lambda_j T_{\phi_j}\| < 1/4.$$

Our example depends on whether or not one of the self-maps is the identity function. If none of our maps is the identity function, we add $0T_{\phi_0}$, where $\phi_0(z) = z$. Hence, without loss of generality, we may assume that one of our self-maps is the identity function, and we will denote this function by ϕ_0 in the future. We denote the coefficient multiplying T_{ϕ_0} by λ.

Now, since our functions are analytic and distinct, they can coincide at most on a set of measure zero on the circle. Thus, we may choose a value $z_0 \in \partial\mathbf{D}$ such that $\phi_j(z_0) \neq \phi_k(z_0)$ for $j \neq k$ (where $j, k = 0, 1, \dots, n$), and such that $|\lambda - z_0| \geq 1$. Now for each j, we have arranged things so that $\phi_j(z_0) \neq z_0$, and therefore the function $(z + z_0)/(2z_0)$ has modulus less than 1 at each $\phi_j(z_0)$. Let f denote the function defined by

$$f(z) = \overline{z_0}\Big(\frac{z + z_0}{2z_0}\Big)^M,$$

where M is an integer that is chosen so large that $\sum_{j=1}^{n} |\lambda_j f(\phi_j(z_0))| < 1/4$. Now $f(z_0) = \overline{z_0}$ and therefore

$$\begin{aligned}
&\|M_z - \lambda T_z - \sum_{j=1}^{n} \lambda_j T_{\phi_j}\| \\
&\qquad \geq \|(M_z - \lambda T_z - \sum_{j=1}^{n} \lambda_j T_{\phi_j}) f\| \\
&\qquad \geq |z_0 f(z_0) - \lambda(f(z_0)) - \sum_{j=1}^{n} \lambda_j f(\phi_j(z_0))| \\
&\qquad \geq |1 - \lambda \overline{z_0}| - 1/4 = |z_0 - \lambda| - 1/4 \geq 3/4,
\end{aligned}$$

contradicting the fact that $\|M_z - \lambda T_z - \sum_{j=1}^{n} \lambda_j T_{\phi_j}\| < 1/4$.

This contradiction shows that the operator M_z does not lie in the closure of the algebra generated by the endomorphisms on $A(\mathbf{D})$.

3. Interpolation and norm estimates

In this section, we look at norm estimates for linear combinations of endomorphisms. The first lemma, which we will use repeatedly, is taken from [2]. The aforementioned paper is concerned with interpolation in maximal ideal spaces, which is almost—but not quite—what we are doing here. In this paper, given a sequence of points we solve an infinite number of finite interpolation problems while controlling the norm of our interpolating functions, but our work does not actually require the existence of a function in the algebra doing the interpolation for the entire sequence.

While the results in [2] are stated for bounded analytic functions, the basic idea lends itself to proofs in more general situations. We restate Lemma 2 of [2] here for convenience and we include a slight variation of the original result, which we will use in the next section of this paper, along with its proof.

Lemma 3 ([2]). *Let j be an integer greater than 1, and let $a_1, \ldots, a_j, b_1, \ldots, b_j$ be nonnegative numbers. Let $\epsilon_1, \ldots, \epsilon_j$ be nonnegative numbers such that*

$$a_k + b_k < 1 + \epsilon_k \text{ for } k = 1, \ldots, j.$$

Then

$$a_1 + a_2 b_1 + a_3 b_1 b_2 + \cdots + a_j b_1 b_2 \cdots b_{j-1} < (1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_j)$$

and

$$a_1 + a_2 b_1 + \cdots + a_{j-1} b_1 b_2 \cdots b_{j-2} + b_1 b_2 \cdots b_{j-1} < (1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_{j-1}).$$

Proof. The proof of the first formula can be found in [2]. The proof of the second (like that of the first) proceeds by induction. For $j = 2$, it is clear that $a_1 + b_1 < 1 + \epsilon_1$, as desired.

Now suppose that the lemma holds for $j-1$ summands. Then

$$\begin{aligned} &a_1 + a_2 b_1 + \cdots + a_{j-1} b_1 b_2 \cdots b_{j-2} + b_1 b_2 \cdots b_{j-1} \\ &\quad = a_1 + b_1(a_2 + a_3 b_2 + \cdots + a_{j-1} b_2 \cdots b_{j-2} + b_2 \cdots b_{j-1}) \\ &\quad < a_1 + b_1(1+\epsilon_2)(1+\epsilon_3)\cdots(1+\epsilon_{j-1}) \\ &\quad \le a_1(1+\epsilon_2)(1+\epsilon_3)\cdots(1+\epsilon_{j-1}) + b_1(1+\epsilon_2)(1+\epsilon_3)\cdots(1+\epsilon_{j-1}) \\ &\quad < (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)\cdots(1+\epsilon_{j-1}), \end{aligned}$$

completing the proof of the induction step. □

The next lemma has its origin in the proof of Lemma 1 of [2]. It requires using the pseudohyperbolic distance, ρ, defined between two points x and y in $M(A)$ by

$$\rho(x,y) = \sup\{|f(y)| : f \in A, \|f\| \le 1, \text{ and } f(x) = 0\}.$$

Lemma 4. *Let A be a uniform algebra. Let $x_1, \ldots, x_n$ be distinct points in $M(A)$ and let $\rho_{jk} = \rho(x_j, x_k)$. Let $\epsilon > 0$ be chosen so that $\epsilon < \min_{j \ne k} \rho_{jk}$. Then, for $j = 1, \ldots, n$, there exist functions f_j and g_j in A satisfying*

$$f_j(x_j) = 1, f_j(x_k) = 0 \text{ for } k \ne j,$$

$$g_j(x_j) = 0, g_j(x_k) = 1 \text{ for } k \ne j,$$

and

$$|f_j| + |g_j| < \left(\frac{1}{\prod_{k \ne j}(\rho_{jk} - \epsilon)}\right)^2 \left(1 + \sqrt{1 - \prod_{k \ne j}(\rho_{jk} - \epsilon)^2}\right) \text{ on } M(A). \tag{1}$$

These pairs of functions are crucial to all the proofs in this paper. If g is a function that satisfies the conditions above, then we will say that g is f's partner. Thus, g is small where f is big, and we can keep a good bound on the norm of the sum $|f| + |g|$ on $M(A)$.

Proof. Fix $j \in \{1, 2, \ldots, n\}$. Then, by definition of the pseudohyperbolic distance, for $k \ne j$ we can find $h_{jk} \in A$ with

$$h_{jk}(x_j) = 1, h_{jk}(x_k) = 0, \text{ and } \|h_{jk}\| < (\rho_{jk} - \epsilon)^{-1}.$$

If $H_j = \prod_{k \ne j} h_{jk}$, then

$$H_j(x_j) = 1, \text{ and for } k \ne j \text{ we have } H_j(x_k) = 0,$$

while

$$1 \le \|H_j\| < \prod_{k \ne j} (\rho_{jk} - \epsilon)^{-1}.$$

We will construct f_j and g_j using H_j to define a new function F_j, but the definition of this function depends on whether $\|H_j\| = 1$ or $\|H_j\| > 1$.

If $\|H_j\| = 1$ we define functions $F_{j,t}$, for t a positive real number satisfying $t < 1$, by

$$F_{j,t} = \frac{t - \frac{2t}{1+t^2} H_j}{1 - t \frac{2t}{1+t^2} H_j}.$$

Then $F_{j,t} \in A$, $\|F_{j,t}\| \le 1$, and a computation shows that

$$F_{j,t}(x_j) = -t, \text{ and for } k \ne j \text{ we have } F_{j,t}(x_k) = t.$$

We will choose t so that

$$1/t^2 < \left(\frac{1}{\prod_{k\neq j}(\rho_{jk}-\epsilon)}\right)^2 \left(1+\sqrt{1-\prod_{k\neq j}(\rho_{jk}-\epsilon)^2}\right).$$

In this case, and for this value of t, we will take $F_j = F_{j,t}$.

If $\|H_j\| > 1$, we let $H_j^\star = H_j/\|H_j\|$ and $t = \|H_j\| - \sqrt{\|H_j\|^2 - 1}$. Note that since $t = 1/(\|H_j\| + \sqrt{\|H_j\|^2-1})$ and $\|H_j\| > 1$, the real number $t < 1$ and therefore $(1-tH_j^\star)^{-1} \in A$. We now consider the function

$$F_j = \frac{t - H_j^\star}{1 - tH_j^\star},$$

and we note that F_j lies in the (closed) unit ball of A. Since $H_j(x_k) = 0$ for $k \neq j$ we have

$$F_j(x_k) = t,$$

and since $H_j(x_j) = 1$, we also have

$$F_j(x_j) = \frac{t - 1/\|H_j\|}{1 - t/\|H_j\|} = \frac{\|H_j\|^2 - \|H_j\|\sqrt{\|H_j\|^2-1} - 1}{\sqrt{\|H_j\|^2-1}} = -t.$$

Thus, in both cases ($\|H_j\| = 1$ and $\|H_j\| > 1$) we can define the partners f_j and g_j by

$$f_j = \left(\frac{t - F_j}{2t}\right)^2 \text{ and } g_j = \left(\frac{t + F_j}{2t}\right)^2. \tag{2}$$

Then, since $F_j(x_k) = t$ and $F_j(x_j) = -t$, we have

$$f_j(x_j) = 1, f_j(x_k) = 0 \text{ for } k \neq j,$$

and

$$g_j(x_j) = 0, g_j(x_k) = 1 \text{ for } k \neq j.$$

In the case that $\|H_j\| = 1$, we have

$$\begin{aligned} |f_j| + |g_j| &= \frac{1}{4t^2}\left(|t-F_j|^2 + |t+F_j|^2\right) \\ &= \frac{1}{2t^2}\left(t^2 + |F_j|^2\right) \\ &\leq 1/t^2, \text{ since } |F_j| \leq 1. \end{aligned}$$

By our choice of t we get

$$|f_j| + |g_j| \leq 1/t^2 \leq \left(\frac{1}{\prod_{k\neq j}(\rho_{jk}-\epsilon)}\right)^2 \left(1+\sqrt{1-\prod_{k\neq j}(\rho_{jk}-\epsilon)^2}\right)$$

on $M(A)$.

In the case that $\|H_j\| > 1$ on $M(A)$ we have

$$\begin{aligned} |f_j| + |g_j| &= \frac{1}{4t^2}\left(|t - F_j|^2 + |t + F_j|^2\right) \\ &= \frac{1}{2t^2}\left(t^2 + |F_j|^2\right) \\ &\leq \frac{1}{2t^2}(1 + t^2) \text{ (since } |F_j| \leq 1) \\ &= \frac{1}{t}\|H_j\| \text{ (by our definition of } t) \\ &= \|H_j\|\left(\|H_j\| + \sqrt{\|H_j\|^2 - 1}\right). \end{aligned}$$

But $\|H_j\| < \prod_{k \neq j}(\rho_{jk} - \epsilon)^{-1}$ and therefore, if we replace $\|H_j\|$ above with this upper estimate we obtain

$$|f_j| + |g_j| \leq \left(\frac{1}{\prod_{k \neq j}(\rho_{jk} - \epsilon)}\right)^2 \left(1 + \sqrt{1 - \prod_{k \neq j}(\rho_{jk} - \epsilon)^2}\right) \text{ on } M(A),$$

completing the proof of this lemma. □

We note (and we will use later) that it follows from the definitions of the functions f_j and g_j given in equation (2) that if (x_n) is a sequence such that $f_j(x_n) \to 0$, then $g_j(x_n) \to 1$. Using these pairs of functions, we can obtain an estimate on the norm of a linear combination of endomorphisms. In what follows, when we refer to a self-map ϕ, we will always mean a self-map of $M(A)$, the spectrum of the uniform algebra A.

Theorem 5. *Let A be a uniform algebra. Let $N \geq 2$ be an integer, and let $T_{\phi_1}, \ldots, T_{\phi_N}$ be endomorphisms of A with associated self-maps $\phi_1, \ldots, \phi_N$. If*

$$s = \sup_{x \in M(A)} \min_{j \neq k} \rho(\phi_j(x), \phi_k(x)) > 0,$$

then

$$\sum_{j=1}^{N} |\lambda_j| \left(\frac{s^{2N-2}}{1 + \sqrt{1 - s^{2N-2}}}\right)^{N-1} \leq \|\sum_{j=1}^{N} \lambda_j T_{\phi_j}\| \leq \sum_{j=1}^{N} |\lambda_j|.$$

Proof. The upper inequality is a triviality, so we turn our attention to the lower inequality. Let $\epsilon > 0$ be given, and assume that $\epsilon < s$. By the definition of s there exists $x \in M(A)$ such that $\rho(\phi_j(x), \phi_k(x)) > s - \epsilon$ (for $j \neq k$). Using the facts that the function $(1 + \sqrt{1 - \eta^2})/\eta^2$ is decreasing on $(0, 1)$, as well as Lemma 4, we see that for each j we can find f_j and g_j satisfying

$$\begin{gathered} f_j(\phi_j(x)) = 1, f_j(\phi_k(x)) = 0 \text{ for } k \neq j, \\ g_j(\phi_j(x)) = 0, g_j(\phi_k(x)) = 1 \text{ for } k \neq j, \\ \text{and } |f_j| + |g_j| < \left(\frac{1}{s - \epsilon}\right)^{2N-2} \left(1 + \sqrt{1 - (s - \epsilon)^{2N-2}}\right) \text{ on } M(A). \end{gathered}$$

If we write $\lambda_j = r_j e^{i\theta_j}$, and we let

$$H = e^{-i\theta_1} f_1 + e^{-i\theta_2} f_2 g_1 + \cdots + e^{-i\theta_{N-1}} f_{N-1} g_1 \ldots g_{N-2} + e^{-i\theta_N} g_1 \ldots g_{N-1},$$

(we note that we have deliberately omitted functions of the form f_j from the final summand above) we may use Lemma 3 to conclude that

$$\|H\| \leq \left(\frac{1+\sqrt{1-(s-\epsilon)^{2N-2}}}{(s-\epsilon)^{2N-2}} \right)^{N-1}.$$

So

$$\begin{aligned} \|\sum_{j=1}^N \lambda_j T_{\phi_j}\| &\geq \|\sum_{j=1}^N \lambda_j T_{\phi_j}(H/\|H\|)\| \\ &= \|\sum_{j=1}^N \lambda_j H(\phi_j)\|/\|H\| \\ &\geq \left|\sum_{j=1}^N \lambda_j H(\phi_j(x))\right|/\|H\| \\ &\geq \left(\frac{(s-\epsilon)^{2N-2}}{1+\sqrt{1-(s-\epsilon)^{2N-2}}} \right)^{N-1} \sum_{j=1}^N |\lambda_j|. \end{aligned}$$

Letting $\epsilon \to 0$ yields the result. □

Letting $N = 2$ in the theorem above leads to the following corollary.

Corollary 6. *Let A be a uniform algebra and let T_{ϕ_1} and T_{ϕ_2} be endomorphisms of A with associated self-maps ϕ_1 and ϕ_2. Then*

$$(|\lambda_1| + |\lambda_2|)(1-\sqrt{1-s^2}) \leq \|\lambda_1 T_{\phi_1} + \lambda_2 T_{\phi_2}\| \leq |\lambda_1| + |\lambda_2|,$$

where $s = \sup_{x \in M(A)} \rho(\phi_1(x), \phi_2(x)) > 0$.

4. The essential norm of a linear combination of endomorphisms

Recall that a sequence (H_m) of functions in a uniform algebra A is *weakly null* if for every element $\psi \in A^\star$ we have $\psi(H_m) \to 0$. A compact operator K always takes weakly null sequences to norm null sequences. Weakly null sequences are helpful in computing the essential norm of an operator, as such sequences make compact operators "disappear." For this reason, it will be important to be able to construct sequences of weakly null functions in A. We will use the following lemma to do so. This useful lemma appeared in [3]. Though this lemma is known, their short proof appears here for completeness.

Lemma 7. *Let A be a uniform algebra on a compact Hausdorff space X, and let (f_n) be a sequence of functions in A. If there exists a constant M satisfying $\sum_{j=1}^N |f_j(x)| < M$ for all $x \in X$ and for all $N \in \mathbb{N}$, then (f_n) is a weakly null sequence.*

Proof. Let $\psi \in A^\star$. Then for each N and for an appropriate choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_N$ of modulus one we have

$$\sum_{j=1}^N |\psi(f_j)| = \sum_{j=1}^N \alpha_j \psi(f_j) = \psi(\sum_{j=1}^N \alpha_j f_j).$$

Since ψ is bounded we conclude that

$$\sum_{j=1}^{N} |\psi(f_j)| < \|\psi\| \| \sum_{j=1}^{N} \alpha_j f_j\| \le \|\psi\|\, M.$$

Therefore, the series $\sum_{j=1}^{\infty} |\psi(f_j)|$ converges, and so the terms $(\psi(f_n))$ must tend to zero. Consequently, (f_n) converges to zero weakly. $\square$

Theorem 8. *Let A be a uniform algebra and let $T_{\phi_1}, \ldots, T_{\phi_N}$ be endomorphisms of A with associated self-maps $\phi_1, \ldots, \phi_N$. Suppose that there exists a sequence (x_n) of points in $M(A)$ with a cluster point x such that for all j and k we have both $\rho(\phi_j(x_n), \phi_k(x_n)) \to 1$ (for $k \neq j$) and $\rho(\phi_j(x_n), \phi_j(x)) \to 1$ as $n \to \infty$. Then*

$$\| \sum_{j=1}^{N} \lambda_j T_{\phi_j} \|_e = \sum_{j=1}^{N} |\lambda_j|.$$

Proof. In this proof, for each $\epsilon > 0$, we will simultaneously construct partners f_m and g_m, and a sequence of points (y_m). Writing each $\lambda_j = r_j e^{i\theta_j}$, we will then use these functions to construct a sequence of functions (H_m) with

$$\|H_m\| < 1 + \epsilon,$$

such that for each k we have

$$|H_m(\phi_k(y_m)) - e^{-i\theta_k}| \to 0 \text{ and}$$
$$(H_m) \text{ converges weakly to zero as } m \to \infty.$$

The sequence (y_n) will be a subsequence of the given sequence, (x_n). Using these new functions and the fact that a compact operator takes weakly null sequences to norm null sequences, we can compute the essential norm of the operator, $T = \sum_{j=1}^{N} \lambda_j T_{\phi_j}$, as follows: Let K be an arbitrary compact operator. For each m,

$$\begin{aligned}
\|T + K\| &\ge \|T(H_m) + K(H_m)\|/(1+\epsilon) \\
&\ge \left(\| \sum_{j=1}^{N} \lambda_j H_m(\phi_j) \| - \|K(H_m)\| \right) /(1+\epsilon) \\
&\ge \left(| \sum_{j=1}^{N} \lambda_j H_m(\phi_j(y_m)) | - \|K(H_m)\| \right) /(1+\epsilon) \\
&\ge \left(\sum_{j=1}^{N} |\lambda_j| - \sum_{j=1}^{N} |\lambda_j| \left| H_m(\phi_j(y_m)) - e^{-i\theta_j} \right| - \|K(H_m)\| \right) /(1+\epsilon).
\end{aligned}$$

As $m \to \infty$, the second summand above tends to 0, and since K is compact and (H_m) weakly null, $\|K(H_m)\| \to 0$. Letting $\epsilon \to 0$ and taking the infimum over all compact operators, we see that $\|T\|_e \ge \sum_{j=1}^{N} |\lambda_j|$. But $\|T\|_e \le \|T\| \le \sum_{j=1}^{N} |\lambda_j|$. Thus, if we can construct the functions H_m the way we promised, we will have proved our theorem.

We now turn to the construction of the interpolating functions, H_m. Let $\epsilon > 0$. Choose a sequence (ϵ_j) such that for each j, $\epsilon_j > 0$ and

$$\Big(\prod_{j=1}^{\infty} (1 + \epsilon_j) \Big)^{N+1} < 1 + \epsilon. \tag{3}$$

Constructing f_1, g_1, and y_1. We will first set ourselves up to use Lemma 4 on a certain set consisting of at most $N+1$ points.

Since

$$\rho(\phi_j(x_n), \phi_j(x)) \to 1 \text{ and } \rho(\phi_j(x_n), \phi_k(x_n)) \to 1$$

for $k \neq j$ and for all $j \in \{1, \dots, N\}$, we may choose a point y_1 and a value ρ_1 (independent of j) with

$$\rho_1 < \min\{\rho(\phi_j(y_1), \phi_j(x)), \min_{k \neq j} \rho(\phi_j(y_1), \phi_k(y_1))\}$$

so close to 1 that

$$\frac{1}{\rho_1^{2N}}\left(1 + \sqrt{1 - \rho_1^{2N}}\right) < 1 + \epsilon_1.$$

According to Lemma 4 applied to the points $\{\phi_1(y_1), \dots, \phi_N(y_1), \phi_j(x)\}$, we may choose $f_{1,j}$ and $g_{1,j}$ satisfying

$$f_{1,j}(\phi_j(y_1)) = 1, f_{1,j}(\phi_k(y_1)) = 0 \text{ for } k \neq j, \text{ and } f_{1,j}(\phi_j(x)) = 0,$$
$$g_{1,j}(\phi_j(y_1)) = 0, g_{1,j}(\phi_k(y_1)) = 1 \text{ for } k \neq j, \text{ and } g_{1,j}(\phi_j(x)) = 1,$$

and since the expression $(1 + \sqrt{1-\eta^2})/\eta^2$ decreases on $(0, 1)$, the upper bound estimate of $|f_{1,j}| + |g_{1,j}|$ given by Lemma 4 (see inequality (1)) implies that on $M(A)$ we have

$$|f_{1,j}| + |g_{1,j}| < \frac{1}{\rho_1^{2N}}\left(1 + \sqrt{1 - \rho_1^{2N}}\right) < 1 + \epsilon_1, \text{ for all } j.$$

This completes the first step in the construction.

Constructing $f_{m,j}$, $g_{m,j}$, and y_m, assuming $f_{n,j}, g_{n,j}$, and y_n have been chosen for $n = 1, \dots, m-1$ and $j = 1, \dots, N$. Suppose that for $n = 1, \dots, m-1$, we have constructed $f_{n,j}$, $g_{n,j}$, and y_n satisfying

$$f_{n,j}(\phi_j(y_n)) = 1, f_{n,j}(\phi_k(y_n)) = 0 \text{ for } k \neq j, \text{ and } f_{n,j}(\phi_j(x)) = 0,$$
$$g_{n,j}(\phi_j(y_n)) = 0, g_{n,j}(\phi_k(y_n)) = 1 \text{ for } k \neq j, \text{ and } g_{n,j}(\phi_j(x)) = 1,$$

and on $M(A)$ we have

$$|f_{n,j}| + |g_{n,j}| < 1 + \epsilon_n, \text{ for all } j.$$

We use the fact that x is in the cluster set of (x_n), that $g_{n,j}(\phi_j(x)) = 1$ and that as $n \to \infty$,

$$\min\{\rho(\phi_j(x_n), \phi_j(x)), \min_{k \neq j} \rho(\phi_j(x_n), \phi_k(x_n))\} \to 1$$

to select y_m so that for each $j = 1, \dots, N$ we have

$$|1 - g_{n,j}(\phi_j(y_m))| < 1/m^2 \tag{4}$$

for $n = 1, \dots, m-1$ and to make both

$$\rho(\phi_j(y_m), \phi_j(x)) \text{ and } \min_{k \neq j} \rho(\phi_j(y_m), \phi_k(y_m))$$

as close to 1 as we like. Now we again apply Lemma 4, this time to the points $\{\phi_1(y_m), \dots, \phi_N(y_m), \phi_j(x)\}$ to obtain $f_{m,j}$ and $g_{m,j}$ satisfying

$$f_{m,j}(\phi_j(y_m)) = 1, f_{m,j}(\phi_k(y_m)) = 0 \text{ for } k \neq j, \text{ and } f_{m,j}(\phi_j(x)) = 0,$$
$$g_{m,j}(\phi_j(y_m)) = 0, g_{m,j}(\phi_k(y_m)) = 1 \text{ for } k \neq j, \text{ and } g_{m,j}(\phi_j(x)) = 1,$$

and

$$|f_{m,j}| + |g_{m,j}| < 1 + \epsilon_m, \text{ for all } j.$$

Constructing the interpolating functions H_m. For each m we define

$$H_m = \sum_{j=1}^{N} e^{-i\theta_j} f_{m,j}(g_{1,j} \cdots g_{m-1,j})(g_{m,1} \cdots g_{m,j-1}).$$

The summands of H_m are thus a product of three terms. The first two will force H_m to converge weakly to zero, and the third will assist us in controlling the norm of H_m. First we will compute the norm of H_m and then we will show that (H_m) tends to zero weakly.

To compute the norm, note that since $\|g_{k,j}\| < 1 + \epsilon_k$ we may replace the term $g_{1,j} \cdots g_{m-1,j}$ with an upper bound on its norm to obtain

$$|H_m| \le \Big(\prod_{k=1}^{m-1} (1+\epsilon_k)\Big)\big(|f_{m,1}| + |f_{m,2}g_{m,1}| + \cdots + |f_{m,N}g_{m,1}\cdots g_{m,N-1}|\big).$$

Now Lemma 3 shows that $|H_m| \le (1+\epsilon_m)^N \prod_{k=1}^{m-1}(1+\epsilon_k)$. By our choice of the sequence (ϵ_m) (see inequality (3)), we find that $\|H_m\| < 1 + \epsilon$.

To see that the functions tend weakly to zero, we note that for fixed j we may replace $g_{m,1} \cdots g_{m,j-1}$ with an upper bound on its norm to obtain

$$\sum_{m=1}^{\infty} |f_{m,j}(g_{1,j} \cdots g_{m-1,j})(g_{m,1} \cdots g_{m,j-1})| \le (1+\epsilon_m)^N \sum_{m=1}^{\infty} |f_{m,j}g_{1,j} \cdots g_{m-1,j}|$$

and the sum $\sum_{m=1}^{\infty} |f_{m,j}g_{1,j} \cdots g_{m-1,j}|$ is less than $\prod_{k=1}^{\infty}(1+\epsilon_k)$ by Lemma 3. Thus, this sum is finite and we may use Lemma 7 to conclude that for fixed j functions of the form $k_{m,j} = f_{m,j}g_{1,j}\cdots g_{m-1,j}g_{m,1}\cdots g_{m,j-1}$ converge weakly to zero. Now, for each m, we have $H_m = \sum_{j=1}^{N} k_{m,j}$, and we know that $(k_{m,j})_{m=1}^{\infty}$ is weakly null for each $j = 1, \dots, N$. Thus we conclude that (H_m) is weakly null.

Checking that (H_m) does everything it needs to do. We have already checked that $\|H_m\| < 1 + \epsilon$ and that (H_m) is a weakly null sequence. Thus, it remains to check that for each k we have $|H_m(\phi_k(y_m)) - e^{-i\theta_k}| \to 0$ as $m \to \infty$. Now $H_m(\phi_k(y_m)) = \sum_{j=1}^{N} e^{-i\theta_j}(f_{m,j}(g_{1,j}\cdots g_{m-1,j})(g_{m,1}\cdots g_{m,j-1}))(\phi_k(y_m))$. But we know that $f_{m,j}(\phi_k(y_m)) = 0$ for $k \ne j$, so

$$H_m(\phi_k(y_m)) = e^{-i\theta_k}(f_{m,k}(g_{1,k}\cdots g_{m-1,k})(g_{m,1}\cdots g_{m,k-1}))(\phi_k(y_m)).$$

Furthermore, $f_{m,k}(\phi_k(y_m)) = 1$ and $g_{m,j}(\phi_k(y_m)) = 1$ for $k \ne j$, so

$$H_m(\phi_k(y_m)) = e^{-i\theta_k}(g_{1,k}\cdots g_{m-1,k})(\phi_k(y_m)).$$

Finally, by inequality (4), we have

$$|1 - g_{n,k}(\phi_k(y_m))| < 1/m^2$$

for $n = 1, 2, \dots, m-1$.

Now, for complex numbers $a_1, \dots, a_{m-1}$ with $|1 - a_n| \le 1/m^2$ for $n = 1, \dots, m-1$, induction shows that

$$|1 - \prod_{n=1}^{m-1} a_n| \le (1 + 1/m^2)^{m-1} \sum_{n=1}^{m-1} |1 - a_n| \le e/m.$$

Consequently,

$$|1 - \prod_{n=1}^{m-1} g_{n,k}(\phi_k(y_m))| \le e/m \to 0,$$

showing that

$$|H_m(\phi_k(y_m)) - e^{-i\theta_k}| = |g_{1,k} \cdots g_{m-1,k}(\phi_k(y_m)) - 1| \to 0$$

as $m \to \infty$. □

5. Other essential norm estimates

In the previous section we were able to compute the essential norm of a linear combination of composition operators assuming that there was a sequence, denoted (x_n) in Theorem 8, satisfying the hypothesis of the theorem. For the case of two functions, these hypotheses become

(a) $\rho(\phi(x_n), \psi(x_n)) \to 1$,
(b) $\rho(\phi(x_n), \phi(x)) \to 1$, and
(c) $\rho(\psi(x_n), \psi(x)) \to 1$, as $n \to \infty$.

On uniform algebras with connected spectrum, compact operators have range contained in a single Gleason part; that is, if ϕ is the map associated with our composition operator, there exists a positive constant $r < 1$ such that $\rho(\phi(x), \phi(y)) \le r$ (see [9] or [10]) for all x and y in the spectrum $M(A)$. Thus, we expect noncompactness to be associated with pseudohyperbolic distance tending to 1. However, in many cases pairs of functions may satisfy just two of the three conditions mentioned above. In that case, we can only obtain a lower bound on the essential norm. We begin this section by presenting that lower bound, and we conclude by giving an example to show that this lower bound is the best possible.

Theorem 9. *Let A be a uniform algebra and T_ϕ and T_ψ be endomorphisms defined on A with associated self-maps ϕ and ψ. Suppose that there exists a sequence (x_n) of points in $M(A)$ with cluster point x such that $\rho(\phi(x_n), \psi(x_n)) \to 1$ and either $\rho(\phi(x_n), \phi(x)) \to 1$ or $\rho(\psi(x_n), \psi(x)) \to 1$. Then $\|T_\phi - T_\psi\|_e \ge 1$.*

Proof. Without loss of generality, we may assume that $\rho(\phi(x_n), \phi(x)) \to 1$. If ϵ is an arbitrary but small positive number, we choose a sequence (ϵ_n) of positive real numbers tending to 0 such that $1 + \epsilon > \prod_{j=1}^{\infty}(1 + \epsilon_j)$. We will also choose a sequence, (δ_n), such that $\prod_{n=1}^{\infty}(1 - \delta_n) > 1 - \epsilon$. Our plan, as before, is to choose a subsequence, (y_n), of (x_n) and corresponding functions f_n, g_n, and k_n which we will use to construct a sequence (F_n) of functions in A that have the property that

$$F_n \to 0 \text{ weakly}, \ \|F_n\| \le 1 + \epsilon, \ F_n(\psi(y_n)) = 0,$$

and there exists a sequence (ρ_n), tending to 1 from below, such that

$$|F_n(\phi(y_n))| > (\rho_n - \epsilon_n)(1 - \epsilon).$$

Suppose, for the moment, that we have constructed such sequences. Let K be an arbitrary compact operator. Then

$$\begin{aligned} & \|T_\phi - T_\psi + K\| \\ \ge\ & (1/(1+\epsilon))\Big(\|T_\phi(F_n) - T_\psi(F_n)\| - \|K(F_n)\|\Big) \\ \ge\ & (1/(1+\epsilon))\Big(|F_n(\phi(y_n)) - F_n(\psi(y_n))| - \|K(F_n)\|\Big) \\ \ge\ & (1/(1+\epsilon))\Big((\rho_n - \epsilon_n)(1 - \epsilon) - \|K(F_n)\|\Big). \end{aligned}$$

Now let $n \to \infty$ and ϵ tend to zero, use the fact that compact operators take weakly null sequences to norm null sequences, and that K was an arbitrary compact operator to obtain the desired conclusion that

$$\|T_\phi - T_\psi\|_e \geq 1.$$

We turn now to the construction of our functions F_n:

Choosing y_1. We will apply Lemma 4 to a certain set of points. For this proof, we need a bit less from the lemma than we did in the previous one. We now begin the sequence at the first point, which we call y_1, where we choose $\rho_1 < \min\{\rho(\phi(y_1), \phi(x)), \rho(\phi(y_1), \psi(y_1))\}$ so close to 1 that

$$\frac{1 + \sqrt{1 - \rho_1^2}}{\rho_1^2} < 1 + \epsilon_1.$$

Thus, when we apply Lemma 4 to $\{\phi(y_1), \phi(x)\}$, we obtain partners f_1 and g_1 in A with

$$f_1(\phi(y_1)) = 1 \text{ and } g_1(\phi(x)) = 1$$

and (once again using the fact that $(1+\sqrt{1-\eta^2})/\eta^2$ decreases on $(0,1)$) we obtain

$$|f_1| + |g_1| < 1 + \epsilon_1$$

on $M(A)$. This is all we will need for the first step.

The sequence (y_n). Suppose that we have obtained points $y_1, y_2, \ldots, y_n$, with $y_l = x_{m_l}$ and $m_l < m_k$ if $l < k$, and corresponding functions $f_1, f_2, \ldots, f_n$ and $g_1, g_2, \ldots, g_n$ such that for $j = 1, \ldots, n$ we have

(a) $|f_j| + |g_j| < 1 + \epsilon_j$ on $M(A)$,
(b) $f_j(\phi(y_j)) = 1$,
(c) $g_j(\phi(x)) = 1$,
(d) $|g_k(\phi(y_j))| > 1 - \delta_k$ for $k < j$, and
(e) $\rho(\phi(y_j), \psi(y_j)) \geq \rho_j$, where $\rho_j \to 1$ (and ρ_j depends on ϵ_j).

Choosing y_{n+1}. We choose y_{n+1} using the following: Since $|g_j(\phi(x))| = 1$, and x is a cluster point of (x_m), we may choose a point y_{n+1} with

$$|g_j(\phi(y_{n+1}))| > 1 - \delta_j \text{ for } j = 1, \ldots, n. \tag{5}$$

We may also choose y_{n+1} so that $\rho(\phi(y_{n+1}), \phi(x))$ and $\rho(\phi(y_{n+1}), \psi(y_{n+1}))$ are as close to 1 as we please. Thus, by applying Lemma 4 again, this time to $\phi(y_{n+1})$ and $\phi(x)$, we may construct our partners f_{n+1} and g_{n+1} satisfying (a)–(c) above (with j replaced by $n+1$). We remark that inequality (5) shows that our functions will satisfy (d), and finally, (e) will be satisfied by our choice of y_{n+1}.

Now that we have our sequence (y_n), and our partners f_n and g_n, we construct our functions F_n. Since $\rho(\phi(y_n), \psi(y_n)) \geq \rho_n$, we may choose $k_n \in A$ with $\|k_n\| \leq 1$, $k_n(\psi(y_n)) = 0$ and $k_n(\phi(y_n)) > \rho_n - \epsilon_n$. Let

$$F_n = k_n f_n g_1 g_2 \cdots g_{n-1}.$$

It follows from Lemma 3 (the k_n do not appear in that lemma, but they can be bounded by their norm before applying the lemma) that on $M(A)$ we have

$$\sum_{j=1}^{n} |F_j| \leq \prod_{j=1}^{n} (1 + \epsilon_j) < 1 + \epsilon,$$

for all n. By Lemma 7 we know that $F_n \to 0$ weakly and $\|F_n\| < 1 + \epsilon$.

We now check to see what $F_n(\phi(y_n))$ and $F_n(\psi(y_n))$ are: Since $k_n(\psi(y_n)) = 0$, we see that $F_n(\psi(y_n)) = 0$. Now consider $F_n(\phi(y_n))$. By construction, the partners f_{n+1} and g_{n+1} satisfy conditions (a)–(d). We chose y_{n+1} (in inequality (5)) so that $|g_j(\phi(y_{n+1}))| > 1 - \delta_j$, for $j = 1, \dots, n$. Thus,

$$\begin{aligned}|F_n(\phi(y_n))| &= |k_n(\phi(y_n))|\,|f_n(\phi(y_n))|\,|g_1 g_2 \dots g_{n-1}(\phi(y_n))| \\ &> (\rho_n - \epsilon_n)\prod_{j=1}^{n-1}(1-\delta_j) \\ &\geq (\rho_n - \epsilon_n)(1-\epsilon),\end{aligned}$$

where the penultimate inequality follows from the fact that $|k_n(\phi(y_n))| > \rho_n - \epsilon_n$, property (d) of our sequence (g_j), and the fact that $f_n(\phi(y_n)) = 1$.

Therefore, F_n satisfies everything we needed to compute the essential norm, completing the proof of this theorem. □

Before proceeding to the next example, we remind the reader that if A is a uniform algebra on a compact Hausdorff space X, a closed subset E of X is said to be a *peak set* for A if there is a function $f \in A$ such that $f(x) = 1$ for all $x \in E$ and $|f(x)| < 1$ for $x \in X \setminus E$. Such a function f is said to be a *peak function* for A.

Example 10. Consider the disk algebra, $A := A(\mathbf{D})$ and its maximal ideal space, $M(A) = \overline{\mathbf{D}}$. Let $\phi_1, \phi_2 \in A$ be self-maps of $\overline{\mathbf{D}}$. Suppose that there exist disjoint peak sets E_1 and E_2 such that $\{|\phi_j| = 1\} \subset E_j$ for $j = 1, 2$. Then $\|T_{\phi_1} - T_{\phi_2}\|_e = 1$.

Proof. We first note that the hypotheses of the previous theorem apply in this case (as is easily checked) and therefore we know that $\|T_{\phi_1} - T_{\phi_2}\|_e \geq 1$.

We now turn to showing that $\|T_{\phi_1} - T_{\phi_2}\|_e \leq 1$. If f_j is a peak function corresponding to E_j, then we claim that the operator M_j defined by $M_j = (1 - f_j)T_{\phi_j}$ is compact. To see this, choose a sequence (g_n) in the unit ball of A. Passing to a subsequence, there exists a function g such that $g_{n_k} \to g$ uniformly on compact subsets of $\mathbf{D}$. Subtracting, we may suppose that our sequence converges to zero uniformly on compact subsets of $\mathbf{D}$. Let $\epsilon > 0$ be given. Then there exists a set $\mathcal{O}_j$, open in $M(A)$, such that $E_j \subseteq \mathcal{O}_j$ and $|1 - f_j| < \epsilon$ on $\mathcal{O}_j$. Since $M(A) \setminus \mathcal{O}_j$ is a compact subset of $M(A)$ on which $|\phi_j| < 1$, we see that $g_{n_k} \circ \phi_j$ tends to zero uniformly on this set. Thus, for n_k sufficiently large, we have $\|(1 - f_j)(g_{n_k} \circ \phi_j)\| < \epsilon$. So $(M_j(g_{n_k}))$ converges uniformly to 0, and M_j is compact.

Note that a peaking function to any power is still a peaking function, so the operator $M_{n,j}$ defined by $M_{n,j} = (1 - f_j^n)T_{\phi_j}$ is still compact. Now $\{f_1 = 1\} \cap \{f_2 = 1\} = \emptyset$, so for each $\epsilon > 0$ there exists N such that

$$|f_1(x)|^N + |f_2(x)|^N < 1 + \epsilon \text{ for all } x \in M(A).$$

Thus,

$$\begin{aligned}\|T_{\phi_1} - T_{\phi_2}\|_e &\leq \|T_{\phi_1} - T_{\phi_2} - M_{N,1} - M_{N,2}\| \\ &= \|f_1^N T_{\phi_1} - f_2^N T_{\phi_2}\| \\ &= \sup_{\{f : \|f\| = 1\}} \|f_1^N (f \circ \phi_1) - f_2^N (f \circ \phi_2)\| \\ &\leq \sup_{x \in M(A)} |f_1(x)|^N + |f_2(x)|^N \leq 1 + \epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ yields $\|T_{\phi_1} - T_{\phi_2}\|_e \leq 1$. Putting this together with the lower bound on the essential norm that we obtained in the beginning of this proof yields $\|T_{\phi_1} - T_{\phi_2}\|_e = 1$. $\square$

6. When is a linear combination compact?

Since we have obtained estimates on the essential norm of a linear combination of unital endomorphisms, it is reasonable to ask if we can use our techniques to characterize compact linear combinations. We begin with a general uniform algebra setting, and move to the more specific setting of $C(X)$ in the final two results in this paper.

A uniform algebra A with spectrum $M(A)$ is called a *K-algebra* if for every $x \in M(A)$ and $r \in (0,1)$ the set $\{y \in M(A) : \rho(y,x) \leq r\}$ is norm compact. In [10] it is shown that many familiar algebras are K-algebras. These include algebras of the form $C(X)$ for a compact Hausdorff space X, as well as the disk algebra and the algebra of bounded analytic functions on the open unit disk, $H^\infty(\mathbf{D})$. In fact, every logmodular algebra is a K-algebra.

Of course the pseudohyperbolic distance plays an important role here. We will only need to use the well-known fact that

$$\frac{\|x-y\|}{2} \leq \rho(x,y) \leq \|x-y\| \text{ for all } x, y \in M(A).$$

(A short proof of this can be found in [10], for example, and dates back to [11].)

The next theorem is concerned with *hyperbolically separated* sequences; that is, a sequence (x_n) in $M(A)$ for which there exists a positive number η satisfying $\rho(x_n, x_m) \geq \eta$ for all $n \neq m$. In [5] it is shown that in some uniform algebras hyperbolically separated sequences always have interpolating subsequences for the double dual of A, and in other algebras they do not.

Theorem 11. *Let A be a K-algebra and let $T_{\phi_1}, \dots, T_{\phi_N}$ be endomorphisms of A with associated self-maps $\phi_1, \dots, \phi_N$. If there exists a sequence of points (x_n) in $M(A)$ and a positive number η with the property that for some j,*

$$\inf_{n,m; n\neq m} \{\rho(\phi_j(x_n), \phi_j(x_m))\} \geq \eta$$

and for all n

$$\min_{k\neq j} \{\rho(\phi_j(x_n), \phi_k(x_n))\} \geq \eta,$$

then $\sum_{k=1}^N \lambda_k T_{\phi_k}$ is not compact for any choice of nonzero complex numbers $\lambda_1, \dots, \lambda_N$.

Proof. We first claim that, because A is a K-algebra and $(\phi_j(x_m))$ is hyperbolically separated, for j and for each k and p we have

$$\liminf_{m\to\infty} \rho(\phi_j(x_m), \phi_k(x_p)) = 1.$$

To see this, consider a positive number r with $0 < r < 1$. Now, by our assumption, for $x_0 \in M(A)$ each disk $D_r = \{x \in M(A) : \rho(x, x_0) \leq r\}$ is norm compact. Thus, each sequence in D_r has a norm convergent subsequence. Since $(\phi_j(x_m))$ is a hyperbolically separated sequence and

$$\frac{\|x-y\|}{2} \leq \rho(x,y) \leq \|x-y\|$$

for all x and y in $M(A)$, it cannot have a norm convergent subsequence. So, we see that D_r can contain only a finite number of points from the sequence $(\phi_j(x_m))$. Thus, for each r with $0 < r < 1$ we see that $\rho(\phi_j(x_m), \phi_k(x_p)) \geq r$ for all but finitely many m. Consequently,

$$\liminf_{m\to\infty} \rho(\phi_j(x_m), \phi_k(x_p)) = 1,$$

establishing the claim.

Let $\delta > 0$ and let (δ_j) be a sequence of positive numbers such that

$$\prod_j (1-\delta_j) > \delta.$$

Because of the claim established above, we may choose a subsequence of (x_n) satisfying the following: Choose $x_{n_1} = x_1$. Then choose x_{n_2} with

$$\rho(\phi_j(x_{n_2}), \phi_k(x_{n_1})) > (1-\delta_1)^{1/N} \text{ for all } k \in \{1, \dots, N\}.$$

Now assume that $x_{n_1}, \dots, x_{n_{m-1}}$ have been chosen such that for p satisfying $m-1 \geq p > l$ we have

$$\rho(\phi_j(x_{n_p}), \phi_k(x_{n_l})) > (1-\delta_l)^{1/N} \text{ for all } k \in \{1, \dots, N\}.$$

Choose x_{n_m} such that for $l < m$ and all $k \in \{1, \dots, N\}$ we have

$$\rho(\phi_j(x_{n_m}), \phi_k(x_{n_l})) > (1-\delta_l)^{1/N}.$$

We now construct a sequence of functions (F_p) in the unit ball of A such that

$$\begin{gathered} F_p(\phi_j(x_{n_p})) \geq (\eta/2)^{2N-1}\delta, \\ F_p(\phi_k(x_{n_p})) = 0 \text{ for } k \neq j, \text{ and} \\ F_p(\phi_k(x_{n_m})) = 0 \text{ for } m < p \text{ and all } k. \end{gathered}$$

Our construction proceeds as follows:

F_1 : Since $\rho(\phi_j(x_{n_1}), \phi_k(x_{n_1})) \geq \eta$ for $k \neq j$, we may choose F_1 in the unit ball of A so that

$$F_1(\phi_j(x_{n_1})) \geq (\eta/2)^{N-1} \text{ and } F_1(\phi_k(x_{n_1})) = 0 \text{ for } k \neq j.$$

F_2 : Since $\rho(\phi_j(x_{n_2}), \phi_k(x_{n_2})) \geq \eta$ for $k \neq j$, we may choose f_2 in the unit ball of A so that

$$f_2(\phi_j(x_{n_2})) \geq (\eta/2)^{N-1} \text{ and } f_2(\phi_k(x_{n_2})) = 0 \text{ for } k \neq j.$$

Since $\rho(\phi_j(x_{n_2}), \phi_k(x_{n_1})) > (1-\delta_1)^{1/N}$ for all $k \in \{1, \dots, N\}$, we see that we may multiply by another function, g_2, in the unit ball of A so that

$$\begin{gathered} g_2(\phi_j(x_{n_2})) \geq 1-\delta_1 \text{ and} \\ g_2(\phi_k(x_{n_1})) = 0 \text{ for all } k \in \{1, \dots, N\}. \end{gathered}$$

Multiplying these two functions yields a function F_2 in the unit ball of A satisfying

$$\begin{gathered} F_2(\phi_j(x_{n_2})) \geq (\eta/2)^{N-1} \cdot (1-\delta_1) > (\eta/2)^{N-1}\delta, \\ F_2(\phi_k(x_{n_2})) = 0 \text{ for } k \neq j, \text{ and} \\ F_2(\phi_k(x_{n_1})) = 0 \text{ for all } k \in \{1, \dots, N\}. \end{gathered}$$

F_3 : Since $\rho(\phi_j(x_{n_3}), \phi_k(x_{n_3})) \geq \eta$ for $k \neq j$, we may choose f_3 in the unit ball of A so that

$$f_3(\phi_j(x_{n_3})) \geq (\eta/2)^{2N-1} \text{ and } f_3(\phi_k(x_{n_3})) = 0 \text{ for } k \neq j.$$

Now, for all $k \in \{1, \dots, N\}$ we have

$$\rho(\phi_j(x_{n_3}), \phi_k(x_{n_1})) \geq (1-\delta_1)^{1/N} \text{ and } \rho(\phi_j(x_{n_3}), \phi_k(x_{n_2})) \geq (1-\delta_2)^{1/N},$$

so we may choose a function g_3, corresponding to

$$\{\phi_j(x_{n_3}), \phi_1(x_{n_1}), \dots, \phi_N(x_{n_1})\},$$

and a function h_3 corresponding to

$$\{\phi_j(x_{n_3}), \phi_1(x_{n_2}), \dots, \phi_N(x_{n_2})\}$$

such that both g_3 and h_3 are in the unit ball of A,

$$g_3(\phi_j(x_{n_3})) \geq 1 - \delta_1,$$
$$g_3(\phi_k(x_{n_1})) = 0 \text{ for all } k \in \{1, \dots, N\},$$

and

$$h_3(\phi_j(x_{n_3})) \geq 1 - \delta_2,$$
$$h_3(\phi_k(x_{n_2})) = 0 \text{ for all } k \in \{1, \dots, N\}.$$

Multiplying these three functions yields a function F_3 in the unit ball of A satisfying

$$F_3(\phi_j(x_{n_3})) \geq (\eta/2)^{2N-1}(1-\delta_1)(1-\delta_2) \geq (\eta/2)^{2N-1}\delta.$$

Continuing on in this fashion produces a bounded sequence of functions satisfying

$$F_p(\phi_j(x_{n_p})) \geq (\eta/2)^{2N-1}\delta,$$
$$F_p(\phi_k(x_{n_p})) = 0 \text{ for } k \neq j, \text{ and}$$
$$F_p(\phi_k(x_{n_m})) = 0 \text{ for } m < p \text{ and all } k,$$

as desired.

Suppose that $T = \sum_{k=1}^N \lambda_k T_{\phi_k}$ is compact. Then $(T(F_p))$ has a convergent subsequence, denoted $(T(F_{p_m}))$. Choose ϵ with $0 < \epsilon < \delta|\lambda_j|\eta^{(2N-1)}/2^{2N}$. Thus, there exists an integer M such that $\|T(F_{p_m}) - T(F_{p_l})\| < 2\epsilon$ for $l, m > M$. Fix $m > M$ for the moment, and choose $l > m > M$. Then our choice of F_{p_m} forces $F_{p_m}(\phi_k(x_{n_{p_m}})) = 0$ for all $k \neq j$, and our choice of F_{p_l} forces $F_{p_l}(\phi_k(x_{n_{p_m}})) = 0$ for all k. Thus

$$\begin{aligned}
\epsilon &> \|\sum_{k=1}^N \lambda_k T_{\phi_k}(F_{p_m} - F_{p_l})/2\| \\
&\geq \left|\lambda_j F_{p_m}(\phi_j(x_{n_{p_m}})) - \sum_{k=1}^N \lambda_k F_{p_l}(\phi_k(x_{n_{p_m}}))\right|/2 \\
&\geq |\lambda_j|(\eta^{2N-1}/2^{2N})\delta > \epsilon.
\end{aligned}$$

But this is impossible, and therefore T cannot be compact. □

Now we restrict ourselves to a compact Hausdorff space X and the algebra $C(X)$. Let $\phi_1, \dots, \phi_N$ be self-maps of X and for $N > 1$ let

$$E = \{x \in X : \phi_j(x) \neq \phi_k(x) \text{ for some } j \text{ and } k\}.$$

If $N = 1$, we interpret $\phi_1(E) = \phi_1(X)$.

Lemma 12. *Let X be a compact Hausdorff space and let $T_{\phi_1}, \dots, T_{\phi_N}$ be endomorphisms of $C(X)$ with associated self-maps $\phi_1, \dots, \phi_N$. If $N = 1$, we suppose that $\lambda_1 \neq 0$. For $N > 1$, we assume $\lambda_1, \dots, \lambda_N \in \mathbf{C} \setminus \{0\}$ and $\sum_{j=1}^N \lambda_j = 0$. Under these assumptions, if $\bigcup_{j=1}^N \phi_j(E)$ is finite, then the operator $T = \sum_{j=1}^N \lambda_j T_{\phi_j}$ is a compact operator on $C(X)$.*

Proof. Let (f_n) be a bounded sequence in $C(X)$. For each $x \in X \setminus E$, we know that $\phi_j(x) = \phi_k(x)$ for all j and k. Writing $y_x = \phi_j(x)$ (for all j) we find that

$$(Tf_n)(x) = \sum_{j=1}^N \lambda_j f_n(\phi_j(x)) = f_n(y_x)(\sum_{j=1}^N \lambda_j) = 0.$$

Thus $Tf_n = Tf_m$ on $X \setminus E$.

Since $\bigcup_{j=1}^N \phi_j(E)$ is finite, some subsequence of $(f_n \circ \phi_j)$ must converge uniformly on E for each j. Putting this together with our work above, we conclude that for each j some subsequence of $(T_{\phi_j} f_n)$ converges uniformly on X. Thus, a subsequence of (Tf_n) converges uniformly, and therefore T is compact. $\square$

Theorem 13. *Let X be a compact Hausdorff space. Let $T_{\phi_1}, \dots, T_{\phi_N}$ be endomorphisms of $C(X)$ with associated self-maps $\phi_1, \dots, \phi_N$. For $N = 1$ we suppose that $\lambda_1 \neq 0$ and for $N > 1$ we suppose $\lambda_1, \dots, \lambda_N$ are complex numbers satisfying $\sum_{j=1}^N \lambda_j = 0$ and $\sum_{j \in I} \lambda_j \neq 0$ for every nonempty proper subset I of $\{1, \dots, N\}$. Then $\sum_{j=1}^N \lambda_j T_{\phi_j}$ is compact if and only if $\bigcup_{j=1}^N \phi_j(E)$ is finite.*

Proof. For $N = 1$, if $\phi_1(X)$ is finite, the result follows from Lemma 12. If $\lambda_1 T_{\phi_1}$ is compact, the result is well known (and can be found in [9]).

So suppose that $N > 1$. Again, if $\bigcup_{j=1}^N \phi_j(E)$ is finite, the result follows from Lemma 12.

So suppose that T is compact. If $\bigcup_{j=1}^N \phi_j(E)$ is infinite, then there exists j_0 and a sequence (x_n) in E such that $(\phi_{j_0}(x_n))$ is a sequence of distinct points in X and, by the definition of the set E, we have $\phi_{j_0}(x_n) \neq \phi_k(x_n)$ for some k depending on n. Passing to a subsequence, if necessary, we may suppose that no point $\phi_{j_0}(x_n)$ is a cluster point of the remaining points in the sequence.

Let $I_n = \{k \in \{1, \dots, N\} : \phi_{j_0}(x_n) = \phi_k(x_n)\}$. Note that I_n is a proper, nonempty subset of $\{1, \dots, N\}$. Choose a weakly null sequence (f_n) in $C(X)$ with $f_n(\phi_{j_0}(x_n)) = 1$ and $f_n(\phi_j(x_n)) = 0$ if $j \notin I_n$. Then

$$\begin{aligned} |(Tf_n)(x_n)| &= |\sum_{j \in I_n} \lambda_j f_n(\phi_j(x_n)) + \sum_{j \notin I_n} \lambda_j f_n(\phi_j(x_n))| \\ &= |\sum_{j \in I_n} \lambda_j f_n(\phi_{j_0}(x_n))| \\ &= |\sum_{j \in I_n} \lambda_j|, \end{aligned}$$

and thus $|(Tf_n)(x_n)| \geq \min\{|\sum_{j\in J}\lambda_j| : \emptyset \subset J \subset \{1,\dots,N\}\} > 0$. So (Tf_n) is not a null sequence, and we conclude that T is not compact. $\square$

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