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NONDEGENERATE q-BIRESOLVING TEXTILE SYSTEMS AND EXPANSIVE AUTOMORPHISMS OF ONESIDED FULL SHIFTS

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Abstract. We study nondegenerate, q-biresolving textile systems and using properties of them, we prove a conjecture of Boyle and Maass on arithmetic constraints for expansive automorphisms of onesided full shifts and positively expansive endomorphisms of mixing topological Markov shifts. A similar result is also obtained for expansive leftmost-permutive endomorphisms of onesided full shifts.

1. Introduction

In [BM, Conjectures 7.1, 7.2 and 7.3], Boyle and Maass gave three conjectures on expansive automorphisms of onesided full shifts. The first conjecture was the one that the dynamical system defined by an expansive automorphism of a onesided full shift is topologically conjugate to a topological Markov shift. It was proved in [N4]. Therefore, if $\tilde{\varphi}$ is an expansive automorphism of the onesided full N-shift $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ over the alphabet A of N symbols, then there exists an N-to-one, positively expansive endomorphism φ_0 of a (twosided) topological Markov shift (X_0, σ_0) such that $(A^{\mathbf{N}}, \tilde{\varphi})$ and (X_0, σ_0) are topologically conjugate and so are $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ and (X_0, φ_0) through a common homeomorphism of $A^{\mathbf{N}}$ onto X_0. Boyle and Maass proved that in this situation the topological Markov shift (X_0, σ_0) is shift equivalent to some full J-shift such that (i) J and N are divisible by the same primes, and gave the second conjecture that in the same situation $(A^{\mathbf{N}}, \tilde{\varphi})$ is topologically conjugate to a full shift. The third conjecture of Boyle and Maass is stated as follows. In the same situation, the left and right multipliers l_{φ_0} and r_{φ_0} (developed by Boyle [B]) are integers such that all N, l_{φ_0} and r_{φ_0} are divisible by the same primes, and in particular (ii) if a prime p divides N, then p^2 divides N. (l_{φ_0} and r_{φ_0} are independent of the choice of φ_0 and have the property that $N = l_{\varphi_0} r_{\varphi_0}$ ([B]).) Boyle and Maass [BM] proved this conjecture for the case that N is a power of a prime. The result of Boyle and Maass above and the third conjecture imply that (i) and (ii) in them are a necessary condition for positive integers J and N to have the property that there is an automorphism of the onesided full N-shift which is conjugate to the full J-shift ([BM]). Boyle and Maass [BM] showed that (i) and (ii) are a sufficient condition for J and N to have the property.

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In this paper, we study nondegenerate, q-biresolving textile systems. Using properties of them (Proposition 3.9), we prove the third conjecture of Boyle and Maass and obtain a similar result for expansive leftmost-permutive endomorphisms of onesided full shifts. The second conjecture of Boyle and Maass is still unsettled as far as the author knows.

The results of Boyle and Maass above and their second and third conjectures have alternative equivalent statements (see [N4, Section 6]). The result stated first is equivalent to the statement "A mixing topological Markov shift having a positively expansive N-to-one endomorphism is shift equivalent to some full J-shift, where J and N are divisible by the same primes". The second conjecture is equivalent to the statement "A mixing topological Markov shift having a positively expansive endomorphism is conjugate to a full shift". The third conjecture is equivalent to the statement "If a positively expansive endomorphism φ of a mixing topological Markov shift is N-to-one, then l_φ and r_φ are integers such that all N, l_φ and r_φ are divisible by the same primes, and in particular if a prime p divides N, then p^2 divides N".

The reader is referred to [Ki] or [LM] for a comprehensive introduction to symbolic dynamics, and to [AH] for information on topological dynamics.

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2. Preliminaries

Let A be an alphabet (i.e., a nonempty finite set of symbols). Let $A^{\mathbf{Z}} = \{(a_j)_{j\in\mathbf{Z}} \mid a_j \in A\}$ be endowed with a metric compatible with the product topology of the discrete topology on A. Let $\sigma_A : A^{\mathbf{Z}} \to A^{\mathbf{Z}}$ be defined by

$$\sigma_A((a_j)_{j\in\mathbf{Z}}) = (a_{j+1})_{j\in\mathbf{Z}}.$$

The dynamical system $(A^{\mathbf{Z}}, \sigma_A)$ is called the *full shift* over A. Let X be a closed subset of $A^{\mathbf{Z}}$ with $\sigma_A(X) = X$. Let $\sigma = \sigma_A|X$. Then we have a dynamical system (X, σ), which is called a *subshift* over A. Let $A^{\mathbf{N}} = \{(a_j)_{j\in\mathbf{N}} \mid a_j \in A\}$ be endowed with a metric compatible with the product topology of the discrete topology on A. Let $\tilde{\sigma}_A : A^{\mathbf{N}} \to A^{\mathbf{N}}$ be defined by

$$\tilde{\sigma}_A((a_j)_{j\in\mathbf{N}}) = (a_{j+1})_{j\in\mathbf{N}}.$$

The dynamical system $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ is called the *onesided full shift* over A. For a subshift (X, σ) over A , let $\tilde{X} = \{(a_j)_{j\in\mathbf{N}} \mid \exists (a_j)_{j\in\mathbf{Z}} \in X\}$. Then we have a dynamical system $(\tilde{X}, \tilde{\sigma})$ with $\tilde{\sigma} = \tilde{\sigma}_A|\tilde{X}$, which is called a *onesided subshift* over A and is said to be *induced* by (X, σ).

Let G be a graph. Here a *graph* means a directed graph which may have multiple arcs and multiple loops. Let A_G and V_G denote the arc-set of G and the vertex-set of G, respectively. Let $i_G : A_G \to V_G$ and $t_G : A_G \to V_G$ be the mappings such that for $a \in A_G$, $i_G(a)$ and $t_G(a)$ are the initial and terminal vertices, respectively, of a in G. Hence the graph G is represented by $V_G \xleftarrow{i_G} A_G \xrightarrow{t_G} V_G$. Let X_G be the set of all points $(a_j)_{j\in\mathbf{Z}}$ in $A_G^{\mathbf{Z}}$ such that $t_G(a_j) = i_G(a_{j+1})$ for all $j \in \mathbf{Z}$. Then we have a subshift (X_G, σ_G), which is called the *topological Markov shift* defined by G.

For graphs Γ and G, a *graph-homomorphism* h of Γ into G, written by $h : \Gamma \to G$, is a pair (h_A, h_V) of mappings $h_A : A_\Gamma \to A_G$ (*arc-map*) and $h_V : V_\Gamma \to V_G$ (*vertex-map*) such that the following diagram is commutative:

$$\begin{array}{ccccc} V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ {\scriptstyle h_V}\downarrow & & {\scriptstyle h_A}\downarrow & & \downarrow{\scriptstyle h_V} \\ V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \end{array}$$

A graph-homomorphism $h : \Gamma \to G$ gives *1-block maps* $\phi_h : X_\Gamma \to X_G$ and $\tilde{\phi}_h : \tilde{X}_\Gamma \to \tilde{X}_G$ by

$$\phi_h((\alpha_j)_{j\in\mathbf{Z}}) = (h_A(\alpha_j)_{j\in\mathbf{Z}}), \quad (\alpha_j)_{j\in\mathbf{Z}} \in X_\Gamma,\ \alpha_j \in A_\Gamma,$$
$$\tilde{\phi}_h((\alpha_j)_{j\in\mathbf{N}}) = (h_A(\alpha_j)_{j\in\mathbf{N}}), \quad (\alpha_j)_{j\in\mathbf{N}} \in \tilde{X}_\Gamma,\ \alpha_j \in A_\Gamma.$$

Now we recall the notion of a textile system introduced in [N2].

A *textile system* T (*defined*) *over* a graph G is defined to be an ordered pair of graph-homomorphisms $p : \Gamma \to G$ and $q : \Gamma \to G$ such that $\alpha \in A_\Gamma$ is uniquely determined by the quadruple $(i_\Gamma(\alpha), t_\Gamma(\alpha), p_A(\alpha), q_A(\alpha))$. We write

$$T = (p, q : \Gamma \to G).$$

We have the following commutative diagram:

$$\begin{array}{ccccc} V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \\ \uparrow{\scriptstyle p_V} & & {\scriptstyle p_A}\uparrow & & {\scriptstyle p_V}\uparrow \\ V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ \downarrow{\scriptstyle q_V} & & {\scriptstyle q_A}\downarrow & & {\scriptstyle q_V}\downarrow \\ V_G & \xleftarrow{i_G} & A_G & \xrightarrow{t_G} & V_G \end{array}$$

If we observe this diagram vertically, then we have the ordered pair of graph-homomorphisms

$$\begin{array}{ccc} V_G & \xleftarrow{i_G} & A_G \\ \uparrow{\scriptstyle p_V} & & {\scriptstyle p_A}\uparrow \\ V_\Gamma & \xleftarrow{i_\Gamma} & A_\Gamma \\ \downarrow{\scriptstyle q_V} & & {\scriptstyle q_A}\downarrow \\ V_G & \xleftarrow{i_G} & A_G \end{array} \quad \text{and} \quad \begin{array}{ccc} A_G & \xrightarrow{t_G} & V_G \\ {\scriptstyle p_A}\uparrow & & {\scriptstyle p_V}\uparrow \\ A_\Gamma & \xrightarrow{t_\Gamma} & V_\Gamma \\ {\scriptstyle q_A}\downarrow & & {\scriptstyle q_V}\downarrow \\ A_G & \xrightarrow{t_G} & V_G \end{array}.$$

This defines another textile system

$$T^* = (p^*, q^* : \Gamma^* \to G^*)$$

called the *dual* of T, where $i_{\Gamma^*} = p_A, t_{\Gamma^*} = q_A$, $i_{G^*} = p_V$ and $t_{G^*} = q_V$.

Let $T = (p, q : \Gamma \to G)$ be a textile system. Let $\xi = \phi_p$ and let $\eta = \phi_q$. A two-dimensional configuration $(\alpha_{ij})_{i,j\in\mathbf{Z}}, \alpha_{ij} \in A_\Gamma$, is called a *textile* woven by T if $(\alpha_{ij})_{j\in\mathbf{Z}} \in X_\Gamma$ and $\eta((\alpha_{i-1,j})_{j\in\mathbf{Z}}) = \xi((\alpha_{ij})_{j\in\mathbf{Z}})$ for all $i \in \mathbf{Z}$. Let U_T denote the set of all textiles woven by T. Define

$$Z_T = \{(\alpha_{0j})_{j\in\mathbf{Z}} \mid \exists(\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T\}, \quad X_T = \{\xi((\alpha_{0j})_{j\in\mathbf{Z}}) \mid \exists(\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T\}.$$

Then we have subshifts (Z_T, ς_T) and (X_T, σ_T). We call (X_T, σ_T) the *woof shift* of T and (X_{T^*}, σ_{T^*}) the *warp shift* of T. We say that T is *nondegenerate* if $(X_T, \sigma_T) = (X_G, \sigma_G)$. We define onto maps $\xi_T : Z_T \to X_T$ and $\eta_T : Z_T \to X_T$ to be the restrictions of ξ and η, respectively. If T is *onesided 1-1*, i.e., ξ_T is 1-1, then an onto endomorphism φ_T of (X_T, σ_T) is defined by

$$\varphi_T = \eta_T \xi_T^{-1}.$$

If T is *1-1*, i.e., both ξ_T and η_T are 1-1, then φ_T is an automorphism of (X_T, σ_T). We also have the onesided subshifts $(\tilde{Z}_T, \tilde{\varsigma}_T)$ and $(\tilde{X}_T, \tilde{\sigma}_T)$ induced by (Z_T, ς_T) and (X_T, σ_T), respectively. We define maps $\tilde{\xi}_T : \tilde{Z}_T \to \tilde{X}_T$ and $\tilde{\eta}_T : \tilde{Z}_T \to \tilde{X}_T$ by the restrictions of $\tilde{\phi}_p$ and $\tilde{\phi}_q$, respectively. If T is onesided 1-1, then an onto continuous map $\bar{\chi}_T : X_T \to \tilde{X}_{T^*}$ is defined by

$$\bar{\chi}_T(\xi_T((\alpha_{1j})_{j\in\mathbf{Z}})) = \tilde{\xi}_{T^*}((\alpha_{i1})_{i\in\mathbf{N}}), \quad (\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T,\ \alpha_{ij} \in A_\Gamma.$$

If both $\tilde{\xi}_{T^*}$ and $\tilde{\eta}_{T^*}$ are 1-1, then an onto continuous map $\bar{\chi}_T^* : \tilde{X}_{T^*} \to X_T$ is defined by

$$\bar{\chi}_T^*(\tilde{\xi}_{T^*}((\alpha_{i1})_{i\in\mathbf{N}})) = \xi_T((\alpha_{1j})_{j\in\mathbf{Z}}), \quad (\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T,\ \alpha_{ij} \in A_\Gamma.$$

If $\tilde{\xi}_{T^*}$ is 1-1, then an onto continuous map $\hat{\chi}_{T^*} : \tilde{X}_{T^*} \to \tilde{X}_T$ is defined by

$$\hat{\chi}_{T^*}(\tilde{\xi}_{T^*}((\alpha_{i1})_{i\in\mathbf{N}})) = \tilde{\xi}_T((\alpha_{1j})_{j\in\mathbf{N}}), \quad (\alpha_{ij})_{i,j\in\mathbf{Z}} \in U_T,\ \alpha_{ij} \in A_\Gamma.$$

Theorem 2.1 (Theorem 2.11 of [N2])**.** *Let T be a textile system.*

(1) *If T is onesided 1-1, then φ_T is positively expansive if and only if both $\tilde{\xi}_{T^*}$ and $\tilde{\eta}_{T^*}$ are 1-1.*
(2) *If $\tilde{\xi}_{T^*}$ is 1-1, then $\tilde{\varphi}_{T^*}$ is an expansive continuous map if and only if T is onesided 1-1.*
(3) *If T is onesided 1-1 and both $\tilde{\xi}_{T^*}$ and $\tilde{\eta}_{T^*}$ are 1-1, then $\bar{\chi}_T$ is a homeomorphism with $\bar{\chi}_T^* = \bar{\chi}_T^{-1}$ and not only (X_T, σ_T) and $(\tilde{X}_{T^*}, \tilde{\varphi}_{T^*})$ but also (X_T, φ_T) and $(\tilde{X}_{T^*}, \tilde{\sigma}_{T^*})$ are topologically conjugate through $\bar{\chi}_T$.*

Let G be a graph. For $n \geq 0$, let $L_n(G)$ denote the set of all *paths* of length n in G and let $L(G)$ be the set of all paths in G, that is,

$$L_n(G) = \{a_1 \dots a_n \mid a_j \in A_G,\ t_G(a_j) = i_G(a_{j+1}) \text{ for } j = 1, \dots, n-1\} \quad \text{for } n \geq 1,$$

$$L_0(G) = V_G \quad \text{and} \quad L(G) = \bigcup_{n=0}^{\infty} L_n(G).$$

We extend $i_G : A_G \to V_G$ and $t_G : A_G \to V_G$ to $i_G : L(G) \to V_G$ and $t_G : L(G) \to V_G$, respectively, as follows. Define $i_G(v) = t_G(v) = v$ for $v \in V_G$. If $n \geq 1$, then for $w = a_1 \dots a_n \in L_n(G)$ with $a_j \in A_G$, we define $i_G(w) = i_G(a_1)$ and $t_G(w) = t_G(a_n)$. We also extend h_V and h_A of a graph-homomorphism $h : \Gamma \to G$ to $h : L(\Gamma) \to L(G)$ as follows. We define $h(\bar{v}) = h_V(\bar{v})$ for $\bar{v} \in V_\Gamma$. If $n \geq 1$, then for $\bar{w} = \alpha_1 \dots \alpha_n \in L_n(\Gamma)$ with $\alpha_j \in A_\Gamma$, we define $h(\bar{w}) = h_A(\alpha_1) \dots h_A(\alpha_n)$.

For $n \geq 2$, let $G^{[n]}$ be the graph defined as follows: $A_{G^{[n]}} = L_n(G), V_{G^{[n]}} = L_{n-1}(G)$, and $i_{G^{[n]}}$ and $t_{G^{[n]}}$ are the mappings such that for $w = a_1 \dots a_n \in A_{G^{[n]}}$ with $a_j \in A_G$, $i_{G^{[n]}}(w) = a_1 \dots a_{n-1}$ and $t_{G^{[n]}}(w) = a_2 \dots a_n$. We define $G^{[1]} = G$. We call $G^{[n]}$ the *higher block graph of order n* of G. For a graph-homomorphism $h : \Gamma \to G$ and $n \geq 1$, we define the *higher block homomorphism of order n* of h, written $h^{[n]} : \Gamma^{[n]} \to G^{[n]}$, by $h_A^{[n]}(w) = h(w), w \in L_n(\Gamma)$. For $n \geq 1$ the *n-th*

power G^n of G is defined to be the graph such that $A_{G^n} = L_n(G), V_{G^n} = V_G$ and $i_{G^n}(w) = i_G(w)$ and $t_{G^n}(w) = t_G(w)$ for $w \in L_n(G)$. For a graph-homomorphism $h : \Gamma \to G$ and $n \geq 1$, we define the *n-th product-power* $h^{(n)} : \Gamma^n \to G^n$ of h by $h_A^{(n)}(w) = h(w), w \in L_n(G)$.

For a textile system $T = (p, q : \Gamma \to G)$ such that the following definitions are well defined (see [N2, pp. 17, 18]) and for $n \geq 1$, we define the *higher block system* $T^{[n]}$ of order n of T by $T^{[n]} = (p^{[n]}, q^{[n]} : \Gamma^{[n]} \to G^{[n]})$, define the *n-th product-power* $T^{(n)}$ of T by $T^{(n)} = (p^{(n)}, q^{(n)} : \Gamma^n \to G^n)$, and define the *n-th composition-power* T^n of T by $T^n = ((T^*)^{(n)})^*$.

3. Nondegenerate q-biresolving textile systems

A graph G is said to be *nondegenerate* if both i_G and t_G are onto. A graph-homomorphism $h : \Gamma \to G$ is said to be *onto* if both h_A and h_V are onto.

An onto graph-homomorphism $h : \Gamma \to G$ between nondegenerate graphs is said to be *right resolving* if for each $u \in V_\Gamma$, the restriction of h_A on $i_\Gamma^{-1}(\{u\})$ is a bijection onto $i_G^{-1}(\{h_V(u)\})$. It is said to be *left resolving* if for each $u \in V_\Gamma$, the restriction of h_A on $t_\Gamma^{-1}(\{u\})$ is a bijection onto $t_G^{-1}(\{h_V(u)\})$.

A textile system $T = (p, q : \Gamma \to G)$ is said to be *q-biresolving* if q is both left resolving and right resolving. It is said to be *LL* if both p and q are left resolving, or equivalently T^* is q-biresolving.

For a graph G, let M_G denote its *adjacency matrix*, i.e., $M_G = \big(M_G(u, v)\big)_{u,v \in V_G}$, where $M_G(u, v)$ is the number of arcs starting from vertex u and ending in vertex v. Let λ_G denote the spectral radius of M_G.

Proposition 3.1. *Let $T = (p, q : \Gamma \to G)$ be a (nondegenerate) q-biresolving textile system with G irreducible, ϕ_p bounded-to-one and ϕ_q N-to-one.*

1. *If $(l(u))_{u \in V_G}$ and $(r(u))_{u \in V_G}$ are left and right eigenvectors of M_G corresponding to λ_G, then $(1, \ldots, 1)$ indexed by V_G and $(l(u)r(u))_{u \in V_G}$ are left and right eigenvectors of M_{G^*} corresponding to $\lambda_{G^*} = N$, where G^* is the graph over which T^* is defined.*
2. *If $\sigma_{(T^{[k]})^*}$ is topologically transitive for all $k \geq 1$ and G is aperiodic, then λ_G is a rational integer.*
3. *If T is onesided 1-1 and σ_{T^*} is topologically transitive, then $\bar{\chi}_T$ is measure preserving with respect to the Parry measures.*
4. *If T^* is 1-1 and σ_{T^*} is topologically transitive, then $\bar{\chi}_T^*$ is measure preserving with respect to the Parry measures.*
5. *If T^* is onesided 1-1 and σ_{T^*} is topologically transitive, then $\hat{\chi}_{T^*}$ is measure preserving with respect to the Parry measures.*

To prove Proposition 3.1, we need a lemma, which should be compared with [B, Lemma 2.1] and [H, Theorem 5.4].

Let $h : \Gamma \to G$ be a graph-homomorphism. Let $w \in L(G)$. Let $M_G(w)$ be the matrix $\big(M_G(w)(u, v)\big)_{u,v \in V_G}$ defined as follows. If $w \in V_G$, then $M_G(w)(u, v) = 1$ if $u = v = w$, and otherwise $M_G(w)(u, v) = 0$. If $w \in L_n(G)$ with $n \geq 1$, then $M_G(w)(u, v) = 1$ if $i_G(w) = u$ and $t_G(w) = v$, and otherwise $M_G(w)(u, v) = 0$. Define $M_h(w)$ to be the matrix $\big(M_h(w)(\bar{u}, \bar{v})\big)_{\bar{u},\bar{v} \in V_\Gamma}$ such that if $w \in V_G$, then $M_h(w)(\bar{u}, \bar{v}) = 1$ if $\bar{u} = \bar{v}$ and $h_V(\bar{u}) = w$, and otherwise $M_h(w)(\bar{u}, \bar{v}) = 0$. If $w \in L_n(G)$ with $n \geq 1$, then $M_h(w)(\bar{u}, \bar{v})$ is the number of paths $\bar{w} \in L_n(\Gamma)$ such that $i_\Gamma(\bar{w}) = \bar{u}, t_\Gamma(\bar{w}) = \bar{v}$ and $h(\bar{w}) = w$.

Lemma 3.2. *If $h : \Gamma \to G$ is a graph-homomorphism between irreducible graphs with $\lambda_G = \lambda_\Gamma = \lambda$ such that the 1-block map ϕ_h given by h is onto, then*

$$\frac{\sum_{\bar{w}\in h^{-1}(\{w\})} \bar{l}(i_\Gamma(\bar{w}))\bar{r}(t_\Gamma(\bar{w}))}{l(i_G(w))r(t_G(w))} = \frac{\bar{l}\bar{r}}{lr} \qquad \forall w \in L(G),$$

where $l = (l(u))_{u\in V_G}$ and $r = (r(u))_{u\in V_G}$ are left and right eigenvectors of M_G corresponding to λ, and $\bar{l} = (\bar{l}(\bar{u}))_{\bar{u}\in V_\Gamma}$ and $\bar{r} = (\bar{r}(\bar{u}))_{\bar{u}\in V_\Gamma}$ are those of M_Γ corresponding to λ.

Proof. We use the extremal argument of Boyle [B].

Since G and Γ are irreducible graphs with the same spectral radius and ϕ_h is onto, $h : L(\Gamma) \to L(G)$ is bounded-to-one and onto ([CP]). Therefore it follows that

$$\left\{ \frac{\bar{l}M_h(w)\bar{r}}{lM_G(w)r} \;\middle|\; w \in L(G) \right\}$$

is a finite set of positive numbers. Let s and t be respectively the minimum and maximum numbers among them. Let $w_1 \in L(G)$ be such that $(\bar{l}M_h(w_1)\bar{r})/(lM_G(w_1)r) = s$. We claim that

$$\frac{\bar{l}M_h(xw_1)\bar{r}}{lM_G(xw_1)r} = s \quad \forall x \in L(G) \text{ with } xw_1 \in L(G).$$

To prove this, let $L_1 = \{x \in L_k(G) \mid t_G(x) = i_G(w_1)\}$ with $k \geq 0$. Then

$$\begin{aligned} s &= \frac{\lambda^k \bar{l}M_h(w_1)\bar{r}}{\lambda^k lM_G(w_1)r} = \frac{\bar{l}M_\Gamma^k M_h(w_1)\bar{r}}{lM_G^k M_G(w_1)r} = \frac{\bar{l}(\sum_{x\in L_k(G)} M_h(x))M_h(w_1)\bar{r}}{l(\sum_{x\in L_k(G)} M_G(x))M_G(w_1)r} \\ &= \frac{\sum_{x\in L_1} \bar{l}M_h(xw_1)\bar{r}}{\sum_{x\in L_1} lM_G(xw_1)r}. \end{aligned}$$

Since $(\bar{l}M_h(xw_1)\bar{r})/(lM_G(xw_1)r) \geq s$ for all $x \in L_1$, if there were $x \in L_1$ with $(\bar{l}M_h(xw_1)\bar{r})/(lM_G(xw_1)r) > s$, then it would follow that

$$\frac{\sum_{x\in L_1} \bar{l}M_h(xw_1)\bar{r}}{\sum_{x\in L_1} lM_G(xw_1)r} > s.$$

Hence the claim is proved. Similarly we see that if $w_2 \in L(G)$ is such that $(\bar{l}M_h(w_2)\bar{r})/(lM_G(w_2)r) = t$, then

$$\frac{\bar{l}M_h(w_2y)\bar{r}}{lM_G(w_2y)r} = t \quad \forall y \in L(G) \text{ with } w_2y \in L(G).$$

Since G is irreducible, there is a path $z \in L(G)$ such that $w_2zw_1 \in L(G)$. By the above we have

$$s = \frac{\bar{l}M_h(w_2zw_1)\bar{r}}{lM_G(w_2zw_1)r} = t.$$

Therefore, for any $w \in L(G)$ and any $u \in V_G$, we have

$$\frac{\bar{l}M_h(w)\bar{r}}{lM_G(w)r} = \frac{\bar{l}M_h(u)\bar{r}}{lM_G(u)r} = \frac{\sum_{u\in V_G} \bar{l}M_h(u)\bar{r}}{\sum_{u\in V_G} lM_G(u)r} = \frac{\bar{l}\bar{r}}{lr}.$$

□

Proof of Proposition 3.1. (1) Since q is biresolving with ϕ_q N-to-one, q_V is N-to-one and onto. Hence in G^* every vertex has exactly N arcs ending in it, so that $(1,\dots,1)$ indexed by $V_G(=V_{G^*})$ is a left eigenvector of M_{G^*} corresponding to $\lambda_{G^*}=N$. Since $q:\Gamma\to G$ is biresolving with G irreducible, Γ is the disjoint union of finitely many irreducible graphs, say Γ_i's, with $\lambda_{\Gamma_i}=\lambda_G$ for all i. Since ϕ_p is bounded-to-one, if $p_i:\Gamma_i\to G$ is the restriction of p, then ϕ_{p_i} is bounded-to-one and onto. From [N1, Proposition 2.3], it follows that the row and column vectors $\bar{l}=(l(q_V(\bar{u})))_{\bar{u}\in V_\Gamma}$ and $\bar{r}=(r(q_V(\bar{u})))_{\bar{u}\in V_\Gamma}$ are left and right eigenvectors of M_Γ corresponding to $\lambda_\Gamma(=\lambda_G)$. Therefore, applying Lemma 3.2 to every p_i, we have

$$\frac{\sum_{\bar{u}\in p_V^{-1}(\{u\})} l(q_V(\bar{u}))r(q_V(\bar{u}))}{l(u)r(u)}=N \qquad \forall u\in V_G.$$

This implies that the strictly positive column vector $(l(u)r(u))_{u\in V_G}$ is a right eigenvector of M_{G^*} corresponding to $\lambda_{G^*}=N$.

(2) For $k\geq 1$, $V_{G^{[k]}}=L_{k-1}(G)$, and $(l(i_G(w)))_{w\in V_{G^{[k]}}}$ and $(r(t_G(w)))_{w\in V_{G^{[k]}}}$ are the left and right eigenvectors of $M_{G^{[k]}}$ corresponding to $\lambda_{G^{[k]}}=\lambda_G=N$. Hence applying the preceding equation (for T) to $T^{[k]}$ for every $k\geq 1$, we have

$$\frac{\sum_{\bar{w}\in p^{-1}(\{w\})} l(i_G(q(\bar{w})))r(t_G(q(\bar{w})))}{l(i_G(w))r(t_G(w))}=N \qquad \forall w\in L(G), \tag{3.1}$$

since for $h=p,q,(h^{[k]})_V(\bar{w})=h(\bar{w})$ for $\bar{w}\in V_{\Gamma^{[k]}}=L_{k-1}(\Gamma)$. Let G_k^* be the graph over which the textile system $(T^{[k]})^*$ is defined. Then $A_{G_k^*}=L_{k-1}(\Gamma), V_{G_k^*}=L_{k-1}(G)$ and for $\bar{w}\in A_{G_k^*}$, $i_{G_k^*}(\bar{w})=p(\bar{w})$ and $t_{G_k^*}(\bar{w})=q(\bar{w})$. Hence (3.1) implies that the strictly positive column vector $(l(i_G(w))r(t_G(w)))_{w\in L_{k-1}(G)}$ is a right eigenvector of $M_{G_k^*}$ corresponding to $\lambda_{G_k^*}=N$. Since $T^{[k]}$ is q-biresolving, $(T^{[k]})^*$ is LL, so that $(T^{[k]})^*$ is nondegenerate. Therefore, since $\sigma_{(T^{[k]})^*}$ is transitive for all $k\geq 1$, G_k^* is irreducible for all $k\geq 1$. Hence for all $k\geq 1$ the eigenvector $(l(i_G(w))r(t_G(w)))_{w\in L_{k-1}(G)}$ is a constant multiple of some column vector with all components in $\mathbf{Q}$. Let $u_0\in V_G$. Since G is irreducible and aperiodic, there exists $k\geq 1$ such that for every $u\in V_G$, there exists a path $w\in L_{k-1}(G)$ with $i_G(w)=u_0$ and $t_G(w)=u$. Therefore the column vector $(l(u_0)r(u))_{u\in V_G}$ is a constant multiple of a column vector with all components in $\mathbf{Q}$. Thus $\lambda_G\in\mathbf{Q}$. Since λ_G is an algebraic integer, it is a rational integer.

Before proving (3), (4) and (5), we make some preliminary arguments.

Let $s\geq -1, s'\geq 1$ and $t\geq 1$ be integers. A finite configuration of the form $(\alpha_{ij})_{1\leq i\leq t,-s\leq j\leq s'}$ with $\alpha_{i,j}\in A_\Gamma$ is called a *cloth* of *size* $t\times(s+s'+1)$ (woven by T), if $\alpha_{i,-s}\dots\alpha_{i,s'}\in L_{s+s'+1}(\Gamma)$ for $1\leq i\leq t$ and $q(\alpha_{i,-s}\dots\alpha_{i,s'})=p(\alpha_{i+1,-s}\dots\alpha_{i+1,s'})$ for all $1\leq i\leq t-1$. For a cloth $c=(\alpha_{ij})_{1\leq i\leq t,-s\leq j\leq s'}$, its uppermost G-path $p(\alpha_{1,-s}\dots\alpha_{1,s'})$, its lowermost G-path $q(\alpha_{t,1}\dots\alpha_{t,s'})$ and its axis G^*-path $(i_\Gamma(\alpha_{i,1}))_{1\leq i\leq t}$ are denoted by $\mathrm{p}(c),\mathrm{q}(c)$ and $\mathrm{o}^*(c)$, respectively. Let $C_{t,s,s'}$ denote the set of all cloths of size $t\times(s+s'+1)$. Since q is biresolving, $c\in C_{t,s,s'}$ is uniquely determined by $\mathrm{q}(c)$ and $\mathrm{o}^*(c)$.

If $x\in L_{s+s'+1}(G)$ and $C(x)=\{c\in C_{t,s,s'}\mid \mathrm{p}(c)=x\}$, then by (3.1) we have

$$\frac{\sum_{c\in C(x)} l(i_G(\mathrm{q}(c)))r(t_G(\mathrm{q}(c)))}{l(i_G(x))r(t_G(x))}=N^t.$$

For $x\in L_{s+s'+1}(G)$ and $y\in L_t(G^*)$, let $\Delta(x,y)=\{\mathrm{q}(c)\mid c\in C(x,y)\}$ with $C(x,y)=\{c\mid c\in C_{t,s,s'},\mathrm{p}(c)=x,\mathrm{o}^*(c)=y\}$. Then since $\mathrm{q}(c)\in\Delta(x,y)$ uniquely

determines $c \in C(x,y)$, it follows that

$$\frac{\sum_{y\in L_t(G^*)}\sum_{w\in\Delta(x,y)} l(i_G(w))r(t_G(w))}{l(i_G(x))r(t_G(x))} = N^t \quad \text{for } x \in L_{s+s'+1}(G). \tag{3.2}$$

For $y \in L_t(G^*)$, let $\Delta(y) = \{a_{-s}\dots a_{s'} \in L_{s+s'+1}(G) \mid a_j \in A_G, i_G(a_1) = t_{G^*}(y)\}$. Let $\Delta_1(y)$ be the set of all paths $w_1 \in L_{s+1}(G)$ with $t_G(w_1) = t_{G^*}(y)$ and let $\Delta_2(y)$ be the set of all paths $w_2 \in L_{s'}(G)$ with $i_G(w_2) = t_{G^*}(y)$. Then $\Delta(y) = \{w_1w_2 \mid w_1 \in \Delta_1(y), w_2 \in \Delta_2(y)\}$. Thus

$$\begin{aligned}\sum_{w\in\Delta(y)} l(i_G(w))r(t_G(w)) &= (\sum_{w_1\in\Delta_1(y)} l(i_G(w_1)))(\sum_{w_2\in\Delta_2(y)} r(t_G(w_2))) \\ &= \lambda_G^{s+1} l(t_G(w_1))\lambda_G^{s'} r(i_G(w_2)) = \lambda_G^{s+s'+1} l(t_{G^*}(y))r(t_{G^*}(y)).\end{aligned} \tag{3.3}$$

(3) Let $y \in L_t(G^*)$ with $t \geq 1$. Since $\bar{\chi}_T$ is continuous, for the clopen set

$$[y] = \{(b_i)_{i\in\mathbf{N}} \in \tilde{X}_{G^*} \mid b_1\dots b_t = y\},$$

there exist $s \geq 0, s' \geq 1$ and distinct $x_1,\dots,x_m \in L_{s+s'+1}(G)$ such that

$$\bar{\chi}_T^{-1}([y]) = [_{-s}\ x_1] \cup \dots \cup [_{-s}\ x_m],$$

where for $x \in L_{s+s'+1}(G)$, $[_{-s}\ x]$ denotes the cylinder set $\{(a_j)_{j\in\mathbf{Z}} \in X_G \mid a_{-s}\dots a_{s'} = x\}$. Since q is biresolving, for any $w \in \Delta(y)$ there exists a unique $c \in C_{t,s,s'}$ such that $\mathrm{q}(c) = w$ and $\mathrm{o}^*(c) = y$, and for any such c, $\mathrm{p}(c) \in \{x_1,\dots,x_m\}$ because c can be extended to a textile (because q is biresolving and T is nondegenerate). Therefore, we have $\Delta(y) = \bigcup_{k=1}^m \Delta(x_k,y)$ with $\Delta(x_k,y) \cap \Delta(x_{k'},y) = \emptyset$, $1 \leq k < k' \leq m$. Thus using (3.2) we have

$$\begin{aligned}\sum_{w\in\Delta(y)} l(i_G(w))r(t_G(w)) &= \sum_{k=1}^m \sum_{w\in\Delta(x_k,y)} l(i_G(w))r(t_G(w)) \\ &= \sum_{k=1}^m N^t l(i_G(x_k))r(t_G(x_k)).\end{aligned}$$

Hence by (3.3) we have

$$\frac{l(t_{G^*}(y))r(t_{G^*}(y))}{N^t} = \frac{\sum_{k=1}^m l(i_G(x_k))r(t_G(x_k))}{\lambda_G^{s+s'+1}}. \tag{3.4}$$

Since G and G^* are irreducible, the Parry measures μ_G and $\tilde{\mu}_{G^*}$ can be considered for σ_G and $\tilde{\sigma}_{G^*}$. The above implies that

$$\tilde{\mu}_{G^*}([y]) = \sum_{k=1}^m \mu_G([_{-s}\ x_k]) = \mu_G(\bar{\chi}_T^{-1}([y])).$$

Thus $\bar{\chi}_T$ is measure preserving with respect to the Parry measures.

(4) Let $x \in L_{s+s'+1}(G)$ with $s \geq 0$ and $s' \geq 1$. Since $\bar{\chi}_T^*$ is continuous, there exist $t \geq 1$ and distinct $y_1,\dots,y_n \in L_t(G^*)$ such that

$$(\bar{\chi}_T^*)^{-1}([_{-s}\ x]) = [y_1] \cup \dots \cup [y_n].$$

Since $\Delta(x, y_k) = \Delta(y_k)$ for $1 \le k \le n$, using (3.2) and (3.3) we have

$$N^t l(i_G(x))r(t_G(x)) = \sum_{k=1}^{n} \sum_{w \in \Delta(x,y_k)} l(i_G(w))r(t_G(w)) = \sum_{k=1}^{n} \sum_{w \in \Delta(y_k)} l(i_G(w))r(t_G(w))$$
$$= \sum_{k=1}^{n} \lambda_G^{s+s'+1} l(t_{G^*}(y_k))r(t_{G^*}(y_k)),$$

so that we have

$$\frac{l(i_G(x))r(t_G(x))}{\lambda_G^{s+s'+1}} = \frac{\sum_{k=1}^{n} l(t_{G^*}(y_k))r(t_{G^*}(y_k))}{N^t}.$$

Thus we have

$$\mu_G([_{-s}\ x]) = \sum_{k=1}^{n} \tilde{\mu}_{G^*}([y_k]) = \tilde{\mu}_{G^*}((\bar{\chi}_T^*)^{-1}([_{-s}\ x])).$$

(5) Let $x \in L_{s'}(G)$ with $s' \ge 1$. Since $\hat{\chi}_{T^*}$ is continuous, there exist $t \ge 1$ and distinct $y_1, \dots, y_n \in L_t(G^*)$ such that

$$(\hat{\chi}_{T^*})^{-1}([x]) = [y_1] \cup \cdots \cup [y_n].$$

If we consider the argument in the proof of (4) for the case that $s = -1$, we see that

$$\frac{l(i_G(x))r(t_G(x))}{\lambda_G^{s'}} = \frac{\sum_{k=1}^{n} l(t_{G^*}(y_k))r(t_{G^*}(y_k))}{N^t}. \tag{3.5}$$

Thus we have

$$\tilde{\mu}_G([x]) = \sum_{k=1}^{n} \tilde{\mu}_{G^*}([y_k]) = \tilde{\mu}_{G^*}((\bar{\chi}_T^*)^{-1}([x])).$$

□

We remark that (3) ((4)) of Proposition 3.1 combined with Theorem 4.1 (in Section 4) and [Wa, the proof of Theorem 8.8] give another proof of the result of Boyle and Maass [BM]: if φ is a positively expansive endomorphism of a mixing topological Markov shift (X, σ), then σ and φ have the same measure of maximal entropy.

Corollary 3.3. *Let $T = (p, q : \Gamma \to G)$ be a (nondegenerate) q-biresolving textile system with G irreducible and aperiodic, ϕ_p bounded-to-one and ϕ_q N-to-one.*

1. *If T is onesided 1-1 and φ_T is topologically transitive, then λ_G is an integer, say J, and every prime dividing N divides J.*
2. *If T^* is onesided 1-1, then λ_G is an integer, say J, and every prime dividing J divides N.*
3. *If both T and T^* are onesided 1-1, then λ_G is an integer, say J, and J and N are divisible by the same primes.*

Proof. (1) Since φ_T is topologically transitive, it follows that $\sigma_{(T^{[k]})^*}$ is topologically transitive for all $k \ge 1$. Hence λ_G is an integer, say J, by (2) of Proposition 3.1. Thus we may assume that all the components of $(l(u))_{u \in V_G}$ and $(r(u))_{u \in V_G}$ in the proposition are integers. By (3.4) we have

$$J^{s+s'+1} l(t_{G^*}(y))r(t_{G^*}(y)) = N^t \sum_{k=1}^{m} l(i_G(x_k))r(t_G(x_k)).$$

Since the length t of y can be arbitrarily large without changing $l(t_{G^*}(y))r(t_{G^*}(y))$, every prime dividing N must divide J.

(2) It follows from [N4, Proposition 5.1] that (X_{T^*}, σ_{T^*}) is conjugate to a full shift. Since T^* is onesided 1-1, $(X_{(T^{[k]})^*}, \sigma_{(T^{[k]})^*})$ and (X_{T^*}, σ_{T^*}) are topologically conjugate for all $k \geq 1$, so that $\sigma_{(T^{[k]})^*}$ is topologically transitive for all $k \geq 1$. Thus (2) is proved by a discussion similar to that in (1) by using (3.5).

(3) By the proofs of (1) and (2), (3) is proved. □

We also remark that Corollary 3.3(3) combined with Theorem 4.1 (in Section 4) recovers a small part of the result of Boyle and Maass [BM] stated in the Introduction.

Marcus proved in [M] the theorem that if G is an irreducible graph whose spectral radius is an integer J, then there exists an irreducible graph G' such that (X_G, σ_G) and $(X_{G'}, \sigma_{G'})$ are topologically conjugate, each row sum of $M_{G'}$ is J and each column sum of $M_{G'}$ is J. His theorem and his proof are the right things to make use of in order to prove the third conjecture of Boyle and Maass in [BM]. His proof will be described in a way suitable to our needs.

For a mapping $f : A \to B$ between finite sets, we define a (rectangular) matrix $M_f = \big(M_f(u,v)\big)_{u\in A, v\in B}$ such that $M_f(u,v) = 1$ if $f(u) = v$, and otherwise $M_f(u,v) = 0$. For a matrix M, M^t denotes its transpose. Hence, for example, $M_G = M_{i_G}^t M_{t_G}$ for a graph G.

Let G be a nondegenerate graph. Let $\overset{e}{\sim}$ be an equivalence relation on A_G such that for $a_1, a_2 \in A_G$, if $a_1 \overset{e}{\sim} a_2$, then $i_G(a_1) = i_G(a_2)$, where e denotes the canonical map of $\overset{e}{\sim}$ which maps each $a \in A_G$ to the equivalence class $e(a)$ containing a. Let C_e denote the set of equivalence classes of $\overset{e}{\sim}$. Let $\overline{C}_e = \{\overline{e}(a) \mid a \in A_G\}$, where $\overline{e}(a)$ is merely a copy of $e(a)$. Let H_e be a bipartite graph defined as follows. Let $A_{H_e} = A_G \cup \overline{C}_e$ and let $V_{H_e} = V_G \cup C_e$. We define i_{H_e} and t_{H_e} by

$$i_G(a) \overset{i_{H_e}}{\longleftarrow\!\shortmid} \overline{e}(a) \overset{t_{H_e}}{\longmapsto} e(a), \qquad e(a) \overset{i_{H_e}}{\longleftarrow\!\shortmid} a \overset{t_{H_e}}{\longmapsto} t_G(a), \qquad a \in A_G.$$

Let $L_e = M_e^t$. We define $\check{e} : C_e \to V_G$ by $e(a) \mapsto i_G(a)$. Let $L_{\check{e}} = M_{\check{e}}^t$. Let $R = M_{t_G}$. Then M_{H_e} is written in the form

$$M_{H_e} = \begin{pmatrix} 0 & L_{\check{e}} \\ L_e R & 0 \end{pmatrix},$$

where the indexing set is $V_G \cup C_e$. The bipartite graph H_e induces a pair of graphs $\acute{H}_e$ and $\grave{H}_e$ such that the arcs of $\acute{H}_e$ are the paths of length 2 in H_e each starting from a vertex in V_G and ending in a vertex in V_G, and the arcs of $\grave{H}_e$ are the paths of length 2 in H_e starting from a vertex in C_e and ending in a vertex in C_e. There exists a graph-isomorphism $k_e : G \to \acute{H}_e$ with $(k_e)_A : a \mapsto \overline{e}(a)a, a \in A_G$. There exists a "bipartite" conjugacy $\zeta_e : (X_{\acute{H}_e}, \sigma_{\acute{H}_e}) \to (X_{\grave{H}_e}, \sigma_{\grave{H}_e})$ defined by

$$(\overline{e}(a_j)a_j)_{j\in\mathbf{Z}} \overset{\zeta_e}{\mapsto} (a_j\overline{e}(a_{j+1}))_{j\in\mathbf{Z}}, \qquad (a_j)_{j\in\mathbf{Z}} \in X_G.$$

Define $G_e = \grave{H}_e$. Define $\psi_e = \zeta_e\kappa_e$, where κ_e is the "symbolic" conjugacy given by $\kappa_e = \phi_{k_e}$. Then

(i) ψ_e is a conjugacy (a "bipartite code") of (X_G, σ_G) onto (X_{G_e}, σ_{G_e}).

We have

$$\begin{pmatrix} M_G & 0 \\ 0 & M_{G_e} \end{pmatrix} = \begin{pmatrix} M_{\acute{H}_e} & 0 \\ 0 & M_{\grave{H}_e} \end{pmatrix} = (M_{H_e})^2 = \begin{pmatrix} L_{\check{e}} L_e R & 0 \\ 0 & L_e R L_{\check{e}} \end{pmatrix},$$

so that we have $M_G = L_{\check{e}} L_e R$ and $M_{G_e} = L_e R L_{\check{e}}$. Since $(1, \dots, 1) M_G = (1, \dots, 1) R$ and $(1, \dots, 1) M_{G_e} = (1, \dots, 1) R L_{\check{e}}$, we know that

(ii) the set of column sums of M_{G_e} is the same as the set of column sums of M_G.

We also know that if r is a right eigenvector of M_G corresponding to an eigenvalue λ, then $L_e R r$ is a right eigenvector of M_{G_e} corresponding to λ, so that

(iii) if $(r_u)_{u \in V_G}$ is a right eigenvector of M_G corresponding to an eigenvalue λ, then $(s_v)_{v \in C_e}$ with $s_v = \sum_{a \in v} r_{t_G(a)}$ is that of M_{G_e} corresponding to λ.

If G is irreducible and the spectral radius of G is an integer, say J, then M_G has the smallest positive integer-valued right eigenvector corresponding to J, which Marcus called the *smallest right eigenvector* corresponding to J.

Noting (iii), we see that Marcus's proof in [M] proves:

Proposition 3.4 (Marcus, [M])**.** *Let G be an irreducible graph whose spectral radius is an integer J. Assume that the smallest right eigenvector $(r_u)_{u \in V_G}$ of M_G corresponding to J has a component $r_u > 1$. Then there exists an equivalence relation $\overset{e}{\sim}$ on A_G such that*

1. *for $a_1, a_2 \in A_G$, $a_1 \overset{e}{\sim} a_2$ implies $i_G(a_1) = i_G(a_1)$, and hence* (i) *and* (ii) *above hold for $\lambda = J$;*
2. *M_{G_e} has the smallest right eigenvector $(s_v)_{v \in C_e}$ corresponding to J such that*
   (a) $\sum_{v \in C_e} s_v = \sum_{u \in V_G} r_u$,
   (b) $s_v \le \max_{u \in V_G} r_u$ *for all $v \in C_e$, and*
   (c) $\sharp \{ v \in C_e \mid s_v = \max_{u \in V_G} r_u \} < \sharp \{ u \in V_G \mid r_u = \max_{u \in V_G} r_u \}$.

Marcus's proof of his theorem is completed in the following way.

Let G be an irreducible graph whose spectral radius is an integer J. Then using the "transpose" version of the above proposition a necessary number of times, we obtain an irreducible graph G' such that $(X_{G'}, \sigma_{G'})$ and (X_G, σ_G) are topologically conjugate and $M_{G'}$ has the left eigenvector corresponding to J with all components equal to 1 (i.e., each column sum of $M_{G'}$ is J). Then using the proposition above a necessary number of times, we obtain from G' a graph G'' such that $(X_{G''}, \sigma_{G''})$ and (X_G, σ_G) are topologically conjugate and $M_{G''}$ has the left and right eigenvectors corresponding to J with all components equal to 1.

Textile systems T and T' are said to be *topologically conjugate* if there exist topological conjugacies $\psi : (X_T, \sigma_T) \to (X_{T'}, \sigma_{T'})$ and $\Psi : (Z_T, \varsigma_T) \to (Z_{T'}, \varsigma_{T'})$ such that $\psi \xi_T = \xi_{T'} \Psi$ and $\psi \eta_T = \eta_{T'} \Psi$.

For $k \ge 1$, a left (resp. right) resolving graph-homomorphism $h : \Gamma \to G$ is said to be *k-definite* if for any path $\bar{w} \in L_k(\Gamma)$, $h(\bar{w})$ uniquely determines $i_\Gamma(\bar{w})$ (resp. $t_\Gamma(\bar{w})$). As is easily seen, a left or right resolving graph-homomorphism $h : \Gamma \to G$ is k-definite for some $k \ge 1$ if and only if the 1-block map $\phi_h : (X_\Gamma, \sigma_\Gamma) \to (X_G, \sigma_G)$ is a conjugacy.

Lemma 3.5. *Let $T = (p, q : \Gamma \to G)$ be a nondegenerate, q-biresolving textile system such that p^* is 1-definite, where $T^* = (p^*, q^* : \Gamma^* \to G^*)$. Let $\overset{e}{\sim}$ be an equivalence relation on A_G such that for $a_1, a_2 \in A_G$, if $a_1 \overset{e}{\sim} a_2$, then $i_G(a_1) =$*

$i_G(a_2)$. Then (G_e and ψ_e being as above) there exists a nondegenerate q-biresolving textile system $T_e = (p_e, q_e : \Gamma_e \to G_e)$ satisfying the following:

(1) *T and T_e are topologically conjugate, more precisely, there exists a conjugacy (a bipartite code) $\Psi_e : Z_T \to Z_{T_e}$ such that $\psi_e \xi_T = \xi_{T_e} \Psi_e$ and $\psi_e \eta_T = \eta_{T_e} \Psi_e$.*
(2) *T_e^* is onesided 1-1 and if T^* is 1-1, then so is T_e^*.*

Proof. We use the same method as in [N2, the proof of Lemma 7.14].

Let $\overset{E}{\sim}$ be the equivalence relation on A_Γ defined as follows. For $\alpha_1, \alpha_2 \in A_\Gamma$,

$$\alpha_1 \overset{E}{\sim} \alpha_2 \overset{\text{def}}{\Leftrightarrow} (i_\Gamma(\alpha_1) = i_\Gamma(\alpha_2)) \wedge (q_A(\alpha_1) \overset{e}{\sim} q_A(\alpha_2)).$$

Let E be the canonical map of $\overset{E}{\sim}$ and let C_E be the set of equivalence classes $E(\alpha), \alpha \in A_\Gamma$. Let $\overline{C}_E = \{\overline{E}(\alpha) \mid \alpha \in A_\Gamma\}$, where $\overline{E}(\alpha)$ denotes a copy of $E(\alpha)$. Let Δ be the bipartite graph defined for $\overset{E}{\sim}$ in the same way that H_e was defined for $\overset{e}{\sim}$. Actually, let $A_\Delta = A_\Gamma \cup \overline{C}_E$, let $V_\Delta = V_\Gamma \cup C_E$ and define i_Δ and t_Δ by

$$i_\Gamma(\alpha) \overset{i_\Delta}{\longleftarrow\!\!\shortmid} \overline{E}(\alpha) \overset{t_\Delta}{\longmapsto} E(\alpha), \qquad E(\alpha) \overset{i_\Delta}{\longleftarrow\!\!\shortmid} \alpha \overset{t_\Delta}{\longmapsto} t_\Gamma(\alpha), \qquad \alpha \in A_\Gamma.$$

We define a "bipartite" textile system (see [As]) $T_e^b = (p_e^b, q_e^b : \Delta \to H_e)$ by

$$(p_e^b)_A(\overline{E}(\alpha)) = \overline{e}(p_A(\alpha)), \qquad (p_e^b)_A(\alpha) = p_A(\alpha),$$
$$(q_e^b)_A(\overline{E}(\alpha)) = \overline{e}(q_A(\alpha)), \qquad (q_e^b)_A(\alpha) = q_A(\alpha)$$

with $\alpha \in A_\Gamma$. These p_e^b and q_e^b are well defined. For if $\alpha_1 \overset{E}{\sim} \alpha_2$ with $\alpha_1, \alpha_2 \in A_\Gamma$, then $i_\Gamma(\alpha_1) = i_\Gamma(\alpha_2)$ and $q_A(\alpha_1) \overset{e}{\sim} q_A(\alpha_2)$, the former of which implies $p_A(\alpha_1) = p_A(\alpha_2)$ because p^* is a 1-definite left resolving graph homomorphism.

Consider the second product-power $(T_e^b)^{(2)}$. This is considered to be the disjoint union of the textile systems $\acute{T}_e^b = (\acute{p}_e^b, \acute{q}_e^b : \acute{\Delta} \to \acute{H}_e)$ and $\grave{T}_e^b = (\grave{p}_e^b, \grave{q}_e^b : \grave{\Delta} \to \grave{H}_e)$, where $\acute{\Delta}$ is the graph whose arcs are the paths of length 2 in Δ going from a vertex in V_Γ to a vertex in V_Γ and $\grave{\Delta}$ is the graph whose arcs are the paths of length 2 in Δ going from a vertex in C_E to a vertex in C_E. Hence $\Delta^2 = \acute{\Delta} \sqcup \grave{\Delta}$. Each arc in $\acute{\Delta}$ is of the form $\overline{E}(\alpha)\alpha, \alpha \in A_\Gamma$, for which we have

$$(\acute{p}_e^b)_A(\overline{E}(\alpha)\alpha) = \overline{e}(p_A(\alpha))p_A(\alpha) \in A_{\acute{H}_e} \text{ and } (\acute{q}_e^b)_A(\overline{E}(\alpha)\alpha) = \overline{e}(q_A(\alpha))q_A(\alpha) \in A_{\acute{H}_e}.$$

Each arc in $\grave{\Delta}$ is of the form $\alpha\overline{E}(\beta), \alpha\beta \in L_2(\Gamma), \alpha, \beta \in A_\Gamma$, for which we have

$$(\grave{p}_e^b)_A(\alpha\overline{E}(\beta)) = p_A(\alpha)\overline{e}(p_A(\beta)) \in A_{\grave{H}_e} \text{ and } (\grave{q}_e^b)_A(\alpha\overline{E}(\beta)) = q_A(\alpha)\overline{e}(q_A(\beta)) \in A_{\grave{H}_e}.$$

There exists a graph-isomorphism $k_E : \Gamma \to \acute{\Delta}$ given by $(k_E)_A(\alpha) = \overline{E}(\alpha)\alpha, \alpha \in A_\Gamma$. T and $\acute{T}_e^b$ are "isomorphic" through the "textile-system-isomorphism" (k_E, k_e), because $\acute{p}_e^b k_E = k_e p$ and $\acute{q}_e^b k_E = k_e q$ (see [As]).

If we define a "bipartite" conjugacy $\zeta_E : (X_{\acute{\Delta}}, \sigma_{\acute{\Delta}}) \to (X_{\grave{\Delta}}, \sigma_{\grave{\Delta}})$ by

$$(\overline{E}(\alpha_j)\alpha_j)_{j\in\mathbf{Z}} \overset{\zeta_E}{\mapsto} (\alpha_j\overline{E}(\alpha_{j+1}))_{j\in\mathbf{Z}}, \qquad (\alpha_j)_{j\in\mathbf{Z}} \in X_\Gamma,$$

then we have the commutative diagram

$$\begin{array}{ccc} X_{\acute{H}_e} & \xrightarrow{\zeta_e} & X_{\grave{H}_e} \\ {\scriptstyle \phi_{\acute{p}_e^b}}\uparrow & & \uparrow{\scriptstyle \phi_{\grave{p}_e^b}} \\ X_{\acute{\Delta}} & \xrightarrow{\zeta_E} & X_{\grave{\Delta}} \\ {\scriptstyle \phi_{\acute{q}_e^b}}\downarrow & & \downarrow{\scriptstyle \phi_{\grave{q}_e^b}} \\ X_{\acute{H}_e} & \xrightarrow{\zeta_e} & X_{\grave{H}_e} \end{array}$$

Therefore if we define $T_e = \grave{T}_e^b$ and $\Psi_e = \zeta_E \kappa_E$ with $\kappa_E = \phi_{k_E}$, then we have $\psi_e \xi_T = \xi_{T_e} \Psi_e$ and $\psi_e \eta_T = \eta_{T_e} \Psi_e$, and hence T_e satisfies (1).

To see that $T_e = \grave{T}_e^b$ is nondegenerate, it suffices to note that T_e^b is nondegenerate, which follows from the fact that $\acute{T}_e^b$ is nondegenerate.

Next we prove that T_e is q-biresolving. To do this, it suffices to show that T_e^b is q-biresolving. Let $\bar{u} \in V_\Delta$. Then $\bar{u} \in V_\Gamma$ or $\bar{u} \in C_E$. First consider the case that $\bar{u} \in V_\Gamma$. Then $(q_e^b)_V(\bar{u}) = q_V(\bar{u})$. Each arc in H_e starting from $q_V(\bar{u})$ is of the form $\overline{e}(a)$ with $a \in A_G$ and $i_G(a) = q_V(\bar{u})$. Since q is right resolving, there exists a unique arc $\alpha \in A_\Gamma$ such that $i_\Gamma(\alpha) = \bar{u}$ and $q_A(\alpha) = a$. It follows that $\overline{E}(\alpha)$ is a unique arc in Δ such that $i_\Delta(\overline{E}(\alpha)) = \bar{u}$ and $(q_e^b)_A(\overline{E}(\alpha)) = \overline{e}(a)$. Each arc in H_e ending in $q_V(\bar{u})$ is of the form $b \in A_G$ with $t_G(b) = q_V(\bar{u})$. Since q is left resolving, there exists a unique arc $\beta \in A_\Gamma$ such that $t_\Gamma(\beta) = \bar{u}$ and $q_A(\beta) = b$. This β is a unique arc in A_Δ such that $t_\Delta(\beta) = \bar{u}$ and $(q_e^b)_A(\beta) = b$. Next consider the case that $\bar{u} \in C_E$. Then $\bar{u} = E(\alpha)$ for some $\alpha \in A_\Gamma$, and $(q_e^b)_V(E(\alpha)) = e(q_A(\alpha))$. In H_e only the arc $\overline{e}(q_A(\alpha))$ ends in the vertex $e(q_A(\alpha))$, in Δ only the arc $\overline{E}(\alpha)$ ends in the vertex $E(\alpha)$ and $(q_e^b)_A(\overline{E}(\alpha)) = \overline{e}(q_A(\alpha))$. Each arc in H_e starting from the vertex $e(q_A(\alpha))$ is of the form $a \in A_G$ with $a \overset{e}{\sim} q_A(\alpha)$. In G, a starts from $i_G(q_A(\alpha)) = q_V(i_\Gamma(\alpha))$. Since q is right resolving, there exists a unique arc γ in Γ satisfying $i_\Gamma(\gamma) = i_\Gamma(\alpha)$ and $q_A(\gamma) = a$, which are equivalent to $\gamma \overset{E}{\sim} \alpha$ and $q_A(\gamma) = a$. Hence, in Δ, γ is a unique arc starting from $E(\alpha)$ with $(q_e^b)_A(\gamma) = a$. Thus we have seen that q_e^b is biresolving.

Since $(\acute{T}_e^b)^*$ is onesided 1-1 (because T^* is onesided 1-1), it follows that $(T_e^b)^*$ is onesided 1-1, so that $(\grave{T}_e^b)^* = T_e^*$ is onesided 1-1. Similarly, if T^* is 1-1, then T_e^* is 1-1. Thus T_e satisfies (2). □

The following lemma is easily proved.

Lemma 3.6. *If T_1 and T_2 are topologically conjugate textile systems, then the n-th composition-powers T_1^n and T_2^n are topologically conjugate for all $n \geq 1$.*

Proposition 3.7. *Let $T = (p, q : \Gamma \to G)$ be a nondegenerate, q-biresolving textile system such that G is irreducible, $\lambda_G = J \in \mathbf{Z}$ and T^* is onesided 1-1. Then there exist $n \geq 1$ and a nondegenerate, q-biresolving textile system $\breve{T}$ over a graph $\breve{G}$ such that*

(i) *T^n and $\breve{T}$ are topologically conjugate,*
(ii) *$M_{\breve{G}}$ has the right eigenvector with all components 1 corresponding to J, and*
(iii) *$\breve{T}^*$ is onesided 1-1.*

If in addition T^* is 1-1 with ϕ_p bounded-to-one and ϕ_q N-to-one, then there exist $n \geq 1$ and a nondegenerate, q-biresolving textile system $\breve{T}$ over a graph $\breve{G}$ such that

(I) *T^n and $\breve{T}$ are topologically conjugate,*
(II) *$M_{\breve{G}}$ has the left and right eigenvectors with all components 1 corresponding to J, and*
(III) *$\breve{T}^*$ is a 1-1 textile system over a graph $\breve{G}^*$ such that $M_{\breve{G}^*}$ has the left and right eigenvectors with all components 1 corresponding to N^n.*

Proof. Using Marcus's algorithm which is based on Proposition 3.4, for some $m \geq 0$ we obtain irreducible graphs $G_0, \dots, G_m$ and equivalence relations $\overset{e_i}{\sim}$ on A_{G_i}, $i = 0, \dots, m-1$, such that $G_0 = G$, $G_i = (G_{i-1})_{e_{i-1}}$ for $i = 1, \dots, m$, the smallest right eigenvector of M_{G_m} corresponding to J is equal to $(1, \dots, 1)^t$. Let $T_0 = T$. Since $T_0^* = (p_0^*, q_0^* : \Gamma_0^* \to G_0^*)$ is a onesided 1-1, LL textile system, p_0^* is a k_0-definite left resolving graph-homomorphism for some $k_0 \geq 1$. Let $T_0^{k_0} = (\bar{p}_0, \bar{q}_0 : \bar{\Gamma}_0 \to G_0)$ be the k_0-th composition-power of T_0. Then $T_0^{k_0}$ is a nondegenerate q-biresolving textile system such that $\bar{p}_0^*$ is 1-definite, where $(T_0^{k_0})^* = (\bar{p}_0^*, \bar{q}_0^* : \bar{\Gamma}_0^* \to \bar{G}_0^*)$. Hence we can apply Lemma 3.5 to $T_0^{k_0}$ and obtain a nondegenerate q-biresolving textile system $T_1 = (T_0^{k_0})_{e_0}$ over $G_1 = (G_0)_{e_0}$. By Lemma 3.5, T_1^* is onesided 1-1 and T_1 and $T_0^{k_0}$ are topologically conjugate. Since T_1^* is onesided 1-1, we similarly obtain a nondegenerate q-biresolving textile system $T_2 = (T_1^{k_1})_{e_1}$ over $G_2 = (G_1)_{e_1}$ for some $k_1 \geq 1$. By Lemma 3.5, T_2^* is onesided 1-1 and T_2 and $T_1^{k_1}$ are topologically conjugate. Continuing this way, we have nondegenerate q-biresolving textile systems T_i over G_i, $i = 0, \dots, m$, such that for $i = 0, \dots, m$, T_i^* is onesided 1-1 and T_i and $(T_{i-1})^{k_{i-1}}$ are topologically conjugate for some $k_{i-1} \geq 1$. If we let $n = k_0 \dots k_{m-1}$, then it follows from Lemma 3.6 that T_m is topologically conjugate to T^n. Therefore, if we let $\breve{T} = T_m$ and $\breve{G} = G_m$, then the first part of the proposition is proved.

If in addition T^* is 1-1 in the above, then it follows from Lemma 3.5 that T_i^* is 1-1 for $i = 1, \dots, m$. Using Marcus's algorithm based on the "transpose" version of Proposition 3.4, we obtain irreducible graphs $G'_0, \dots, G'_{m'}$ and equivalence relations $\overset{e'_i}{\sim}$ on $A_{G'_i}$, $i = 0, \dots, m'-1$, such that $G'_0 = G_m$, $G'_i = (G'_{i-1})_{e'_{i-1}}$ for $i = 1, \dots, m'$, the smallest right eigenvector of $M_{G'_{m'}}$ corresponding to J is equal to $(1, \dots, 1)^t$ and the smallest left eigenvector corresponding to J is equal to $(1, \dots, 1)$. Using the "transpose" version of Lemma 3.5, we have nondegenerate q-biresolving textile systems T'_i over G'_i, $i = 0, \dots, m'$, with $T'_0 = T_m$ such that for $i = 0, \dots, m'$, $(T'_i)^*$ is 1-1 and T'_i and $(T'_{i-1})^{k'_{i-1}}$ are topologically conjugate for some k'_{i-1}. Let $n = k_0 \dots k_{m-1} k'_0 \dots k'_{m'-1}$. Then it follows from Lemma 3.6 that $T_{m'}$ is topologically conjugate to T^n. Therefore, if we let $\breve{T} = T_{m'}$ and $\breve{G} = G_{m'}$, then the second part of the proposition is proved by using Proposition 3.1(1). $\square$

Here we describe a method given in [N2, Section 3] to compute "multipliers" of Boyle [B] in terms of textile systems.

Let $h : \Gamma \to G$ be a graph-homomorphism between irreducible graphs with $\lambda_G = \lambda_\Gamma = \lambda$, and assume that ϕ_h is onto. For $U \subset V_\Gamma$ and $w \in L_G$, define

$$S_+(U, w) = \{t_\Gamma(z) \mid z \in L(\Gamma), i_\Gamma(z) \in U, h(z) = w\},$$
$$S_-(w, U) = \{i_\Gamma(z) \mid z \in L(\Gamma), t_\Gamma(z) \in U, h(z) = w\}.$$

We call $S_+(U,w)$ and $S_-(w,U)$ a *successor* of U and a *predecessor* of U, respectively. A *right-compatible set* (for h) and a *left-compatible set* (for h) are defined to be a nonempty successor of a singleton of V_Γ and a nonempty predecessor of a singleton of V_Γ, respectively. Then any successor of a maximal right-compatible set is also a maximal right-compatible set, and any predecessor of a maximal left-compatible set is also a maximal left-compatible set ([N1]).

Let $l = (l(v))_{v\in V_G}$ and $r = (r(v))_{v\in V_G}$ be left and right eigenvectors of M_G corresponding to λ. Let $\bar{l} = (\bar{l}(u))_{u\in V_\Gamma}$ and $\bar{r} = (\bar{r}(u))_{u\in V_\Gamma}$ be left and right eigenvectors of M_Γ corresponding to λ. By a similar argument (using the extremal argument of Boyle [B]) to that in the proof of Lemma 3.2, we see that for any maximal left-compatible sets U, U' for h and any maximal right-compatible sets V, V' for h,

$$\sum_{u\in U}\frac{\bar{l}(u)}{l(h_V(u))} = \sum_{u\in U'}\frac{\bar{l}(u)}{l(h_V(u))} \quad \text{and} \quad \sum_{u\in V}\frac{\bar{r}(u)}{r(h_V(u))} = \sum_{u\in V'}\frac{\bar{r}(u)}{r(h_V(u))}.$$

Thus we define $L(h,\bar{l},l)$ to be the first value and $R(h,\bar{r},r)$ to be the second. Corresponding to the theorem of L. R. Welch ([H, Theorem 14.9]), by similar arguments to those in [H, Section 14] (cf. Lemma 3.2) we have

$$L(h,\bar{l},l)R(h,\bar{r},r)\deg\phi_h = \frac{\bar{l}\bar{r}}{lr},$$

where $\deg\phi_h = \sharp\,\phi_h^{-1}(\{x\})$ for any bilaterally transitive point $x \in X_G$.

Let $T = (p,q : \Gamma \to G)$ be a nondegenerate textile system with G and Γ irreducible and $\lambda_G = \lambda_\Gamma$. We define

$$l_T = \frac{L(p,\bar{l},l)}{L(q,\bar{l},l)}, \quad r_T = \frac{R(p,\bar{r},r)}{R(q,\bar{r},r)}, \quad \text{and} \quad \deg T = \frac{\deg\phi_q}{\deg\phi_p}.$$

Then these are independent of the choice of $l, \bar{l}, r$ and $\bar{r}$, and they are invariants of topological conjugacy of textile systems ([N2, Proposition 5.4]). We have $l_T r_T = \deg T$.

Let m, n be nonnegative integers. Let φ be an onto endomorphism of (m,n)-*type* of an irreducible topological Markov shift (X_G,σ_G), that is, there exists a "local rule" $f : L_{m+n+1}(G) \to A_G$ such that $\varphi((a_j)_{j\in\mathbf{Z}}) = (f(a_{j-m})\dots f(a_{j+n}))_{j\in\mathbf{Z}}$ for $(a_j)_{j\in\mathbf{Z}} \in X_G, a_j \in A_G$. We define a textile system $T_\varphi = (p_\varphi, q_\varphi : G^{[m+n+1]} \to G)$ such that for $w = a_1\dots a_{m+n+1} \in L_{m+n+1}(G), a_j \in A_G$, $p_\varphi(w) = a_{m+1}$ and $q_\varphi(w) = f(w)$. Then T_φ is onesided 1-1 and nondegenerate and $\varphi_{T_\varphi} = \varphi$. The "multipliers" l_φ and r_φ of Boyle [B] are equal to l_{T_φ} and r_{T_φ}, respectively. In fact, if we choose $\bar{l}$ and $\bar{r}$ such that $\bar{l}(u) = l(i_G(u))$ and $\bar{r}(u) = r(t_G(u))$ for $u \in V_{G^{[m+n+1]}} = L_{m+n}(G)$, then we see that

$$l_\varphi = \frac{\lambda_G^m}{L(q_\varphi,\bar{l},l)} = l_{T_\varphi} \quad \text{and} \quad r_\varphi = \frac{\lambda_G^n}{R(q_\varphi,\bar{r},r)} = r_{T_\varphi}.$$

If T is a onesided 1-1, nondegenerate textile system over an irreducible graph G, then

$$l_{\varphi_T} = l_T \quad \text{and} \quad r_{\varphi_T} = r_T,$$

because T and T_{φ_T} are topologically conjugate textile systems. If $\psi : (X_G,\sigma_G) \to (X_{G'},\sigma_{G'})$ is a topological conjugacy, where G' is an irreducible graph, then there exists a onesided 1-1, nondegenerate textile system T' over G' such that T and

T' are topologically conjugate and $\varphi_{T'} = \psi\varphi_T\psi^{-1}$. Hence $T_{\psi\varphi_T\psi^{-1}}$ and T are topologically conjugate.

Theorem 3.8 (Boyle, [B])**.** *Let φ be an onto endomorphism of a topologically transitive topological Markov shift (X, σ). Then*

(1) $l_\varphi r_\varphi = \deg \varphi$.
(2) *If φ' is an onto endomorphism of (X, σ), then $l_{\varphi\varphi'} = l_\varphi l_{\varphi'}$ and $r_{\varphi\varphi'} = r_\varphi r_{\varphi'}$.*
(3) *l_φ and r_φ are invariants of topological conjugacy of endomorphisms, that is, for any topological conjugacy $\psi : (X, \sigma) \to (X', \sigma')$ onto a topological Markov shift, $l_{\psi\varphi\psi^{-1}} = l_\varphi$ and $r_{\psi\varphi\psi^{-1}} = r_\varphi$.*

Proposition 3.9. *Let $T = (p, q : \Gamma \to G)$ be a onesided 1-1, nondegenerate, q-biresolving textile system with G irreducible and aperiodic, φ_T N-to-one and T^* onesided 1-1. Then*

(1) *l_{φ_T} and r_{φ_T} are integers, and N and r_{φ_T} are divisible by the same primes.*
(2) *If T^* is 1-1, then l_{φ_T} and r_{φ_T} are integers such that all N, l_{φ_T} and r_{φ_T} are divisible by the same primes, and in particular if a prime p divides N, then p^2 divides N.*

Proof. We use the same argument as in the proof of Proposition 5.4 of [N4].

Since T is q-biresolving, $T^* = (p^*, q^* : \Gamma^* \to G^*)$ is LL. Since ξ_{T^*} is 1-1, p^* is a k-definite left resolving graph-homomorphism for some $k \geq 1$. Let $\bar{T} = (\bar{p}, \bar{q} : \bar{\Gamma} \to G)$ be the k-th composition-power T^k of T. Then $\bar{q}$ is biresolving. Since p^* is a k-definite left resolving graph-homomorphism, it follows that for any arcs α, α' in $\bar{\Gamma}$, $i_{\bar{\Gamma}}(\alpha) = i_{\bar{\Gamma}}(\alpha')$ implies $\bar{p}_A(\alpha) = \bar{p}_A(\alpha')$. Hence we define $F : V_{\bar{\Gamma}} \to A_G$ by

$$F(i_{\bar{\Gamma}}(\alpha)) = \bar{p}_A(\alpha), \qquad \alpha \in A_{\bar{\Gamma}}.$$

Let $\bar{\bar{T}} = (\bar{\bar{p}}, \bar{\bar{q}} : \bar{\bar{\Gamma}} \to G)$ be the second composition-power of $\bar{T}$. Then each arc γ in $\bar{\bar{\Gamma}}$ is of the form $\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}$, where α and β are arcs in $\bar{\Gamma}$ with $\bar{q}_A(\alpha) = \bar{p}_A(\beta)$. For $\gamma = \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}$, $\bar{\bar{p}}_A(\gamma) = \bar{p}_A(\alpha)$ and $\bar{\bar{q}}_A(\gamma) = \bar{q}_A(\beta)$. Since $\bar{q}$ is biresolving, so is $\bar{\bar{q}}$. Let $\hat{T} = (\hat{p}, \hat{q} : \hat{\Gamma} \to G)$ be the textile system such that $\hat{\Gamma}$ and $\hat{q}$ are the same as $\bar{\bar{\Gamma}}$ and $\bar{\bar{q}}$, respectively, but $\hat{p}$ is defined as follows. For $\gamma = \begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}$ in $A_{\hat{\Gamma}} = A_{\bar{\bar{\Gamma}}}$, let $\hat{p}_A(\gamma) = F(t_{\bar{\Gamma}}(\alpha))$. Then it is observed that $\bar{\bar{T}}$ weaves a textile $(\gamma_{ij})_{i,j\in\mathbf{Z}}$ if and only if $\hat{T}$ weaves a textile $(\gamma_{i,j-i})_{i,j\in\mathbf{Z}}$. Thus $\hat{T}$ is a onesided 1-1, nondegenerate, q-biresolving textile system over the irreducible, aperiodic graph G such that

$$\varphi_{\hat{T}} = \varphi_{\bar{T}}^2 \sigma_{\bar{T}}^{-1} = \varphi_T^{2k} \sigma_T^{-1}. \tag{3.6}$$

Moreover, the same proof as in [N4, the proof of Proposition 5.4] proves that $\hat{T}^*$ is 1-1.

By Corollary 3.3, λ_G is an integer, say J. Hence M_G and M_Γ have left and right eigenvectors corresponding to J such that all the components are in $\mathbf{Q}$. Therefore, l_{φ_T} and r_{φ_T} are in $\mathbf{Q}$.

By Proposition 3.7, there exist $n \geq 1$ and a nondegenerate, q-biresolving textile system $\breve{T} = (\breve{p}, \breve{q} : \breve{\Gamma} \to \breve{G})$ such that T^n and $\breve{T}$ are topologically conjugate and $r = (1, \ldots, 1)^t$ is a right eigenvector of $M_{\breve{G}}$ corresponding to J. Since T is a onesided

1-1, nondegenerate textile system over an irreducible graph, it follows that $\breve{T}$ is onesided 1-1 and $\breve{G}$ and $\breve{\Gamma}$ are irreducible. Since $\breve{q}$ is biresolving, it follows from Proposition 2.3 of [N1] that the column vector $\bar{r} = (r(\breve{q}_V(u)))_{u \in V_{\breve{\Gamma}}}$, the components of which are all 1, is a right eigenvector of $M_{\breve{\Gamma}}$ corresponding to $J = \lambda_{\breve{\Gamma}} = \lambda_{\breve{G}}$. Hence we have $R(\breve{q}, \bar{r}, r) = 1$. Thus $r_{\varphi_{\breve{T}}} = r_{\breve{T}} = R(\breve{p}, \bar{r}, r)/R(\breve{q}, \bar{r}, r) = R(\breve{p}, \bar{r}, r)$. Since $R(\breve{p}, \bar{r}, r)$ is equal to the cardinality of the maximal right-compatible set for $\breve{p}$, we see that $r_{\varphi_{\breve{T}}} \in \mathbf{Z}$. Since $\breve{T}$ and T^n are topologically conjugate, using Theorem 3.8 we have $(r_{\varphi_T})^n = r_{\varphi_T^n} = r_{\varphi_{T^n}} = r_{T^n} = r_{\breve{T}} = r_{\varphi_{\breve{T}}}$. Therefore, since $r_{\varphi_T} \in \mathbf{Q}$, we have $r_{\varphi_T} \in \mathbf{Z}$.

Applying this result (for T) to $\hat{T}$, we also have $r_{\varphi_{\hat{T}}} \in \mathbf{Z}$. Applying Theorem 3.8 to (3.6), we have

$$r_{\varphi_{\hat{T}}} = r_{\varphi_T}^{2k} r_{\sigma_T^{-1}} = r_{\varphi_T}^{2k} J^{-1}.$$

Hence J divides $r_{\varphi_T}^{2k}$. Therefore, it follows from Corollary 3.3(3) that every prime dividing N divides r_{φ_T}.

In this stage, we prove (2) preceding the completion of the the proof of (1). To prove (2), assume that T^* is 1-1. Then by the above, r_{φ_T} is an integer which is divisible by every prime dividing N. Moreover, by the "transpose" version of the arguments in the proof above (using the property that η_{T^*} is 1-1 instead of the property that ξ_{T^*} is 1-1), we know that l_{φ_T} is an integer which is divisible by every prime dividing N. Since $l_{\varphi_T} r_{\varphi_T} = N$, we obtain the proof of (2).

Now we complete the proof of (1). It remains to show that $l_{\varphi_T} \in \mathbf{Z}$ and that every prime dividing r_{φ_T} divides N with only the assumption that T^* is onesided 1-1. Applying (2) to $\hat{T}$, we know that $l_{\varphi_{\hat{T}}} \in \mathbf{Z}$. Therefore, since $l_{\varphi_{\hat{T}}} = l_{\varphi_T}^{2k} J$ by (3.6) and Theorem 3.8, we have $l_{\varphi_T}^{2k} J \in \mathbf{Z}$. Since this holds for all sufficiently large k and since $l_{\varphi_T} \in \mathbf{Q}$, it follows that $l_{\varphi_T} \in \mathbf{Z}$. Since $l_{\varphi_T} r_{\varphi_T} = N$, every prime dividing r_{φ_T} divides N. □

By the proof above, we have the following. (Cf. [N4, Proposition 5.4].)

Proposition 3.10. *If T is a onesided 1-1, nondegenerate, q-biresolving textile system with σ_T topologically mixing and T^* onesided 1-1, then there exist $k \geq 1$ and a onesided 1-1, nondegenerate, q-biresolving textile system $\hat{T}$ such that $\varphi_{\hat{T}} = \varphi_T^k \sigma_T^{-1}$ and this is positively expansive.*

4. Expansive automorphisms of onesided full shifts

An endomorphism φ of a dynamical system (X, τ), written by $\varphi : (X, \tau) \circlearrowleft$, and another $\varphi' : (X', \tau') \circlearrowleft$ are said to be *topologically conjugate* if there exists a topological conjugacy $\psi : (X, \tau) \to (X', \tau')$ with $\psi\varphi = \varphi'\psi$.

Kůrka [Ku] proved that a positively expansive endomorphism of a transitive topological Markov shift is topologically conjugate to a onesided topological Markov shift. (His proof can be stated as follows. Such an endomorphism is biclosing, so that it is open (see [N1, Theorems 6.3 and 6.5]). Thus use [P, Theorem 1].) Combining this result with [N2, Theorem 3.9(1)] , we have:

Theorem 4.1. *Let φ be a positively expansive endomorphism of a topologically transitive topological Markov shift (X, σ). Then there exists a onesided 1-1, nondegenerate, q-biresolving textile system T with T^* 1-1 such that the endomorphisms $\varphi_T : (X_T, \sigma_T) \circlearrowleft$ and φ are topologically conjugate.*

The following theorem proves the third conjecture of Boyle and Maass.

Theorem 4.2. *If φ is a positively expansive endomorphism of a mixing topological Markov shift and is N-to-one, then l_φ and r_φ are integers such that all N, l_φ and r_φ are divisible by the same primes, and in particular if a prime p divides N, then p^2 divides N.*

Proof. By Theorem 4.1, Proposition 3.9 and Theorem 3.8. □

As was shown in [N4], if $\tilde{\varphi}$ is an expansive automorphism of a onesided full shift $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ over an alphabet A, then $(A^{\mathbf{N}}, \tilde{\varphi})$ is topologically conjugate to a mixing topological Markov shift. Let $\bar{\chi}^*$ be any topological conjugacy of $(A^{\mathbf{N}}, \tilde{\varphi})$ onto a mixing topological Markov shift (X_0, σ_0) and let $\varphi_0 = \bar{\chi}^* \tilde{\sigma}_A (\bar{\chi}^*)^{-1}$. Define $l_{\tilde{\varphi}} = l_{\varphi_0}$ and $r_{\tilde{\varphi}} = r_{\varphi_0}$ ([BM]). Then φ_0 is positively expansive and N-to-one, where $N = \sharp A$. By Theorem 3.8, l_{φ_0} and r_{φ_0} are independent of choice of $\bar{\chi}^*$, so that $l_{\tilde{\varphi}}$ and $r_{\tilde{\varphi}}$ are well defined.

Corollary 4.3. *Let $\tilde{\varphi}$ be an expansive automorphism of the onesided full shift $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ over the alphabet A of cardinality N. Then $l_{\tilde{\varphi}}$ and $r_{\tilde{\varphi}}$ are integers such that all N, $l_{\tilde{\varphi}}$ and $r_{\tilde{\varphi}}$ are divisible by the same primes, and in particular if a prime p divides N, then p^2 divides N.*

An endomorphism $\tilde{\varphi} : (\tilde{X}_{G^*}, \tilde{\sigma}_{G^*}) \circlearrowleft$ of a onesided topological Markov shift over a graph G^* is said to be *leftmost permutive* if $\tilde{\varphi} = \tilde{\varphi}_f$ for some mapping (local rule) $f : L_{k+1}(G^*) \to A_{G^*}$ with $k \geq 0$, which induces a left resolving graph-homomorphism of $(G^*)^{[k+1]}$ onto G^*, where $\tilde{\varphi}_f$ is defined by

$$\tilde{\varphi}_f((a_i)_{i\in\mathbf{N}}) = (f(a_i \dots a_{i+k}))_{i\in\mathbf{N}}, \qquad (a_i)_{i\in\mathbf{N}} \in \tilde{X}_{G^*},\ a_i \in A_{G^*}.$$

Any automorphism of a onesided topological Markov shift is a leftmost permutive endomorphism of the onesided shift. If $\tilde{\varphi} : (\tilde{X}, \tilde{\sigma}) \circlearrowleft$ is a leftmost-permutive endomorphism of a onesided topological Markov shift and $\tilde{\varphi}^m$ is topologically transitive for all $m \geq 1$, then there exists a leftmost-permutive endomorphism $\tilde{\varphi}_0 : (A^{\mathbf{N}}, \tilde{\sigma}_A) \circlearrowleft$ of the onesided full shift over an alphabet A such that $\tilde{\varphi}$ and $\tilde{\varphi}_0$ are topologically conjugate ([N4, Corollary 5.3]). Here a *leftmost-permutive endomorphism* of the onesided full shift $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ means that of $(\tilde{X}_{G_A}, \tilde{\sigma}_{G_A})$, where G_A is the one-vertex graph with $A_{G_A} = A$. An endomorphism $\tilde{\varphi} : (\tilde{X}_{G^*}, \tilde{\sigma}_{G^*}) \circlearrowleft$ of a onesided topological Markov shift is leftmost permutive if and only if $\tilde{\varphi}$ is an *LL endomorphism* of the shift, that is, there exists a onesided 1-1, LL textile system T^* over G^* with $\tilde{\varphi} = \tilde{\varphi}_{T^*}$ ([N4, Remark 5.2]).

Remark 4.4. Leftmost permutive or LL endomorphisms of onesided topological Markov shifts are an invariant for topological conjugacy between endomorphisms of onesided subshifts.

Proof. Let $T^* = (p^*, q^* : \Gamma^* \to G^*)$ be a onesided 1-1, LL textile system and let $\tilde{\psi} : (\tilde{X}_{G^*}, \tilde{\sigma}_{G^*}) \to (\tilde{X}_{H^*}, \tilde{\sigma}_{H^*})$ be a topological conjugacy between onesided subshifts, where H^* is a nondegenerate graph. Then there exists $k \geq 1$ and $g : L_k(G^*) \to L_1(H^*)$ such that g induces a left resolving graph homomorphism of $(G^*)^{[k]}$ onto H^* and for $(a_i)_{\in\mathbf{N}} \in \tilde{X}_{G^*}$ with $a_i \in A_{G^*}$, $\tilde{\psi}((a_i)_{i\in\mathbf{N}}) = (g(a_i \dots a_{i+k-1}))_{i\in\mathbf{N}}$. Let $(T^*)^{[k]} = ((p^*)^{[k]}, (q^*)^{[k]} : (\Gamma^*)^{[k]} \to (G^*)^{[k]})$ be the higher block textile system of T^* of order k. Then the textile system $\hat{T}^* = (g \cdot (p^*)^{[k]}, g \cdot (q^*)^{[k]} : (\Gamma^*)^{[k]} \to H^*)$ is a onesided 1-1, LL textile system with $\tilde{\varphi}_{\hat{T}^*} = \tilde{\psi}\tilde{\varphi}_{T^*}\tilde{\psi}^{-1}$. □

Theorem 4.5. *Let $\tilde{\varphi} : (A^{\mathbf{N}}, \tilde{\sigma}_A) \circlearrowleft$ be an expansive leftmost-permutive endomorphism of the onesided full shift over the alphabet A of cardinality N. Then there exists a onesided 1-1, nondegenerate, q-biresolving textile system T over an irreducible and aperiodic graph such that T^* is onesided 1-1, φ_T is N-to-one and $\tilde{\varphi}_{T^*} : (\tilde{X}_{T^*}, \tilde{\sigma}_{T^*}) \circlearrowleft$ and $\tilde{\varphi}$ are topologically conjugate.*

Proof. Let $f : A^{k+1} \to A$ be a leftmost-permutive local rule with $\tilde{\varphi} = \tilde{\varphi}_f$. Let $T_0^* = (p_0^*, q_0^* : \Gamma_0^* \to G_A)$ be the textile system such that $\Gamma_0^* = G_A^{[k+1]}$ and for $w = a_1 \dots a_{k+1}, a_i \in A$, $p_0^*(w) = a_1$ and $q_0^*(w) = f(w)$. Then T_0^* is onesided 1-1 and LL. Let $T_0 = (p_0, q_0 : \Gamma_0 \to G_B)$ be the dual of T_0^* with $B = A^k$. Then T_0 is onesided 1-1 and q-biresolving, G_B is irreducible and aperiodic, and $(X_{T_0^*}, \sigma_{T_0^*}) = (A^{\mathbf{Z}}, \sigma_A)$ is topologically mixing. Therefore, by Proposition 3.5 of [N4], (X_{T_0}, σ_{T_0}) is topologically conjugate to a mixing topological Markov shift.

By the proof of Corollary 3.6 of [N4], there exist $n \geq 1$ and a onesided 1-1, nondegenerate, q-biresolving textile system $T = (p, q : \Gamma \to G)$ with G irreducible and aperiodic such that $U_T = U_{T_0^{[n]}}$. Since $\tilde{\xi}_{T_0^*}$ is one-to-one, there are topological conjugacies $\tilde{\psi} : (\tilde{X}_{T^*}, \tilde{\sigma}_{T^*}) \to (\tilde{X}_{T_0^*}, \tilde{\sigma}_{T_0^*})$ and $\tilde{\Psi} : (\tilde{Z}_{T^*}, \tilde{\varsigma}_{T^*}) \to (\tilde{Z}_{T_0^*}, \tilde{\varsigma}_{T_0^*})$ with $\tilde{\psi}\tilde{\xi}_{T_0^*} = \tilde{\xi}_{T^*}\tilde{\Psi}$ and $\tilde{\psi}\tilde{\eta}_{T_0^*} = \tilde{\eta}_{T^*}\tilde{\Psi}$ (cf. [N2, the proof of Lemma 2.8]). Hence T^* is onesided 1-1 and the endomorphisms $\tilde{\varphi} = \tilde{\varphi}_{T_0^*}$ and $\tilde{\varphi}_{T^*}$ are topologically conjugate.

Since q is biresolving, ϕ_q is N'-to-one for some $N' \geq 1$, so that φ_T is N'-to-one because T is onesided 1-1 and nondegenerate. Since q_V is N'-to-one, the topological entropy of (X_{T^*}, σ_{T^*}) is $\log N'$. The subshift (X_{T^*}, σ_{T^*}) is topologically conjugate to the full N-shift $(X_{T_0^*}, \sigma_{T_0^*}) = (A^{\mathbf{Z}}, \sigma_A)$, so that $N' = N$. Thus φ_T is N-to-one. $\square$

Let $\tilde{\varphi} : (\tilde{X}, \tilde{\sigma}) \circlearrowleft$ be an onto endomorphism of a onesided subshift. Let $(\mathcal{O}_{\tilde{\varphi}}, \Sigma_{\tilde{\varphi}})$ be the inverse limit system of $\tilde{\varphi}$, that is, $\mathcal{O}_{\tilde{\varphi}}$ is the compact metric space given by $\mathcal{O}_{\tilde{\varphi}} = \{(x_j)_{j\in\mathbf{Z}} \mid \forall j, x_j \in \tilde{X}, \tilde{\varphi}(x_j) = x_{j+1}\}$ and $d_{\mathcal{O}_{\tilde{\varphi}}}((x_j)_{j\in\mathbf{Z}}, (y_j)_{j\in\mathbf{Z}}) = \sup\{2^{-|j|} d_{\tilde{X}}(x_j, y_j) \mid j \in \mathbf{Z}\}$, and $\Sigma_{\tilde{\varphi}} : \mathcal{O}_{\tilde{\varphi}} \to \mathcal{O}_{\tilde{\varphi}}$ maps $(x_j)_{j\in\mathbf{Z}}$ to $(x_{j+1})_{j\in\mathbf{Z}}$. We consider the onto endomorphism $\Phi_{\tilde{\varphi}} : (x_j)_{j\in\mathbf{Z}} \mapsto (\tilde{\sigma}(x_j))_{j\in\mathbf{Z}}$ of $(\mathcal{O}_{\tilde{\varphi}}, \Sigma_{\tilde{\varphi}})$. If $\tilde{\varphi}$ is expansive (i.e., there is $\delta > 0$ such that for any $(x_j)_{j\in\mathbf{Z}}, (y_j)_{j\in\mathbf{Z}} \in \mathcal{O}_{\tilde{\varphi}}$, if $d_{\tilde{X}}(x_j, y_j) \leq \delta$ for all $j \in \mathbf{Z}$, then $(x_j)_{j\in\mathbf{Z}} = (y_j)_{j\in\mathbf{Z}}$), then there exists an onto endomorphism $\varphi_0 : (X_0, \sigma_0) \circlearrowleft$ of a subshift which is topologically conjugate to $\Phi_{\tilde{\varphi}} : (\mathcal{O}_{\tilde{\varphi}}, \Sigma_{\tilde{\varphi}}) \circlearrowleft$. If onto endomorphisms $\tilde{\varphi} : (\tilde{X}, \tilde{\sigma}) \circlearrowleft$ and $\tilde{\varphi}' : (\tilde{X}', \tilde{\sigma}') \circlearrowleft$ of onesided subshifts are topologically conjugate, then so are $\Phi_{\tilde{\varphi}} : (\mathcal{O}_{\tilde{\varphi}}, \Sigma_{\tilde{\varphi}}) \circlearrowleft$ and $\Phi_{\tilde{\varphi}'} : (\mathcal{O}_{\tilde{\varphi}'}, \Sigma_{\tilde{\varphi}'}) \circlearrowleft$. If in addition these are respectively topologically conjugate to onto endomorphisms $\varphi_0 : (X_0, \sigma_0) \circlearrowleft$ and $\varphi_0' : (X_0', \sigma_0') \circlearrowleft$ of irreducible topological Markov shifts, then $l_{\varphi_0} = l_{\varphi_0'}$ and $r_{\varphi_0} = r_{\varphi_0'}$, by Theorem 3.8. Thus for an onto endomorphism $\tilde{\varphi} : (\tilde{X}, \tilde{\sigma}) \circlearrowleft$ of a onesided subshift, if $\Phi_{\tilde{\varphi}} : (\mathcal{O}_{\tilde{\varphi}}, \Sigma_{\tilde{\varphi}}) \circlearrowleft$ is topologically conjugate to an onto endomorphism $\varphi_0 : (X_0, \sigma_0) \circlearrowleft$ of an irreducible topological Markov shift, then we can define $l_{\tilde{\varphi}}$ and $r_{\tilde{\varphi}}$ by $l_{\tilde{\varphi}} = l_{\varphi_0}$ and $r_{\tilde{\varphi}} = r_{\varphi_0}$ and they are invariants for topological conjugacy between such onto endomorphisms of onesided subshifts.

Theorem 4.6. *Let $\tilde{\varphi}$ be an expansive leftmost-permutive endomorphism of the onesided full shift $(A^{\mathbf{N}}, \tilde{\sigma}_A)$ over the alphabet A of cardinality N. Then $l_{\tilde{\varphi}}$ and $r_{\tilde{\varphi}}$ are integers, and N and $r_{\tilde{\varphi}}$ are divisible by the same primes.*

Proof. By Theorem 4.5, there exists a onesided 1-1, nondegenerate, q-biresolving textile system T over an irreducible and aperiodic graph such that T^* is onesided

1-1, φ_T is N-to-one and $\tilde{\varphi}_{T^*} : (\tilde{X}_{T^*}, \tilde{\sigma}_{T^*}) \circlearrowleft$ and $\tilde{\varphi}$ are topologically conjugate. Since $l_{\tilde{\varphi}} = l_{\varphi_T}$ and $r_{\tilde{\varphi}} = r_{\varphi_T}$, the theorem follows from Proposition 3.9. $\square$

Let $A = \{0, 1\}$ and let $B = \{0, 1, 2\}$. Let $T_1 = (p_1, q_1 : G_A^{[3]} \to G_A)$ and $T_2 = (p_2, q_2 : G_B^{[2]} \to G_B)$ be the textile systems defined by $(p_1)_A : a_1a_2a_3 \mapsto a_2$, $(q_1)_A : a_1a_2a_3 \mapsto a_1 + a_3 \bmod 2$ with $a_j \in A$ and $(p_2)_A : b_1b_2 \mapsto b_1$, $(q_2)_A : b_1b_2 \mapsto b_1 + b_2 \bmod 3$ with $b_j \in B$. Then T_1 and T_2 are onesided 1-1, nondegenerate and q-biresolving. T_1^* and T_2^* are onesided 1-1 and LL. If $\varphi_0 = \varphi_{T_1} \times \varphi_{T_2}$ and $\tilde{\varphi} = \tilde{\varphi}_{T_1^*} \times \tilde{\varphi}_{T_2^*}$, then $\tilde{\varphi}$ is an expansive LL endomorphism of the onesided full 12-shift with $l_{\tilde{\varphi}} = l_{\varphi_0} = 2$ and $r_{\tilde{\varphi}} = r_{\varphi_0} = 6$.

If T is a onesided 1-1, nondegenerate, q-biresolving textile system, then we call $\varphi_T : (X_T, \sigma_T) \circlearrowleft$ a *q-biresolving endomorphism* of the topological Markov shift (X_T, σ_T) ([N4]). We are interested in topologically mixing q-biresolving endomorphisms of topological Markov shifts, which are also topologically mixing by [N3, Theorem 1.6]. A positively expansive endomorphism of a mixing topological Markov shift is conjugate to a topologically mixing q-biresolving endomorphism. The example above gives an example of a topologically mixing q-biresolving endomorphism which is not positively expansive. However note Proposition 3.10. It is seen by using Theorem 4.2 that if $\varphi_1 : (A^{\mathbf{Z}}, \sigma_A) \circlearrowleft$ and $\varphi_2 : (B^{\mathbf{Z}}, \sigma_B) \circlearrowleft$ are defined by $\varphi_1((a_j)_{j\in\mathbf{Z}}) = (a_{j-1} + a_j \bmod 2)_{j\in\mathbf{Z}}$ and $\varphi_2((b_j)_{j\in\mathbf{Z}}) = (b_j + b_{j+1} \bmod 3)_{j\in\mathbf{Z}}$ with $A = \{0, 1\}$ and $B = \{0, 1, 2\}$, then $\varphi = \varphi_1 \times \varphi_2$ is an example of a topologically mixing q-biresolving endomorphism such that $\varphi^m\sigma^n$ is not positively expansive for all $m \in \mathbf{N}$ and $n \in \mathbf{Z}$, where $\sigma = \sigma_A \times \sigma_B$.

Since topologically mixing, q-biresolving endomorphisms of topological Markov shifts are topologically mixing, constant-to-one endomorphisms of mixing topological Markov shifts, constraints for these (see [B], [T1] and [T2]) are imposed on them. There are further constraints for them, in particular on their underlying mixing topological Markov shifts, which have not been sufficiently understood. In fact, suppose that a mixing topological Markov shift (X_G, σ_G) has an N-to-one, topologically mixing, q-biresolving endomorphism φ. Then, by Corollary 3.3(1) we know that λ_G is a rational integer J such that every prime dividing N divides J. The result of Boyle and Maass [BM] stated in the Introduction is equivalent to the following. If φ is positively expansive, then (X_G, σ_G) is shift equivalent to some full J-shift such that J and N are divisible by the same primes. The second conjecture of Boyle and Maass says that in this result "shift equivalent" can be replaced by "topologically conjugate". Is the result of Boyle and Maass extended to topological Markov shifts having topologically mixing q-biresolving endomorphisms? Furthermore, the following is a corrected version of Question 6.5 of [N4].

Question 4.7. Is every topological Markov shift having a topologically mixing q-biresolving endomorphism topologically conjugate to a full shift?

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