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STEINBERG SYMBOLS MODULO THE TRACE CLASS, HOLONOMY, AND LIMIT THEOREMS FOR TOEPLITZ DETERMINANTS

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Abstract. Suppose that $\phi = \psi z^\gamma$ where $\gamma \in Z_+$ and $\psi \in \mathrm{Lip}_\beta$, $\frac{1}{2} < \beta < 1$, and the Toeplitz operator T_ψ is invertible. Let $D_n(T_\phi)$ be the determinant of the Toeplitz matrix $((\hat{\phi}_{i,j})) = ((\hat{\phi}_{i-j}))$, $0 \le i,j \le n$, where $\hat{\phi}_k = \frac{1}{2\pi}\int_0^{2\pi}\phi(\theta)e^{-ik\theta}\,d\theta$. Let P_n be the orthogonal projection onto $\ker S^{*n+1} = \bigvee\{1, e^{i\theta}, e^{2i\theta}, \ldots, e^{in\theta}\}$, where $S = T_z$; set $Q_n = 1 - P_n$, let H_ω denote the Hankel operator associated to ω, and set $\tilde{\omega}(t) = \omega(\frac{1}{t})$ for $t \in \mathbb{T}$. For the Wiener-Hopf factorization $\psi = f\bar{g}$ where f, g and $\frac{1}{f}, \frac{1}{g} \in \mathrm{Lip}_\beta \cap H^\infty(\mathbb{T})$, $\frac{1}{2} < \beta < 1$, put $E(\psi) = \exp\sum_{k=1}^\infty k(\log f)_k(\log\bar{g})_{-k}$, $G(\psi) = \exp(\log\psi)_0$.

Theorem A.
$$D_n(T_\phi) = (-1)^{(n+1)\gamma}G(\psi)^{n+1}E(\psi)G(\tfrac{\bar{g}}{f})^\gamma \cdot \det\left((T_{\frac{f}{\bar{g}}z^{n+1}}\cdot[1-H_{\frac{\bar{g}}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar{g}})^{\sim}}]^{-1}z^{\alpha-1}, z^{\tau-1})\right)_{\gamma\times\gamma}\cdot[1+O(n^{1-2\beta})].$$

Let $H^2(\mathbb{T}) = \mathcal{X} \dot{+} \mathcal{Y}$ be a decomposition into $T_\phi T_{\phi^{-1}}$ invariant subspaces, $\mathcal{X} = \bigcap_{n=1}^\infty \mathrm{ran}(T_\phi T_{\phi^{-1}})^n$ and $\mathcal{Y} = \bigcup_{n=1}^\infty \ker(T_\phi T_{\phi^{-1}})^n$, so that $T_\phi T_{\phi^{-1}}$ restricted to $\mathcal{X}$ is invertible, $\mathcal{Y}$ is finite dimensional, and $T_\phi T_{\phi^{-1}}$ restricted to $\mathcal{Y}$ is nilpotent. Let $\{w_\alpha\}_1^\gamma$ be the basis $\{T_f z^\alpha\}_0^{\gamma-1}$ for the null space of $T_\phi T_{\phi^{-1}}$, and let u_α be the top vector in a Jordan root vector chain of length $m_\alpha + 1$ lying over $(-1)^{m_\alpha}w_\alpha$, i.e., $(T_\phi T_{\phi^{-1}})^{m_\alpha}u_\alpha = (-1)^{m_\alpha}w_\alpha$ where $m_\alpha = \max\{m \in Z_+ : \exists x \text{ so that} (T_\phi T_{\phi^{-1}})^m x = w_\alpha\}^{-1}$.

Theorem B.
$$E(\psi)G(\tfrac{\bar{g}}{f})^\gamma = \frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}}\lambda}{\det(u_\alpha, T_{\frac{1}{g}}z^{\tau-1})} = \left(\bar{g}\cup f \times \tfrac{\bar{g}}{f}\cup z^\gamma\right)(\mathbb{T}),$$
the holonomy of a Deligne bundle with connection defined by the factorization $\phi = f\bar{g}z^\gamma$.

Note that the generalizations of the Szegö limit theorem for $D_n(T_\phi)$ which have appeared in the literature with 1 instead of $[1 - H_{\frac{\bar{g}}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar{g}})^{\sim}}]^{-1}$ have the defect that the limit of $\frac{D_n(T_\phi)}{(-1)^{(n+1)\gamma}G(\psi)^{n+1}\det(T_{\frac{f}{\bar{g}}z^{n+1}}z^{\alpha-1}, z^{\tau-1})}$ does not exist in general. An example is given with $D_n(T_\phi) \neq 0$ yet $D_{\gamma-1}(T_{\frac{f}{\bar{g}}z^{n+1}}) = 0$ for infinitely many n.

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1. Introduction

Jacobi's theorem on the conjugate minors of the adjugate matrix formed from the cofactors of $D_n(T_\phi)$ has been the main tool of previous attempts to generalize the classical strong Szegö limit theorem. But Toeplitz operators are defined by an algebraic relation with the shift, $S^{*n+1}T_\phi S^{n+1} = T_\phi$, and it is natural instead to probe the implications of the Toeplitz property algebraically. The result gives, for the first time to our knowledge, a full extension of the Szegö theorem to the case where wind $(\phi, 0) \neq 0$,

$$\lim_{n\to\infty} \frac{D_n(T_\phi)}{(-1)^{(n+1)\gamma} G(\psi)^{n+1} \cdot \det\left((T_{\frac{f}{\bar g} z^{n+1}} \cdot [1 - H_{\frac{\bar g}{f}} Q_{n-\gamma} H^{-1}_{(\frac{f}{\bar g})} z^{\alpha-1}, z^{\tau-1}) \right)_{\gamma\times\gamma}}$$
$$= E(\psi) G(\frac{\bar g}{f})^\gamma = \frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, T_{\frac{1}{g}} z^{\tau-1})_{\gamma\times\gamma}},$$

and shows that the limit splits into two parts, a "torsion", $\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda$, the product of the non-zero eigenvalues of $T_\phi T_{\phi^{-1}}$ and a denominator reflecting the Jordan chain structure of the finite dimensional nilpotent operator $T_\phi T_{\phi^{-1}}$ acting on $\mathcal{Y}$.

The methods we employ have wider implications. In *Orthogonal polynomials and periodic recurrence relations*, perturbation vectors are related to linear systems on Riemann surfaces to give the orthogonal polynomials corresponding to periodic Jacobi matrices.

Steinberg symbols. The relative algebraic K group $K_1(\mathcal{L}(H), \mathcal{L}^1(H))$ may be identified with the quotient group of the invertible elements of $\mathcal{L}(H)$ of the form $1+J$ where J is in $\mathcal{L}^1(H)$, by the commutator subgroup $\{1 + \mathcal{L}^1(H), \mathcal{L}(H)\}$ generated by elements of the form $\{1 + J, A\} \equiv (1 + J)A(1 + J)^{-1}A^{-1}$ where A and $1 + J$ are invertible and J is in $\mathcal{L}^1(H)$. If J is trace class and $1 + J$ is invertible, then write $[1 + J]_1$ for the corresponding element in $K_1(\mathcal{L}(H), \mathcal{L}^1(H))$. Every element in $K_1(\mathcal{L}(H), \mathcal{L}^1(H))$ has such a representation. The map det is a group homomorphism of the invertible operators of the form $1+J$, where $J \in \mathcal{L}^1(H)$, onto C^*. The homomorphism $\det_*$ is then defined by $\det_*[1+J]_1 = \det(1+J)$. It is shown in [1] that the map $\det_*$ is the projection onto the first factor in the isomorphism $K_1(\mathcal{L}(H), \mathcal{L}^1(H)) \cong C^* \oplus V$ in which V has uncountable linear dimension.

Furthermore, there is a connecting map ∂, so that there is a homomorphism to the non-zero complex numbers C^* given by the composition

$$\det_* \circ \partial : K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \to C^*,$$

where, for invertible elements $\alpha, \beta \in \mathcal{L}(H)/\mathcal{L}^1(H)$ which happen to commute, there is the Steinberg symbol

$$\{\alpha, \beta\} \in K_2(\mathcal{L}(H)/\mathcal{L}^1(H)) \xrightarrow{\partial} K_1(\mathcal{L}(H); \mathcal{L}^1(H)) \cong C^* \oplus V.$$

Let R, S, and T denote regularizers for A, B and AB respectively. Thus $RA - I, AR - I \in \mathcal{L}^1(H)$, etc. The elements $a, b \in \mathcal{L}(H)/\mathcal{L}^1(H)$ are lifted to A, B and a^{-1}, b^{-1}, $(ab)^{-1}$ are lifted to R, S and T.

The definition of Steinberg symbols leads to (see [15])

$$\partial\{a,b\} = \Bigg[\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ -R & I \end{pmatrix} \cdot \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \cdot \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix} \cdot \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & -AB \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \cdot \begin{pmatrix} I & -AB \\ 0 & I \end{pmatrix} \Bigg] \quad \mathrm{mod}\mathcal{L}^1(H),$$

where there is latitude in the choice of regularizers, since different regularizers lead to the same class, and therefore define liftings with the same determinant.

Thus, for example, when T_ϕ is invertible, take $A = T_\phi$, $B = S^*$ so that $AB = T_\phi S^*$ with regularizers $T = S{T_\phi}^{-1}$, $R = {T_\phi}^{-1}$. Then, by substitution as above,

$$\begin{aligned} &\partial\{T_\phi + \mathcal{L}^1(H), S^* + \mathcal{L}^1(H)\} \\ &= \Bigg[\begin{pmatrix} I & T_\phi \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ -T_\phi^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} I & T_\phi \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \cdot \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \\ &\cdot \begin{pmatrix} I & S^* \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix} \cdot \begin{pmatrix} I & S^* \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & -T_\phi S^* \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ ST_\phi^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} I & -T_\phi S^* \\ 0 & I \end{pmatrix} \Bigg]. \end{aligned}$$

If we multiply these twelve factors together and use $SS^* = 1 - P_0$, we get

$$\det{}_* \circ \partial\{T_\phi + \mathcal{L}^1(H), S^* + \mathcal{L}^1(H)\} = \det({T_\phi}^{-1} S T_\phi S^* + {T_\phi}^{-1} P_0).$$

Thus,

$$\begin{aligned} &\frac{\det(P_0 T_\phi P_0)}{\det_* \circ \partial\{T_\phi + \mathcal{L}^1(H), S + \mathcal{L}^1(H)\}} \\ &= \det(P_0 T_\phi P_0) \cdot \det{}_* \circ \partial\{T_\phi + \mathcal{L}^1(H), S^* + \mathcal{L}^1(H)\} \\ &= \det(P_0^\perp + P_0 T_\phi) \cdot \det({T_\phi}^{-1} S T_\phi S^* + {T_\phi}^{-1} P_0) \\ &= \det(P_0^\perp {T_\phi}^{-1} S T_\phi S^* + P_0^\perp {T_\phi}^{-1} P_0 + P_0) = \det(P_0^\perp {T_\phi}^{-1} S T_\phi S^* + P_0) \\ &= \det(P_0^\perp {T_\phi}^{-1} S T_\phi S^* + 1 - SS^*) = \det(1 + (P_0^\perp {T_\phi}^{-1} S T_\phi - S) S^*) \\ &= \det(1 + S^*(P_0^\perp {T_\phi}^{-1} S T_\phi - S)) = \det(1 + S^*({T_\phi}^{-1} S T_\phi - S)) = \det(S^* {T_\phi}^{-1} S T_\phi). \end{aligned}$$

The "same" calculation carried out with S^* replaced by ${S^*}^{n+1}$ and P_0 replaced by P_n gives

$$\frac{\det(P_n T_\phi P_n)}{\det_* \circ \partial\{T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\}} = \det({S^*}^{n+1} {T_\phi}^{-1} S^{n+1} T_\phi).$$

Proposition 1.1 (Equation (35) of [15]). *If $\phi \in \mathcal{L}^\infty(0, 2\pi)$ with T_ϕ Fredholm, injective, and $D_n(\phi) \neq 0$, then*

$$\frac{D_n(T_\phi)}{\det_* \circ \partial\{T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\}} = \det(T_\phi {S^*}^{n+1} R(T_\phi) S^{n+1} + P(T_{\bar\phi}) S^{n+1}).$$

The Steinberg symbol determinant in the denominator makes sense when T_ϕ is Fredholm and not invertible and as already noted can be expressed explicitly. It is shown in §1.2, that if T_ψ is invertible, then $\det_* \circ \partial\{T_{\psi z^\gamma} + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\} = \mathbf{G}(\phi)^{n+1} = G(\psi)^{n+1} \cdot (-1)^{(n+1)\gamma}$.

Thus, when T_ϕ is invertible and $\gamma = 0$, since $\mathbf{G}(\phi) = G(\psi)$, the geometric mean, the proposition becomes the very special case proved above:

$$(*) \qquad \frac{D_n(T_\phi)}{G(\phi)^{n+1}} = \det(S^{*n+1} T_\phi^{-1} S^{n+1} T_\phi).$$

This immediately gives the strong Szegö limit theorem

$$\lim_{n\to\infty} \frac{D_n(T_\phi)}{G(\phi)^{n+1}} = \det(T_\phi T_{\phi^{-1}}).$$

Proposition 1.1 was proved in [15] in order to treat the case where wind$(\phi, 0) \neq 0$. Nevertheless, since there have now been a number of papers (published subsequent to [15]) giving variant proofs of a formula of A. Borodin and A. Okunkov [9]:

$$\det(P_n T_\phi P_n) = G(\phi)^{n+1} \frac{\det\left[1 - Q_n H_{\frac{\bar g}{f}} H_{\frac{\tilde f}{\tilde g}} Q_n\right]}{\det\left[1 - H_{\frac{\bar g}{f}} H_{\frac{\tilde f}{\tilde g}}\right]},$$

for the case of invertible T_ϕ, $\phi = f\bar g$, we note the following:

Remark 1.1. **The Borodin-Okunkov formula is equivalent to the special case $(*)$ of Proposition 1.1.**

This is immediate.

Since $T_{ab} = T_a T_b + H_a H_{\tilde b}$, and T_ϕ is invertible, we have, with $\tilde b(t) \equiv b(\bar t)$ for $t \in \mathbb{T}$, and $\langle \alpha, \beta \rangle = \alpha\beta\alpha^{-1}\beta^{-1}$ denoting the multiplicative commutator of the operators α and β,

$$\begin{aligned}
\det(T_\phi S^{*n+1} T_\phi^{-1} S^{n+1}) &= \det(T_{\bar g} T_f S^{*n+1} T_{f^{-1}} T_{\bar g^{-1}} S^{n+1}) \\
&= \det(T_f \langle T_{f^{-1}}, T_{\bar g}\rangle T_{\bar g} S^{*n+1} T_{f^{-1}} T_{\bar g^{-1}} S^{n+1}) \\
&= \det(\langle T_{f^{-1}}, T_{\bar g}\rangle)(S^{*n+1} T_{\bar g} T_{f^{-1}} T_{\bar g^{-1}} T_f S^{n+1}) \\
= \frac{\det(S^{*n+1} T_{\frac{\bar g}{f}} T_{\frac{f}{\bar g}} S^{n+1})}{\det(\langle T_{\bar g}, T_{f^{-1}}\rangle)} &= \frac{\det\left[1 - Q_n H_{\frac{\bar g}{f}} H_{\frac{\tilde f}{\tilde g}} Q_n\right]}{\det\left[1 - H_{\frac{\bar g}{f}} H_{\frac{\tilde f}{\tilde g}}\right]}.
\end{aligned}$$

However, the goal here is to describe a more comprehensive framework for wind $(\phi, 0) \neq 0$. Thus, in [15] we introduced perturbation vectors $\sigma_{A,B}$ associated to a pair of Fredholm operators A, B with $A - B$ in the trace ideal. These non-zero vectors are a substitute for the perturbation determinants studied by Aronszjan, Weinstein, M. G. Krein, and many others. In the special case where A is the identity, $\sigma_{A,B}$ splits and the scalar part of these perturbation vectors amounts to taking the product of the non-zero eigenvalues, while the vector itself comes from a section of an appropriate bundle.

To create this object the authors proved that there exists a trivialization of the pullback $i_M^*(\wp)$ of a determinant bundle, given by a section, σ, which they called the perturbation section. Let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on a Hilbert space H. Let $\mathcal{L}^1(\mathcal{H})$ denote the ideal of compact operators T with trace $|T| < \infty$. Let $\mathcal{F}$ denote the Fredholm operators on H. Let $\mathcal{Q} \to \mathcal{F}$ be the Quillen det bundle. Form $\mathcal{Q} + \mathcal{Q}^*$ as a bundle over $\mathcal{F} \times \mathcal{F}$. Let $\wp = \det(\mathcal{Q} + \mathcal{Q}^*)$ be the determinant bundle. Let $M \equiv \{(A, B) : (A, B) \in \mathcal{F} \times \mathcal{F} \text{ and } A - B \in \mathcal{L}^1(H)\}$ with $i_M : M \to \mathcal{F} \times \mathcal{F}$ the inclusion map.

The perturbation vector $\sigma_{A,B}$ is the value of the section at $(A, B) \in M$.

When wind $(\phi, 0)$ is not zero and the normalized Szegö determinant $\frac{D_n(T_\phi}{G(\phi)^{n+1}}$ converges to zero, the perturbation vector $\sigma_{1,T_\phi S^{*n+1}R(T_\phi)S^{n+1}}$ gives a replacement[1] because

$$\sigma_{1,T_\phi S^{*n+1}R(T_\phi)S^{n+1}} \to \sigma_{1,T_\phi T_{\phi^{-1}}}.$$

The analysis of the convergence of this sequence of perturbation vectors in combination with Proposition 1.1 will be our basic study. A review of the perturbation vector construction and some of their properties is given in §2.

We will see in §2 that if $1 - X \in \mathcal{L}^1$, then

$$\|\sigma_{1,X}\| = \frac{\prod_{\lambda\in\sigma(X)\setminus\{0\}} |\lambda|}{|\det(u_i, y_j)|} = \prod_{s_\alpha(X)\neq 0} \mathbf{s}_\alpha(X),$$

where $\{y_j\}$ is any orthonormal basis for $\ker X^*$, $\{u_i\}$ is any set of maximal root vectors relative to any orthonormal basis of $\ker X$ and $\{\mathbf{s}_\alpha(X)\}$ are the singular values of X. For example this implies that $dd^c \prod_{\mathbf{s}_\alpha\neq 0} \mathbf{s}_\alpha(T_{z-\lambda}T_{(z-\lambda)^{-1}}) = dd^c\|\sigma_{1,T_{z-\lambda}}T_{(z-\lambda)^{-1}}\| = -\frac{1}{(1-|\lambda|^2)^2} d\lambda \wedge d\bar{\lambda}$ is the Gaussian curvature of the Hermitian bundle whose fibre at the point λ is $\ker T_{z-\lambda}{}^*$.

The perturbation vector version of Proposition 1.1. Proposition 1.1 is the scalar part of a vector result proved in [15].

The fact that the ratio

$$\frac{D_n(T_\phi)}{\det(T_\phi S^{*n+1} R(T_\phi) S^{n+1} + P(T_{\bar\phi}) S^{n+1})} = \det{}_* \circ \partial\{T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\}$$

depends only on the cosets of T_ϕ and the shift S modulo the trace ideal is part of a statement made in terms of perturbation vectors:

For $\gamma = 0$,

$$\begin{aligned}\sigma_{S^{*n+1},\, T_\phi S^{*n+1} T_\phi{}^{-1}} &= \frac{\det(T_\phi S^{*n+1} T_\phi{}^{-1} S^{n+1})}{D_n(T_\phi)} \rho_n^* \otimes T_\phi \rho_n \\ &= \frac{\rho_n^* \otimes T_\phi\rho_n}{\det{}_* \circ \partial\{T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\}},\end{aligned}$$

where ρ_n is any non-zero vector in the exterior product space $\det P_n H \equiv \bigwedge^{n+1} P_n H$, i.e., the Grassmann product of any set of basis vectors in $\ker S^{*n+1}$. And when $\gamma \neq 0$,

$$\begin{aligned}&\sigma_{S^{*n+1}, T_\phi S^{*n+1} R(T_\phi)} \\ &\quad= \frac{\det(T_\phi S^{*n+1} R(T_\phi) S^{n+1} + P(T_{\bar\phi}) S^{n+1})}{D_n(T_\phi)} \rho_n^* \otimes T_\phi \rho_n \wedge x \otimes (\pi_{T_\phi} x)^*,\end{aligned}$$

where $0 \neq x \in \det\ker T_{\bar\phi}$ and $\pi_{T_\phi} : H \to H/T_\phi(H)$is the quotient map.

The Szegö Problem. For $\phi \in \mathcal{L}^\infty(\mathbb{T})$, the Toeplitz operator T_ϕ is defined on $H^2(\mathbb{T})$ by $T_\phi f = P\phi \cdot f$ where P is the orthogonal projection of $\mathcal{L}^2(\mathbb{T})$ onto $H^2(\mathcal{L}(\mathbb{T}))$. Let P_n be the orthogonal projection onto

$$\ker S^{*n+1} = \bigvee\{1, e^{i\theta}, e^{2i\theta}, \ldots, e^{in\theta}\} = \bigvee\{e_j\},$$

[1]For definiteness we henceforth take $R(T_\phi) = (T_{\bar\phi} T_\phi)^{-1} T_{\bar\phi}$, the Moore-Penrose inverse of T_ϕ.

where $S = T_z$; set $Q_n = 1 - P_n$ with $D_n(T_\phi) \equiv \det(P_n T_\phi P_n)$. Let $G(|\phi|) \equiv \exp \frac{1}{2\pi}\int_0^{2\pi} \log|\phi(e^{i\theta})|\, d\theta$.

The study of the asymptotics of $D_n(T_\phi)$ originated with the discovery in 1915 by G. Szegö that powers of the geometric mean serve as normalizing factors for the determinants $D_n(T_\phi) \equiv \det(P_n T_\phi P_n)$. Finally, in 1952 he proved the following statement:

The sharp Szegö Theorem ([30]). *If $\phi(e^{i\theta}) > 0$ in $[0, 2\pi]$ and ϕ' satisfies a Lipschitz condition with exponent β, $0 < \beta \le 1$, then*

$$\lim_{n\to\infty} \frac{D_n(T_\phi)}{[G(\phi)]^{n+1}} = \exp\{\frac{1}{4}\sum_1^\infty n|k_n|^2\},$$

where $\sum_1^\infty k_n z^n = \frac{1}{2\pi}\int_0^{2\pi} \log\phi(\theta)\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\, d\theta$, and $D_n(T_\phi)$ is the determinant of the Toeplitz matrix $((\hat{\phi}_{i,j})) = ((\hat{\phi}_{i-j}))$, $\quad 0 \le i, j \le n$, with $\hat{\phi}_k = \frac{1}{2\pi}\int_0^{2\pi}\phi(\theta)e^{-ik\theta}\, d\theta$.

Szegö's theorem was extended by Baxter [5], [6], Devinatz [20], Hirschman [25], Ibragimov [26], Kac [28], Widom [31], [32] and others with fewer restrictive smoothness conditions, and finally with complex symbols but originally with wind $(\phi, 0) = 0$.

However, something different is required when wind $(\phi, 0) \neq 0$. For then the normalized Szegö determinant $\frac{|D_n(T_\phi)|}{G(|\phi|)^{n+1}} \to 0$ for smooth ϕ. Thus, to solve the problem of finding a full analog of the Szegö theorem when wind$(\phi, 0) \neq 0$ it is necessary to

a) produce a normalization divisor A_n so that the sequence $\frac{D_n(T_\phi)}{A_n}$ has a natural limit,

b) explain the limit in terms of topological and analytic data implicit in the symbol.

The main results below are contained in Theorems 1.1, 1.2, 1.3, and 1.6.

If $\phi = \psi z^\gamma$, wind $(\psi, 0) = 0$, put

$$\bar{g} \equiv \exp\big(\sum_{n=1}^\infty (\log\psi)_{-n} t^{-n}\big), \qquad f \equiv \exp\big(\sum_{n=0}^\infty (\log\psi)_n t^n\big)$$

so that $\psi = f\bar{g}$. Let H_ω be the Hankel operator with symbol ω. Then, with $E(\psi) \equiv \exp\sum_{k=1}^\infty k(\log\psi)_k(\log\psi)_{-k}$, and $\mathbf{G}(\phi) \equiv \det_*\partial\{T_{\psi z^\gamma} + \mathcal{L}^1(H), S + \mathcal{L}^1(H)\}$, a Steinberg symbol determinant defined in terms of the cosets relative to the trace ideal of the Toeplitz operator and the shift:

$$\alpha)\quad A_n \sim \mathbf{G}(\phi)^{n+1}\ \det\bigg((T_{\frac{f}{\bar{g}}z^{n+1}} \cdot [1 - H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}]^{-1} z^{\alpha-1}, z^{\tau-1})\bigg)_{\gamma\times\gamma}, \quad \text{and}$$

$$\beta)\quad \lim_{n\to\infty}\frac{D_n(T_\phi)}{A_n} = E(\psi)G(\frac{\bar{g}}{f})^\gamma = \frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}}\lambda}{\det(u_\alpha, T_{\frac{1}{\bar{g}}} z^{\tau-1})} = \bigg(\bar{g}\cup f \times \frac{\bar{g}}{f}\cup z^\gamma\bigg)(\mathbb{T}).$$

When $\phi = \psi z^\gamma$ and wind $(\psi, 0) = 0$, we will prove that

$$\mathbf{G}(\phi)^{n+1} = \det_*\partial\{T_{\psi z^\gamma} + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\} = (-1)^{(n+1)\gamma}G(\psi)^{n+1}.$$

Although it is known (see §1.0.1 and §5) that

$$\lim_{n\to\infty}\frac{D_n(T_\phi)}{(-1)^{(n+1)\gamma}G(\psi)^{n+1}\det(T_{\frac{f}{\bar{g}}z^{n+1}} z^{\alpha-1}, z^{\tau-1})} = E(\psi)G(\frac{\bar{g}}{f})^\gamma$$

for many symbols, the limit on the left does not exist in general. There are smooth symbols ϕ so that $D_n(T_\phi) \neq 0$, yet $D_{\gamma-1}(T_{\frac{f}{\bar g}} z^{n+1}) = \det(T_{\frac{f}{\bar g} z^{n+1}} z^{\alpha-1}, z^{\tau-1}) = 0$ for infinitely many n. See §6 below. Thus the corrected nomalization provided by α) is necessary to find the leading term in the asymptotics of $D_n(T_\phi)$ and thereby get the general limit theorem in β).

The topological formula $\left(\bar g \cup f \times \frac{\bar g}{f} \cup z^\gamma\right)(\mathbb{T})$ denotes the holonomy of a flat bundle taken over the unit circle $\mathbb{T}$. Both the bundle and its connection are defined by the factorization $\phi = z^\gamma f \bar g$. This will be discussed in §1.2.

Theorem 1.1. *Suppose $\phi \in \mathcal{L}^\infty(\mathbb{T})$ where T_ϕ is Fredholm, injective, $T_\phi T_{\phi^{-1}} - 1 \in \mathcal{L}^1(H^2(\mathbb{T}))$. Let $\{w_\alpha\}_1^\gamma$ be any basis for $\ker T_{\phi^{-1}}$ and let $\{u_\alpha\}_1^\gamma$ be any set of maximal root vectors relative to $\{(-1)^{m_\alpha} w_\alpha\}_1^\gamma$ and $T_\phi T_{\phi^{-1}}$, i.e., $(T_\phi T_{\phi^{-1}})^{m_\alpha} u_\alpha = (-1)^{m_\alpha} w_\alpha$ where $m_\alpha = \max\{m \in Z_+ : \exists x$ so that $(T_\phi T_{\phi^{-1}})^m x = w_\alpha\}$. Then, if $\{w_\alpha^{(n)}\}_1^\gamma$ is any sequence of bases in $\ker S^{*n+1} R(T_\phi) S^{n+1}$ for which $w_\alpha^{(n)}$ converges to w_α for $1 \le \alpha \le \gamma$ and if $\{t_\beta\}_1^\gamma$ is any basis of $\ker T_{\bar\phi}$, then for $n \gg 0$,*

$$\text{i)}\quad D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \cdot \det(w_\alpha^{(n)}, S^{*n+1} t_\tau)_{\gamma\times\gamma} \cdot \left[\frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, t_\tau)}\right]\left[1 + o(1)\right].$$

If in addition, $\phi \in \mathrm{Lip}_\beta$, $\frac{1}{2} < \beta < 1$, then for $n \gg 0$,

$$\text{ii)}\quad D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \cdot \det(w_\alpha^{(n)}, S^{*n+1} t_\tau)_{\gamma\times\gamma} \cdot \left[\frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, t_\tau)}\right]\left[1 + O(n^{(1-2\beta)})\right].$$

Theorem 1.2. *Let $\phi = \psi z^\gamma$ for integer-valued $\gamma > 0$ with normalized Wiener-Hopf factorization $\psi = f \bar g$ and let f and g be outer functions in $K_{2,2}^{\frac12,\frac12} = \{\phi : \phi \in L^\infty, \sum_{-\infty}^{\infty} |\hat\phi(k)|^2 |k| < \infty\}$. Let $\{e_\alpha\}_1^\gamma \equiv \{1, z, \ldots, z^{\gamma-1}\}$ denote the standard basis for $\ker T_{\bar z^\gamma}$, and let $u_1, u_2, \ldots, u_\gamma$ be associated root vectors of the nilpotent part of the operator $T_\phi T_{\phi^{-1}}$ so that u_α has maximal order with respect to $(-1)^{m_\alpha} T_f e_\alpha$, , for $\alpha = 1, 2, \ldots, \gamma$, i.e., $(T_\phi T_{\phi^{-1}})^{m_\alpha} u_\alpha = (-1)^{m_\alpha} T_f e_\alpha$ where $m_\alpha = \max\{m : \exists x$ so that $(T_\phi T_{\phi^{-1}})^m x = T_f e_\alpha\}$. Then with $E(\psi) = \det(T_\psi T_{\psi^{-1}}) = \exp \sum_{k=1}^\infty k (\log f)_k (\log \bar g)_{-k}$, we have*

$$\text{a)}\quad \sum_1^\gamma m_\alpha = \dim \text{root space}\,[Q_{\gamma-1}\big(I - H_\psi H_{\frac{1}{\psi}}\big|_{\mathrm{ran}(Q_{\gamma-1})}\big)],$$

$$\text{b)}\quad E(\psi) G\left(\frac{\bar g}{f}\right)^\gamma = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, T_{\frac{1}{g}} e_\tau)},$$

$$\text{c)}\quad \textit{when } m_\alpha = 0 \textit{ for } \alpha = 1, \ldots, \gamma, \quad \textit{then } E(\psi) G\left(\frac{\bar g}{f}\right)^\gamma = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{D_{\gamma-1}\left(T_{\frac{f}{\bar g}}\right)}.$$

Note that $m_\alpha = 0$ if ϕ is unimodular.

Theorem 1.3. *Suppose $\phi = \psi z^\gamma$ where $\gamma \in Z_+$ and $\psi \in \mathrm{Lip}_\beta$, $\frac12 < \beta < 1$ and T_ψ is invertible. Then we have $\psi = f \bar g$ where f, g and $\frac{1}{f}, \frac{1}{g} \in \mathrm{Lip}_\beta \cap H^\infty(\mathbb{T})$, $\frac12 < \beta < 1$,* *if* $e_\alpha = z^\alpha$, $\alpha = 1, 2, \dots, \gamma$, *and* $b = \frac{\bar{g}}{f}$,

$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} E(\psi) G(b)^\gamma \cdot \det\left((T_{\frac{f}{\bar{g}} z^{n+1}} \cdot [1 - H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}]^{-1} e_\alpha, e_\tau)\right)_{\gamma\times\gamma} [1 + O(n^{1-2\beta})].$$

If $\phi \in K_{2,2}^{\frac{1}{2},\frac{1}{2}}$, *then*

$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} E(\psi) G(b)^\gamma \cdot \det\left((T_{\frac{f}{\bar{g}} z^{n+1}} \cdot [1 - H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}]^{-1} e_\alpha, e_\tau)\right)_{\gamma\times\gamma} [1 + o(1)].$$

The proof uses Theorem 1.1 and the identity

$$\det\left((S^{n+1}[S^{*n+1-\gamma} T_\psi^{-1} S^{n+1-\gamma}]^{-1} e_\alpha, T_{\frac{1}{g}} e_\tau)\right) = \det\left((T_{\frac{f}{\bar{g}} z^{n+1}} \cdot [1 - H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}]^{-1} T_{\bar{g}} e_\alpha, e_\tau)\right)_{\gamma\times\gamma}.$$

1.0.1. Relation to other results. Undefined factors in the Böttcher-Silbermann approximation. An important step towards the non-zero index case, was made when M. Fisher and E. Hartwig [21] used Jacobi's theorem on the conjugate minors of the adjugate matrix formed from the cofactors of $D_n(T_\phi)$ to give asymptotic information even when ϕ is piecewise continuous and wind $(\phi, 0) \neq 0$.

The pioneering Fisher-Hartwig results were refined by Böttcher and Silbermann [10] where it was assumed that $\phi = az^\gamma$, with $a \in \mathrm{Lip}_\beta$, $\beta > \frac{1}{2}$, with the Wiener-Hopf factorization $a = a_+ a_-$, and invertible T_a. For integer $\gamma \geq 0$ and $n \gg 0$, the Böttcher-Silbermann approximation is:

$$D_n(T_{az^\gamma}) = (-1)^{(n+1)\gamma} G(a)^{n+1} E(a) G(b)^\gamma \cdot \left[\det\begin{pmatrix} \hat{c}_{-n-1} & \cdots & \hat{c}_{-n+\gamma-2} \\ \vdots & \vdots & \vdots \\ \hat{c}_{-n-\gamma} & \cdots & \hat{c}_{-n-1} \end{pmatrix} + O(n^{-3\beta})\right] [1 + O(n^{1-2\beta)})],$$

with $c = \frac{a_+}{a_-}$, $b = \frac{a_-}{a_+}$, and $\hat{c}_j$ denoting the jth Fourier coefficient, and $E(a) = \exp \sum_{k=1}^\infty k(\log a_+)_k (\log a_-)_{-k}$. As noted above, this approximation fails to give a limit theorem. In §6 an example is given of a rational symbol for which the factor $\det(C_{n,\gamma}) = 0$ for infinitely many n and yet $D_n(\phi) \neq 0$ for $n \gg 0$. In this case, for a subsequence $\{n_j\}$ for which $\det(C_{n_j,\gamma}) = 0$, the Fisher-Hartwig, Böttcher-Silbermann assertion becomes the statement that there is an unspecified denominator, $\left[O(n_j^{-3\beta})\right]$, so that

$$\frac{D_{n_j}(T_{az^\gamma})}{(-1)^{(n_j+\gamma)\gamma} G(a)^{n_j+1}\left[O(n_j^{-3\beta})\right]} = E(a) G(b)^\gamma [1 + O(n_j^{1-2\beta)}]), \quad j = 1, \dots, \infty.$$

Note that the geometric series for

$$[1 - H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}]^{-1} = 1 + \sum_{k=1}^\infty \left(H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}\right)^k$$

of Theorem 1.3, which is valid for large n, provides a correction for $\det C_{n,\gamma}$ since we have the following remark.

Remark 1.2. If $a = a_+ a_- \in L^\infty(\mathbb{T})$ with a_+ and $\bar{a}_-$ outer in $H^2(\mathbb{T})$ and with T_a invertible, then

$$\ker T_{\bar{a}\bar{z}^\gamma} = \bigvee\left\{\frac{1}{\bar{a}_-}, \frac{z}{\bar{a}_-}, \dots, \frac{z^{\gamma-1}}{\bar{a}_-}\right\},$$
$$\ker T_{(az^\gamma)^{-1}} = \bigvee\{a_+, za_+, z^2 a_+, \dots, z^{\gamma-1}a_+\},$$

and if we set $w_\alpha = z^{\alpha-1}a_+$ and $y_\tau = \frac{z^{\tau-1}}{\bar{a}_-}$ for $\alpha, \tau = 1, \dots, \gamma$, then

$$\det(C_{n,\gamma}) = \det(w_\alpha, S^{*n+1}y_\tau)$$
$$= \det\begin{pmatrix} \hat{c}_{-n-1} & \cdots & \hat{c}_{-n+\gamma-2} \\ \vdots & \vdots & \vdots \\ \hat{c}_{-n-\gamma} & \cdots & \hat{c}_{-n-1} \end{pmatrix} = \det(T_{\frac{a_+}{\bar{a}_-}z^{n+1}} z^\alpha, z^\tau).$$

Identification of factors in the Widom approximation. Harold Widom [32] treats symbols having the form $\phi = f\bar{g}z^\gamma$ where f is piecewise C^∞ but g is not in C^∞, i.e., there are finitely many points at which certain higher order derivatives are piecewise continuous. The aim of his work is to produce an asymptotic formula for $D_n(T_\phi)$ in the presence of such higher order singularities and non-vanishing index. His main result is

$$D_n(T_\phi) = G(f\bar{g})^{n+1}n^{-e}(c_n + o(1)),$$

where e is a non-negative exponent depending on the nature of the singularities of ϕ (these are points with no neighborhoods in which $\phi \in C^\infty$) and c_n is a bounded sequence given by "a rather complicated but (in principle) perfectly explicit formula."

Comparison of the Widom result and Theorem 1.1 gives

$$n^{-e}[c_n + o(1)] = (-1)^{(n+1)\gamma}\det(S^{n+1}w_\alpha^{(n)}, t_\tau)\frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}}\lambda}{\det(u_\alpha, t_\tau)}.$$

For a certain class of generating functions ϕ Widom succeeds in showing that for almost all x not in the range of ϕ, the sequence $c_n = c_n(x)$ corresponding to the symbol $\phi - x$ remains bounded away from zero and he says that "we have no doubt that this is always true"—but "we have not been able to prove that this is always the case".

Note that for fixed x the sequence $c_n(x)$ may approach zero very rapidly. For example, suppose that $f \in \mathrm{Lip}(\beta)$ with $\beta > \frac{1}{2}$ and is piecewise in C^∞, while $g \in C^\infty$. Since $\ker T_{\bar{\phi}} = T_{\frac{1}{g}}\{1, z, z^2, \dots, z^{\gamma-1}\}$, it follows that $\| S^{*n+1}y_\tau \| = o(n^{-p})$, $\forall p > 0$, and therefore $c_n + o(1) = o(n^{-p})$ for any $p > 0$.

On the other hand, if c_n is bounded away from zero, Theorem 1.3 says

$$c_n \sim (-1)^{(n+1)\gamma}n^e E(f\bar{g})G(\frac{\bar{g}}{f})^\gamma \cdot \det\left((T_{\frac{f}{\bar{g}}z^{n+1}} \cdot [1 - H_{\frac{\bar{g}}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar{g}})}]^{-1}e_\alpha, e_\tau)\right)_{\gamma\times\gamma}.$$

Some key relations. The vectors $\sigma_{S^{*n+1}, T_\phi S^{*n+1}T_{\phi^{-1}}}$ and $\sigma_{1, T_\phi S^{*n+1}R(T_\phi)S^{n+1}}$ were calculated in [15]. The following is one consequence of these calculations that is fundamental for the present investigation.

The appearance of root vectors. The limit in Theorem 1.1 above is obtained from the convergence of the sequence of perturbation vectors

$$\lim_{n\to\infty} \sigma_{1,T_\phi S^{*n+1}R(T_\phi)S^{n+1}} = \sigma_{1,T_\phi T_{\phi^{-1}}} = \frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, y_\beta)} \big(\bigwedge_{k=1}^{\gamma} w_k\big) \otimes \big(\bigwedge_{k=1}^{\gamma} [y_k + \operatorname{ran} T_\phi]\big)^*.$$

This form for the right-hand side of the limiting perturbation vector $\sigma_{1,T_\phi T_{\phi^{-1}}}$ follows from an apparently new observation about finite matrices given in Proposition 2.1 below:

If X is an operator on Hilbert space so that $X = \text{identity} + \text{trace class}$, and $\{w_\alpha\}_1^\gamma$ is any basis for $\ker X$, $\{y_\alpha\}_1^\gamma$ is any orthonormal basis for $\ker X^*$, and if $\{u_\alpha\}_1^\gamma$ is a corresponding set of root vectors of maximal height for $\{w_\alpha\}_1^\gamma$, i.e., $(X)^{\max} u_\alpha = w_\alpha$, $\alpha = 1, 2, \dots, \gamma$. Then with $m_\alpha + 1$ denoting the algebraic multiplicty of w_α,

$$(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \cdot \prod_{\lambda\in\sigma(X)\setminus\{0\}} \lambda \cdot \frac{\|\bigwedge w_j\|^2}{\det(u_j, y_i)} = \det\Big(X + \sum_{j=1}^{\gamma} y_j \otimes w_j\Big).$$

Three examples. Consider the matrix given by $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ with minimal polynomial $x(x-1)^3$. Then an orthonormal basis for $\ker A$ is $w_1 = \big(0,0,0,1\big)$ and an orthonormal basis for $\ker A^*$ is $y_1 = \big(\frac{1}{\sqrt{31}}, -\frac{2}{\sqrt{31}}, \frac{5}{\sqrt{31}}, -\frac{1}{\sqrt{31}}\big)$. Since $m_\alpha = 0$, $u_1 = w_1$ is a root vector of maximal height lying over the kernel vector of A, i.e., $(A)^0 u_1 = w_1$.

Then, $\det(w_1, y_1) = -\frac{1}{\sqrt{31}}$ so that

$$\frac{1}{\det(u_1, y_1)} = -\sqrt{31} = \det\big(A + y_1 \otimes w_1\big) = \det \begin{pmatrix} 1 & 2 & 1 & \frac{1}{\sqrt{31}} \\ 0 & 1 & 3 & -\frac{2}{\sqrt{31}} \\ 0 & 0 & 1 & \frac{5}{\sqrt{31}} \\ 1 & 0 & 0 & -\frac{1}{\sqrt{31}} \end{pmatrix}.$$

For another example, take $B = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$. The minimal polynomial is $x^3(x+1)$, an orthonormal basis for $\ker B$ is $w_1 = (\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}})$ while $y_1 = (0,0,1,0)$ is an orthonormal basis for $\ker X^*$. $u_1 = (0, 0, \frac{\sqrt{2}}{12}, 0)$ is a root vector of maximal height lying over w_1, i.e., $B^2 u_1 = w_1$. The length $m_\alpha + 1$ of the corresponding Jordan chain is 3, but $\prod_{\lambda\in\sigma(X)\setminus\{0\}} \lambda = -1$. Hence we have

$$y_1 \otimes w_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1/2)\cdot\sqrt{2} & 0 & 0 & (1/2)\cdot\sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\det(B + y_1 \otimes w_1) = \det \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ (1/2)\cdot\sqrt{2} & 0 & 0 & (1/2)\cdot\sqrt{2} \\ 1 & 0 & 0 & -1 \end{pmatrix}$$
$$= -6\sqrt{2} = \frac{(-1)^3}{\det(u_1, y_1)} = \frac{(-1)^3}{\frac{\sqrt{2}}{12}}.$$

For a Hilbert space example, let us consider $B \equiv 1 - a \otimes b$ where $(a, b) = 1$. Then $\ker B$ has basis $\{a\}$ and a is a maximal root vector lying over itself. Hence, since the non-zero eigenvalues of B are all equal to 1, we get $\sigma_{1,B} = a \otimes (a + \operatorname{ran} B)^*$. Thus, $\|a + \operatorname{ran} B\| = \|P(B^*)a\|$. Now $P(B^*) = \frac{b}{\|b\|} \otimes \frac{b}{\|b\|}$, and $(a, b) = 1$, so we have $\|P(B^*)a\| = \frac{1}{\|b\|}$. Consequently, $\|\sigma_{1,B}\| = \|a\| \cdot \|b\|$.

Now let $\phi = z - \lambda$, $\|\lambda\| < 1$. Then $T_\phi T_{\phi^{-1}} = 1 - e_0 \otimes e_{\bar\lambda}$, where $e_{\bar\lambda} = \sum_{k=0}^{\infty} \bar\lambda^k\, z^k$. But $\|e_0\| = 1$ and $\|e_{\bar\lambda}\| = \frac{1}{\sqrt{1-|\lambda|^2}}$. Thus, by the paragraph above, $\|\sigma_{1,T_\phi T_{\phi^{-1}}}\| = \frac{1}{\sqrt{1-|\lambda|^2}}$.

The invariant vector. Suppose that for a Fredholm operator B we have $1 - B \in \mathcal{L}^1$. We observe now that if we pick any basis $\{(-1)^{m_\alpha} w_\alpha\}$ for $\ker B$, and any corresponding basis of top root vectors $\{u_\alpha\}$ for Jordan chains lying over the kernel vectors, we may form the vector

$$\mathcal{I}(B) \equiv (\bigwedge w_\alpha)^* \otimes (\bigwedge u_\alpha + H/BH).$$

a) $\mathcal{I}(B)$ is invariant under the choice of the root vectors $\{u_\alpha\}$ for a fixed choice of $\{w_\alpha\}$.
b) $\mathcal{I}(B)$ is invariant under the choice of null vectors $\{w_\alpha\}$.

Pairing the invariant vector with the perturbation vector gives $\prod_{\lambda\in\sigma(B)\setminus\{0\}} \lambda$ so that

$$\sigma_{1,B} = \prod_{\lambda\in\sigma(B)\setminus\{0\}} \lambda \ \cdot \mathcal{I}(B).$$

But Theorem 1.1 exploits the fact that we have the freedom to choose a different basis $\{t_\tau\}$ for the cokernel space. We obtain then the same vector $\sigma_{1,B}$, but another numerical coefficient multiplying the corresponding tensor product. In the preceding theorems and Theorem 1.6 below we exploit this freedom to choose bases for the cokernel space corresponding to the factorization $\phi = \psi z^\gamma$, and in Theorem A we obtain an additional meaning for the resulting coefficient in terms of holonomy.

The separation property for finite matrices or for operators of the form 1+ trace class. Suppose that A and B are stationary[2] Fredholm operators. Then we have the Riesz reduction $H^2(\mathbb{T}) = R_\infty(A) + N_\infty(A) = R_\infty(B) + N_\infty(B)$, where now we use the notation $N_\infty(T) = \bigcup_{n=1}^{\infty} \ker T^n$ and $R_\infty(T) = \bigcap_{n=1}^{\infty} \operatorname{ran} T^n$.

[2]Recall that T is stationary if there exist a finite p so that $N_p \equiv \ker T^p = \ker T^{p+1} = \cdots = N_\infty(T)$ and $R_p(T) \equiv \operatorname{ran} T^p = R_{p+1}(T) = \cdots = R_\infty(T)$.

Select a basis $\bigwedge_1^\gamma(-1)^{m_j} w_j^A$ for $\det\ker A$ with associated maximal root vectors $\{u_j^A\}_{j=1}^\gamma$ and form the invariant vector

$$\mathcal{I}(A) = (\bigwedge_1^\gamma w_j^A)^* \otimes [\bigwedge_1^\gamma u_j^A + \operatorname{ran} A].$$

Similarly, select a basis $\bigwedge_1^\gamma(-1)^{m_\tau} w_\tau^B$ for $\det\ker B$ and associated maximal root vectors $\{u_\tau^B\}_{\tau=1}^\mu$ lying over these basis vectors and form the invariant vector

$$\mathcal{I}(B) = (\bigwedge_1^\mu w_j^B)^* \otimes [\bigwedge_1^\mu u_j^B + \operatorname{ran} B].$$

Let M be the linear map on H so that $M = 0$ on $R_\infty(A)$ and M maps w_j^A to u_j^A for all $j = 1, \dots, \gamma$.

Let L be the linear map on H so that $L = 0$ on $R_\infty(B)$ and L maps w_τ^B to u_τ^B for all $\tau = 1, \dots, \mu$. Then

$$\sigma_{A,B} = \det[(A+M)^{-1}(B+L)] \cdot \mathcal{I}(A) \otimes \mathcal{I}(B)^*.$$

Factorization. Suppose A and B are finite matrices (alternately, A and B are of the form $1 + \mathcal{L}^1$). The indices are both zero and the given description of the vector $\sigma_{A,B}$ applies. The coefficient factors as $\frac{\det(B+L)}{\det(A+M)} = \frac{\prod_{\lambda_B \in \sigma(B)\setminus\{0\}} \lambda_B}{\prod_{\lambda_A \in \sigma(A)\setminus\{0\}} \lambda_A}$, and therefore

$$\sigma_{A,B} = \sigma_{1,A}^* \otimes \sigma_{1,B} = \sigma_{A,1} \otimes \sigma_{1,B}.$$

1.1. Perturbation vectors and the Szegö limit theorem.

Lemma 1.1. *Suppose T_ϕ is injective and $T_\phi T_{\phi^{-1}} - 1 \in \mathcal{L}^1(\mathbb{T})$. Let $w_1, \dots, w_\gamma$ be any basis for $\ker T_{\phi^{-1}}$. Then there is a basis $\{w_k^{(n)}\}$ in $\ker[S^{*n+1}R(T_\phi)S^{n+1}]$ for $n \gg 0$ so that $w_k^{(n)}$ converges to w_k for $1 \le k \le \gamma$.*

Lemma 1.2. *Suppose $D_n(T_\phi) \neq 0$, and $\{t_\tau\}_1^\gamma$ is a basis for $\ker T_{\bar\phi}$, while $\{w_k^{(n)}\}_{k=1}^\gamma$ is any basis for $\ker[S^{*n+1}R(T_\phi)S^{n+1}]$. Then*

$$\sigma_{1,T_\phi S^{*n+1}R(T_\phi)S^{n+1}} = \frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1} \cdot \det\big((w_\alpha^{(n)}, S^{*n+1}t_\tau)\big)} \cdot \big(\bigwedge_{k=1}^\gamma w_k^{(n)}\big) \otimes \big(\bigwedge_{k=1}^\gamma [t_k + \operatorname{ran} T_\phi]\big)^*,$$

and if top root vectors u_α of $T_\phi T_{\phi^{-1}}$ are chosen to lie over the kernel vectors $(-1)^{m_\alpha} w_\alpha \in \ker T_{\phi^{-1}}$, i.e., $(T_\phi T_{\phi^{-1}})^{m_\alpha} u_\alpha = (-1)^{m_\alpha} w_\alpha$ where $m_\alpha = \max\{m \in Z_+ : \exists x$ so that $(T_\phi T_{\phi^{-1}})^m x = w_\alpha\}$, we have

$$\sigma_{1,T_\phi T_{\phi^{-1}}} = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, t_\tau)} \big(\bigwedge_{k=1}^\gamma w_k\big) \otimes \big(\bigwedge_{k=1}^\gamma [t_k + \operatorname{ran} T_\phi]\big)^*.$$

If $w_k^{(n)} \to w_k$ it is clear that this representation of $\sigma_{1,T_\phi T_{\phi^{-1}}}$ immediately gives a limit theorem for the coefficients of the two perturbation vectors

$$\lim_{n\to\infty} \frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1} \cdot \det(w_\alpha^{(n)}, S^{*n+1}t_\tau)} = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, t_\tau)}.$$

The factor $\mathbf{G}(\phi)^{n+1}$ is explictly evaluated using Theorem 1.5 below. If $\phi = \psi z^\gamma$, then $\mathbf{G}(\phi)^{n+1} = (-1)^{\gamma(n+1)} \cdot G(\psi)^{n+1} = \det_* \partial\{T_\phi + \mathcal{L}^1(H), T_{z^{n+1}} + \mathcal{L}^1(H)\} = \phi \cup z^{n+1}(\Sigma)$, the holonomy of a bundle $\phi \cup z^{n+1}$ over the unit circle $\mathbb{T}$ which is computed by an integral.

The form of the perturbation vector in Lemma 1.2. Using Proposition 1.1 it was shown in [15], equation (73), that when T_ϕ is injective, Fredholm, $\det(P_n T_\phi P_n) \neq 0$, and $\{v_\alpha^{(n)}\}_1^\gamma$ is a basis for $\ker S^{*n+1} R(T_\phi) S^{n+1}$, then

$$\sigma_{1, T_\phi S^{*n+1} R(T_\phi) S^{n+1}} = \frac{D_n(T_\phi)}{\det_* \partial\{T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\}} \cdot \Big(\bigwedge_{\alpha=1}^{\gamma} v_\alpha^{(n)}\Big) \otimes \Big(\bigwedge_{\alpha=1}^{\gamma} [S^{n+1} v_\alpha^{(n)} + \operatorname{ran} T_\phi]\Big)^*. \tag{1.1}$$

Since the cosets $S^{n+1} v_\alpha^{(n)} + \operatorname{ran} T_\phi = \sum_1^\gamma (S^{n+1} v_\alpha^{(n)} y_\beta) y_\beta + \operatorname{ran} T_\phi$, for any orthonormal basis $\{y_\alpha\}_1^\gamma \subset \ker T_{\bar\phi}$, it follows that $\big(\bigwedge_{\alpha=1}^\gamma [S^{n+1} v_\alpha^{(n)} + \operatorname{ran} T_\phi]\big)^* = \frac{(\bigwedge_{\alpha=1}^\gamma [y_\alpha + \operatorname{ran} T_\phi])^*}{\det(v_\alpha^{(n)}, S^{*n+1} y_\tau)}$, and therefore (1.1) becomes the stated result in Lemma 1.2.

$$\sigma_{1, T_\phi S^{*n+1} R(T_\phi) S^{n+1}} = \frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1} \cdot \det(v_\alpha^{(n)}, S^{*n+1} y_\beta)} \cdot \Big(\bigwedge_{\alpha=1}^{\gamma} v_\alpha^{(n)}\Big) \otimes \Big(\bigwedge_{\alpha=1}^{\gamma} [y_\alpha + \operatorname{ran} T_\phi]\Big)^*.$$

The second equation in Lemma 1.2 follows by substitution of the formulas of Proposition 2.1 into the definition of the perturbation vector.

1.2. **Bundles, holonomy, Steinberg symbols.** Recall the definition of the *tame symbol*: if $x \in X$, a complex curve in C^n, and f and h are meromorphic functions at x, the function $\big(\frac{f^{\operatorname{order}_x h}}{h^{\operatorname{order}_x f}}\big)$ is of order 0 at x and we can say it has a value at x. The value

$$c_x(f,h) = (-1)^{\operatorname{order}_x h \cdot \operatorname{order}_x f} \cdot \Big(\frac{f^{\operatorname{order}_x h}}{h^{\operatorname{order}_x f}}\Big)$$

is called the *tame symbol* at x.

Beilinson [8] and Deligne [19] investigated the "universal" bundle (m, ∇) with connection on $C^* \times C^*$, and $f \cup g$ the pull-back $(f,g)^*(m,\nabla)$. It is known that for any Riemann surface $\mathcal{Y}$ there is an element of the group $H^1(Y, \mathcal{O}^*_{\mathcal{Y}})$ obtained from the universal line bundle by the pull-back

$$r_0(f,g) = (f,g)^*(m,\nabla) \quad \text{where} \quad (f,g) : \mathcal{Y} \to C^* \times C^*, \quad x \to ((f(x), g(x)).$$

The regulator map r_0 is associated with the residue map, so that if $S = Z(f) \cup Z(g)$ is a finite subset of $\mathcal{Y}$ there is the long exact Gysin sequence:

$$0 \to H^1(\mathcal{Y}, C^*) \to H^1(\mathcal{Y} - S), C^*) \xrightarrow{\coprod_{\lambda \in S} \partial_\lambda} \coprod_{\lambda \in S} C^* \to H^2(\mathcal{Y}, C^*) \longrightarrow \cdots$$

where ∂_λ is the residue map at λ, $\coprod_{\lambda \in S} C^*$ denotes the direct product and $\partial_\lambda r_0(f,g) = c_\lambda(f,g)$.

The bundle construction. For smooth complex valued ϕ and ψ defined and non-vanishing on $\mathbb{T}$, the line bundle $\phi \cup \psi$ over the loop $\mathbb{T}$ is defined by specifying transition functions

$$\left\{\psi^{\frac{1}{2\pi i}(\log_\alpha \phi - \log_\beta \phi)}\right\} \quad \text{together with the connection form} \quad \left\{ -\frac{1}{2\pi i}\log_\alpha \phi \cdot \frac{d\psi}{\psi}\right\}$$

for a Čech cover $\{U_\alpha\}$ such that $\log \phi|_{U_\alpha}$ is defined (denoted by $\log_\alpha \phi$). In other words, the relation between local sections $s_\mu = \{\log_\mu \phi, \psi\}$ defined by different branches of the logarithm on $U_\mu \cap U_\nu$ is

$$\left\{\psi^{\frac{1}{2\pi i}(\log_\mu \phi - \log_\nu \phi)}\right\} \cdot \{\log_\nu \phi, \psi\} = \{\log_\mu \phi, \psi\}$$

and the connection ∇ on the bundle E defined by these transition functions is characterized by

$$\nabla(\{\log(\phi), \psi\}) = \frac{1}{2\pi i}\log \phi \cdot \frac{d\psi}{\psi} \otimes \{\log(\phi), \psi\}.$$

There is an identification $H^1(\mathbb{T}, C^*) \cong \mathrm{Hom}(\pi_1(\mathbb{T}), C^*) \cong C^*$ determined by lifting a generator ρ of $\pi_1(\mathbb{T})$ to the bundle $\mathcal{E}$ defined by these transition functions.

Fix base points $p \in \mathbb{T}$ and $\zeta \in C^*$ so that $[(p,\zeta)] \in \mathcal{E}$. Let ρ be a closed path in $\mathbb{T}$ starting at p and winding counterclockwise to p. Let $\tilde{\rho}$ be a horizontal lift of ρ to $\mathcal{E}$ starting at $[(p,\zeta)]$. Parallel transporting via the connection around a loop leads to $[(p, \tau\cdot\zeta)]$, where $\tau \in C^*$. The map $\rho \to \tau$ defines an element of $\mathrm{Hom}(\pi_1(\mathbb{T}), C^*)$.

The holonomy τ is computed by choosing a disjoint union of open arcs obtained by decomposing $\{\phi^{-1}(U_\alpha) \cap \psi^{-1}(U_\beta)\}_{\alpha,\beta}$ into a disjoint union of open arcs $\{I_j^{\alpha,\beta}\}_{j=1}^{m(\alpha,\beta)}$ on $\mathbb{T}$ so that each arc I_j has a branch $\log_j$ of the logarithm defined, and then further refining this union into a covering by open arcs $\{J_j\}_{j=1}^n$, of $\mathbb{T}$, so that $p \in J_1$ and $p_\mu \in J_\mu \cap J_{\mu+1}$ for $2 \le \mu \le n-1$ while $p_n \in J_1 \cap J_n$. Then as we proceed around the circle counterclockwise we first hit J_2, then J_3, etc. Thus for $1 \le \mu \le n-1$ we have $J_\mu \cap J_{\mu+1} \neq \emptyset$, and $J_1 \cap J_n \neq \emptyset$.

The horizontal lift on J_μ is

$$\psi(p_{\mu+1})^{\frac{1}{2\pi i}[\log_{\mu+1}\phi(p_{\mu+1}) - \log_\mu \phi(p_{\mu+1})]} \cdot \exp \frac{1}{2\pi i}\int_{p_\mu}^{p_{\mu+1}} \log_\mu \phi \, \frac{d\psi}{\psi},$$

so that the total lift around $\mathbb{T}$ is

$$\tau = \prod_{\mu=1}^{n-1}\left(\psi(p_{\mu+1})^{\frac{1}{2\pi i}[\log_{\mu+1}\phi(p_{\mu+1}) - \log_\mu \phi(p_{\mu+1})]} \cdot \exp \frac{1}{2\pi i}\int_{p_\mu}^{p_{\mu+1}} \log_\mu \phi \, \frac{d\psi}{\psi}\right)$$
$$\times\ \psi(p_n)^{\left(\frac{1}{2\pi i}\log_1 \phi(p_n) - \log_n \phi(p_n)\right)} \cdot \exp\left(\frac{1}{2\pi i}\int_{p_n}^{p_1} \log_1 \phi \, \frac{d\psi}{\psi}\right).$$

Keeping track of telescoping factors [8], [16] shows that this product can be rewritten:

Theorem 1.4. *If the unit circle taken positively is a generator Σ of $\pi_1(\mathbb{T})$, then for functions ϕ and ψ smooth on $\mathbb{T}$, for example in the Krein algebra $K_{2,2}^{\frac{1}{2},\frac{1}{2}}$, the element $\phi \cup \psi$ of $H^1(\mathbb{T}, C^*) \cong \mathrm{Hom}\,(\pi_1(\mathbb{T}), C^*)$ is given by*

$$\phi \cup \psi(\Sigma) = \exp \frac{1}{2\pi i}\left(\int_{|z|=1} \log \phi \cdot \frac{d\psi}{\psi} - \log \psi(p)\int_{|z|=1}\frac{d\phi}{\phi}\right).$$

Here p is a base point of the unit circle $\mathbb{T}$, the branches of the logarithms are continuous except possibly at p, and the integrals are taken positively over the circle $\mathbb{T}$ starting at p.

The present authors proved an Index Theorem [14], [16] which deals with the Steinberg symbols of equivalence classes of Toeplitz operators modulo the trace ideal and smooth functions ϕ, ψ defined on the unit circle $\mathbb{T}$:

Theorem 1.5. *For $\phi, \psi \in K_{2,2}^{\frac{1}{2},\frac{1}{2}}$, Fredholm T_ψ and T_ϕ,*

$$\det{}_*\partial\{T_\phi + \mathcal{L}^1(H), T_\psi + \mathcal{L}^1(H)\} = \phi \cup \psi(\Sigma).$$

where Σ is the unit circle taken positively.

Theorem 1.5 is applied to give $\det_*\partial\{T_\phi+\mathcal{L}^1(H), T_{z^{n+1}}+\mathcal{L}^1(H)\} = \phi\cup z^{n+1}(\Sigma)$.

Remark 1.3. Taking $p = 1$ and using Theorem 1.4 and the Index Theorem 1.5 above we get

$$\begin{aligned}\mathbf{G}(\phi)^{n+1} &\equiv \det{}_*\partial\{T_{\psi z^\gamma} + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\} = \psi z^\gamma \cup z^{n+1}(\Sigma)\\ &= \exp\frac{1}{2\pi i}\left(\int_{|z|=1} \log\psi z^\gamma \cdot \frac{dz^{n+1}}{z^{n+1}} - \log p^{n+1}\int_{|z|=1}\frac{d\psi z^\gamma}{\psi z^\gamma}\right).\end{aligned}$$

Accordingly, since

$$\frac{1}{2\pi i}\oint_{\mathbb{T}} \log z^\gamma \, d\log z = \frac{\gamma}{2\pi i}\int_0^{2\pi} i\theta \, d(i\theta) = \pi\gamma i,$$

we have

$$\begin{aligned}\mathbf{G}(\phi)^{n+1} &= \phi \cup z^{n+1}(\Sigma) = \det{}_*\partial\{T_\phi + \mathcal{L}^1(H), T_{z^{n+1}} + \mathcal{L}^1(H)\}\\ &= G(\psi)^{n+1}\cdot\exp[-(n+1)\pi\cdot\gamma i].\end{aligned}$$

The connection of the Steinberg symbols of operator cosets and the geometric mean of an essentially bounded symbol ϕ is intrinsic and does not depend on the smoothness of the symbol. A relationship persists even when the smoothness assumptions used here to define the bundle $\phi \cup z^{n+1}$ are not satisfied:

Recall the following result [17], [16]. If $\phi \in \mathcal{L}^\infty(\mathbb{T})$ and T_ϕ is Fredholm, then

$$|\det{}_*\{T_\phi + \mathcal{L}^1(H), S^{n+1} + \mathcal{L}^1(H)\}| = \exp\frac{n+1}{2\pi}\int_0^{2\pi}\log|\phi(e^{i\theta})|\,d\theta.$$

The classical Szegö Theorem as a holonomy result. It is of some interest to see now that the original Szegö result may be viewed geometrically:

Remark 1.4. Suppose $\phi \in K_{2,2}^{\frac{1}{2},\frac{1}{2}} \cap \mathcal{W}$, where $\mathcal{W}$ is the Wiener algebra, and wind $(\phi, 0) = 0$. Then with the Wiener- Hopf factorization $\phi = \phi_+\phi_-$, the foregoing considerations give

$$\begin{aligned}\lim_{n\to\infty}\frac{D_n(T_\phi)}{\phi\cup z^{n+1}(\Sigma)} &= \phi_- \cup \phi_+(\Sigma) = \det{}_*\circ\partial\{T_{\phi_-} + \mathcal{L}^1(H), T_{\phi_+} + \mathcal{L}^1(H)\}\\ &= \exp\sum_{k=1}^{\infty} k(\log\phi_+)_k(\log\phi_-)_{-k}.\end{aligned}$$

The limit is holonomy when the index is non-zero. Note that for $\gamma \neq 0$, $E(\psi) = \bar{g} \cup f(\Sigma)$ and by definition $G(b) = \frac{\bar{g}}{f} \cup z(\Sigma)$ is a holonomy. The bundles over $\mathbb{T}$ form a group. Thus with $\times$ the group multiplication, the holonomy of the product bundle is

$$E(\psi)G(b)^\gamma = \left(\bar{g} \cup f \times \frac{\bar{g}}{f} \cup z^\gamma\right)(\Sigma).$$

There is a relationship between the Jordan chains lying over null vectors, and the holonomy of a product bundle.

Proposition 1.2. *Let $\phi = \psi z^\gamma$ for integer-valued $\gamma > 0$ with normalized Wiener-Hopf factorization $\psi = f\bar{g}$, and let f and g be outer functions in $K_{2,2}^{\frac{1}{2},\frac{1}{2}} \cap \mathcal{W}$. Then*

$$(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, T_{\frac{1}{g}} e_\tau)_{1\le\alpha,\tau\le\gamma}} = E(\psi)G\left(\frac{\bar{g}}{f}\right)^\gamma = \left(\bar{g} \cup f \times \frac{\bar{g}}{f} \cup z^\gamma\right)(\Sigma).$$

1.3. **Inner outer factorization and normalization.** We are free to take any factorization for the symbol. There are several reasons for considering the factorization in the form $\phi = \Theta_1 \bar{\Theta}_2 f \cdot \bar{g}$ with $\Theta_1 = \prod_{i=1}^{q} \frac{z-\nu_i}{1-\bar{\nu}_i z}$ and $\Theta_2 = \prod_{k=1}^{\ell} \frac{z-\mu_k}{1-\bar{\mu}_k z}$ coprime finite Blaschke products, and f and g outer functions. Among these we note:

- Any $\phi \in \mathcal{L}^\infty$ with $\inf|\phi| > 0$ factors[3] as $\Theta_1\bar{\Theta}_2 hk$ where h is outer and k is continuous; T_ϕ is Fredholm iff $T_{\Theta_1\bar{\Theta}_2}$ is Fredholm and index $T_\phi = \operatorname{index} T_{\Theta_1\bar{\Theta}_2}$. This factorization[4] is unique (up to constants) when $k = 1$.
- The factorization is unique if ϕ is rational.
- Factoring ϕ into $\Theta_1\bar{\Theta}_2 f\bar{g}$ defines a generator of $\det H(T_\phi T_{\phi^{-1}})$ as follows: pick a basis $\{x_j\}$ of $\ker T_{\Theta_1\bar{\Theta}_2}$. The tensor product $\bigwedge T_f x_j \otimes (T_{\frac{1}{g}} x_j + \operatorname{ran} T_\phi)^*$ is a non-zero element of $\det(T_\phi T_{\phi^{-1}})$ which does not depend on the choice of $\{x_j\}$. Pairing with the perturbation vector produces the coefficient

$$\frac{(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda \| \bigwedge T_{\frac{1}{g}} \big| \ker T_{\Theta_1\bar{\Theta}_2} \|^2}{\det(u_\alpha, T_{\frac{1}{g}} e_\tau)_{1\le\alpha,\tau\le\gamma}},$$

where $\{u_\alpha\}$ are the top root vectors associated with $\{T_f x_a\}$.

There are two prominent generators for $\det H(T_\phi S^{*n+1} R(T_\phi) S^{n+1})$, if $D_n(T_\phi) \neq 0$. One is the perturbation vector $\sigma_{1,T_\phi} S^{*n+1} R(T_\phi) S^{n+1}$ and the other is $\bigwedge w_j^n \otimes \bigwedge (S^{n+1} w_j^n + \operatorname{ran} T_\phi)^*$, where w_j^n is any basis for $\ker S^{*n+1} R(T_\phi) S^{n+1}$. Again, we have independence of the choice of basis. Pairing these two yields the Szegö sequence $\frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1}}$, with the generalized geometric mean in the denominator.

- When ϕ is both unimodular and rational we have previously [17] explored a relation between the geometry of subspaces in Hilbert space and hyperbolic plane geometry by evaluating the Szegö limit in terms of the Bolyai-Lobachevsky angles of parallelism associated with pairs of zeros of the Blaschke factors. Recall that these angles are defined by limiting directions.

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[3] N.K. Nikol'skii, *Treatise on the shift operator*, Springer-Verlag, Grundlehren 273, Berlin (1986).

[4] It is natural to consider the Szegö limit problem for more general inner functions. For example, when Θ_1 is singular and the supports of Θ_1 and Θ_2 are identical, as in the work of Lee and Sarason (The spectra of some Toeplitz operators, J. Anal. Appl. **33** (1971), 529–543) where Θ_1 is singular with mass one and Θ_2 is a Blaschke product with zeros clustering at 1.

• The limit result for the factorization $\phi = \Theta_1 \bar{\Theta}_2 f \bar{g}$ clearly displays the way in which the perturbation vector is related to algebraic K-theory, Steinberg symbols, and the tame symbols formed from the zeros of the symbols.

Steinberg symbols were originally introduced for the study of arithmetic because of their natural connection to tame symbols. We are interested in another relationship to geometry in Hilbert space involving the notion of separation of subspaces when degree Θ_1 = degree Θ_2. The purely geometric object $\chi^2(P(T_{\bar{\Theta}_1}), P(T_{\bar{\Theta}_2})) \equiv \det(P(T_{\bar{\Theta}_1})P(T_{\bar{\Theta}_2})\big|_{\ker T_{\bar{\Theta}_1}})$ can be computed in terms of tame symbols and was found to express angular relationships between the indicated kernel spaces (see [15]).

Theorem 1.6. *Let $\phi = \Theta_1 \cdot \bar{\Theta}_2 f \cdot \bar{g}$ with $\Theta_1 = \prod_{i=1}^{q} \frac{z-\nu_i}{1-\bar{\nu}_i z}$ and $\Theta_2 = \prod_{k=1}^{\ell} \frac{z-\mu_k}{1-\bar{\mu}_k z}$ coprime finite Blaschke products, and let f and g be outer functions in $K_{2,2}^{\frac{1}{2},\frac{1}{2}} \cap \mathcal{W}$, the intersection of the Krein algebra $\{a \in \mathcal{L}^\infty(\mathbb{T}) : \sum_{n\in Z}(|n|+1)|a_n|^2 < \infty$ and the Wiener algebra, $\mathcal{W}$. Let $\gamma \equiv q - \ell = \mathrm{wind}(\phi, 0) \geq 0$. Let $w_\alpha = T_f x_\alpha$ and let u_α be the associated top root vectors lying over $(-1)^{m_\alpha} T_f x_\alpha$ where $\{x_\alpha\}_1^\gamma$ is an orthonormal basis for $\ker T_{\bar{\Theta}_1 \Theta_2} = T_{\bar{\Theta}_2}(K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp})$. Then for any sequence of vectors $\{v_\alpha^{(n)}\}_1^\gamma$ in $\ker S^{*n+1} R(T_\phi) S^{n+1}$ for which $\lim_{n\to\infty} v_\alpha^{(n)} = T_f x_\alpha$, for $n \gg 0$,*

i)
$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \cdot \det(v_\alpha^{(n)}, S^{*n+1} T_{\frac{1}{g}} x_\tau)_{\gamma\times\gamma} \cdot \left[\frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, T_{\frac{1}{g}} x_\tau)}\right]\left[1 + O(n^{(1-2\beta)})\right].$$

ii)
$$\lim_{n\to\infty} \frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1} \cdot \det(v_\alpha^{(n)}, S^{*n+1} T_{\frac{1}{g}} x_\tau)} = \chi^2(P(T_{\bar{\Theta}_1}), P(T_{\bar{\Theta}_2})) \cdot \prod_{k=1}^{\ell} [\frac{f(\mu_k)}{\bar{g}(\mu_k)}] \prod_{i=1}^{q} [\frac{\bar{g}(\nu_i)}{f(\nu_i)}]$$
$$\cdot \exp \sum_{k=1}^{\infty} k(\log f)_k (\log \bar{g})_{-k} \cdot \det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2})$$
$$\cdot T_{|f|^2}\big|_{K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp}})^{\frac{1}{2}} \cdot \det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2}\big|K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp})^{\frac{1}{2}},$$

where explicitly for simple zeros $\{\mu_\alpha\}$ and $\{\nu_\tau\}$,

$$\chi^2(P(T_{\bar{\Theta}_1}), P(T_{\bar{\Theta}_2})) = \frac{\prod_{1\leq\alpha,\tau\leq\ell} |1 - \bar{\mu}_\alpha \mu_\tau|}{\prod_{\substack{1\leq\alpha\leq\ell \\ 1\leq\beta\leq q}} |1 - \bar{\mu}_\alpha \nu_\tau|^2} \cdot \left| \sum_N \frac{\prod_{\substack{\mu_\alpha\in\bar{N} \\ 1\leq\beta\leq q}} (\mu_\alpha - \nu_\tau)}{\prod_{\substack{\mu_\alpha\in N \\ \mu_\tau\in\bar{N}}} (\mu_\alpha - \mu_\tau)} \cdot \frac{\prod_{\substack{\frac{1}{\bar{\mu}_\alpha}\in N \\ 1\leq\tau\leq q}} (\bar{\mu}_\alpha - \bar{\nu}_\tau)}{\prod_{\substack{\frac{1}{\bar{\mu}_\alpha}\in N \\ \frac{1}{\bar{\mu}_\tau}\in\bar{N}}} (\bar{\mu}_\alpha - \bar{\mu}_\tau)} \cdot \frac{\prod_{\substack{\frac{1}{\bar{\mu}_\alpha}\in\bar{N} \\ 1\leq\tau\leq q}} (1 - \bar{\mu}_\alpha \nu_\tau)}{\prod_{\substack{\frac{1}{\bar{\mu}_\alpha}\in\bar{N} \\ \mu_\tau\in N}} (1 - \bar{\mu}_\alpha \mu_\tau)} \cdot \frac{\prod_{\substack{\mu_\alpha\in N \\ 1\leq\tau\leq q}} (1 - \mu_\alpha \bar{\nu}_\tau)}{\prod_{\substack{\mu_\alpha\in\bar{N} \\ \frac{1}{\bar{\mu}_\tau}\in N}} (\mu_\alpha \bar{\mu}_\tau - 1)} \right|.$$

The sum is extended over all subsets $N \subset \{\mu_1, \cdots, \mu_\ell; \frac{1}{\bar{\mu}_1}, \cdots, \frac{1}{\bar{\mu}_\ell}\}$ *of cardinality* ℓ, *and* $\bar{N} = \{\mu_1, \cdots, \mu_\ell; \frac{1}{\bar{\mu}_1}, \cdots, \frac{1}{\bar{\mu}_\ell}\} \setminus N$.

For an inner function Θ we use the notation $K_\Theta = H^2 \ominus \Theta H^2$. The numbers $\det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|f|^2} \big| K_{\Theta_1} \cap K_{\Theta_2}^{\perp})^{\frac{1}{2}}$ and $\det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2} \big|_{K_{\Theta_1} \cap K_{\Theta_2}^{\perp}})^{\frac{1}{2}}$ are fixed finite order determinants. These normalization factors are computed from the eigenvectors associated with the zeros.

2. Perturbation vectors

Let H be an n-dimensional vector space over a field F. Let T be an endomorphism on H. Let $p = \dim \ker T$. Then we have the exact sequence

$$\mathcal{T}: \quad 0 \longrightarrow \operatorname{Ker} T \xrightarrow{i} H \xrightarrow{T} H \xrightarrow{\pi_T} \operatorname{Coker} T \longrightarrow 0.$$

The torsion vector of this complex is the element in $\det \mathcal{T}$ defined as follows: Pick non-zero vectors $s_1 \in \det \ker T$, $s_2 \in \bigwedge^{n-p} H$ and $s_3 \in \bigwedge^p H$, where $s_1 \wedge s_2 \neq 0$ and $Ts_2 \wedge s_3 \neq 0$. Then

$$(-1)^{p(n-p)} s_1^* \otimes (s_1 \wedge s_2) \otimes (Ts_2 \wedge s_3)^* \otimes \pi_T s_3$$

defines a generator $\sigma(T)$ of $\det \mathcal{T}$ that is independent of the choice of vectors s_1, s_2, s_3.

In the same way, a pair of maps S and T acting between finite dimensional spaces H_1 and H_2 produces torsion vectors $\sigma(S)$ and $\sigma(T)$. The product

$$\sigma_{S,T} \equiv \sigma(S) \otimes \sigma(T)^*$$

is then a well-defined element in $\det \mathcal{S} \otimes (\det \mathcal{T})^*$.

In infinite dimensions, $\sigma(S)$ and $\sigma(T)$ are not defined, so the coupling represented by $\sigma_{S,T}$ may not factor. But in [15] it was shown how the construction extends to the case where H_1 and H_2 are infinite dimensional, with S and T algebraically Fredholm and $S - T$ finite rank, or in the case of Banach spaces where S and T are Fredholm in the usual sense and $S - T$ is nuclear.

The following properties of the norms of perturbation vectors were proved in [15]:

Let H_1 and H_2 be Hilbert spaces. Then

i) $\| \sigma_{T,S} \| = \| \sigma_{S,T} \|^{-1}$.

ii) If index $T \geq 0$, then $\| \sigma_{T,S} \|^2 = \det \left(\big(TT^* + P(T^*)\big)^{-1} \cdot \big(SS^* + P(S^*)\big) \right)$.

iii) If index $T \leq 0$, then $\| \sigma_{T,S} \|^2 = \det \left(\big(T^*T + P(T)\big)^{-1} \big(S^*S + P(S)\big) \right)$.

iv) Suppose $H_1 = H_2 = H$ is a complex Hilbert space and $T^*T - TT^* \in \mathcal{L}^1(H)$. Then $\| \sigma_{T,S} \| = \| \sigma_{T^*,S^*} \|$.

v) $\sigma_{S,T} = \sigma_{T,S}^*$, $\sigma_{T,S} \otimes \sigma_{S,R} = \sigma_{T,R}$.

The equality in (v) is to be understood in the sense that the vectors on the right and left-hand sides are images of each other under the canonical isomorphism $H \otimes H^* \cong C$.

Example. Suppose $H = H_1 = H_2$ is a separable Hilbert space and $1 - X$ is a trace class operator. Let x be any vector in $\det \ker X$. Then x is decomposable and there is a basis $\{x_j\}$ for $\ker X$ so that $x = \bigwedge x_j$. Let $\{u_j\}$ be a set of root vectors of maximal height lying over $\{(-1)^{m_j} x_j\}$ relative to $X\big|_{\bigcup_{n=1}^\infty \ker(X)^n}$ with algebraic multiplicity m_j+1. Recall that the Riesz-Schauder theorem implies there is a direct sum decomposition of the Hilbert space $H = \mathcal{X} \dot{+} \mathcal{Y}$ into X invariant subspaces, $\mathcal{X} = \bigcap_{n=1}^{\infty} \operatorname{ran}(X)^n$ and $\mathcal{Y} = \bigcup_{n=1}^{\infty} \ker(X)^n$, of X so that X restricted to $\mathcal{X}$ is invertible, $\mathcal{Y}$ is finite dimensional, and X restricted to $\mathcal{Y}$ is nilpotent. Furthermore,

$$\sigma_{1,X} = \prod_{\lambda\in\sigma(X)\setminus\{0\}} \lambda \ \cdot\ \bigwedge x_j \otimes \Big(\bigwedge (u_j + \operatorname{ran} X)\Big)^*$$

and

$$\|\sigma_{1,X}\| = \frac{\prod_{\lambda\in\sigma(X)\setminus\{0\}} |\lambda|}{|\det(u_i, y_j)|} = \prod_{s_\alpha(X)\neq 0} \mathbf{s}_\alpha(X),$$

where $\{y_j\}$ is any orthonormal basis for $\ker X^*$, $\{u_i\}$ is any set of maximal root vectors relative to any orthonormal basis of $\ker X$ and $\{\mathbf{s}_\alpha(\mathbf{X})\}$ are the singular values of X. If $\sigma(X)\setminus\{0\} = \emptyset$, the product is understood to be 1.

Proposition 2.1. *Let X be an operator on Hilbert space H so that $X =$ identity + trace class. Suppose $\{w_\alpha\}_1^\gamma$ is any basis for $\ker X$ and $\{y_\alpha\}_1^\gamma$ is any orthonormal basis for $\ker X^*$. Let $\{u_\alpha\}_1^\gamma$ be a corresponding set of maximal root vectors for $\{w_\alpha\}_1^\gamma$, i.e., $(X)^{\max} u_\alpha = w_\alpha$, $\alpha = 1, 2, \ldots, \gamma$. Then*

i) $\sigma_{1,X} = (-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \cdot \prod_{\lambda\in\sigma(X)\setminus\{0\}} \lambda \ \cdot (\bigwedge_{k=1}^{\gamma} w_k) \otimes (\bigwedge_{\tau=1}^{\gamma} [u_\tau + \operatorname{ran} X])^*,$

ii) $(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \cdot \prod_{\lambda\in\sigma(X)\setminus\{0\}} \lambda \ \cdot \dfrac{\|\bigwedge w_j\|^2}{\det(u_j, y_i)} = \det(X + \sum_{j=1}^{\gamma} y_j \otimes w_j).$

iii) *Let $\{u'_\alpha\}_1^\gamma$ be the maximal root vectors corresponding to $\{w'_\alpha\}_1^\gamma$, the orthonormal vectors obtained from $\{w_\alpha\}_1^\gamma$ by the Gram-Schmidt process. Then* $\|\bigwedge_{\alpha=1}^{\gamma} w_\alpha\| \cdot \det(u'_\alpha, y_\tau) = \det(u_\alpha, y_\tau)$.

Proof. By definition of the perturbation vector $\sigma_{1,X}$ (see §2 in [15]), we have

$$\sigma_{1,X} = \det\left(X + \sum_{j=1}^{\gamma} u_j \otimes w_j\right)\left(\bigwedge_1^\gamma t_j\right) \otimes \left(\bigwedge_1^\gamma (u_j + \operatorname{ran} X)\right)^*,$$

where * denotes the dual vector and $\{t_j\}_1^\gamma$ is a dual basis to $\{w_j\}_1^\gamma$, i.e., $(t_j, w_k) = \delta_{jk}$ and $\{t_j\}_1^\gamma$ spans $\ker X$. Note that $\bigwedge_1^\gamma t_j = \|\bigwedge_1^\gamma w_j\|^{-2} \bigwedge_1^\gamma w_j$.

But

$$\det(X + \sum u_j \otimes w_j) = \det(X\big|_{\mathcal{X}}) \det(X + \sum u_j \otimes w_j\big|_{\mathcal{Y}}),$$

and

$$\det(X\big|_{\mathcal{X}}) = \prod_{\lambda\in\sigma(X)\setminus\{0\}} \lambda,$$

$$\det(X + \sum_1^\gamma u_j \otimes w_j\big|_{\mathcal{Y}}) = (-1)^{\sum_{j=1}^{\gamma} m_j} \|\bigwedge_1^\gamma w_j\|^2.$$

Thus i) follows by substitution.

Now we prove ii). Again by definition of the perturbation vector $\sigma_{1,X}$, we have

$$\sigma_{1,X} = \det\left(X + \sum_{j=1}^{\gamma} y_j \otimes w_j\right)\left(\bigwedge_1^\gamma t_j\right) \otimes \left(\bigwedge_1^\gamma (y_j + \operatorname{ran} X)\right)^*.$$

With $u_j = \sum_{k=1}^{\gamma}(u_j, y_k)y_k \mod \operatorname{ran} X$ we have

$$\left(\bigwedge_1^{\gamma}(u_j + \operatorname{ran} X\right)^* = \frac{\left(\bigwedge_1^{\gamma}(y_j + \operatorname{ran} X)\right)^*}{\det(u_j, y_k)}.$$

Therefore,

$$\frac{\det(X + \sum_1^{\gamma} u_j \otimes w_j)}{\det(u_j, y_k)} = \det(X + \sum_{j=1}^{\gamma} y_j \otimes w_j),$$

so that

$$\left[(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \prod_{\lambda \in \sigma(X \setminus \{0\}} \lambda\right] \frac{\| \bigwedge_1^{\gamma} w_j \|^2}{\det(u_j, y_k)} = \det(X + \sum_1^{\gamma} y_j \otimes w_j).$$

Next we prove iii). Replacing $\{w_j\}_1^{\gamma}$ by $\{w_j'\}_1^{\gamma}$ and $\{u_j\}_1^{\gamma}$ by $\{u_j'\}_1^{\gamma}$ in part ii) gives

$$\frac{(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \prod_{\lambda \in \sigma(X \setminus \{0\}} \lambda}{\det(u_j', y_k)} = \det(X + \sum_1^{\gamma} y_j \otimes w_j').$$

By equating expressions for the perturbation vector, we get

$$\det\left(X + \sum_1^{\gamma} y_j \otimes w_j'\right)\left(\bigwedge_1^{\gamma} w_j'\right) \otimes \left(\bigwedge_1^{\gamma}(y_j + \operatorname{ran} X\right)^*$$
$$= \det\left(X + \sum_1^{\gamma} y_j \otimes w_j\right)\left(\bigwedge_1^{\gamma} t_j\right) \otimes \left(\bigwedge_1^{\gamma}(y_j + \operatorname{ran} X)\right)^*.$$

Again $\bigwedge_1^{\gamma} t_j = \| \bigwedge_1^{\gamma} w_j \|^{-2} \bigwedge_1^{\gamma} w_j$ and $\bigwedge_1^{\gamma} w_j' = \| \bigwedge_1^{\gamma} w_j \|^{-1} \bigwedge_1^{\gamma} w_j$. Therefore,

$$\det\left(X + \sum_1^{\gamma} y_j \otimes w_j'\right) = \| \bigwedge_1^{\gamma} w_j \|^{-1} \det\left(X + \sum_1^{\gamma} y_j \otimes w_j\right).$$

Consequently,

$$(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda \cdot \frac{1}{\det(u_j', y_k)}$$
$$= \| \bigwedge_1^{\gamma} w_j \|^{-1} \left[(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \prod_{\lambda \in \sigma(X) \setminus \{0\}} \lambda\right] \frac{\| \bigwedge_1^{\gamma} w_j \|^2}{\det(u_j, y_k)}.$$

Canceling like factors gives iii). □

3. Proof of Theorem 1.1, part i)

The proof of Theorems 1.1, 1.2, and 1.3 are centered around equation (35) of [15] cited earlier:

$$\frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1}} = \det(T_\phi S^{*n+1} R(T_\phi) S^{n+1} + P(T_{\bar{\phi}}) S^{n+1}).$$

We proceed under the assumptions of Theorem 1.1, part i). In particular, since $T_\phi T_{\phi^{-1}}$ is of the form $1 + J$, where J is in the trace ideal, $T_{\phi^{-1}}$ is a regularizer for T_ϕ and $R(T_\phi) - T_{\phi^{-1}}$ is in $\mathcal{L}^1(H)$. Let $\gamma = \operatorname{Index} T_\phi = \dim \ker T_{\bar{\phi}} \geq 1$. Thus $P(T_{\bar{\phi}})$ has rank γ.

Lemma 1.1 of §1.1 is contained in the following lemma.

Lemma 3.1. *Suppose $\phi \in \mathcal{L}^\infty(\mathbb{T})$ and T_ϕ is injective and Fredholm with $T_\phi T_{\phi^{-1}} - 1 \in \mathcal{L}^1(H^2(\mathbb{T}))$. Then as $n \to \infty$, $P(S^{*n+1}R(T_\phi)S^{n+1})$ converges to $P(T_{\phi^{-1}})$ in trace norm and $D_n(T_\phi) = 0 \iff \det\left(S^{n+1}v_\alpha^{(n)}, y_\tau\right) = 0$, where $\{v_\alpha^{(n)}\}_1^\gamma$ (respectively $\{y_\tau\}_1^\gamma$) is a basis for $\ker(S^{*n+1}R(T_\phi)S^{n+1})$ (respectively $\ker(T_{\bar\phi})$).*

Proof. Since $T_\phi T_{\phi^{-1}} - 1$ is compact, $R(T_\phi)$ has the form $T_{\phi^{-1}} + F$ where F is also compact. Consequently, $S^{*n+1}R(T_\phi)S^{n+1}$ converges in the uniform norm to $T_{\phi^{-1}}$. Since $T_{\phi^{-1}}$ is surjective, the same is true for $S^{*n+1}R(T_\phi)S^{n+1}$ provided that $n \gg 0$. The first assertion now follows since $P(S^{*n+1}R(T_\phi)S^{n+1}) = 1 - S^{*n+1}R(T_\phi)^*S^{n+1}[S^{*n+1}R(T_\phi)S^{n+1}S^{*n+1}R(T_\phi)^*S^{n+1}]^{-1}S^{*n+1}R(T_\phi)S^{n+1}$.

Now index $S^{*n+1}R(T_\phi)S^{n+1} = \gamma$ and therefore $\dim\ker S^{*n+1}R(T_\phi)S^{n+1} = \gamma$. Now fix $n \gg 0$. When $D_n(\phi) \neq 0$ we have from Proposition 1.1

$$\frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1}} = \det\left(T_\phi S^{*n+1}R(T_\phi)S^{n+1} + P(T_{\bar\phi})S^{n+1}\right).$$

If we replace ϕ by $\phi - \lambda$, then continuity in the generating symbol implies that this equation holds also when $D_n(T_\phi) = 0$. Thus $D_n(T_\phi) = 0$ if and only if there is a non-zero vector $x \in \ker(S^{*n+1}R_\phi S^{n+1})$ and $P(T_{\bar\phi})S^{n+1}x = 0$. For $n \gg 0$ this means x is a linear combination $x = \sum_{\alpha=1}^\gamma \lambda_\alpha v_\alpha^{(n)}$ and $(S^{n+1}x, y_\tau) = 0$ for all τ. By Cramer's rule this is the same as $\det\left((S^{n+1}v_\alpha^{(n)}, y_\tau)\right) = 0$. □

Lemma 3.2. *Suppose that $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ and A are invertible, and are of the form 1+ trace class. Then*

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \frac{\det A}{\det \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1}\Big|\operatorname{ran}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}}.$$

Proof. We have $\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1\end{pmatrix} \cdot \begin{pmatrix} A & B \\ 0 & -CA^{-1}B+D\end{pmatrix}$. Taking determinants we have

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det A \cdot \det(-CA^{-1}B + D).$$

Now (cf. Lemma 2.9 in [12]), $D - CA^{-1}B = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1}\Big|\operatorname{ran}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\right)^{-1}$. Hence

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \frac{\det A}{\det \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1}\Big|\operatorname{ran}\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}}. \qquad \square$$

The Riesz Schauder decomposition applies to $X = T_\phi T_{\phi^{-1}}$ so that $H^2(\mathbb{T}) = \mathcal{X} \dotplus \mathcal{Y}$ where $\mathcal{X} = \bigcap_{n=1}^\infty \operatorname{ran}(T_\phi T_{\phi^{-1}})^n$ and $\mathcal{Y} = \bigcup_{n=1}^\infty \ker(T_\phi T_{\phi^{-1}})^n$. Let $P_\mathcal{X}$ be the projection onto $\mathcal{X} = \bigcap_{n=1}^\infty \operatorname{ran}(X)^n$ parallel to $\mathcal{Y}$, and set $P_\mathcal{Y} = 1 - P_\mathcal{X}$.

Then we have a block matrix decomposition

$$X_n = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \equiv T_\phi S^{*n+1}R(T_\phi)S^{n+1} + P(T_{\bar\phi})S^{n+1}.$$

Since $R(T_\phi) = T_{\phi^{-1}} - E$ where E is trace class, X_n converges to $T_\phi T_{\phi^{-1}}$ in the uniform norm. It then follows that $P(\mathcal{X})X_n\big|_\mathcal{X}$ is invertible provided that $n \gg 0$.

Now suppose that $n \gg 0$ is chosen so that X_n is invertible, e.g., $D_n(T_\phi) \neq 0$. Then by Proposition 1.1 and Lemma 3.2

$$\frac{D_n(T_\phi)}{\mathbf{G}(\phi)^{n+1}} = \frac{\det(P_{\mathcal{X}} X_n\big|_{\mathcal{X}})}{\det(P_{\mathcal{Y}} X_{n^{-1}}\big|_{\mathcal{Y}})}. \tag{3.1}$$

Since the numerator of this quotient converges to

$$\det(P_{\mathcal{X}} T_\phi T_{\phi^{-1}}\big|_{\mathcal{X}}) = \prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda,$$

the product of the non-zero eigenvalues—using the well-known theorem of V.B. Lidskii—it suffices to consider the denominator $\det(P_{\mathcal{Y}} X_{n^{-1}}\big|_{\mathcal{Y}})$.

Choose a basis $\{w_\alpha\}_1^\gamma$ for the kernel of $T_{\phi^{-1}}$. Let $u_1, u_2, \ldots, u_\gamma$ be associated vectors so that u_α has maximal order with respect to $\{w_\alpha\}$, i.e., $(T_\phi T_{\phi^{-1}})^{m_\alpha} u_\alpha = w_\alpha$ where $m_\alpha = \max\{m : \exists\, x \text{ so that} (T_\phi T_{\phi^{-1}})^m x = w_\alpha\}$. Then the collection $\{(T_\phi T_{\phi^{-1}})^r u_\alpha : 0 \leq r \leq m_\alpha, 1 \leq \alpha \leq \gamma\}$ is a basis for $\mathcal{Y}$. The set of algebraic multiplicities $\{m_\alpha + 1\}$ is independent of the choice of basis for $\ker T_\phi T_{\phi^{-1}}$ and $\sum_1^\gamma m_\alpha + \gamma = \dim \mathcal{Y}$.

In what follows we take the collection $\{u_\alpha\}_1^\gamma$ so that for $j \geq \alpha \geq 1$ we have $m_\alpha \geq 1$ and for $\gamma \geq \alpha \geq j+1$ we have $m_\alpha = 0$. Let $\mathcal{Z}$ be obtained from $\mathcal{Y}$ by replacing $\bigvee_{\alpha=1}^\gamma \{u_\alpha\}$ with the kernel of $T_\phi = \ker(T_\phi T_{\phi^{-1}})^*$. Then we have the direct sum decomposition $H^2(\mathbb{T}) = \mathcal{X} \dotplus \mathcal{Z}$.

Moreover, $\det(P_{\mathcal{Y}} X_n^{-1}\big|_{\mathcal{Y}}) = \det(P_{\mathcal{Z}} X_n^{-1}\big|_{\mathcal{Z}})$. So we can work with $\mathcal{Z}$ instead of $\mathcal{Y}$. The presence of the projection $P(T_{\bar\phi})$ in the right-hand side of Proposition 1.1 makes this change desirable.

For Fredholm operators T let $R(T)$ denote the regularizer of T with respect to the trace ideal so that $TR(T) = 1 - P(T^*)$, $R(T)T = 1 - P(T)$, with $P(T)$ denoting the orthogonal projection to the kernel of T. The Moore-Penrose inverse is such a regularizer.

Lemma 3.3. *Suppose $\{B_n\}_1^\infty$ is a sequence of Fredholm operators which converges in the uniform norm to an operator B, and suppose also that $P(B_n)$ converges in the uniform norm to $P(B)$. Then $R(B_n)$ converges in the uniform norm to $R(B)$.*

Proof. Recall that for any operator A with closed range the Moore-Penrose inverse $A^\dagger$ has the form $R(A) = [A^*A + P(A)]^{-1}A^*$. Thus the assertion follows by continuity of the inverse map in the uniform topology. □

Corollary 3.1.

$$\lim_{n\to\infty} R(T_\phi S^{*n+1} R(T_\phi) S^{n+1}) = R(T_\phi T_{\phi^{-1}}) \quad \textit{uniformly.}$$

Proof. Since T_ϕ is injective $\ker T_\phi S^{*n+1} R(T_\phi) S^{n+1} = \ker S^{*n+1} R(T_\phi) S^{n+1}$. Also $S^{*n+1} R(T_\phi) S^{n+1}$ converges uniformly to $T_{\phi^{-1}}$. So the result follows by Lemmas 3.1 and 3.3. □

Now we will prove Lemma 1.1 which asserts that if $w_1, \ldots, w_\gamma$ is a basis for $\ker T_{\phi^{-1}}$, there is a basis $\{w_\tau^{(n)}\}$ in $\ker[S^{*n+1} R(T_\phi) S^{n+1}]$ so that $w_\tau^{(n)}$ converges to w_τ for $1 \leq \tau \leq \gamma$.

Proof. Recall that by Lemma 3.1 $P(S^{*n+1}R(T_\phi)S^{n+1})$ converges to $P(T_{\phi^{-1}})$. Hence we can take $w_\alpha^{(n)} = P(S^{*n+1}R(T_\phi)S^{n+1})w_\alpha$. □

Note that if $\{v_\beta^{(n)}\}_1^\gamma$ are the vectors obtained by applying the Gram-Schmidt procedure to $\{w_\alpha^n\}_{\alpha=1}^\gamma$, then $v_\alpha^{(n)} = \frac{w_\alpha^n - \mathrm{proj}_{\bigvee_{1\le j<\alpha} w_j^n} w_\alpha^n}{\|w_\alpha^n - \mathrm{proj}_{\bigvee_{1\le j<\alpha} w_j^n} w_\alpha^n\|}$ and $\|w_\alpha^n - \mathrm{proj}_{\bigvee_{1\le j<\alpha} w_j^n} w_\alpha^n\|^2 = \frac{\det(w_i^n, w_j^n)_{1\le i,j\le\alpha}}{\det(w_i^n, w_j^n)_{1\le i,j<\alpha}} \longrightarrow 1$, $\lim_{n\to\infty} v_\alpha^{(n)} = v_\alpha$, where the v_α are obtained by applying the Gram-Schmidt procedure to w_α.

The action of $P_{\mathcal{Z}}X_n^{-1}\big|_{\mathcal{Z}}$. We consider first the action of $P_{\mathcal{Z}}X_n^{-1}$ on the vectors $\{(T_\phi T_{\phi^{-1}})^r u_\alpha$. For $m_\alpha \ge r \ge 1$, define $z_{r\alpha}(n) = X_n^{-1}(T_\phi T_{\phi^{-1}})^r u_\alpha$. Then $(T_\phi T_{\phi^{-1}})^r u_\alpha = X_n z_{r\alpha}(n)$. Since $r \ge 1$, we have $P(T_{\bar\phi})S^{n+1}z_{r\alpha}(n) = 0$ so that

$$\begin{aligned}(T_\phi T_{\phi^{-1}})^r u_\alpha &= T_\phi S^{*n+1}R(T_\phi)S^{n+1}z_{r\alpha}(n)\\ &= T_\phi S^{*n+1}R(T_\phi)S^{n+1}[1 - P(S^{*n+1}R(T_\phi)S^{n+1})]z_{r\alpha}(n)\end{aligned}$$

and

$$R(T_\phi S^{*n+1}R(T_\phi)S^{n+1})(T_\phi T_{\phi^{-1}})^r u_\alpha = [1 - P(S^{*n+1}R(T_\phi S^{n+1})]z_{r\alpha}(n).$$

By Lemma 3.1, $P(S^{*n+1}R(T_\phi)S^{n+1})$ converges in the trace norm to $P(T_{\phi^{-1}}) = P(T_\phi T_{\phi^{-1}})$.

Therefore, by Corollary 3.1 it follows that

$$[1 - P(S^{*n+1}R(T_\phi)S^{n+1})]z_{r\alpha}(n) \quad \text{converges to} \quad [1 - P(T_\phi T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1}u_\alpha,$$

and $\tilde z_{r\alpha}(n) \equiv X_n[1 - P(S^{*n+1}R(T_\phi)S^{n+1})]z_{r\alpha}(n)$ converges to the basis vector $(T_\phi T_{\phi^{-1}})^r u_\alpha$. Note that $\mathcal{Z} = \bigvee_{1\le r\le m_\alpha, 1\le\alpha\le\gamma}\{\tilde z_{r\alpha}(n)\} \dotplus \ker T_{\bar\phi}$. Furthermore,

$$\begin{aligned}P_{\mathcal{Z}}X_n^{-1}\tilde z_{r\alpha}(n) &= P_{\mathcal{Z}}[1 - P(S^{*n+1}R(T_\phi)S^{n+1})]z_{r\alpha}(n)\\ &\longrightarrow P_{\mathcal{Z}}[1 - P(T_\phi T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1}u_\alpha.\end{aligned}$$

Next we consider $P_{\mathcal{Z}}X_n^{-1}$ acting on $\ker T_{\bar\phi}$. Let $y_1, \ldots, y_\gamma$ be an orthonormal basis for $\ker T_{\bar\phi}$. Set $z_\alpha(n) = X_n^{-1}y_\alpha$. Then $X_n z_\alpha(n) = y_\alpha$ so that

$$S^{*n+1}R(T_\phi)S^{n+1}z_\alpha(n) = 0 \quad \text{and} \quad (S^{n+1}z_\alpha(n), y_\tau) = \delta_{\alpha\tau}.$$

Let $\{w_\alpha^{(n)}\}_1^\gamma$ be any basis for $\ker(S^{*n+1}T_\phi S^{n+1})$. and suppose that $\{v_\alpha^{(n)}\}_1^\gamma$ denotes the associated orthonormal basis derived by the Gram-Schmidt procedure. Then, since $z_\alpha(n) = \sum_{\alpha=1}^\gamma (z_\alpha(n), v_\tau^{(n)})v_\tau^{(n)})$, we have

$$\delta_{\alpha\mu} = (S^{n+1}z_\alpha(n), y_\mu) = \sum_{\tau=1}^{\gamma}(S^{n+1}v_\tau^{(n)}, y_\mu)(z_\alpha(n), v_\tau^{(n)}), \tag{3.2}$$

and it follows that

$$\det((z_\alpha(n), v_\tau^{(n)})) = \frac{1}{\det((S^{n+1}v_\alpha^{(n)}, y_\tau))}.$$

Let C_n and K_n denote the linear maps defined on $\mathcal{Z}$ so that for $\alpha = 1, 2, \ldots, \gamma$,

1) $C_n(y_\alpha) = \sum_{\ell=1}^\gamma (S^{*n+1}v_\alpha^{(n)}, y_\ell)y_\ell$,
2) $C_n(T_\phi T_{\phi^{-1}})^r u_\alpha = \tilde z_{r\alpha}(n) = (T_\phi T_{\phi^{-1}})^r u_\alpha - d_{r\alpha}(n)$, where $d_{r\alpha}(n) \in \ker T_{\bar\phi}$,
3) $K_n(y_\alpha) = P_{\mathcal{Z}}(v_\alpha^{(n)})$,
4) $K_n(T_\phi T_{\phi^{-1}})^r u_\alpha = P_{\mathcal{Z}}R(T_\phi S^{*n+1}R_\phi S^{n+1})(T_\phi T_{\phi^{-1}})^r u_\alpha$ for $m_\alpha \ge r \ge 1$.

Then

$$K_n = P_{\mathcal{Z}} X_n^{-1} P_{\mathcal{Z}} C_n.$$

To check this, first consider the action on y_α. We have

$$\begin{aligned} X_n^{-1} P_{\mathcal{Z}} C_n(y_\alpha) &= X_n^{-1}[\sum_{\ell=1}^{\gamma}(S^{n+1}v_\alpha^{(n)}, y_\ell)y_\ell] = \sum_{\ell=1}^{\gamma}(S^{n+1}v_\alpha^{(n)}, y_\ell)z_\ell(n) \\ &= \sum_{\ell=1}^{\gamma}(S^{n+1}v_\alpha^{(n)}, y_\ell)[\sum_{\tau=1}^{\gamma}(z_\ell(n), v_\tau^{(n)})v_\tau^{(n)}] \\ &= \sum_{\tau=1}^{\gamma}\left[\sum_{\ell=1}^{\gamma}(S^{n+1}v_\alpha^{(n)}, y_\ell)\cdot(z_\ell(n), v_\tau^{(n)})\right]v_\tau^{(n)}. \end{aligned}$$

After taking the transpose of equation (3.2), the inner sum becomes $\delta_{\alpha\tau}$ and therefore $P_{\mathcal{Z}} X_n^{-1} C_n y_\alpha = P_{\mathcal{Z}} v_\alpha^{(n)}$.

Next we consider the action on vectors $(T_\phi T_{\phi^{-1}})^r u_\alpha$. It follows that

$$\begin{aligned} X_n^{-1} P_{\mathcal{Z}} C_n (T_\phi T_{\phi^{-1}})^r u_\alpha &= [1 - P(S^{*n+1}R(T_\phi)S^{n+1})]z_{r_\alpha}(n) \\ &= R(T_\phi S^{*n+1}R(T_\phi)S^{n+1})(T_\phi T_{\phi^{-1}})^r u_\alpha; \end{aligned}$$

thus $K_n = P_{\mathcal{Z}} X_n^{-1} P_{\mathcal{Z}} C_n$.

Now, since $\det(C_n) = \det(S^{n+1}v_\alpha^{(n)}, y_\tau)$, it follows that

$$\det(P_{\mathcal{Y}} X_n^{-1}|_{\mathcal{Y}}) = \det(P_{\mathcal{Z}} X_n^{-1}|_{\mathcal{Z}}) = \frac{\det(K_n)}{\det(C_n)} = \frac{\det(K_n)}{\det(S^{n+1}v_\alpha^{(n)}, y_\tau)},$$

and therefore,

$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1}\frac{\det P_{\mathcal{X}} X_{n|\mathcal{X}})}{\det(K_n)}\cdot\det(S^{n+1}v_\alpha^{(n)}, y_\tau) \quad \text{provided} \quad D_n(\phi) \neq 0.$$

On the other hand, since $\dim\ker(S^{*n+1}R(T_\phi)S^{n+1}) = \gamma$ when $n \gg 0$, the operator K_n is defined even when X_n is singular. By Gram-Schmidt, if $w_\alpha^{(n)} \to w_\alpha$, then $v_\alpha^{(n)} \to v_\alpha$ where $\{v_\alpha\}_1^\gamma$ is derived from $\{w_\alpha\}_1^\gamma$ by the Gram-Schmidt procedure. In that case, K_n converges to the operator K defined by

$$K(y_\alpha) = P_{\mathcal{Z}} v_\alpha,$$

$$K(T_\phi T_{\phi^{-1}})^r u_\alpha = P_{\mathcal{Z}}[1 - P(T_\phi T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1}u_\alpha \quad \text{for} \quad m_\alpha \geq r \geq 1.$$

Since $\mathcal{Z}$ is finite dimensional, K_n will be invertible provided that K is invertible.

Lemma 3.4. *$\det(K) = (-1)^{\sum_1^\gamma m_\alpha}\det(b_\alpha, y_\ell)$ where $\{b_\alpha\}_1^\gamma$ is any set of $T_\phi T_{\phi^{-1}}$ maximal vectors for $\{v_\alpha\}_1^\gamma$. In particular, K is invertible.*

Proof. Let $L : \mathcal{Z} \to \mathcal{Y}$ be the linear map where for $\alpha = 1, 2, \ldots, \gamma$, $Ly_\alpha = v_\alpha$ and $L(T_\phi T_{\phi^{-1}})^r u_\alpha = [1 - P(T_\phi T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1}u_\alpha$, for $1 \leq r \leq m_\alpha$. Then $K = P_{\mathcal{Z}} L$. Hence, $\det K = \det P_{\mathcal{Z}} L = \det LP_{\mathcal{Z}}\big|_{\mathcal{Y}}$.

Let

$$\mathcal{Y}_1 = \bigvee_{\alpha=1}^{\gamma}\{u_\alpha\} \quad \text{and} \quad \mathcal{Y}_2 = \bigvee_{\alpha=1}^{\gamma}\bigvee_{r=1}^{m_\alpha}\{(T_\phi T_\phi^{-1})^r u_\alpha\},$$

and let

$$\mathcal{Y}_3 = \bigvee_{\alpha=1}^{\gamma}\{w_\alpha\} \quad \text{and} \quad \mathcal{Y}_4 = \operatorname{ran}(1 - P(T_\phi T_{\phi^{-1}}))\mathcal{Y}.$$

Then

$$\mathcal{Y} = \mathcal{Y}_1 \dotplus \mathcal{Y}_2 = \mathcal{Y}_3 \dotplus \mathcal{Y}_4.$$

Note that $\mathcal{Y}_4 = \mathrm{rp}\,[(T_\phi T_{\phi^{-1}})^*]\mathcal{Y}$. Let $P(\mathcal{Y}_3)$ be the linear map on $\mathcal{Y}$ which acts as the projection onto $\mathcal{Y}_3$ parallel to $\mathcal{Y}_4$. Define the linear map $M : \mathcal{Y} \to \mathcal{Y}$ by setting

$$M = (P(\mathcal{Y}_3)LP_{\mathcal{Z}}{}_{\big|\mathcal{Y}_1})^{-1}P(\mathcal{Y}_3) + (LP_{\mathcal{Z}}{}_{\big|\mathcal{Y}_2})^{-1}(1 - P(\mathcal{Y}_3)).$$

Computation shows that $MLP_{\mathcal{Z}}{}_{\big|\mathcal{Y}} = I + N$ where $N\mathcal{Y}_2 = 0$, and $N : \mathcal{Y}_1 \to \mathcal{Y}_2$. Therefore, $\det LP_{\mathcal{Z}}{}_{\big|\mathcal{Y}} = \det M^{-1}$. Further computation shows $M^{-1}u_\alpha = \sum_{\rho=1}^{\gamma} a_{\alpha\rho} w_\rho$ where $a_{\alpha\rho} = \sum_{\tau=1}^{\gamma} (u_\alpha, y_\tau)\Gamma_{\tau\rho}$ with $(\Gamma_{\tau,\rho})$ the matrix relative to $|w_\tau\}_1^\gamma$ for the map defined by $\Gamma(w_\tau) = v_\tau$, and we also have

$$M^{-1}(T_\phi T_{\phi^{-1}})^r u_\alpha = [1 - P(T_\phi T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1} u_\alpha.$$

Recall that the $\{u_\alpha\}_1^\gamma$ have been so ordered that $1 \le m_\alpha$ for $1 \le \alpha \le j$, and $m_\alpha = 0$ for $j+1 \le \alpha \le \gamma$. Thus, if $X = T_\phi T_{\phi^{-1}}$, then M^{-1} has the following matrix representation:

$$\begin{array}{c} \\ u_1 \\ Xu_1 \\ X^2u_1 \\ \vdots \\ X^{m_1-1}u_1 \\ u_2 \\ Xu_2 \\ X^2u_2 \\ \vdots \\ X^{m_2-1}u_2 \\ \vdots \\ \vdots \\ u_j \\ Xu_j \\ X^2u_j \\ \vdots \\ X^{m_j-1}u_j \\ X^{m_1}u_1 \\ X^{m_2}u_2 \\ \vdots \\ X^{m_j}u_j \\ u_{j+1} \\ \vdots \\ u_\gamma \end{array}
\begin{array}{cccccccccccccccccc}
Xu_1 & X^2u_1 & \cdots & X^{m_1}u_1 & Xu_2 & X^2u_2 & \cdots & X^{m_2}u_2 & \cdots & \cdots & Xu_j & X^2u_j & \cdots & X^{m_j}u_j & u_1 & u_2 & \cdots & u_\gamma \\
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
* & * & \cdots & * & * & * & \cdots & * & \cdots & \cdots & * & * & \cdots & * & a_{11} & a_{21} & \cdots & a_{\gamma 1} \\
* & * & \cdots & * & * & * & \cdots & * & \cdots & \cdots & * & * & \cdots & * & a_{12} & a_{22} & \cdots & a_{\gamma 2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & \cdots & * & * & * & \cdots & * & \cdots & \cdots & * & * & \cdots & * & a_{1j} & a_{2j} & \cdots & a_{\gamma j} \\
* & * & \cdots & * & * & * & \cdots & * & \cdots & \cdots & * & * & \cdots & * & a_{1\,j+1} & a_{2\,j+1} & \cdots & a_{\gamma\,j+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & \cdots & * & * & * & \cdots & * & \cdots & \cdots & * & * & \cdots & * & a_{1\gamma} & \cdots & \cdots & a_{\gamma\gamma}
\end{array}$$

Now $\det \Gamma = \frac{1}{\| \bigwedge w_\alpha \|}$. Therefore by ii) of Proposition 2.1 we have

$$\det K = \det M^{-1} = (-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \det(u_\alpha, y_\tau) \cdot \det(\Gamma) = (-1)^{\sum_{\alpha=1}{=}^{\gamma} m_\alpha} \det(b_\alpha, y_\tau),$$

where the $\{b_\alpha\}_1^\gamma$ are maximal vectors for $\{v_\alpha\}_1^\gamma$. □

Now, since K_n is invertible, and $D_n(T_\phi) = 0$ if and only if $\det(S^{n+1}v_\alpha^{(n)}, y_\tau) = 0$, we have shown that for all $n \gg 0$,

$$D_n(\phi) = G(\phi)^{n+1} \frac{\det(P_{\mathcal{X}} X_{n|\mathcal{X}})}{\det(K_n)} \cdot \det(S^{n+1}v_\alpha^{(n)}, y_\tau). \tag{3.3}$$

Let $\Gamma^{(n)}$ be the linear map on $\ker(S^{*n+1}R(T_\phi)S^{n+1})$ so that $\Gamma^{(n)}(w_\alpha^{(n)}) = v_\alpha^{(n)}$. Then $\det(\Gamma^{(n)}) \to \det(\Gamma) = \frac{1}{\|\bigwedge w_\alpha\|}$, where as above $\Gamma(w_\alpha) = v_\alpha$.

Let $\{t_\tau\}_1^\gamma$ be any basis for $\ker T_\phi$. Then upon substitution into (3.3), Proposition 2.1 gives

$$D_n(\phi) = \mathbf{G}(\phi)^{n+1} \cdot \det(S^{n+1}w_\alpha^{(n)}, t_\tau) \cdot \left[(-1)^{\sum_1^\gamma m_\alpha} \frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, t_\tau)}\right] \cdot [1 + o(1)],$$

where the eigenvalues λ are repeated according to their multiplicity. Again the equality $\det T_\phi T_{\phi^{-1}}\big|_{\mathcal{X}} = \prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda$ follows from the Riesz-Schauder decomposition and the well-known result of V.B. Lidskii. This completes the proof of part i) of Theorem 1.1.

4. Proofs of Theorem 1.2 and Theorem 1.6

Assertion a) of Theorem 1.2 is contained in Proposition 4.1 below. Assertion c) follows from b) while b) is a special case of Proposition 4.3 with $\Theta_1 = z^\gamma$ and $\Theta_2 = 1$.

Let $\phi = \Theta_1 f \cdot \bar{\Theta}_2 \bar{g}$ with Θ_1 and Θ_2 coprime finite Blaschke products, and let f and g be outer functions in $K_{2,2}^{\frac12,\frac12} \cap \mathcal{W}$, the intersection of the Krein algebra $\{a \in \mathcal{L}(\mathbb{T})^\infty : \sum_{n\in Z}(|n|+1)|a_n|^2 < \infty$ and the Wiener algebra, $\mathcal{W}$. Note that if $a \in K_{2,2}^{\frac12,\frac12} \cap \mathcal{W}$ and has no zeros on $\mathbb{T}$, then $a^{-1} \in K_{2,2}^{\frac12,\frac12} \cap \mathcal{W}$. In this case the Hankel operators $H(a)$ and $H(a^{-1})$ are in Hilbert-Schmidt class; cf. [12]. Let $\gamma \equiv \operatorname{wind}(\phi, 0) = \deg\Theta_1 - \deg\Theta_2 > 0$. Note that $\ker T_{\bar\phi} = T_{\frac{1}{g}}[\ker T_{\bar\Theta_1\Theta_2}]$, $\ker T_{\phi^{-1}} = T_f \ker T_{\bar\Theta_1\Theta_2}$ and $\ker T_{\bar\Theta_1\Theta_2} = T_{\bar\Theta_2}[K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp]$.

Lemma 4.1. *For $h \in \mathcal{L}^\infty(\mathbb{T})$ and the multiplication operator M_h on $\mathcal{L}^2(\mathbb{T})$*

$$\begin{array}{ccc} K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp & \xrightarrow{P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) M_h} & K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp \\ {\scriptstyle T_{\bar\Theta_2}}\downarrow & & {\scriptstyle T_{\bar\Theta_2}}\downarrow \\ \ker T_{\bar\Theta_1\Theta_2} & \xrightarrow{P(T_{\bar\Theta_1\Theta_2})M_h} & \ker T_{\bar\Theta_1\Theta_2} \end{array}$$

Proof. The result is a consequence of the following two facts:

1) $P(T_{\bar\Theta_1\Theta_2}) = T_{\bar\Theta_2} P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) T_{\Theta_2}$,
2) $P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot M_{\Theta_2} P = P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot M_{\Theta_2}$. □

As a consequence of the lemma we see that

$$\begin{aligned} \det P(T_{\bar\Theta_1\Theta_2})M_h\Big|_{\ker T_{\bar\Theta_1\Theta_2}} &= \det P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2})M_h\Big|_{K_{\Theta_1}\cap K_{\bar\Theta_2}^\perp}, \\ \dim\ker P(T_{\bar\Theta_1\Theta_2})M_h\Big|_{\ker T_{\bar\Theta_1\Theta_2}} &= \dim\ker P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2})M_h\Big|_{K_{\Theta_1}\cap K_{\bar\Theta_2}^\perp}. \end{aligned}$$

Fix orthonormal bases $\{w_\alpha\}_1^\gamma$ and $\{y_\alpha\}_1^\gamma$ of $\ker T_{\phi^{-1}}$ and $\ker T_{\bar\phi}$. Pick $\{s_\alpha\}$ to be orthonormal eigenvectors of $P(T_{\bar\Theta_1\Theta_2})T_{|f|^2}\Big|_{\ker T_{\bar\Theta_1\Theta_2}}$ so that

$$P(T_{\bar\Theta_1\Theta_2})T_{|f|^2}\Big|_{\ker T_{\bar\Theta_1\Theta_2}} s_\alpha = \mu_\alpha s_\alpha. \tag{4.1}$$

Then $\{\frac{1}{\sqrt{\mu_\beta}} T_f s_\tau\}_1^\gamma$ is a complete orthonormal set in $\ker T_{\phi^{-1}}$. Similarly, pick $\{r_\alpha\}_1^\gamma$ to be orthonormal eigenvectors of $P(T_{\bar\Theta_1\Theta_2})T_{|\frac{1}{g}|^2}\big|_{\ker T_{\bar\Theta_1\Theta_2}}$ so that

$$P(T_{\bar\Theta_1\Theta_2})T_{|\frac{1}{g}|^2}\Big|_{\ker T_{\bar\Theta_1\Theta_2}} r_\alpha = \lambda_\alpha r_\alpha. \tag{4.2}$$

Then $\{\frac{1}{\sqrt{\lambda_\beta}} T_{\frac{1}{g}} r_\beta\}_1^\gamma$ is a complete orthonormal set in $\ker T_{\bar\phi}$. Let $W : \ker P(T_{\bar\Theta_1\Theta_2}) \to \ker P(T_{\bar\Theta_1\Theta_2})$ be the unitary map so that $Wr_\alpha = s_\alpha$. Similarly, let $V : \ker T_{\phi^{-1}} \to \ker T_{\phi^{-1}}$ be the unitary map such that $VT_f \frac{s_\beta}{\sqrt{\mu}_\beta} = w_\beta$ and let $U : \ker T_{\bar\phi} \to \ker T_{\bar\phi}$ be the unitary map so that $UT_{\frac{1}{g}} \frac{r_\beta}{\sqrt{\lambda\beta}} = y_\beta$.

Proposition 4.1. a) $\#\{\alpha : m_\alpha \neq 0\} = \dim\ker P(T_{\bar\Theta_1}) \wedge \mathrm{rp}\,(T_{\Theta_2})M_{\frac{f}{\bar g}}\big|_{K_{\Theta_1}\cap K_{\bar\Theta_2}^\perp} = \dim\ker T_\phi T_{\phi^{-1}}$. *In particular, if ϕ is unimodular, then $m_\alpha = 0$ for all α.*

b) $\sum_1^\gamma m_\alpha = \dim$ root space $[Q_{\gamma-1}\big(I - H_\psi H_{\frac{1}{\bar\psi}}\big|_{\mathrm{ran}\, S^\gamma}\big]$,

Proof. a) Since $T_{\phi^{-1}}$ is surjective,

$$j = \text{cardinality}\,\{\alpha : m_\alpha \neq 0\} = \dim[\mathrm{ran}\, T_\phi \cap \ker T_{\phi^{-1}}].$$

Now we may pick the basis $\{w_\alpha\}_1^\gamma$ so that $w_1, \ldots, w_j$ is a basis for $[\mathrm{ran}\, T_\phi \cap \ker T_{\phi^{-1}}]$ and then $w_{j+1}, \ldots, w_\gamma$ is a basis for $[\ker T_{\phi^{-1}} \ominus \mathrm{ran}\, T_\phi \cap \ker T_{\phi^{-1}}]$. Since the inner product $(w_\alpha, y_\beta) = (u_\alpha, y_\beta)$ for $j+1 \leq \alpha \leq \gamma$ and $(w_\alpha, y_\beta) = 0$ for $1 \leq \alpha \leq j$, it follows that

$$\begin{aligned}
\gamma - j = \#\{\alpha : m_\alpha = 0\} &= \text{ column rank } ((w_\alpha, y_\beta)) = \text{ column rank } ((T_f s_\alpha, T_{\frac{1}{g}} r_\beta)) \\
&= \text{ column rank } ((T_f r_\alpha, T_{\frac{1}{g}} r_\beta)) = \dim\mathrm{ran}\, P(T_{\bar\Theta_1\Theta_2})T_{\frac{f}{\bar g}}\big|_{\ker P(T_{\bar\Theta_1\Theta_2})} \\
&= \gamma - \dim\ker P(T_{\bar\Theta_1\Theta_2})T_{\frac{f}{\bar g}}\big|_{\ker P(T_{\bar\Theta_1\Theta_2})} \\
&= \gamma - \dim\ker P(T_{\bar\Theta_1}) \wedge \mathrm{rp}\,(T_{\Theta_2})M_{\frac{f}{\bar g}}\Big|_{K_{\Theta_1}\cap K_{\bar\Theta_2}^\perp}.
\end{aligned}$$

If ϕ is unimodular, then $f = c \cdot \frac{1}{g}$ for some $c \in C^*$, and then $j = 0$. □

Proof. b) With $\phi = f\bar g z^\gamma$, and $\langle\,,\,\rangle$ denoting multiplicative commutator we have

$$T_\phi T_{\phi^{-1}} = T_{f\bar g} Q_{\gamma-1} T_{\frac{1}{f\bar g}} = T_f \big\langle T_{\frac{1}{f}}, T_{\bar g}\big\rangle T_{\bar g} Q_{\gamma-1} T_{\frac{1}{\bar g}} T_{\frac{1}{f}}.$$

Therefore, $T_\phi T_{\phi^{-1}}$ is similar to

$$T_{\frac{1}{\bar g}} \big\langle T_{\frac{1}{f}}, T_{\bar g}\big\rangle T_{\bar g} Q_{\gamma-1} = T_{\frac{1}{f\bar g}} T_{f\bar g} Q_{\gamma-1} = [1 - H_{\frac{1}{f\bar g}} H_{\tilde f\bar g}] Q_{\gamma-1}.$$

This last operator has the block matrix form $\begin{pmatrix} M & 0 \\ L & 0 \end{pmatrix}$ relative to the decomposition $H^2(\mathbb{T}) = \mathrm{ran}\, S^\gamma \oplus \ker S^{*\gamma}$. Since $H_{\frac{1}{f\bar g}} H_{\tilde f\tilde{\bar g}}$ is trace class, we have for sufficiently small $\delta > 0$

$$\gamma + \sum_1^\gamma m_\alpha = \dim \text{root space}\, T_\phi T_{\phi^{-1}} = \dim \text{root space} \begin{pmatrix} M & 0 \\ L & 0 \end{pmatrix}$$
$$= \operatorname{tr} \frac{1}{2\pi i} \int_{|z|=\delta} \left(z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} M & 0 \\ L & 0 \end{pmatrix} \right)^{-1} dz$$
$$= \operatorname{tr} \frac{1}{2\pi i} \int_{|z|=\delta} \begin{pmatrix} (z-M)^{-1} & 0 \\ \frac{L}{z}(z-M)^{-1} & \frac{1}{z} \end{pmatrix} dz$$
$$= \frac{1}{2\pi i} \int_{|z|=\delta} \operatorname{tr} \begin{pmatrix} (z-M)^{-1} - (z-1)^{-1} & 0 \\ \frac{L}{z}(z-M)^{-1} & \frac{1}{z} \end{pmatrix} dz$$
$$= \gamma + \operatorname{tr} \frac{1}{2\pi i} \int_{|z=\delta} (z-M)^{-1}\, dz = \gamma + \dim \text{ root space} \, [Q_{\gamma-1}\big(I - H_\psi H_{\frac{1}{\bar\psi}} |\operatorname{ran} S^\gamma\big)].$$

□

Proposition 4.2. *Suppose that* $m_\alpha = 0, \quad \forall \alpha$. *Then*

$$\text{i)} \quad \sigma_{1, T_\phi T_{\phi^{-1}}} = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda}{\det(v_\alpha, y_\tau)} \bigwedge v_\alpha \otimes \big(\bigwedge y_\tau + \operatorname{ran} T_\phi\big)^*$$

where $\{v_\alpha\}_1^\gamma$ *and* $\{y_\tau\}_1^\gamma$ *are respectively orthonormal bases for* $\ker T_{\phi^{-1}}$ *and* $\ker T_{\bar\phi}$. *In particular,*

$$\|\sigma_{1, T_\phi T_{\phi^{-1}}}\| = \left| \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda}{\chi(P(T_{\phi^{-1}}), P(T_{\bar\phi}))} \right|.$$

ii) $\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda = \chi^2(P(T_{\bar\Theta_1}), P(T_{\bar\Theta_2}))$

$$\times \prod_{|x|<1} c_x\Big(\frac{f}{\bar g}, \frac{\Theta_2}{\Theta_1}\Big) \cdot \det(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{\frac{f}{\bar g}}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp}) \times \det{}_* \partial\{T_{\bar g} + \mathcal{L}^1, T_f + \mathcal{L}^1\}.$$

iii) *With* $e^{i\rho} = \det U^* \det V \det W$,

$$\det(v_j, y_k) = \frac{\det(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{\frac{f}{\bar g}}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp})e^{i\rho}}{\det(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|f|^2}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp})^{\frac12} \cdot \det(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp})^{\frac12}}.$$

iv)

$$\frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda}{\det(v_j, y_k)} = \chi^2(P(T_{\bar\Theta_1}), P(T_{\bar\Theta_2})) \cdot \prod_{|x|<1} c_x\Big(\frac{f}{\bar g}, \frac{\Theta_2}{\Theta_1}\Big)$$
$$\cdot \det{}_* \partial\{T_{\bar g} + \mathcal{L}^1, T_f + \mathcal{L}^1\} e^{-i\rho} \det(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|f|^2}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp})^{\frac12}$$
$$\cdot \det\big(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^\perp}\big)^{\frac12}.$$

In particular, if $\Theta_1 = z^\gamma$, $\Theta_2 = 1$, *then*

$$E(\psi) G\Big(\frac{\bar g}{f}\Big)^\gamma = \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda}{D_{\gamma-1}(T_{\frac{f}{\bar g}})}.$$

v) *Suppose* $\{x_\alpha\}_1^\gamma$ *is an orthonormal basis for* $\ker T_{\Theta_1\bar{\Theta}_2}$. *Let* $\{w_\alpha\}_1^\gamma$ *be obtained from the vectors* $\{T_f x_\alpha\}_1^\gamma$ *via the Gram-Schmidt orthogonalization procedure. Similarly, let* $\{y_\alpha\}_1^\gamma$ *be the result of orthogonalizing the vectors* $\{T_{\frac{1}{g}} x_\alpha\}_1^\gamma$. *Then, with this choice of bases for* $\ker T_{\phi^{-1}}$ *and* $\ker T_{\bar{\phi}}$ *respectively, statements* iii) *and* iv) *hold with* $\rho = 0$.

Proof. Part i) follows immediately from Proposition 2.1 by taking $X = T_\phi T_{\phi^{-1}}$. To prove ii) recall that since T_ϕ is injective, $T_{\phi^{-1}}$ is surjective and so $\operatorname{ran} T_\phi T_{\phi^{-1}} = \operatorname{ran} T_\phi$. Consequently, $\mathcal{X} = \operatorname{ran} T_\phi$ and so

$$\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda = \det(T_\phi T_{\phi^{-1}}\big|_{\mathcal{X}}) = \det(T_\phi T_{\phi^{-1}}\big|_{\operatorname{ran} T_\phi}) = \det(T_{\phi^{-1}} T_\phi).$$

Since $\phi^{-1} = \Theta_2\bar{\Theta}_1\frac{1}{f}\frac{1}{\bar{g}}$, assertion ii) is obtained by the arguments given in the proofs of Theorems 3.6 and Theorem 3.7 of [17].

To prove ii) note that $u_j = w_j$ for $j = 1, \ldots, \gamma$.

Recall that $\{s_\alpha\}$ are orthonormal eigenvectors of $P(T_{\bar{\Theta}_1\Theta_2})T_{|f|^2}\big|_{\ker T_{\bar{\Theta}_1\Theta_2}}$ so that

$$P(T_{\bar{\Theta}_1\Theta_2})T_{|f|^2}\big|_{\ker T_{\bar{\Theta}_1\Theta_2}} s_\alpha = \mu_\alpha s_\alpha,$$

and $\{\frac{1}{\sqrt{\mu_\beta}} T_f s_\tau\}$ is a complete orthonormal set in $\ker T_{\phi^{-1}}$. Also, the $\{r_\alpha\}$ are an orthonormal set of eigenvectors of $P(T_{\bar{\Theta}_1\Theta_2})T_{|\frac{1}{g}|^2}\big|_{\ker T_{\bar{\Theta}_1\Theta_2}}$ so that

$$P(T_{\bar{\Theta}_1\Theta_2})T_{|\frac{1}{g}|^2}\big|_{\ker T_{\bar{\Theta}_1\Theta_2}} r_\alpha = \lambda_\alpha r_\alpha,$$

and $\{\frac{1}{\sqrt{\lambda_\beta}} T_{\frac{1}{g}} r_\tau\}$ is a complete orthonormal set in $\ker T_{\bar{\phi}}$. Let $W : \ker P(T_{\bar{\Theta}_1\Theta_2}) \to \ker P(T_{\bar{\Theta}_1\Theta_2})$ be the unitary map so that $W r_\alpha = s_\alpha$ and $V : \ker T_{\phi^{-1}} \to \ker T_{\phi^{-1}}$ is the unitary map such that $VT_f \frac{s_\tau}{\sqrt{\mu}_\tau} = w_\tau$ while $U : \ker T_{\bar{\phi}} \to \ker T_{\bar{\phi}}$ is the unitary map so that $UT_{\frac{1}{g}} \frac{r_\tau}{\sqrt{\lambda}_\tau} = y_\tau$.

Then

$$\det(w_j, y_k) = \det(VT_f W\frac{r_j}{\sqrt{\mu}_j}, UT_{\frac{1}{g}}\frac{r_k}{\sqrt{\lambda}_k}) = \det U^* \det V \det W$$

$$\times\frac{\det(P(T_{\bar{\Theta}_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{\frac{f}{g}}\big|_{K_{\Theta_1}\cap K_{\Theta_2}^\perp})}{\det(P(T_{\bar{\Theta}_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|f|^2}\big|_{K_{\Theta_1}\cap K_{\Theta_2}^\perp})^{\frac{1}{2}} \cdot \det(P(T_{\bar{\Theta}_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2}\big|_{K_{\Theta_1}\cap K_{\Theta_2}^\perp})^{\frac{1}{2}}}.$$

Finally, set $\det U^* \det V \det W = e^{i\rho}$. □

Combining ii) and iii) we arrive at iv).

Proof of v). It suffices to prove iii) with $\rho = 0$. We have

$$u_j = w_j = \frac{[T_f x_j - \operatorname{Proj}_{\bigvee_{1\le i<j} T_f x_i} T_f x_j]}{\|T_f x_j - \operatorname{Proj}_{\bigwedge_{1\le i<j} T_f x_i} T_f x_j\|}. \tag{4.3}$$

Therefore,

$$\bigwedge_{j=1}^{\gamma} u_j = \frac{\bigwedge_{j=1}^{\gamma} T_f x_j}{\prod_{j=1}^{\gamma} \|T_f x_j - \operatorname{Proj}_{\bigvee_{1\le i\le j} T_f x_i}\|}.$$

Since $\| \bigwedge u_j \| = 1$,

$$\prod_{j=1}^{\gamma} \|T_f x_j - \mathrm{Proj}_{\bigvee_{1\le i<j} T_f e_i} T_f x_j\|^2 = \| \bigwedge_{j=1}^{\gamma} T_f x_\alpha \|^2 = \det(T_f x_\alpha, T_f x_\tau)_{1\le \alpha,\tau \le \gamma}.$$

Consequently,

$$\bigwedge_{j=1}^{\gamma} u_j = \frac{\bigwedge_{j=1}^{\gamma} T_f x_j}{\det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|f|^2}\big|_{K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp}})^{\frac{1}{2}}},$$

and similarly,

$$\bigwedge_{j=1}^{\gamma} y_j = \frac{\bigwedge_{j=1}^{\gamma} T_{\frac{1}{g}} x_j}{\det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2}\big|_{K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp}})^{\frac{1}{2}}}.$$

Therefore,

$$\det(u_j, y_i) = (\bigwedge_{j=1}^{\gamma} u_j, \bigwedge_{=1}^{\gamma} y_j)$$

$$= \frac{\left(\bigwedge_{j=1}^{\gamma} (T_f x_j, \bigwedge_{j=1}^{\gamma} T_{\frac{1}{g}} x_j \right)}{\det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|f|^2}\big|_{K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp}})^{\frac{1}{2}} \det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2}\big|_{K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp}})^{\frac{1}{2}}}.$$

On the other hand,

$$(\bigwedge_{j=1}^{\gamma} (T_f x_j, \bigwedge_{j=1}^{\gamma} T_{\frac{1}{g}} x_j) = \det(T_{\frac{f}{\bar{g}}} x_j, x_i) = \det(P(T_{\bar{\Theta}_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{\frac{f}{\bar{g}}}\big|_{K_{\Theta_1} \cap K_{\bar{\Theta}_2}^{\perp}}).$$

Combining the last two equations gives the desired result. □

In order to compute $\sigma_{1,T_\phi T_{\phi^{-1}}}$ in general we must remove the restriction that $m_\alpha = 0$, for all α.

Lemma 4.2. *For small enough $t \neq 0 \in C$, the operator $P(T_{\bar{\Theta}_1 \Theta_2}) T_{\frac{1}{\bar{g}-t}}\big|_{\ker T_{\bar{\Theta}_1 \Theta_2}}$ is invertible.*

Proof. $\sigma(T_{\frac{1}{\bar{g}-t}}\big|_{K_{\Theta_1}}) = \{\frac{1}{\bar{g}(z)-t} : z \in \Theta_1^{-1}(0)\}$; moreover for small $t \notin \sigma(T_{\frac{1}{\bar{g}-t}}\big|_{K_{\Theta_1}})$, we have $P(T_{\bar{\Theta}_1 \Theta_2}) T_{\frac{1}{\bar{g}-t}}\big|_{\ker T_{\bar{\Theta}_1 \Theta_2}} = P(T_{\bar{\Theta}_1 \Theta_2})[g(P(T_{\bar{\Theta}_1})S\big|_{K_{\Theta_1}})^* - t]^{-1}\big|_{\ker T_{\bar{\Theta}_1 \Theta_2}}$. Consequently, $\det\left(P(T_{\bar{\Theta}_1 \Theta_2}) T_{\frac{1}{\bar{g}-t}}\big|_{\ker T_{\bar{\Theta}_2 \Theta_1}}\right)$ is a non-zero function of t analytic in a neighborhood of the origin. Thus it has at most an isolated zero at the origin. □

Proposition 4.3. *Suppose that $\{x_j\}_1^\gamma$ is any orthonormal basis for $\ker T_{\bar{\Theta}_1 \Theta_2}$. Suppose that $\{v_j\}_1^\gamma$ is obtained from $\{T_f x_j\}_1^\gamma$ by the Gram-Schmidt procedure. Suppose also that $\{y_j\}_1^\gamma$ is obtained from $\{T_{\frac{1}{g}} x_j\}_1^\gamma$ by the Gram-Schmidt orthogonalization procedure. Then if $\{b_j\}_1^\gamma$ are the maximal root vectors associated to the orthonormal* *basis* $\{v_j\}_1^\gamma$,

$$
\begin{aligned}
&(-1)^{\sum_{\alpha=1}^{\gamma} m_\alpha} \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(b_j, y_k)} \\
&= \chi^2(P(T_{\bar\Theta_1}), P(T_{\bar\Theta_2})) \cdot \prod_{|x|<1} c_x(\frac{f}{\bar g}, \frac{\Theta_2}{\Theta_1}) \cdot \det{}_*\partial\{T_{\bar g} + \mathcal{L}^1(H), T_f + \mathcal{L}^1(H)\} \\
&\times \det(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|f|^2}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^{\perp}})^{\frac{1}{2}} \\
&\cdot \det(P(T_{\bar\Theta_1}) \wedge \operatorname{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g}|^2}\big|_{K_{\Theta_1} \cap K_{\bar\Theta_2}^{\perp}})^{\frac{1}{2}}.
\end{aligned}
$$

Proof. Set $q(t) = \det P(T_{\bar\Theta_1\Theta_2})T_{\frac{1}{\bar g - t}}\big|_{\ker T_{\bar\Theta_1\Theta_2}}$. By Lemma 4.2 for small t the operator $P(T_{\bar\Theta_1\Theta_2})T_{\frac{1}{\bar g - t}}\big|_{\ker T_{\bar\Theta_1\Theta_2}}$ is invertible. For small t and $a \in C$ consider the two parameter family of operators

$$
\begin{aligned}
P(T_{\bar\Theta_1\Theta_2})T_{\frac{f-a}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2} &= P(T_{\bar\Theta_1\Theta_2})T_{\frac{f}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2} - a\, P(T_{\bar\Theta_1\Theta_2})T_{\frac{1}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2} \\
&= P(T_{\bar\Theta_1\Theta_2})\frac{T_{\frac{1}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2}}{q(t)}[q(t)(P(T_{\bar\Theta_1\Theta_2})T_{\frac{1}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2})^{-1} \\
&\qquad \times P(T_{\bar\Theta_1\Theta_2})T_{\frac{f}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2} - q(t)\, a],
\end{aligned}
$$

provided $q(t) \neq 0$.

Now $[q(t)(P(T_{\bar\Theta_1\Theta_2})T_{\frac{1}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2})^{-1}(P(T_{\bar\Theta_1\Theta_2})T_{\frac{f}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2}) - q(t)\, a]$ is analytic in t in a neighborhood $\mathcal{N}$ of zero. Therefore,

$$
\det P(T_{\bar\Theta_1\Theta_2})T_{\frac{f-a}{\bar g - t}}|\ker T_{\bar\Theta_1\Theta_2} = \frac{p(a,t)}{q(t)^{\gamma-1}}
$$

where $p(a,t)$ is a polynomial in a of degree γ with coefficients analytic in $t \in \mathcal{N}$. Since the divisor of the right-hand side is supported on a one-dimensional analytic variety, there exists a sequence of points $\{a_n, t_n\} \subset C^2$ converging to $(0,0)$ where $P(T_{\bar\Theta_1\Theta_2})T_{\frac{f-a_n}{\bar g - t_n}}|\ker T_{\bar\Theta_1\Theta_2}$ is invertible.

For $t_n \in \mathcal{N}$ and small $|a_n|$, let $\phi_n \equiv \Theta_1\bar\Theta_2(f - a_n)(\bar g - t_n)$. The corresponding $m_\alpha = 0$. Therefore, Proposition 4.2 v) holds for ϕ_n. Since the right-hand side of Proposition 4.2 iv) is continuous in the spectral parameters (t, a), it suffices to check the continuity of the left-hand side. To that end we shall use Proposition 2.1. So suppose $\{v_j^{(n)}\}_1^\gamma$ is obtained from $\{T_{f-a_n}x_j\}_1^\gamma$ by the Gram-Schmidt procedure and suppose also that $\{y_j^{(n)}\}_1^\gamma$ is similarly obtained from $\{T_{\frac{1}{g - t_n}}x_j\}_1^\gamma$. By Propositions

4.3 and 4.2 part iv) we have for $n = 1, 2, \ldots,$

$$\det(T_{\phi_n} T_{\phi_n^{-1}} + \sum_1^\gamma y_j^{(n)} \otimes v_j^{(n)})$$
$$= \chi^2(P(T_{\bar\Theta_1}), P(T_{\bar\Theta_2})) \cdot \prod_{|x|<1} c_x(\frac{f-a_n}{\bar g - t_n}, \frac{\Theta_1}{\Theta_2})$$
$$\cdot \det{}_* \partial\{T_{\bar g - t_n} + \mathcal{L}^1(H), T_{f-a_n} + \mathcal{L}^1(H)\}$$
$$\times \det(P(T_{\bar\Theta_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|f-a_n|^2}\big|_{K_{\Theta_1} \cap K_{\Theta_2}^\perp})^{\frac12}$$
$$\cdot \det(P(T_{\bar\Theta_1}) \wedge \mathrm{rp}(T_{\Theta_2}) \cdot T_{|\frac{1}{g-t_n}|^2}\big|_{K_{\Theta_1} \cap K_{\Theta_2}^\perp})^{\frac12}.$$

Now suppose $\{h_j^{(n)}\}_1^\gamma$ is a sequence of linearly independent sets of vectors so that $h_j^{(n)} \to h_j$ for each $j = 1, 2, \ldots, \gamma$. Suppose also that $\{h_j\}_1^\gamma$ is linearly independent with associated orthonormal vectors $\{z_j\}_1^\gamma$. For each $n = 1, 2, \ldots,$ we can apply the Gram-Schmidt procedure to $\{h_j^{(n)}\}_1^\gamma$ to get vectors $\{z_j^{(n)}\}_1^\gamma$.

We claim that $z_j^{(n)} \to z_j$, for $j = 1, 2, \ldots, \gamma$. To check this, note that

$$z_1^{(n)} = \frac{h_1^{(n)}}{\|h_1^{(n)}\|} \to \frac{h_1}{\|h_1\|} = z_1,$$
$$z_2^{(n)} = \frac{h_2^{(n)} - (h_2^{(n)}, z_1^{(n)}) z_1^{(n)}}{\|h_2^{(n)}\|} \to \frac{h_2 - (h_2, z_1) z_1}{\|h_2\|} = z_2.$$

So in general,

$$z_j^{(n)} = \frac{h_j^{(n)} - \sum_{1 \le i < j} (h_j^{(n)}, z_i^{(n)}) z_i^{(n)}}{\|h_j^{(n)} - \sum_{1 \le i < j} (h_j^{(n)}, z_i^{(n)}) z_i^{(n)}\|} \to \frac{h_j - \sum_{1 \le i < j} (h_j, z_i) z_i}{\|h_j - \sum_{1 \le i < j} (h_j, z_i) z_i\|} = z_j.$$

Now with $h_j^{(n)} = T_{f-a_n} x_j$, we get $v_j^{(n)} = z_j^{(n)} \to z_j = v_j$. Similarly, if $h_j^{(n)} = T_{\frac{1}{g-t_n}} x_j$, then $y_j^{(n)} = z_j^{(n)} \to z_j = y_j$. Therefore,

$$\sum_{j=1}^\gamma y_j^{(n)} \otimes v_j^{(n)} \to \sum_{j=1}^\gamma y_j \otimes v_j \quad \text{in trace norm.}$$

Lemma 4.3. *If $\{b_j\}_1^\gamma$ are maximal vectors associated to the orthonormal basis $\{v_j\}_1^\gamma$, then*

$$\lim_{n\to\infty} \frac{\prod_{\lambda \in \sigma(T_{\phi_n} T_{\phi_n^{-1}}) \setminus \{0\}} \lambda}{\det(v_j^{(n)}, y_k^{(n)})} = (-1)^{\sum_{\alpha=1}^\gamma m_\alpha} \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda}{\det(b_j, y_k)}.$$

Proof. By Proposition 2.1 it suffices to show that

$$\lim_{n\to\infty} \det(T_{\phi_n} T_{\phi_n^{-1}} + \sum_{j=1}^\gamma y_j^{(n)} \otimes v_j^{(n)}) = \det(T_\phi T_{\phi^{-1}} + \sum_{j=1}^\gamma y_j \otimes v_j).$$

Since the dyadic sums converge in $\mathcal{L}^1(H)$ it remains to check that $1 - T_{\phi_n} T_{\phi_n}^{-1}$ converges to $1 - T_\phi T_{\phi^{-1}}$ in trace norm.

Note that the Hankel operators $H_{\Theta_1 \bar\Theta_2}$ and $H_{\tilde{\bar\Theta}_1 \tilde\Theta_2}$ are trace class.

Let $\psi = \Theta_1\bar{\Theta}_2 \cdot \omega\bar{\eta}$, where ω, η are outer in $K^{\frac{1}{2},\frac{1}{2}}_{2,2} \cap \mathcal{W}$. Then

$$T_\psi T_{\psi^{-1}} = [T_{\Theta_1\bar{\Theta}_2}T_{\omega\bar{\eta}} + H_{\Theta_1\bar{\Theta}_2}H_{\tilde{\omega}\bar{\tilde{\eta}}}][T_{\omega^{-1}\bar{\eta}^{-1}}T_{\bar{\Theta}_1\Theta_2} + H_{\omega^{-1}\bar{\eta}^{-1}}H_{\tilde{\bar{\Theta}}_1\tilde{\Theta}_2}].$$

With $\omega = f - a_n$ and $\eta = g - \bar{t}_n$ we have $H_{\tilde{\omega}\bar{\tilde{\eta}}} \to H_{\tilde{f}\bar{\tilde{g}}}$ and $H_{\omega^{-1}\bar{\eta}^{-1}} \to H_{f^{-1}\bar{g}^{-1}}$ in the uniform norm. Also, $T_{\Theta_1\bar{\Theta}_2}[T_{\omega\bar{\eta}}T_{\omega^{-1}\bar{\eta}^{-1}}]T_{\bar{\Theta}_1\Theta_2} = T_{\Theta_1\bar{\Theta}_2}[1+[T_{\bar{\eta}}, T_\omega]T_{\bar{\eta}^{-1}}T_{\omega^{-1}}]T_{\bar{\Theta}_1\Theta_2}$. But $[T_{\bar{\eta}}, T_\omega] = [T_{\bar{g}}, T_f] \in \mathcal{L}^1(H^2(\mathbb{T}))$. Since $T_{\bar{\eta}^{-1}}T_{\omega^{-1}} \to T_{\bar{g}}^{-1}T_{f^{-1}}$ in the uniform norm it follows that $1 - T_{\phi_n}T^{-1}_{\phi_n}$ converges to $1 - T_\phi T_{\phi^{-1}}$ in trace norm. □

To complete the proof of Theorem 1.6 we note that when the zeros of Θ_1 and Θ_2 are simple, the equality

$$\chi^2(P(T_{\bar{\Theta}_1}), P(T_{\bar{\Theta}_2})) = \frac{\prod_{1\le\alpha,\beta\le\ell}|1-\bar{\mu}_\alpha\mu_\beta|}{\prod_{\substack{1\le\alpha\le\ell\\1\le\beta\le q}}|1-\bar{\mu}_\alpha\nu_\beta|^2}\left|\sum_N \frac{\prod_{\substack{\mu_\alpha\in\bar{N}\\1\le\beta\le q}}(\mu_\alpha-\nu_\beta)}{\prod_{\substack{\mu_\alpha\in N\\\mu_\beta\in\bar{N}}}(\mu_\alpha-\mu_\beta)}\right.$$
$$\left.\cdot\frac{\prod_{\substack{\frac{1}{\mu_\alpha}\in N\\1\le\beta\le q}}(\bar{\mu}_\alpha-\bar{\nu}_\beta)}{\prod_{\substack{\frac{1}{\mu_\alpha}\in N\\\frac{1}{\mu_\beta}\in\bar{N}}}(\bar{\mu}_\alpha-\bar{\mu}_\beta)}\cdot\frac{\prod_{\substack{\frac{1}{\mu_\alpha}\in\bar{N}\\1\le\beta\le q}}(1-\bar{\mu}_\alpha\nu_\beta)}{\prod_{\substack{\frac{1}{\mu_\alpha}\in\bar{N}\\\mu_\beta\in N}}(1-\bar{\mu}_\alpha\mu_\beta)}\cdot\frac{\prod_{\substack{\mu_\alpha\in N\\1\le\beta\le q}}(1-\mu_\alpha\bar{\nu}_\beta)}{\prod_{\substack{\mu_\alpha\in\bar{N}\\\frac{1}{\mu_\beta}\in N}}(\mu_\alpha\bar{\mu}_\beta-1)}\right|,$$

with the sum extended over all subsets $N \subset \{\mu_1, \cdots, \mu_\ell; \frac{1}{\bar{\mu}_1}, \cdots, \frac{1}{\bar{\mu}_\ell}\}$ of cardinality ℓ, and $\bar{N} = \{\mu_1, \cdots, \mu_\ell; \frac{1}{\bar{\mu}_1}, \cdots, \frac{1}{\bar{\mu}_\ell}\} \setminus N$ is proved in Proposition 3.4 of [17].

Corollary 4.1. *Let $\{x_j\}_1^\gamma$ be an orthonormal basis for $\ker T_{\bar{\Theta}_1\Theta_2}$. Let $\{u_j\}_1^\gamma$ be the top root vectors of the associated Jordan chain lying over $T_f x_j$. Then*

i)
$$\sigma_{1,T_\phi T_{\phi^{-1}}} = (-1)^{\sum_{\alpha=1}^\gamma m_\alpha}\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}}\lambda\cdot\frac{\|\bigwedge T_{\frac{1}{g}}|_{\ker T_{\bar{\Theta}_1\Theta_2}}\|^2}{\det(u_j, T_{\frac{1}{g}}x_\tau)}$$
$$\cdot[\bigwedge T_f x_j \otimes \bigwedge T_{\frac{1}{g}}x_j + \operatorname{ran} T_\phi])^*,$$

ii)
$$(-1)^{\sum_{\alpha=1}^\gamma m_\alpha}\frac{\prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}}\lambda}{\det(u_j, T_{\frac{1}{g}}x_\tau)} = \chi^2(P(T_{\bar{\Theta}_1}), P(T_{\bar{\Theta}_2}))\prod_{\{x:\|x\|<1\}}c_x\left(\frac{f}{\bar{g}}, \frac{\Theta_2}{\Theta_1}\right)$$
$$\cdot\det{}_*\partial\{T_{\bar{g}} + \mathcal{L}^1, T_f + \mathcal{L}^1\}.$$

5. Proofs of Theorem 1.1 part ii) and Theorem 1.3

It is of interest to estimate the speed of convergence in our limit theorems. The proofs of Theorem 1.1 part ii) and Theorem 1.3 are based on a series of lemmas with numerical estimates. In what follows $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm and $\|\cdot\|_1$ denotes the trace norm.

Lemma 5.1. *Suppose $f \in \mathrm{Lip}_\beta$, $\beta > \frac{1}{2}$. Then*

$$\|S^{*n+1}PfQ\|_2 \le \left(\frac{4\pi}{3}\right)^\beta\left(\frac{1}{1-(\frac{1}{2})^{2\beta}}\right)^{\frac{1}{2}}\left(\frac{1}{2\beta-1}\right)^{\frac{1}{2}}\|f\|_{\mathrm{Lip}_\beta}\cdot(n+1)^{\frac{1-2\beta}{2}}.$$

Proof. The proof ultimately depends upon the Bernstein inequality; cf. p. 32, Katznelson [27]. It is known[5] that

$$\sum_{2^m \leq k \leq 2^{m+1}} |\hat{f}(k)|^2 \leq \left(\frac{2\pi}{3 \cdot 2^m}\right)^{2\beta} \|f\|^2_{\mathrm{Lip}_\beta}.$$

So, if $2^m \leq n \leq 2^{m+1}$, we have

$$\begin{aligned}\sum_{n \leq k} |\hat{f}(k)|^2 &\leq \left(\frac{2\pi}{3}\right)^{2\beta} \|f\|^2_{\mathrm{Lip}_\beta} \sum_{r \geq m} \left(\frac{1}{2^r}\right)^{2\beta} \\ &= \left(\frac{2\pi}{3}\right)^{2\beta} \|f\|^2_{\mathrm{Lip}_\beta} \left(\frac{1}{2^m}\right)^{2\beta} \frac{1}{1 - (\frac{1}{2})^{2\beta}}.\end{aligned}$$

Hence $\sum_{n \leq k} |\hat{f}(k)|^2 \leq \frac{\Omega}{n^{2\beta}}$, where $\Omega = \left(\frac{4\pi}{3}\right)^{2\beta} \left(\frac{1}{1-(\frac{1}{2})^{2\beta}}\right) \|f\|^2_{\mathrm{Lip}_\beta}$.

Now,

$$\begin{aligned}\|S^{*n+1} P M_f Q\|_2^2 &= \sum_{k \geq 1} k |\hat{f}(k+n+1)|^2 \\ = \sum_{k \geq 1} |\hat{f}(k+n+1)|^2 &+ \sum_{k \geq 1} |\hat{f}(k+n+2)|^2 + \sum_{k \geq 1} |\hat{f}(k+n+3)|^2 + \cdots \\ \leq \Omega \left[\frac{1}{(n+2)^{2\beta}} + \frac{1}{(n+3)^{2\beta}} + \frac{1}{(n+4)^{2\beta}} + \cdots\right] &\leq \frac{\Omega}{(2\beta - 1)} (n+1)^{1-2\beta}.\end{aligned}$$

□

Since $S^{*\gamma} T_{\frac{1}{f}} T_{\frac{1}{\bar{g}}} [1 - P(T_{\bar{\phi}})] = R(T_\phi)$, it follows that for $n \gg 0$,

$$\ker[S^{*n+1} R(T_\phi) S^{n+1}] = \mathrm{ran}\{S^{*n+1} T_{\frac{1}{f}} T_{\frac{1}{\bar{g}}} [1 - P(T_{\bar{\phi}})] S^{n+1}\}^{-1} P_\gamma.$$

Moreover, $\{S^{*n+1} T_{\frac{1}{f}} T_{\frac{1}{\bar{g}}} [1 - P(T_{\bar{\phi}})] S^{n+1}\}^{-1}$ converges uniformly to $T_f T_{\bar{g}}$.

Lemma 5.2. *Suppose $\phi = z^\gamma \psi$ with $\psi \in \mathrm{Lip}_\beta$, $\frac{1}{2} < \beta < 1$, and T_ψ is invertible. Set $J_n = S^{*n+1} T_\psi^{-1} [1 - P(T_{\bar{\phi}})] S^{n+1}$. Then*

(i) $$\|S^{*n+1} R(T_\phi) S^{n+1} - T_{\phi^{-1}}\| = O(n^{-2\beta}).$$

(ii) $$\|J_n - T_{\psi^{-1}}\| = O(n^{-2\beta}) \quad \textit{and} \quad \|J_n - T_{\psi^{-1}}\|_1 = O(n^{1-2\beta}).$$

In particular, for $n \gg 0$, J_n is invertible and

$$\ker S^{*n+1} R(T_\phi) S^{n+1} = J_n^{-1} P_{\gamma-1}(H^2(\mathbb{T})).$$

(iii) $$\|J_n^{-1} T_{\frac{1}{\bar{g}}} e_\alpha - T_f e_\alpha\| = O(n^{-2\beta}) \quad \textit{for} \quad \alpha = 0, 1, 2, \ldots, \gamma - 1.$$

Proof. We have $R(T_\phi) = S^{*\gamma} T_\psi^{-1} [1 - P(T_{\bar{\phi}})]$, and $\ker P(T_{\bar{\phi}}) = \{\bigvee_{\alpha=0}^{\gamma-1} \frac{1}{g} e_\alpha\}$ where $\psi = f\bar{g}$, $f, g \in H^\infty \cap \mathrm{Lip}_\beta$. In particular, $\|P(T_{\bar{\phi}}) S^{n+1}\| = O(n^{-\beta})$.

The fact that $f, g \in \mathrm{Lip}_\beta \cap H^\infty$ follows because T_ψ is invertible and the conjugate function is bounded on Lip_β, $0 < \beta < 1$; cf. [28]. Accordingly, there are polynomials

[5] See for example equation (6.4), Katznelson.

p_n and q_n of degree $\leq n$ so that $\|\frac{1}{f} - p_n\|_\infty$ and $\|\frac{1}{g} - q_n\|_\infty$ are $O(n^{-\beta})$. Also, since $S^{*n+1}T_\psi^{-1}S^{n+1} = T_{\psi^{-1}} - S^{*n+1}P\frac{1}{f}Q\frac{1}{\bar{g}}S^{n+1}$, we have

$$J_n = S^{*n+1}T_\psi^{-1}[1 - P(T_{\bar{\phi}})]S^{n+1} = S^{*n+1}T_\psi^{-1}S^{n+1} - S^{*n+1}T_\psi^{-1}P(T_{\bar{\phi}})S^{n+1}$$
$$= T_{\psi^{-1}} - S^{*n+1}P\frac{1}{f}Q\frac{1}{\bar{g}}PS^{n+1} - S^{*n+1}T_{\psi^{-1}}P(T_{\bar{\phi}})S^{n+1} + S^{*n+1}P\frac{1}{f}Q\frac{1}{\bar{g}}P(T_{\bar{\phi}})S^{n+1}.$$

Also

$$\|S^{*n+1}P\frac{1}{f}Q\frac{1}{\bar{g}}PS^{n+1}\| \leq \|S^{*n+1}P\frac{1}{f}Q\|\|Q\frac{1}{\bar{g}}PS^{n+1}\|$$
$$= \|S^{*n+1}P(\frac{1}{f} - p_n)Q\|\,\|Q(\frac{1}{\bar{g}} - \bar{q}_n)PS^{n+1}\| \leq \|\frac{1}{f} - p_n\|_\infty\,\|\frac{1}{g} - q_n\|_\infty = O(n^{-2\beta}),$$

and $\|S^{*n+1}P\frac{1}{f}Q\frac{1}{\bar{g}}S^{n+1}\|_1 \leq \|S^{*n+1}P\frac{1}{f}Q\|_2\,\|S^{*n+1}P\frac{1}{\bar{g}}Q\|_2 = O(n^{1-2\beta})$.

Now, $S^{*n+1}T_{\psi^{-1}} = T_{\frac{1}{\bar{g}}}S^{*n+1}T_{\frac{1}{f}}$, and $\ker T_{\bar{\phi}} = \bigvee_{\alpha=0}^{\gamma-1}\{\frac{1}{g}e_\alpha\}$, ran $T_{\frac{1}{f}}P(T_{\bar{\phi}}) = \bigvee_{\alpha=0}^{\gamma-1}\{\frac{1}{fg}e_\alpha\}$. Since $\frac{1}{f}, \frac{1}{g} \in \mathrm{Lip}_\beta$ it follows that

$$\|S^{*n+1}T_{\psi^{-1}}P(T_{\bar{\phi}})S^{n+1}\|_1 = O(n^{-2\beta}).$$

Therefore,

$$\|J_n - T_{\psi^{-1}}\| = O(n^{-2\beta}) \quad \text{and} \quad \|J_n - T_{\psi^{-1}}\|_1 = O(n^{1-2\beta}).$$

The fact that $\ker S^{*n+1}R(T_\phi)S^{n+1} = J_n^{-1}P_{k-1}(H^2(\mathbb{T}))$ follows immediately since $S^{*n+1}R(T_\phi)S^{n+1} = S^{*k}J_n$. Because $\|J_n - T_{\psi^{-1}}\| = O(n^{-2\beta})$ and $T_{\psi^{-1}}$ is invertible we have $\|J_n^{-1} - T_{\psi^{-1}}^{-1}\| = O(n^{-2\beta})$. Furthermore, since for all α we have $T_{\psi^{-1}}^{-1}T_{\frac{1}{\bar{g}}}e_\alpha = T_f e_\alpha$, we have $\|J_n^{-1}T_{\frac{1}{\bar{g}}}e_\alpha - T_f e_\alpha\| = O(n^{-2\beta})$. □

Lemma 5.3.

$$\|R(T_\phi S^{*n+1}R(T_\phi)S^{n+1}) - R(T_\phi T_{\phi^{-1}})\| = O(n^{-2\beta}).$$

Proof. Set $Y_n = T_\phi S^{*n+1}R(T_\phi)S^n$ and $Y = T_\phi T_{\phi^{-1}}$. For $n \gg 0$, $S^{*n+1}R(T_\phi)S^{n+1}$ is surjective. Hence $\mathrm{rp}\,(Y_n) = 1 - P(T_{\bar{\phi}})$. Therefore, since $R(Y_n^*) = [Y_nY_n^* + P(T_{\bar{\phi}})]^{-1}Y_n$, we have $\|R(Y_n^*) - R(Y^*)\| = O(n^{-2\beta})$. Consequently, since $R(A)^* = R(A^*)$, we have

$$\|R(Y_n) - R(Y)\| = \|R(Y_n)^* - R(Y)^*\| = \|R(Y_n^*) - R(Y^*)\| = O(n^{-2\beta}).$$

Fix an orthonormal basis $\{y_\tau\}_{\tau=1}^\gamma$ of $\ker(T_{\bar{\phi}})$. Suppose $n \gg 0$ so that $\dim\ker(S^{*n+1}R(T_\phi)S^{n+1}) = \gamma$ and J_n is invertible. For $\alpha = 1, 2, \ldots, \gamma$ set $w_\alpha^{(n)} = J_n^{-1}T_{\frac{1}{\bar{g}}}e_\alpha$ and let $\{v_\alpha^{(n)}\}_1^\gamma$ be obtained from $\{w_\alpha^{(n)}\}_1^\gamma$ by the Gram-Schmidt orthogonalization procedure. Note that $\|w_\alpha^{(n)} - fe_\alpha\| = O(n^{-2\beta})$.

As above let $\mathcal{Z}$ denote the subspace of $H^2(\mathbb{T})$ given by $\ker(T_{\bar{\phi}}) \oplus \bigvee\{(T_\phi T_{\phi^{-1}})^r u_\alpha : m_\alpha \geq r \geq 1, \alpha = 1, 2, \ldots, \gamma\}$, where $\{u_\alpha\}$is any set of maximal root vectors for $\{T_f e_\alpha\}_1^\gamma$. Let K_n denote the operator on $\mathcal{Z}$ where

$$K_n(y_\tau) = P_{\mathcal{Z}}(v_\alpha^{(n)}),$$
$$K_n(T_\phi T_{\phi^{-1}})^r u_\alpha = P_{\mathcal{Z}}R(T_\phi S^{*n+1}R(T_\phi)S^{n+1})(T_\phi T_{\phi^{-1}})^r u_\alpha.$$

By (ii) of Lemma 5.2 and the continuity of the Gram-Schmidt procedure, $v_\alpha^{(n)} \to v_\alpha$ where $\{v_\alpha^{(n)}\}_1^\gamma$ is obtained from $\{T_f e_\alpha\}_1^\gamma$ by applying the Gram-Schmidt orthogonalization. Let K denote the limit operator on $\mathcal{Z}$ so that

$$K(y_\tau) = P_{\mathcal{Z}} v_\tau,$$
$$K(T_\phi T_{\phi^{-1}})^r u_\alpha = P_{\mathcal{Z}}[1 - P(T_\phi T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1} u_\alpha.$$

□

Lemma 5.4. $\|K_n - K\| = O(n^{-2\beta})$.

Proof. Since $\|J_n^{-1} T_{\frac{1}{\bar{g}}} e_\alpha - T_f e_\alpha\| = O(n^{-2\beta})$, computation shows that $\|v_\alpha^{(n)} - v_\alpha\| = O(n^{-2\beta})$. Now, comparing K_n and K on basis elements of Z, we have with $z_{r_\alpha}(n) = X_n^{-1}(T_\phi T_{\phi^{-1}})^r u_\alpha$ as in §3, for $j = 1, 2, \ldots, \gamma$,

$$\|(K_n - K)y_j\| = \|P_{\mathcal{Z}}(v_\alpha^{(n)} - v_\alpha)\| = O(n^{-2\beta}).$$

Also, on $(T_\phi T_{\phi^{-1}})^r u_\alpha$ we have

$$\begin{aligned}[K_n - K](T_\phi T_{\phi^{-1}})^r u_\alpha = P_{\mathcal{Z}}[1 - P(S^{*n+1} R(T_\phi) S^{n+1})] z_{r_\alpha}(n) \\ - P_{\mathcal{Z}}[1 - P(T_{\phi^{-1}})](T_\phi T_{\phi^{-1}})^{r-1} u_\alpha.\end{aligned}$$

Now,

$$[1 - P(S^{*n+1} R(T_\phi) S^{n+1})] z_{r_{\alpha(n)}} = R[T_\phi S^{*n+1} R(T_\phi) S^{n+1}](T_\phi T_{\phi^{-1}})^r u_\alpha.$$

Hence, by Lemma 5.3 we have

$$\begin{aligned}\|[K_n - K](T_\phi T_\phi^{-1})^r u_\alpha\| &= \|P_{\mathcal{Z}} \left\{R(T_\phi S^{*n+1} R(T_\phi) S^{n+1})(T_\phi T_{\phi^{-1}}) \right. \\ &\qquad \left. -[1 - P(T_{\phi^{-1}})]\right\} (T_\phi T_{\phi^{-1}})^{r-1} u_\alpha\| \\ &= \|P_{\mathcal{Z}} \left\{R(T_\phi S^{*n+1} R(T_\phi) S^{n+1}) - R(T_\phi T_{\phi^{-1}})\right\} (T_\phi T_{\phi^{-1}})^r u_\alpha\| = O(n^{-2\beta}).\end{aligned}$$

Consequently,

$$\|K_n - K\| = O(n^{-2\beta}).$$

□

Recall that for $n \gg 0$,

$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \frac{\det(P_{\mathcal{X}} X_{n|\mathcal{X}})}{\det K_n} \cdot \det(S^{n+1} v_\alpha^{(n)}, y_\tau).$$

Now

$$\begin{aligned}\|X_n - T_\phi T_{\phi^{-1}}\|_1 &= \|T_\phi S^{*\gamma} J_n - P(T_{\bar{\phi}}) S^{n+1} - T_\phi S^{*\gamma} T_{\psi^{-1}}\|_1 \\ &\le \|T_\phi S^{*\gamma}[J_n - T_{\psi^{-1}}]\|_1 + \|P(T_{\bar{\phi}}) S^{n+1}\|_1 = O(n^{1-2\beta}).\end{aligned}$$

Using the inequality

$$|\det[1 + M] - \det[1 + N]| \le \|M - N\|_1 \exp(\|M\|_1 + \|N\|_1 + 1),$$

we see that

$$\begin{aligned}\det P_{\mathcal{X}} X_n P_{\mathcal{X}} &= \det[P_{\mathcal{X}} T_\phi T_{\phi^{-1}}|_{\mathcal{X}}][1 + O(n^{1-2\beta})] \\ &= \prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}} \setminus \{0\}} \lambda \quad [1 + O(n^{1-2\beta})].\end{aligned} \tag{5.1}$$

Also by Lemmas 5.2 and 5.4

$$\frac{\det(S^{n+1}v_\alpha^{(n)}, y_\tau)}{\det K_n} = (-1)^{\sum_1^\gamma m_\alpha} \cdot \frac{\det(S^{n+1}w_\alpha^{(n)}, y_\tau)}{\det(u_\alpha, y_\tau)} \cdot [1 + O(n^{-2\beta})].$$

Consequently, switching vectors $\{y_\tau\}_1^\gamma$ to an arbitrary basis $\{t_\tau\}_1^\gamma$ of $\ker T_{\bar\phi}$, we have

$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \det(S^{n+1}w_\alpha^{(n)}, t_\tau) \frac{(-1)^{\sum_1^\gamma m_\alpha} \prod_{\lambda\in\sigma(T_\phi T_{\phi^{-1}})\setminus\{0\}} \lambda}{\det(u_\alpha, t_\tau)} [1+O(n^{1-2\beta})].$$

This proves statement ii) of Theorem 1.1. We now complete the proof of Theorem 1.3.

Lemma 5.5.

$$\begin{aligned} &\det(S^{n+1}w_\alpha^{(n)}, T_{\frac{1}{g}} e_\tau) \\ &= \frac{1}{\bar g(0)^\gamma} \det(S^{n+1}(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}e_\alpha, T_{\frac{1}{g}} e_\tau)[1 + O(n^{-2\beta})]. \end{aligned}$$

Proof. It is enough to prove the statement with $T_{\frac{1}{g}} e_\alpha$ replaced by any basis of $\ker T_{\bar\phi}$.

Let $\{y_\tau\}_1^\gamma$ be an orthonormal basis for $\ker T_{\bar\phi}$. Since $S^{n+1}w_\alpha^{(n)} = S^{n+1}J_n^{-1}T_{\frac{1}{\bar g}} e_\alpha \in \ker[T_{\bar\phi} \oplus \operatorname{ran} T_\phi P_n] \cap \operatorname{ran} S^{n+1}$, there are complex numbers $\{\lambda_{n,\alpha,\tau}\}_{1\le\alpha,\tau\le\gamma}$ and a vector $g_{n\alpha} \in \operatorname{ran}(T_\phi P_n)$ so that

$$S^{n+1}w_\alpha^{(n)} = \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau} y_\tau + T_\phi g_{n\alpha},$$

where $\lambda_{n,\alpha,\tau} = (S^{n+1}w_\alpha^{(n)}, y_\tau)$. Since $T_\phi = T_\psi S^\gamma$ and $S^{*\gamma}T_\psi S^\gamma = T_\psi$, we have for $n \gg 0$

$$S^{*n+1}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)} = \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau} S^{*n+1}T_\psi^{-1}S^{*\gamma}y_\tau.$$

Multiplying by S^γ and rewriting gives

$$\begin{aligned} S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)} &= \sum_{\tau=1}^\gamma (S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)}, e_\tau)e_\tau \\ &\quad + \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau} S^\gamma S^{*n+1}T_\psi^{-1}S^{*\gamma}y_\tau. \end{aligned}$$

Applying the inverse of $S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}$ to both sides gives

$$\begin{aligned} w_\alpha^{(n)} &= \sum_{\tau=1}^\gamma (S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)}, e_\tau)(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}e_\tau \\ &\quad + \sum_{\tau=1}^\gamma \lambda_{n,\alpha,\tau}(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}S^\gamma S^{*n+1-\gamma}T_\psi^{-1}S^{*\gamma}y_\tau. \end{aligned}$$

Taking the inner product on both sides with $S^{*n+1}y_k$ we get

$$(w_\alpha^{(n)}, S^{*n+1}y_k) = \sum_{\tau=1}^{\gamma}(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)}, e_\tau) \cdot\big([S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}]^{-1}e_\tau, S^{*n+1}y_k\big)$$
$$+\sum_{\ell=1}^{\gamma}(w_\alpha^{(n)}, S^{*n+1}y_\ell)\big([S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}]^{-1}S^\gamma S^{*n+1}T_\psi^{-1}S^{*\gamma}y_\ell, S^{*n+1}y_k\big).$$

Now define $\gamma\times\gamma$ matrices A_n, B_n, E_n, F_n as follows:

$$A_n = ((S^{n+1}w_\alpha^{(n)}, y_\tau)),$$
$$B_n = ((S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}w_\alpha^{(n)}, e_\tau)),$$
$$E_n = ((S^{n+1}(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}e_\alpha, y_\tau)),$$
$$F_n = ((S^{n+1}(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}\cdot S^\gamma S^{*n+1-\gamma}T_\psi^{-1}S^{*\gamma}y_\alpha, y_\tau).$$

The preceding system of equations becomes $A_n = B_nE_n + A_nF_n$. Accordingly, $\det A_n = \dfrac{\det B_n \det E_n}{\det(1-F_n)}$. By Lemma 5.2, $B_n = ((T_{\frac{1}{\bar g}}e_\alpha, e_\tau)) + O(n^{-2\beta})$, and since $(T_\psi)^{-1}$ maps Lip_β to Lip_β we have $F_n = O(n^{-2\beta})$, so that

$$\det A_n = \det(T_{\frac{1}{\bar g}}e_\alpha, e_\tau)\cdot\det E_n\cdot[1+O(n^{-2\beta})] = \frac{1}{\bar g(0)^\gamma}\det E_n[1+O(n^{-2\beta})].$$

Therefore,

$$\det((S^{n+1}w_\alpha^{(n)}, y_\tau)) = \frac{1}{\bar g(0)^\gamma}\det(S^{n+1}(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}e_\alpha, y_\tau)\cdot[1+O(n^{-2\beta})].$$

□

Lemma 5.6.

$$\det((S^{n+1}(S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma})^{-1}e_\alpha, T_{\frac{1}{g}}e_\tau))$$
$$= \bar g(0)^\gamma\cdot\det\left((T_{\frac{f}{\bar g}z^{n+1}}\cdot[1-H_{\frac{\bar g}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar g})}]^{-1}e_\alpha, e_\tau)\right)_{\gamma\times\gamma}$$

Proof. We have

$$S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma} = S^{*n+1-\gamma}T_{f\bar g}^{-1}S^{n+1-\gamma} = S^{*n+1-\gamma}T_{\frac{1}{f}}T_{\frac{1}{\bar g}}S^{n+1-\gamma}$$
$$= S^{*n+1-\gamma}T_{\frac{1}{\bar g}}\langle T_{\bar g}, T_{\frac{1}{f}}\rangle T_{\frac{1}{f}}S^{n+1-\gamma} = T_{\frac{1}{\bar g}}S^{*n+1-\gamma}T_{\frac{\bar g}{f}}T_{\frac{f}{\bar g}}S^{n+1-\gamma}T_{\frac{1}{f}}$$
$$= T_{\frac{1}{\bar g}}(1-S^{*n+1-\gamma}H_{\frac{\bar g}{f}}H_{(\frac{f}{\bar g})}S^{n+1-\gamma})T_{\frac{1}{f}} = T_{\frac{1}{\bar g}}[1-H_{\frac{\bar g}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar g})}]T_{\frac{1}{f}}.$$

Thus

$$[S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}]^{-1} = T_f[1-H_{\frac{\bar g}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar g})}]^{-1}T_{\bar g},$$

and so,

$$\det\big((S^{n+1}[S^{*n+1-\gamma}T_\psi^{-1}S^{n+1-\gamma}]^{-1}e_\alpha, T_{\frac{1}{g}}e_\tau)\big)$$
$$= \det\left((T_{\frac{f}{\bar g}z^{n+1}}\cdot[1-H_{\frac{\bar g}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar g})}]^{-1}T_{\bar g}e_\alpha, e_\tau)\right)_{\gamma\times\gamma}$$
$$= \bar g(0)^\gamma\det\left((T_{\frac{f}{\bar g}z^{n+1}}\cdot[1-H_{\frac{\bar g}{f}}Q_{n-\gamma}H_{(\frac{f}{\bar g})}]^{-1}e_\alpha, e_\tau)\right)_{\gamma\times\gamma}.$$

□

Collecting the results: by equation (3.3) we have for $n \gg 0$,

$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} \frac{\det P_{\mathcal{X}} X_n P_{\mathcal{X}})}{\det(K_n)} \cdot \det(S^{n+1} v_\alpha^{(n)}, y_\tau).$$

By equation (5.1) we have

$$\det P_{\mathcal{X}} X_n P_{\mathcal{X}} = \prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda \quad [1 + O(n^{1-2\beta})],$$

and by Lemmas 3.5 and 5.4

$$\det(K_n) = (-1)^{\sum_1^\gamma m_\alpha} \det(b_\alpha, y_\tau) \cdot [1 + O(n^{-2\beta})],$$

while Lemmas 5.5 and 5.6 give us

$$(S^{n+1} v_\alpha^{(n)}, y_\tau) = \det \Gamma^{(n)} \cdot \det(S^{n+1} w_\alpha^{(n)}, y_\tau)$$
$$= \frac{1}{\| \bigwedge T_f e_\alpha \|} \det(S^{n+1} T_f [1 - H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}]^{-1} e_\alpha, y_\tau) \cdot [1 + O(n^{-2\beta})].$$

On the other hand, Proposition 2.1 ii) gives

$$\| \bigwedge T_f e_\alpha \| \det(b_\alpha, y_\tau) = \det(u_\alpha, y_\tau),$$

and we have "the ratio invariance property"

$$\frac{\det(S^{n+1} w_\alpha^{(n)}, y_\tau)}{\det(u_\alpha, y_\tau)} = \frac{\det(S^{n+1} w_\alpha^{(n)}, T_{\frac{1}{g}} e_\tau)}{\det(u_\alpha, T_{\frac{1}{g}} e_\tau)},$$

while Theorem 1.2 gives us

$$(-1)^{\sum_{\alpha=1}^\gamma m_\alpha} \frac{\prod_{\lambda \in \sigma(T_\phi T_{\phi^{-1}}) \setminus \{0\}} \lambda}{\det(u_\alpha, T_{\frac{1}{g}} e_\alpha)} = E(\psi) G(\frac{\bar{g}}{f})^\gamma.$$

Putting these pieces together yields the approximation theorem

$$D_n(T_\phi) = \mathbf{G}(\phi)^{n+1} E(\psi) G(\frac{\bar{g}}{f})^\gamma$$
$$\cdot \det \left((T_{\frac{f}{\bar{g}} z^{n+1}} \cdot [1 - H_{\frac{\bar{g}}{f}} Q_{n-\gamma} H_{(\frac{f}{\bar{g}})}]^{-1} e_\alpha, e_\tau) \right)_{\gamma \times \gamma} [1 + O(n^{1-2\beta})].$$

Note that if $\phi = z^\gamma f \bar{g}$ where $f, g \in K_{2,2}^{\frac{1}{2},\frac{1}{2}} \cap H^\infty(\mathbb{T})$, the same formula holds with $O(n^{-2\beta})$ replaced by $o(1)$. This concludes the proof of Theorem 1.3. □

6. Rational symbol ϕ with $D_n(T_\phi) \neq 0$ and $D_{\gamma-1}(T_{\frac{f}{\bar{g}}} z^{n+1}) = 0$

Now we construct a rational symbol ϕ with $\phi \neq 0$ on $\mathbb{T}$ so that T_ϕ is injective and Fredholm and $D_n(T_\phi) \neq 0$ for $n \gg 0$, yet for an $r \neq 0$, we have the factorization $\det(P(T_{\bar{\phi}}) S^{n+1} P(T_{\phi^{-1}}) S^{*n+1} \big| \ker T_{\bar{\phi}}) = a_n r^{n+1}$ where a_n is periodic in n, and one of the values it assumes is zero. Let $r \in (0,1)$, $\theta \in (0, \pi)$. Set $\lambda_1 = re^{i\theta}, \lambda_2 = re^{-i\theta} = \bar{\lambda}_1 \neq \lambda_1$, $\mu = r = |\lambda_1| \neq \lambda_1, \lambda_2$. Set $\phi = \Theta_{\lambda_1} \Theta_{\lambda_2} \bar{\Theta}_\mu$. Then $\phi^{-1} =$

$\bar\phi$, T_ϕ is injective and Fredholm. Thus $\det(P(T_{\bar\phi})S^{n+1}P(T_{\phi^{-1}})S^{*n+1})\big|\ker T_{\bar\phi}) = |\det(P(T_{\bar\phi})S^{n+1}\big|\ker T_{\bar\phi})|^2$. In Lemma 52 of [15] we showed that

$$\det(P(T_{\bar\phi})S^{n+1}P(T_{\phi^{-1}})\big|\ker T_{\bar\phi}) = \frac{1}{|1+r^2e^{i\theta}|^2+r^2|e^{i\theta}-1|^2}\frac{1}{(e^{-i\theta}-e^{i\theta})}$$
$$\times\Big[(e^{i\theta}-1)(1-r^2e^{i\theta})(1-r^2e^{-2i\theta})e^{i(n+1)\theta}$$
$$-(e^{-i\theta}-1)(1-r^2e^{-i\theta})(1-r^2e^{2i\theta})e^{-i(n+1)\theta}\Big]r^{n+1}.$$

Set

$$B_n(\theta,r) = (e^{i\theta}-1)(1-r^2e^{i\theta})(1-r^2e^{-2i\theta})e^{i(n+1)\theta},$$

so that

$$\det(P(T_{\bar\phi})S^{n+1}P(T_{\phi^{-1}})\big|\ker T_{\bar\phi})$$
$$= \frac{1}{|1+r^2e^{i\theta}|^2+r^2|e^{i\theta}-1|^2}\frac{r^{n+1}}{(e^{-i\theta}-e^{i\theta})}\big[B_n(\theta,r)-B_n(-\theta,r)\big] = 0,$$

whenever $B_n(\theta,r)$ is real valued. Suppose that $n=0$. For fixed $\theta\in(0,\frac{\pi}{4})$, consider $\Im B_0(\theta,r)$ as a function of $r\in[0,1]$. We have

$$\Im B_0(\theta,0) = \sin 2\theta - \sin\theta > 0,$$

$$\Im B_0(\theta,1) = -|1-e^{i\theta}|^2\sin 2\theta < 0.$$

Hence, by the intermediate value theorem, we get $r(\theta)\in(0,1)$ so that $\Im B_0(\theta,r(\theta)) = 0$.

Now pick $\theta\in(0,\frac{\pi}{4})$ so that $\frac{\theta}{\pi}$ is rational. Then with $r=r(\frac{\theta}{\pi})$ the sequence $\{\Im B_n(\theta,r)\}_{n=1}^\infty$ is periodic with some period P and has zero for one of its values. Hence $\det(P(T_{\bar\phi})S^{kP+1}P(T_{\phi^{-1}})\big|\ker T_{\bar\phi}) = 0$ for $k=0,1,2,\ldots$.

Now we must check that for the indicated choices of r and θ the determinant $D_n(T_\phi)\neq 0$ for $n\gg 0$.

For this purpose we use results of K.M. Day [18],

Theorem 6.1. *Let* $\phi = \frac{c_0\prod_{j=1}^p(z-r_j)}{\prod_{j=1}^h(1-\frac{z}{\rho_j})\prod_{j=1}^\ell(z-\delta_j)}$ *on the circle* $|z|=1$ *with* $|\delta_j|<1$ $(j=1,\ldots,\ell)$, *and* $|\rho_j|>1$ *for* $j=1,\ldots,h$, c_0 *a constant and* $r_1,\ldots,r_p$ *pairwise distinct. Then,*

$$\text{(i)}\qquad \text{For } p\geq \ell+h \text{ and } n\geq 0,\quad D_n(\phi) = (-1)^{(p-\ell)(n+1)}\sum_M A_M r_M^{n+1}$$

(6.1)

$$\text{(ii)}\qquad \text{for } n\geq\ell,\quad D_n(\phi)=0\quad\text{if}\quad p<\ell,$$
$$\text{and } D_n(\phi) = (-1)^{(p-\ell)(n+1)}\sum_M A_M r_M^{n+1}\quad\text{if}\quad p\geq\ell$$

where the sum is taken over all $\binom{q}{\ell}$ *subsets* $M \subset \{1,\ldots,p\}$ *of cardinality* ℓ*, and with* $\overline{M} = \{1,\ldots,p\} \setminus M$ *the* r_M*'s and the* A_M*'s are given as*

$$r_M = c_0 \prod_{j\in\overline{M}} r_j,$$

$$A_M = \prod_{\substack{j\in\overline{M}\\ \alpha\in\{1,\ldots,\ell\}}} (r_j-\delta_\alpha) \prod_{\substack{i\in M\\ \beta\in\{1,\ldots,h\}}} (\rho_\beta - r_i) \prod_{\substack{\alpha\in\{1,\ldots,\ell\}\\ \beta\in\{1,\ldots,h\}}} (\rho_\beta-\delta_\alpha)^{-1} \prod_{\substack{i\in M\\ j\in\overline{M}}} (r_j-r_i)^{-1}.$$

In order to apply this theorem the ϕ of our example is rewritten as

$$\phi = c_0 \frac{(z-r_1)(z-r_2)(z-r_3)}{(1-\frac{z}{\rho_1})(1-\frac{z}{\rho_2})(z-\delta_1)}$$

i.e.,

$$c_0 = -r, r_1 = re_1^{i\rho} = \frac{1}{re^{i\theta}},\ \rho_2 = \frac{1}{re^{-i\theta}}, \delta_1 = r.$$

Thus, $p=3, h=2$ and $\ell=1$. Accordingly, the set M is either $\{1\}$, $\{2\}$ or $\{3\}$ and we have

$$\begin{aligned}
A_{\{1\}}{r_{\{1\}}}^{n+1} &= (r_2-\delta_1)(r_3-\delta_1)(\rho_1-r_1)(\rho_2-r_1)\\
&\quad\cdot(\rho_1-\delta_1)^{-1}(\rho_2-\delta_1)^{-1}(r_2-r_1)^{-1}(r_3-r_1)^{-1}(-r)^{n+1}(r_2r_3)^{n+1}.\\
A_{\{2\}}{r_{\{2\}}}^{n+1} &= (r_1-\delta_1)(r_3-\delta_1)(\rho_1-r_2)(\rho_2-r_2)\\
&\quad\cdot(\rho_1-\delta_1)^{-1}(\rho_2-\delta_1)^{-1}(r_1-r_2)^{-1}(r_3-r_2)^{-1}(-r)^{n+1}(r_1r_3)^{n+1}.\\
A_{\{3\}}{r_{\{3\}}}^{n+1} &= (r_1-\delta_1)(r_2-\delta_1)(\rho_1-r_3)(\rho_2-r_3)\\
&\quad\cdot(\rho_1-\delta_1)^{-1}(\rho_2-\delta_1)^{-1}(r_1-r_3)^{-1}(r_2-r_3)^{-1}(-r)^{n+1}(r_1r_2)^{n+1}.
\end{aligned}$$

A straightforward substitution in the Day formula now gives

$$\begin{aligned}
A_{\{1\}}{r_{\{1\}}}^{n+1} + A_{\{2\}}{r_{\{2\}}}^{n+1} &= (r_3-\delta_1)(\rho_1-\delta_1)^{-1}(\rho_2-\delta_1)^{-1}(r_1-r_2)^{-1}(-r)^{n+1}\\
&\quad\cdot\frac{(1-r^2)}{|1-r^2e^{i\theta}|^2}\cdot\big[(e^{i\theta}-1)(1-r^2e^{i\theta})(1-r^2e^{-2i\theta})e^{i(n+1)\theta}\\
&\quad-(e^{-i\theta}-1)(1-r^2e^{-i\theta})(1-r^2e^{2i\theta})e^{i(n+1)\theta}\big]\\
&= (r_3-\delta_1)(\rho_1-\delta_1)^{-1}(\rho_2-\delta_1)^{-1}(r_1-r_2)^{-1}(-r)^{n+1}\\
&\quad 2\cdot\frac{(1-r^2)}{|1-r^2e^{i\theta}|^2}\Im B_n(\theta,r).
\end{aligned}$$

Hence there are non-zero constants C_1 and C_2 so that $D_n(T_\phi)$ has the form

$$D_n(T_\phi) = \left[C_1 \Im B_n(\theta,r) + C_2 r^{2(n+1)}\right](-r)^{n+1}.$$

Since $0<r<1$ and $\Im B_n(\theta,r)$ assumes only finitely many values, it follows that $D_n(T_\phi)\neq 0$ provided that $n \gg 0$.

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