TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 358, Number 5, Pages 2057–2069
S 0002-9947(05)03973-5
Article electronically published on December 20, 2005

ALMOST COMPLEX MANIFOLDS AND CARTAN'S UNIQUENESS THEOREM

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ABSTRACT. We present a generalization of Cartan's uniqueness theorem to the almost complex manifolds.

1. Introduction

The primary goal of this article is to present a generalization *to the almost complex manifolds* of the following celebrated theorem of H. Cartan, which is usually called Cartan's uniqueness theorem (see p. 66, [13]).

Theorem 1.1 (H. Cartan). *Let Ω be a bounded domain in $\mathbb{C}^n$. If a holomorphic mapping $f : \Omega \to \Omega$ satisfies that $f(p) = p$ and $df_p = \mathrm{Id}$ for some $p \in \Omega$, then f is the identity mapping.*

In order to state the main theorem of this article, we shall introduce the necessary terminology and concepts.

A pair (M, J) is called an *almost complex manifold* if M is a C^∞-smooth real manifold and J is a field of endomorphisms of the tangent bundle TM with $J^2 = -\mathrm{Id}$, i.e. for each $p \in M$, $J_p : T_pM \to T_pM$ is an endomorphism with $J_p^2 = -\mathrm{Id}$. We call J an almost complex structure on M. Throughout this paper, by a smooth almost complex manifold we mean a manifold with a C^∞-smooth almost complex structure.

Given two almost complex manifolds (M, J) and (M', J'), a C^1 mapping f from M to M' is said to be *(J, J')-holomorphic* (or simply *pseudo-holomorphic*, so there is no danger of confusion) if its differential $df : TM \to TM'$ satisfies

$$df \circ J = J' \circ df \tag{1.1}$$

on TM. If (M, J) is a Riemann surface, f is called a pseudo-holomorphic curve. In the case (M, J) is the unit disc $\mathbf{D}$ in $\mathbb{C}$ with the standard complex structure J_{st}, we call f a pseudo-holomorphic disc. We denote by $\mathcal{O}_{(J,J')}(M, M')$ the space of (J, J')-holomorphic mappings from M to M'.

By the existence theorem of pseudo-holomorphic discs (Nijenhuis and Woolf [15]), we can define the Kobayashi pseudo-distance ([8]) and the Kobayashi-Royden pseudo-metric ([16]) for the almost complex manifolds.

Let (M, J) be an almost complex manifold. Given two points p and q in M, a finite sequence of pseudo-holomorphic discs $c = \{\phi_j\}_{j=1,\dots,k} \subset \mathcal{O}_{(J_{st},J)}(\mathbf{D}, M)$

Received by the editors March 12, 2004.
2000 *Mathematics Subject Classification.* Primary 32H02, 32Q60.
This research was upported in part by Grant KRF-2002-070-C00005 (PI: K.T. Kim) from the Korea Research Foundation.

is called a chain of pseudo-holomorphic discs from p to q if there are points $p = p_0, p_1, \ldots, p_k = q$ in M and $a_1, a_2, \ldots, a_k$ in $\mathbf{D}$ such that

$$\phi_j(0) = p_{j-1} \quad \text{and} \quad \phi_j(a_j) = p_j$$

for $j = 1, \ldots, k$. For this chain, we define its length $\ell(c)$ by

$$\ell(c) = \log \frac{1+|a_1|}{1-|a_1|} + \ldots + \log \frac{1+|a_k|}{1-|a_k|} \ .$$

Note that $\log \frac{1+|z|}{1-|z|}$ is the Poincaré distance from 0 to z in $\mathbf{D}$. The *Kobayashi pseudo-distance* $d_{(M,J)}$ on (M, J) is then defined by

$$d_{(M,J)}(p, q) = \inf \ell(c),$$

where the infimum is taken over all chains of pseudo-holomorphic discs from p to q. The *Kobayashi-Royden pseudo-metric* $F_{(M,J)}$ is the infinitesimal version of the Kobayashi pseudo-distance defined by

$$F_{(M,J)}(p, v) = \inf\{\frac{1}{|a|} : \phi \in \mathcal{O}_{(J_{st},J)}(\mathbf{D}, M) \text{ with } \phi(0) = p,\ d\phi(\mathbf{e}) = av\},$$

where $\mathbf{e}$ is the unit vector in $T_0\mathbf{D}$ and $p \in M$ and $v \in T_pM$. We exploit from [10] and [11] the following properties that are exactly the same as in the integrable case ([8] and [16]):

(a) $F_{(M,J)}$ is upper semi-continuous and

$$d_{(M,J)}(p, q) = \inf \int_0^1 F_{(M,J)}(\gamma(t), \gamma'(t))dt,$$

where the infimum is taken over all piecewise smooth paths $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$.

(b) Let $f : (M, J) \to (M', J')$ be a pseudo-holomorphic mapping. For any points p and q in M and tangent vector $v \in T_pM$, we have

$$d_{(M',J')}(f(p), f(q)) \leq d_{(M,J)}(p, q)$$

and

$$F_{(M',J')}(f(p), df_p(v)) \leq F_{(M,J)}(p, v) \ .$$

(c) The Kobayashi pseudo-distance $d_{(M,J)}$ is finite and continuous on $M \times M$.

(d) If $d_{(M,J)}$ is a distance, it induces the standard topology on M.

We say that (M, J) is *(Kobayashi) hyperbolic* if $d_{(M,J)}$ is a proper distance. Note that for any neighborhood U of $p \in M$, there is a constant $r > 0$ such that the Kobayashi ball $\mathbf{B}_{(M,J)}(p, r) = \{q \in M : d_{(M,J)}(p, q) < r\}$ is contained in U when (M, J) is hyperbolic.

Now we state our main theorem.

Theorem 1.2. *Let (M, J) be a C^∞-smooth almost complex manifold. Moreover, M is connected and Kobayashi hyperbolic. Suppose that there is a pseudo-holomorphic mapping $f : M \to M$ with $f(p) = p$ and $df_p = \mathrm{Id}$. Then f is the identity mapping.*

The proof of this theorem appears in Section 5. Sections 2, 3 and 4 contain a regularity theorem and derivative estimates for pseudo-holomorphic mappings which will be used in the proof of Theorem 1.2.

2. Regularity of pseudo-holomorphic mappings

We now study the smoothness of pseudo-holomorphic mappings. Since the problem is local, we assume that our manifold is a domain in a Euclidean space. Let $(\Omega, J) \subset \mathbb{R}^{2n}$ and $(\Omega', J') \subset \mathbb{R}^{2m}$ be domains with almost complex structures $J \in C^\infty(\overline{\Omega})$ and $J' \in C^\infty(\overline{\Omega'})$. (If the underlying space of an almost complex manifold is a domain in a Euclidean space, we will call it the *almost complex domain.*) Assume that Ω is bounded and has smooth boundary. Regard J and J' as matrix-valued functions on Ω and Ω', respectively. In this section $j, k, l, \ldots = 1, 2, \ldots, 2n$ and $\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, 2m$.

Let $f : \Omega \to \Omega'$ be a pseudo-holomorphic mapping of class $C^1(\overline{\Omega})$. Then $J'_f = J' \circ f$ is $2m \times 2m$ matrix-valued function defined on Ω of class $C^1(\overline{\Omega})$. We will fix f and simply denote J'_f by J' for the rest of this section. Let $J = (a^k_j)$ and $J' = (b^\alpha_\beta)$, where $a^k_j \in C^\infty(\overline{\Omega})$ and $b^\alpha_\beta \in C^1(\overline{\Omega})$.

Denote by $L^2(\Omega, \mathbb{R}^{2m})$ (resp. $L^2(\Omega, M_{2m\times 2n}(\mathbb{R}))$) the space of $\mathbb{R}^{2m}$-valued (resp. $2m \times 2n$ matrix-valued) square integrable functions. For $g \in L^2(\Omega, \mathbb{R}^{2m})$ and $\varphi \in L^2(\Omega, M_{2m\times 2n}(\mathbb{R}))$, we write $g = (g_\alpha)$ and $\varphi = (\varphi^\alpha_j)$. Define the inner products of $L^2(\Omega, \mathbb{R}^{2m})$ and $L^2(\Omega, M_{2m\times 2n}(\mathbb{R}))$ by

$$\begin{aligned}(g,h) &= \int_\Omega \Big(\sum_\alpha g_\alpha h_\alpha\Big),\\ (\varphi,\psi) &= \int_\Omega \operatorname{trace}(\varphi^t\psi + (J'\varphi)^t J'\psi)\\ &= \int_\Omega \Big(\sum_{\alpha,j} \varphi^\alpha_j \psi^\alpha_j + \sum_{\alpha,\beta,\gamma,j} \varphi^\alpha_j b^\beta_\alpha b^\beta_\gamma \psi^\gamma_j\Big),\end{aligned}$$

where $g, h \in L^2(\Omega, \mathbb{R}^{2m})$ and $\varphi, \psi \in L^2(\Omega, M_{2m\times 2n}(\mathbb{R}))$.

For fixed f, we can define the densely defined linear differential operator $\overline{\partial}$: $L^2(\Omega, \mathbb{R}^{2m}) \to L^2(\Omega, M_{2m\times 2n}(\mathbb{R}))$ by

$$\overline{\partial} g = dg + J' dg J,$$

where dg denotes the Jacobian matrix of g. Since f satisfies equation (1.1), it follows that $\overline{\partial} f = 0$. The (α, j)-th entry of $\overline{\partial} g$ can be expressed by

$$(\overline{\partial} g)^\alpha_j = \frac{\partial g_\alpha}{\partial x_j} + \sum_{\beta,k} b^\alpha_\beta \frac{\partial g_\beta}{\partial x_k} a^k_j \ . \tag{2.1}$$

We consider the following linear differential operator $\vartheta : L^2(\Omega, M_{2m\times 2n}(\mathbb{R})) \to L^2(\Omega, \mathbb{R}^{2m})$ by

$$(\vartheta\varphi)_\alpha = -\sum_j \frac{\partial \varphi^\alpha_j}{\partial x_j} + \sum_{\beta,j,k} b^\alpha_\beta a^k_j \frac{\partial \varphi^\beta_j}{\partial x_k} \ .$$

In fact, the principal part of the formal adjoint operator of $\overline{\partial}$ is of the form $(I + J'^t J')\vartheta$. Replacing φ by $\overline{\partial} g$, we have

$$(\vartheta\overline{\partial} g)_\alpha = -\sum_j \frac{\partial}{\partial x_j}(\overline{\partial} g)^\alpha_j + \sum_{\beta,j,k} b^\alpha_\beta a^k_j \frac{\partial}{\partial x_k}(\overline{\partial} g)^\beta_j \ .$$

Applying equation (2.1), we have that

$$\begin{aligned}(\vartheta\overline{\partial}g)_\alpha &= -\sum_j \frac{\partial^2 g_\alpha}{\partial x_j \partial x_j} \\ &\quad -\sum_{\beta,j,k} b^\alpha_\beta a^k_j \left(\frac{\partial^2 g_\beta}{\partial x_j \partial x_k} - \frac{\partial^2 g_\beta}{\partial x_k \partial x_j}\right) \\ &\quad +\sum_{\beta,\gamma,j,k,l} b^\alpha_\beta b^\beta_\gamma a^k_j a^l_j \frac{\partial^2 g_\gamma}{\partial x_k \partial x_l} \\ &\quad +(Cg)_\alpha,\end{aligned}$$

where $(Cg)_\alpha$ is part of $(\vartheta\overline{\partial}g)_\alpha$ of lower order given by

$$\begin{aligned}(Cg)_\alpha &= -\sum_{\beta,j,k} \frac{\partial g_\beta}{\partial x_k} \frac{\partial}{\partial x_j}(b^\alpha_\beta a^k_j) \\ &\quad +\sum_{\beta,\gamma,j,k,l} b^\alpha_\beta a^k_j \frac{\partial g_\gamma}{\partial x_l} \frac{\partial}{\partial x_k}(b^\beta_\gamma a^l_j)\ .\end{aligned}$$

Remark 2.1. Since a^k_j, b^α_β and its first derivatives are continuous on $\overline{\Omega}$, it follows that $(Cg)_\alpha \in L^2(\Omega)$ if $g \in W^{1,2}(\Omega, \mathbb{R}^{2m}) = \bigoplus^{2m} W^{1,2}(\Omega)$. In particular, $(Cf)_\alpha \in L^p(\Omega)$ for any $p \geq 1$.

Let $p > 2n$. For any positive integer k, we have $kp > 2n$; hence by Theorem 5.23 in [1], $W^{k,p}(\Omega)$ is a Banach algebra, i.e. $uv \in W^{k,p}(\Omega)$ for any u and v in $W^{k,p}(\Omega)$. Additionally, using the chain rule, $b^\alpha_\beta \in W^{k,p}(\Omega)$ whenever $f_\alpha \in W^{k,p}(\Omega)$ for each α. Moreover, $(Cf)_\alpha \in W^{k-1,p}(\Omega)$.

For convenience, we let $A^k_l = \sum_j a^k_j a^l_j \in C^\infty(\overline{\Omega})$. In fact, A^k_l is the (k,l)-th entry of the matrix JJ^t. Since $\sum_\beta b^\alpha_\beta b^\beta_\gamma = -\delta_{\alpha,\gamma}$, it follows that

$$\begin{aligned}(\vartheta\overline{\partial}g)_\alpha &= -\sum_j \frac{\partial}{\partial x_j}(\overline{\partial}g)^\alpha_j + \sum_{\beta,j,k} b^\alpha_\beta a^k_j \frac{\partial}{\partial x_k}(\overline{\partial}g)^\beta_j \\ &= -\sum_j \frac{\partial^2 g_\alpha}{\partial x_j \partial x_j} - \sum_{k,l} A^k_l \frac{\partial^2 g_\alpha}{\partial x_k \partial x_l} \\ &\quad +(Cg)_\alpha\end{aligned}$$

when each g_α is of class C^∞. For any $h \in C^1_0(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega (\vartheta\overline{\partial}g)_\alpha h &= \sum_j \int_\Omega (\overline{\partial}g)^\alpha_j \frac{\partial h}{\partial x_j} - \sum_{\beta,j,k} \int_\Omega (\overline{\partial}g)^\beta_j \frac{\partial}{\partial x_k}(b^\alpha_\beta a^k_j h) \\ &= \sum_j \int_\Omega \frac{\partial g_\alpha}{\partial x_j} \frac{\partial h}{\partial x_j} + \sum_{k,l} \int_\Omega \frac{\partial g_\alpha}{\partial x_l} \frac{\partial}{\partial x_k}(A^k_l h) \qquad (2.2) \\ &\quad + \int_\Omega (Cg)_\alpha h\ .\end{aligned}$$

Since $C^\infty(\Omega)$ is dense in $W^{1,2}(\Omega)$, we take a sequence f^ν in $C^\infty(\Omega, \mathbb{R}^{2m})$ which converges to f in $W^{1,2}(\Omega, \mathbb{R}^{2m})$. Then $(\overline{\partial}f^\nu)^\alpha_j$, $(Cf^\nu)_\alpha$ and all the remaining first derivatives of f^ν converge to those of f in $L^2(\Omega)$. Since $(\overline{\partial}f)^\alpha_j = 0$, the sequence of equations (2.2) for f^ν converges to

$$(2.3) \qquad -\sum_j \int_\Omega \frac{\partial f_\alpha}{\partial x_j}\frac{\partial h}{\partial x_j} - \sum_{k,l}\int_\Omega \frac{\partial f_\alpha}{\partial x_l}\frac{\partial}{\partial x_k}(A_l^k h) = \int_\Omega (Cf)_\alpha h$$

for any $h \in C_0^1(\Omega)$.

Take the linear partial differential operator $H = \sum_j \frac{\partial^2}{\partial x_j \partial x_j} + \sum_{k,l} A_l^k \frac{\partial^2}{\partial x_k \partial x_l}$. The symbol of H is $\sum_j \zeta_j^2 + \sum_{k,l} \zeta_k A_l^k \zeta_l = |\zeta|^2 + |J\zeta|^2$. So H is strictly elliptic on Ω with smooth coefficients. Equation (2.3) means that

$$Hf_\alpha = (Cf)_\alpha$$

in the weak sense.

By our assumption, it follows that $(Cf)_\alpha \in L^2(\Omega)$ for each α. By the elliptic regularity theorem (Theorem 8.8 in [5]), we have $f_\alpha \in W^{2,2}_{loc}(\Omega)$ for each α.

Let $p > 2n$. Since $(Cf)_\alpha \in L^p(\Omega)$, by the uniqueness of solutions of the Dirichlet problem for the elliptic equation (Corollary 9.18 in [5]), it follows that $f_\alpha \in W^{2,p}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ for each α. From Remark 2.1, we have $(Cf)_\alpha \in W^{1,p}_{loc}(\Omega)$; hence Theorem 9.19 in [5] implies that $f_\alpha \in W^{3,p}_{loc}(\Omega)$ for each α. Simultaneously, $(Cf)_\alpha \in W^{2,p}_{loc}(\Omega)$. Repeating our argument, we show that $f_\alpha \in W^{k,p}_{loc}(\Omega)$ for each positive integer k. By the Sobolev imbedding theorem, we have

Proposition 2.2. *Let (M^{2n}, J) and (M'^{2m}, J') be C^∞-smooth almost complex manifolds. Any C^1 pseudo-holomorphic mapping from M to M' is of class C^∞.*

For the regularity of pseudo-holomorphic curves ($n = 1$), see Theorem 3.2.2 in [12] and Theorem 2.2.1 in [17].

3. First order estimate of pseudo-holomorphic mappings

In this section, we derive the Cauchy estimate for pseudo-holomorphic mappings. For the first order estimate, it suffices to treat the case of pseudo-holomorphic discs.

Proposition 3.1 (Sikorav [17])**.** *Fix $r, \eta \in (0,1)$. Let W be a bounded domain in $\mathbb{C}^n$. Then there exist positive constants ε and C with the following property:*

If $\phi : \mathbf{D} \to W$ is a differentiable mapping such that

$$\frac{\partial \phi}{\partial \overline{z}} + q(\phi)\frac{\partial \phi}{\partial z} = 0,$$

where $q : W \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$ is of class C^r and $\|q\|_{C^r} \le \varepsilon$, then ϕ is of class C^{1+r} on $\mathbf{D}(1-\eta)$. Moreover,

$$\|\phi\|_{C^{1+r}(\mathbf{D}(1-\eta))} \le C\|\phi\|_{L^\infty}\ .$$

The C^0 and C^k norms for a C^k mapping $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is usually defined by $\|f\|_{C^0(U)} = \sum_{j=1}^m \sup_{x\in U} |f_j(x)|$ and $\|f\|_{C^k(U)} = \sum_{j=1}^m \sum_{|\alpha|\le k} \|D^\alpha f_j\|_{C^0(U)}$, where $|\cdot|$ is a standard Euclidean norm. For $0 < r < 1$, the C^{k+r} (Hölder) norm is defined by

$$\|f\|_{C^{k+r}(U)} = \|f\|_{C^k(U)} + \sum_{j=1}^m \sup_{|\alpha|=k} \sup_{\substack{x\neq y\\ x,y\in U}} \frac{|D^\alpha f_j(x) - D^\alpha f_j(y)|}{|x-y|^r}\ .$$

Note that for a C^1 mapping $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $\|f\|_{C^1(U)}$ is equivalent to

$$\|f\|_{C^0(U)} + \sup_{\substack{v \in TU \\ |v| \le 1}} |df(v)| .$$

Now we present:

Theorem 3.2. *Let $(\Omega, J) \subset \mathbb{R}^{2n}$ and $(\Omega', J') \subset \mathbb{R}^{2m}$ be almost complex domains. For each point $p \in \Omega'$, there is a bounded neighborhood U of p in Ω' such that $\{\|f\|_{C^1(K)} : f \in \mathcal{O}_{(J,J')}(\Omega, U)\}$ is uniformly bounded for any compact subset K of Ω.*

Proof. First, let us study the pseudo-holomorphic discs in Ω'. Applying a linear change of coordinates and a translation of $\mathbb{R}^{2m}$, we may assume that $p = 0$ and J' coincides with the canonical complex structure at 0, i.e. $J'_0 = J_{st}$. Take a neighborhood V of 0 such that $J' + J_{st}$ is invertible on V.

Suppose that $\phi : \mathbf{D} \to V \subset \Omega'$ is a pseudo-holomorphic disc. Then the following equation holds:

$$\frac{\partial \phi}{\partial y} = J'_\phi \frac{\partial \phi}{\partial x} .$$

Since $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}}$ and $\frac{\partial \phi}{\partial y} = J_{st}\left(\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial \bar{z}}\right)$, we have

$$(J'_\phi + J_{st}) \frac{\partial \phi}{\partial \bar{z}} = -(J'_\phi - J_{st}) \frac{\partial \phi}{\partial z} . \tag{3.1}$$

Defining the mapping $q : V \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$ by $q(a) = (J'_a + J_{st})^{-1}(J'_a - J_{st})$, we see that (3.1) can be written as

$$\frac{\partial \phi}{\partial \bar{z}} + q(\phi) \frac{\partial \phi}{\partial z} = 0 .$$

Since V is relatively compact in Ω', q has the same (Hölder) regularity as that of J' on V.

Define the renormalization q_β of q by $q_\beta : \beta^{-1}V = \{\beta^{-1}a : a \in V\} \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^m)$ and $q_\beta(a) = q(\beta a)$ for an arbitrary real number $\beta > 0$. Take a sufficiently small β such that $B(0, 1) \subset \beta^{-1}V$, equivalently $B(0, \beta) \subset V$. Then for fixed $0 < r < 1$, we have

$$\begin{aligned} \|q_\beta\|_{C^r(B(0,1))} &= \|q_\beta\|_{C^0(B(0,1))} + \sup_{\substack{x \ne y \\ x,y \in B(0,1)}} \frac{|q_\beta(x) - q_\beta(y)|}{|x - y|^r} \\ &= \|q\|_{C^0(B(0,\beta))} + \sup_{\substack{x \ne y \\ x,y \in B(0,1)}} \frac{|q(\beta x) - q(\beta y)|}{|\beta x - \beta y|^r} \beta^r \\ &\le \|q\|_{C^0(B(0,\beta))} + \sup_{\substack{x \ne y \\ x,y \in V}} \frac{|q(x) - q(y)|}{|x - y|^r} \beta^r . \end{aligned}$$

Since $q(0) = 0$, it follows that $\|q\|_{C^0(B(0,\beta))} \to 0$ as $\beta \to 0$. For a sufficiently small β, we have that $\|q_\beta\|_{C^r(B(0,1))} < \varepsilon$, where ε is in Proposition 3.1 for the case $W = B(0, 1)$. Now a new mapping $\phi_\beta = \beta^{-1}\phi$ satisfies

$$\frac{\partial \phi_\beta}{\partial \bar{z}} + q_\beta(\phi_\beta) \frac{\partial \phi_\beta}{\partial z} = 0 .$$

Let $U = B(0,\beta)$. By Proposition 3.1, we can deduce that

$$\begin{aligned}\|\phi\|_{C^1(\mathbf{D}(1-\eta))} &\leq \beta\|\phi_\beta\|_{C^{1+r}(\mathbf{D}(1-\eta))}\\ &\leq C\beta\|\phi_\beta\|_{L^\infty}\\ &\leq C\|\phi\|_{L^\infty}\end{aligned}$$

for any $\phi \in \mathcal{O}_{(J_{st},J')}(\mathbf{D},U)$.

By 5.4a in [15], there is a constant $R > 0$ such that for any vector $v \in T\Omega$ based on K with $|v| \leq R$, there is a pseudo-holomorphic disc $\phi : \mathbf{D} \to \Omega$ such that $d\phi(\mathbf{e}) = v$, where $\mathbf{e}$ is an unit vector in $T_0\mathbf{D}$. For any $f \in \mathcal{O}_{(J,J')}(\Omega,U)$, $f\circ\phi : \mathbf{D} \to U$ is pseudo-holomorphic; hence it follows that $|df(v)| = |d(f\circ\phi)(\mathbf{e})| \leq \|d(f\circ\phi)_0\| \leq \|f\circ\phi\|_{C^1(\mathbf{D}(1-\eta))} \leq C\|f\circ\phi\|_{L^\infty} \leq C\|f\|_{C^0}$. Therefore we have

$$\begin{aligned}\|f\|_{C^1(K)} &\sim \|f\|_{C^0(K)} + \sup_{x\in K}\sup_{\substack{v\in T_x\Omega\\ |v|\leq R}}\frac{1}{R}|df_x(v)|\\ &\leq \|f\|_{C^0(\Omega)} + \frac{C}{R}\|f\|_{C^0(\Omega)}\\ &\leq (1+\frac{C}{R})\|f\|_{C^0(\Omega)}\ .\end{aligned}$$

This proves the theorem. □

4. Pseudo-holomorphic jet bundles

In order to prove Theorem 1.2, we need some information about the ∞-jet of a certain family of pseudo-holomorphic mappings at a given point. These can be obtained by jet bundles.

Gauduchon ([4]) has shown that there is a natural almost complex structure in a pseudo-holomorphic 1-jet bundle such that the lifting of the pseudo-holomorphic mapping is also pseudo-holomorphic. In the first two subsections, we follow Gauduchon's work (see chapter 4 in [2] and [4]).

4.1. **Horizontal distribution.** Let $\pi : E \to M$ be a vector bundle with a linear connection ∇. For any point $u \in E_x = \pi^{-1}(x)$, the vertical tangent space T^V_uE at u is a subspace of T_uE whose elements are tangent to E_x. Let $T^VE = \bigcup_{u\in E} T^V_uE$.

Fix any section $\xi \in \Gamma(E)$ with $\xi(x) = u$. For each vector $X \in T_xM$, we define a lifting $\widetilde{X}_u$ in T_uE by

$$\widetilde{X}_u = d\xi_x(X) - \nabla_X\xi,$$

where $\nabla_X\xi \in E_x$ is considered as an element of T^V_uE. This definition of $\widetilde{X}_u$ is independent of the choices for ξ. Therefore, the horizontal subspace H^∇_u at u can be uniquely defined as a lifting subspace of T_xM in T_uE up to the linear connection ∇. We call $H^\nabla = \bigcup_{u\in E} H^\nabla_u$ the *horizontal distribution.* It is easy to check that H^∇ is a smooth distribution and that the following properties hold:

(a) $T_uE = H^\nabla_u \oplus T^V_uE$ at each $u \in E$.

(b) Let $v^\nabla : H^\nabla \oplus T^VE \to T^VE$ be a natural projection (vertical projection). If $Y \in T_uE$ with $d\xi_x(X) = Y$ for some section ξ, then

$$v^\nabla(Y) = \nabla_X\xi\ .$$

(c) The vertical projection v^∇ is also smooth. This means that for any smooth vector field X of TE, $v^\nabla(X)$ is a smooth vector field of T^VE.

(d) Given $Y \in T_uE \setminus T_u^VE$, there is a unique vector $X \in T_xM$ such that $d\xi_x(X) = Y$ for some section ξ. Therefore we have the natural projection from T_uE to T_xM and the canonical decomposition $T_uE \simeq T_{\pi(u)}M \times T_u^VE$.

4.2. **Pseudo-holomorphic 1-jet bundle and its almost complex structure.** Given two smooth (C^∞) almost complex manifolds (M^{2n}, J) and (M'^{2m}, J'), a (J, J')-*holomorphic* (or *pseudo-holomorphic*) 1-*jet bundle* over $M \times M'$ is defined by

$$\mathcal{J}^1_{(J,J')}(M, M') = \bigcup_{(x,y)\in M\times M'} \mathrm{Hom}_{(J_x,J'_y)}(T_xM, T_yM'),$$

where $\mathrm{Hom}_{(J_x,J'_y)}(T_xM, T_yM')$ is the space of (J_x, J'_y)-linear transformations from T_xM to T_yM'. Now $\pi = \pi_1 \times \pi_2 : \mathcal{J}^1_{(J,J')}(M, M') \to M \times M'$ is a vector bundle of rank $2nm$. We will frequently use the notation $\mathcal{J}^1(M, M')$ instead of $\mathcal{J}^1_{(J,J')}(M, M')$ for simplicity.

Choose any linear connection ∇ on $\mathcal{J}^1(M, M')$. We have the canonical identification

$$\begin{aligned} T_u\mathcal{J}^1(M, M') &\simeq T_{\pi_1(u)}M \times T_{\pi_2(u)}M' \times T_u^V\mathcal{J}^1(M, M') \\ &\simeq T_{\pi_1(u)}M \times T_{\pi_2(u)}M' \times \mathrm{Hom}_{(J,J')}(T_{\pi_1(u)}M, T_{\pi_2(u)}M') . \end{aligned}$$

By this, any tangent vector $Y \in T_u\mathcal{J}^1(M, M')$ can be decomposed into

$$Y = (X_1, X_2, v^\nabla(Y)),$$

where:

i) X_1 and X_2 are images of the natural projection of Y into $T_{\pi_1(u)}M$ and $T_{\pi_2(u)}M'$, respectively,

ii) $v^\nabla(Y)$ is considered as an element in $\mathrm{Hom}_{(J,J')}(T_{\pi_1(u)}M, T_{\pi_2(u)}M')$.

Now we can define an almost complex structure J^∇ on $\mathcal{J}^1(M, M')$ depending on ∇ by

$$J^\nabla(Y) = (J_{\pi_1(u)}X_1, J'_{\pi_2(u)}X_2, J'_{\pi_2(u)} \circ v^\nabla(Y)). \tag{4.1}$$

It is easy to see $v^\nabla(J^\nabla(Y)) = J'_{\pi_2(u)}\circ v^\nabla(Y)$; hence J^∇ is well defined. Furthermore, J^∇ is a smooth almost complex structure. Hence $(\mathcal{J}^1(M, M'), J^\nabla)$ is also a smooth almost complex manifold.

Theorem 4.1 (Gauduchon [4])**.** *There is a linear connection ∇ on $\mathcal{J}^1(M, M')$ with following property:*

For any pseudo-holomorphic mapping $f : M \to M'$, its lifting $\mathrm{L}(f) : (M, J) \to (\mathcal{J}^1(M, M'), J^\nabla)$ is also pseudo-holomorphic.

4.3. **Higher order jet bundles.** We can define the k-jet bundles over $M \times M'$ inductively. But we need only the local information, so we shall consider the Euclidean case.

Let $(\Omega, J) \subset \mathbb{R}^{2n}$ and $(\Omega', J') \subset \mathbb{R}^{2m}$ be smooth almost complex domains. Let $(x_1, \ldots, x_{2n})$ and $(w_1, \ldots, w_{2m})$ be the standard coordinate systems for $\mathbb{R}^{2n}$ and $\mathbb{R}^{2m}$, respectively. Assume that

$$\{\partial/\partial x_1, \cdots, \partial/\partial x_n\} \text{ is a complex basis of } T_x\Omega \text{ for each } x \in \Omega. \tag{$*$}$$

Condition $(*)$ means that $\{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ and its images under J_x form a real basis of $T_x\Omega$.

By $(*)$ a (J,J')-linear mapping from $T_x\Omega$ to $T_y\Omega'$ is completely determined by the images of $\{\partial/\partial x_1,\ldots,\partial/\partial x_n\}$; hence $\mathcal{J}^1(\Omega,\Omega')$ is a trivial bundle. From now on, we consider $\mathcal{J}^1(\Omega,\Omega')$ as an open set $\Omega\times\Omega'\times\mathbb{R}^{2nm}$ in $\mathbb{R}^{2(n+m+nm)}$. More precisely, a coordinate mapping is given by

$$\tau=\left(\pi_1(\tau),\pi_2(\tau),\left[dw_\alpha(\tau(\frac{\partial}{\partial x_j}))\right]_{\substack{\alpha=1,\ldots,2m\\ j=1,\ldots,n}}\right)\ . \tag{4.2}$$

The lifting $\mathrm{L}(f)$ of a pseudo-holomorphic mapping f is parameterized by

$$\mathrm{L}(f)(x)=\left(x_1,\ldots,x_{2n},f_1(x),\ldots,f_{2m}(x),\left[\frac{\partial f_\alpha}{\partial x_j}(x)\right]_{\substack{\alpha=1,\ldots,2m\\ j=1,\ldots,n}}\right)\ . \tag{4.3}$$

To compare $\|f\|_{C^l}$ with $\|\mathrm{L}(f)\|_{C^{l-1}}$, we have to consider the partial derivatives of f that are missing in the above expression of $\mathrm{L}(f)(x)$. Solving the system of linear equations $J'_f\circ df=df\circ J$ with respect to $\{\partial f_\alpha/\partial x_j\}_{j>n}$, we have

$$\frac{\partial f_\alpha}{\partial x_j}(x)=\sum_{\beta=1}^{2m}\sum_{k=1}^{n}A_{jk}^{\alpha\beta}(x,f(x))\frac{\partial f_\beta}{\partial x_k}(x)\quad\text{ on }\Omega$$

for $j>n$, where $A_{jk}^{\alpha\beta}$ is a globally defined C^∞-smooth function on $\Omega\times\Omega'$. Therefore, for each compact subset K in Ω and any positive integer l, there is a suitable constant M_l depending on K with

$$\left\|\frac{\partial f_\alpha}{\partial x_j}\right\|_{C^l(K)}\le M_l\sum_{\beta=1}^{2m}\sum_{k=1}^{n}\left\|\frac{\partial f_\beta}{\partial x_k}\right\|_{C^l(K)}$$

for $j>n$. We may deduce that

$$\|f\|_{C^l(K)}\lesssim\|\mathrm{L}(f)\|_{C^{l-1}(K)} \tag{4.4}$$

uniformly for $f\in\mathcal{O}_{(J,J')}(\Omega,\Omega')$.

By the expression (4.3), we also obtain

Proposition 4.2. *Let $f,g\in\mathcal{O}_{(J,J')}(\Omega,\Omega')$ and $\nu\ge 1$. If f and g share the same ν-jet at $p\in\Omega$, then $\mathrm{L}(f)$ and $\mathrm{L}(g)$ share the same $(\nu-1)$-jet at p.*

We now go to the 2-jet.

Take any linear connection ∇_1 on $\mathcal{J}^1(\Omega,\Omega')$. From our assumption $(*)$ about Ω, the pseudo-holomorphic 2-jet bundle over $\Omega\times\Omega'$ defined by

$$\mathcal{J}^2(\Omega,\Omega')=\mathcal{J}^1_{(J,J^{\nabla_1})}(\Omega,\mathcal{J}^1(\Omega,\Omega'))$$

is also trivial. Choosing ∇_ν inductively, we can define a pseudo-holomorphic $(\nu+1)$-jet bundle by

$$\mathcal{J}^{\nu+1}(\Omega,\Omega')=\mathcal{J}^1_{(J,J^{\nabla_\nu})}(\Omega,\mathcal{J}^\nu(\Omega,\Omega'))\ .$$

For any choice of ∇_ν at each step, $\mathcal{J}^\nu(\Omega,\Omega')$ is always trivial.

From now on, we fix a suitable linear connection ∇_ν as in Theorem 4.1 at each step. Then for a pseudo-holomorphic mapping $f:\Omega\to\Omega'$, its lifting $\mathrm{L}^\nu(f)=\mathrm{L}(\mathrm{L}^{\nu-1}(f)):\Omega\to\mathcal{J}^\nu(\Omega,\Omega')$ is always (J,J^{∇_ν})-holomorphic.

Given $f\in\mathcal{O}_{(J,J')}(\Omega,\Omega')$ and $p\in\Omega$, a family of mappings defined by

$$\mathcal{F}_p^\nu(f;\Omega,\Omega')=\{g\in\mathcal{O}_{(J,J')}(\Omega,\Omega'):g\text{ has the same }\nu\text{-jet with }f\text{ at }p\}$$

has the following property.

Theorem 4.3. *Let $(\Omega, J) \subset \mathbb{R}^{2n}$ and $(\Omega', J') \subset \mathbb{R}^{2m}$ be hyperbolic almost complex domains. Assume that Ω satisfies condition $(*)$. For any $f \in \mathcal{O}_{(J,J')}(\Omega, \Omega')$, there is a neighborhood V_ν of p such that $\{\mathrm{L}^\nu(g) : g \in \mathcal{F}_p^{\nu-1}(f; \Omega, \Omega')\}$ is uniformly bounded on V_ν. Moreover, we can find V_ν such that $V_{\nu+1} \subset\subset V_\nu$ for each $\nu = 1, 2, \ldots$.*

Proof. Choose $r > 0$ such that the Kobayashi ball $U = \mathbf{B}_{(\Omega',J')}(f(p), r)$ is a bounded neighborhood of $f(p)$ as in Theorem 3.2. Denote $V = \mathbf{B}_{(\Omega,J)}(p, r)$. Since $\mathcal{F}_p^0(f; \Omega, \Omega') = \{g \in \mathcal{O}_{(J,J')}(\Omega, \Omega') : g(p) = f(p)\}$, we have $g(V) \subset U$ for any $g \in \mathcal{F}_p^0(f; \Omega, \Omega')$. Take any relatively compact neighborhood V_1 of p in V. By Theorem 3.2, $\{\|g\|_{C^1(V_1)} : g \in \mathcal{F}_p^0(f; \Omega, \Omega')\}$ is uniformly bounded so that $\{\mathrm{L}(g) : g \in \mathcal{F}_p^0(f; \Omega, \Omega')\}$ is uniformly bounded on V_1. This proves the case $\nu = 1$.

Since (V, J) and (U, J') are also Kobayashi hyperbolic, Theorem 3 in [11] implies that every bounded domain in $\mathcal{J}^1_{(J,J')}(V, U)$ is hyperbolic with respect to J^{∇_1}. Therefore, we may assume that

$$\bigcup_{g \in \mathcal{F}_p^0(f; \Omega, \Omega')} \mathrm{L}(g)(V_1) \subset \Omega_1,$$

where Ω_1 is a hyperbolic neighborhood of $\mathrm{L}(f)(p)$ in $\mathcal{J}^1_{(J,J')}(V, U)$.

Suppose that our theorem holds for the case $\nu \le \lambda$. Since the pair (V_1, J) and (Ω_1, J^{∇_1}) satisfy the assumption of the theorem, there are neighborhoods $V_1', \ldots, V_\lambda'$ of p in V_1 such that $\{\mathrm{L}^\nu(h) : h \in \mathcal{F}_p^{\nu-1}(\mathrm{L}(f); V_1, \Omega_1)\}$ is uniformly bounded on V_ν' for $\nu = 1, \ldots, \lambda$, and such that $V_\lambda' \subset\subset V_{\lambda-1}' \subset\subset \cdots \subset\subset V_1'$. By Proposition 4.2, we have

$$\mathrm{L}\left(\mathcal{F}_p^\nu(f; \Omega, \Omega')\right) \subset \mathcal{F}_p^{\nu-1}(\mathrm{L}(f); V_1, \Omega_1)$$

for any ν. Therefore $\mathrm{L}^{\nu+1}(g) = \mathrm{L}^\nu(\mathrm{L}(g))$ is uniformly bounded on $V_{\nu+1} = V_\nu'$ for $g \in \mathcal{F}_p^\nu(f; \Omega, \Omega')$ and for $\nu = 1, \ldots, \lambda$. This proves the theorem by the induction hypothesis. □

For this sequence $\{V_\nu\}$ of nested neighborhoods of p, we have

Corollary 4.4. $\{\|g\|_{C^\nu(V_\nu)} : g \in \mathcal{F}_p^{\nu-1}(f; \Omega, \Omega')\}$ *is uniformly bounded.*

Proof. From (4.4), we have

$$\|g\|_{C^\nu(V_\nu)} \lesssim \|\mathrm{L}(g)\|_{C^{\nu-1}(V_\nu)} \lesssim \cdots \lesssim \|\mathrm{L}^\nu(g)\|_{C^0(V_\nu)}$$

uniformly for $g \in \mathcal{O}_{(J,J')}(\Omega, \Omega')$. When $g \in \mathcal{F}_p^{\nu-1}(f; \Omega, \Omega')$, the last term of this inequality is bounded by Theorem 4.3. □

5. Proof of Theorem 1.2

Let (M, J) be a connected hyperbolic almost complex manifold of class C^∞. Suppose that there is a pseudo-holomorphic self-mapping $f : M \to M$ with $f(p) = p$ and $df_p = \mathrm{Id}$ for some $p \in M$. From Proposition 2.2, f is of class C^∞ and we can compare all partial derivatives of f with those of the identity mapping. To prove that f is the identity, we need the unique continuation property for pseudo-holomorphic mappings.

Proposition 5.1. *Let (M, J) and (M', J') be smooth almost complex manifolds. Moreover M is connected. Suppose that two pseudo-holomorphic mappings $f, g : M \to M'$ share the same ∞-jet at some point in M. Then $f \equiv g$ on M.*

Proof. It is sufficient to prove that $A = \{p \in M : f$ and g share the same ∞-jet at $p\}$ is open. Then our assertion follows, since A is open, closed and nonempty set.

Suppose that $p \in A$. There is a neighborhood U_p of p such that any point q in U_p can be joined to p by a single pseudo-holomorphic disc ([6] and [10]). Take any q in U_p and suppose that there is a pseudo-holomorphic disc $\phi : \mathbf{D} \to M$ with $\phi(0) = p$ and $\phi(1/2) = q$. Since $p \in A$, the two pseudo-holomorphic discs $f \circ \phi,\ g \circ \phi : \mathbf{D} \to M'$ share the same ∞-jet at 0. By the unique continuation property of pseudo-holomorphic curves (see [3] and [12]), it holds that $f \circ \phi \equiv g \circ \phi$. Furthermore $f(q) = g(q)$. Since q is an arbitrary point in U_p, we have $f|_{U_p} \equiv g|_{U_p}$. Hence $p \in U_p \subset A$, and A is open. This proves the proposition. □

By Proposition 5.1, it is sufficient to prove that $D^\alpha f_j(p) = 0$ for any j and any multi-indices $|\alpha| \geq 2$. Then f has the same ∞-jet with the identity mapping. Therefore f is the identity mapping.

Choose a local coordinate system $\varphi : (V, 0) \to (M, p)$ about p with $\varphi(V) \subset\subset M$. Since the Kobayashi distance function $d_{(M,J)}$ is continuous, we can take a positive real number $r < \min_{q \in \partial\varphi(V)} d_{(M,J)}(p, q)$. Then the Kobayashi ball $\mathbf{B}_{(M,J)}(p, r)$ is contained in $\varphi(V)$. By the distance-decreasing property of the Kobayashi distance, we have $f(\mathbf{B}_{(M,J)}(p, r)) \subset \mathbf{B}_{(M,J)}(p, r)$ for all r. Now we identify $p = 0$, $\varphi(V) = V$ is a bounded domain in $\mathbb{R}^{2n}$ and $J = \varphi^* J = (d\varphi)^{-1} \circ J \circ d\varphi$ is an induced almost complex structure on V. For sufficiently small r we may assume that $(U = \varphi^{-1}(\mathbf{B}_{(M,J)}(p, r)), J)$ satisfies condition $(*)$ in Section 4.

Consider an iterated family $\{f^m = f \circ f^{m-1}\}_{m=1,2,\ldots}$ of f. Note that $f|_U$ is in $\mathcal{O}_{(J,J)}(U, U)$, so is $f^m|_U$. Now we have

Proposition 5.2. *$(D^\alpha(f^m)_j)(0) = m(D^\alpha f_j)(0)$ for $|\alpha| = 2$.*

Suppose that $D^\alpha f_j(0) = 0$ for any $2 \leq |\alpha| < \nu$ and $j = 1, \ldots, 2n$. Then $(D^\beta(f^m)_j)(0) = m(D^\beta f_j)(0)$ for each $|\beta| = \nu$ and each j.

Proof. Since $d(f^m)_0 = (df_0)^m = \mathrm{Id}$, we have

$$\frac{\partial (f^m)_j}{\partial x_k}(0) = \delta_{j,k} \tag{5.1}$$

for $m = 1, 2, \ldots$.

Let $D^\alpha = \frac{\partial^2}{\partial x_{\alpha_1} \partial x_{\alpha_2}}$. Since $(f^m)_j = f_j \circ f^{m-1}$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial x_{\alpha_1} \partial x_{\alpha_2}}(f^m)_j(0) &= \frac{\partial}{\partial x_{\alpha_1}} \left(\sum_{k=1}^{2n} \frac{\partial f_j}{\partial x_k}(f^{m-1}(x)) \frac{\partial (f^{m-1})_k}{\partial x_{\alpha_2}}(x) \right)(0) \\
&= \sum_{k,l=1}^{2n} \frac{\partial^2 f_j}{\partial x_l \partial x_k}(f^{m-1}(0)) \frac{\partial (f^{m-1})_l}{\partial x_{\alpha_1}}(0) \frac{\partial (f^{m-1})_k}{\partial x_{\alpha_2}}(0) \\
&\quad + \sum_{k=1}^{2n} \frac{\partial f_j}{\partial x_k}(f^{m-1}(0)) \frac{\partial^2 (f^{m-1})_k}{\partial x_{\alpha_1} \partial x_{\alpha_2}}(0) \\
&= \frac{\partial^2 f_j}{\partial x_{\alpha_1} \partial x_{\alpha_2}}(0) + \frac{\partial^2 (f^{m-1})_j}{\partial x_{\alpha_1} \partial x_{\alpha_2}}(0),
\end{aligned}$$

where the last equality follows by (5.1). This equation proves the case of $|\alpha| = 2$ by induction.

Suppose that $D^\alpha f_j(0) = 0$ for any $2 \le |\alpha| < \nu$ and $j = 1, \dots, 2n$. Let $|\beta| = \nu$ and $D^\beta = \frac{\partial^\nu}{\partial x_{\beta_1} \cdots \partial x_{\beta_\nu}}$. From (5.1), we obtain

$$\begin{aligned}
&D^\beta (f^m)_j(0) \\
&= \sum_{\gamma_1,\dots,\gamma_\nu=1}^{2n} \frac{\partial^\nu f_j}{\partial x_{\gamma_1} \cdots \partial x_{\gamma_\nu}}(f^{m-1}(0)) \frac{\partial (f^{m-1})_{\gamma_1}}{\partial x_{\beta_1}}(0) \cdots \frac{\partial (f^{m-1})_{\gamma_\nu}}{\partial x_{\beta_\nu}}(0) \\
&\quad + (\text{terms which contain } D^\alpha f_j \text{ for } 2 \le |\alpha| < \nu) \\
&\quad + \sum_{k=1}^{2n} \frac{\partial f_j}{\partial x_k}(f^{m-1}(0)) \frac{\partial^\nu (f^{m-1})_k}{\partial x_{\beta_1} \cdots \partial x_{\beta_\nu}}(0) \\
&= \frac{\partial^\nu f_j}{\partial x_{\beta_1} \cdots \partial x_{\beta_\nu}}(0) + \frac{\partial^\nu (f^{m-1})_j}{\partial x_{\beta_1} \cdots \partial x_{\beta_\nu}}(0) \\
&= D^\beta f_j(0) + D^\beta (f^{m-1})_j(0)\ .
\end{aligned}$$

This proves the proposition. □

We are now ready to complete the proof of Theorem 1.2.

Suppose that $D^\alpha f_j(0) \neq 0$ for some multi-index α with $|\alpha| = 2$ and some j. By Proposition 5.2, we have $|(D^\alpha (f^m)_j)(0)| = m|(D^\alpha f_j)(0)| \to \infty$ as $m \to \infty$. Since $f^m(0) = f(0) = 0$ and $d(f^m)_0 = df_0 = \mathrm{Id}$, we have $f^m \in \mathcal{F}_0^1(f; U, U)$ for each m. Corollary 4.4 implies that $\{|(D^\alpha (f^m)_j)(0)|\}_{m=1,2,\dots}$ must be bounded. Therefore it follows that $D^\alpha f_j(0) = 0$ for each $|\alpha| = 2$ and j.

Inductively let us assume that $D^\beta f_j(0) \neq 0$ and $D^\alpha f_k(0) = 0$ for $2 \le |\alpha| < |\beta| = \nu$ and $k = 1, \dots, 2n$. Proposition 5.2 implies that $(D^\alpha (f^m)_k)(0) = m(D^\alpha f_k)(0) = 0$ for $2 \le |\alpha| < \nu$ and $k = 1, \dots, 2n$. Hence it follows that $f^m \in \mathcal{F}_0^{\nu-1}(f; U, U)$. But Proposition 5.2 also means that $|(D^\beta (f^m)_j)(0)| = m|(D^\beta f_j)(0)| \to \infty$ as $m \to \infty$. It is a contradiction to Corollary 4.4. Therefore we have $D^\alpha f_j(0) = 0$ for any $|\alpha| \ge 2$.

Consequently f has same ∞-jet with the identity mapping at 0. This proves Theorem 1.2. □

Acknowledgment

This work is part of the author's dissertation for his doctoral degree at the Pohang University of Science and Technology. He expresses his gratitude to his advisor Kang-Tae Kim for guidance and encouragement.

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