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SIGNATURE INVARIANTS OF COVERING LINKS

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Abstract. We apply the theory of signature invariants of links in rational homology spheres to covering links of homology boundary links. From patterns and Seifert matrices of homology boundary links, we derive an explicit formula to compute signature invariants of their covering links. Using the formula, we produce fused boundary links that are positive mutants of ribbon links but are not concordant to boundary links. We also show that for any finite collection of patterns, there are homology boundary links that are not concordant to any homology boundary links admitting a pattern in the collection.

1. Introduction

For a link L, the pre-image (of a sublink) of L in a finite cyclic cover of the ambient space branched along a component of L is called a *covering link of L*. In the work of Cochran and Orr [6, 7], it was observed that concordances of links in spheres can be studied via their covering links due to the following facts: If L is a link in a $\mathbf{Z}_p$-homology sphere for some prime p, so is a p^a-fold covering link [2], and corresponding covering links of concordant links are concordant as links in $\mathbf{Z}_p$-homology spheres via a concordance obtained by a similar covering construction. Using the Blanchfield form of covering links, they proved the long-standing conjecture that there are links which are not concordant to boundary links in [6, 7]. Milnor's $\bar{\mu}$-invariants [22, 23] are also generalized for covering links in [9].

In this paper, we view covering links as links in rational homology spheres, and utilize the signature invariant developed by the authors in [4] to study covering links. For homology boundary links, we develop a new systematic method to compute signature jump functions of covering links. Recall that a link L with m components is called a *homology boundary link* if there exists an epimorphism of the fundamental group of the complement of L onto the free group of rank m. An m-tuple $r = (r_1, \ldots, r_m)$ is called a *pattern* for L if r_i is the image of the i-th meridian under the epimorphism [5]. A homology boundary link admits a system of "singular" Seifert surfaces, and Seifert pairings and Seifert matrices on singular Seifert surfaces are defined as in [8]. In Section 2, we prove an explicit formula to compute Seifert matrices and the signature jump functions of covering links of a homology boundary link from its Seifert matrix and pattern (see Theorem 2.1).

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In order to prove the formula, we construct covering links and their generalized Seifert surfaces in the sense of [4] using cut and paste arguments, and compute Seifert matrices and signature jump functions from the generalized Seifert surfaces. The only algebraic tool needed is linear algebra of matrices over the complex field. So our approach is geometric and elementary in contrast to that in [8] where invariants for covering links are obtained from the Blanchfield duality and their invariance under concordance is shown using heavy machinery of homological algebra.

There is another known way to compute Seifert matrices and signature jump functions of covering links. For many concrete examples, the ambient space of a covering link can be calculated as a surgery diagram using the method of [1], and then we can construct a Seifert surface in the surgery diagram and compute a Seifert matrix as illustrated in an example of [4]. Comparing with this, our approach is more practical in the sense that we can compute invariants of a covering link of a given link directly from the given link using a formula without appealing to any diagram.

In Section 3, we apply the above results on covering links to study link concordance in spheres as influenced by [6, 7]. First we study links which are not concordant to boundary links using the method of Section 2. Recall that a link is called a *boundary link* if its components bound disjoint Seifert surfaces. Because boundary link concordance classes can be algebraically classified [10, 18, 24], it has been the center of interest whether all $(2q-1)$-links (with vanishing $\bar{\mu}$-invariants if $q=1$) are concordant to boundary links. The first counterexamples were given by Cochran and Orr [6, 7] as mentioned before. Gilmer and Livingston [11], and Levine [21] showed the same result using different techniques like Casson–Gordon invariants and η-invariants, respectively. We offer another method to detect links not concordant to boundary links using signature jump functions of covering links. Roughly speaking, it is shown that signature jump functions of covering links of links which are concordant to boundary links must have period 2π. Using this, we show that if a homology boundary link has a specific form of pattern and Seifert matrix, then it is not concordant to boundary links (see Theorem 3.2).

Furthermore, we show that there are 1-dimensional links with vanishing $\bar{\mu}$-invariants which are positive mutants of ribbon links but not concordant to boundary links (see Theorem 3.3). For a link L and a 3-ball B in S^3 such that L and ∂B transversally meet at exactly 4 points, the link obtained by pasting $(B, L\cap B)$ and $(S^3-\operatorname{int} B, L-\operatorname{int} B)$ along an orientation preserving involution on $(\partial B, L\cap\partial B)$ whose fixed points are disjoint from $L\cap\partial B$ is called a *mutant* of L. If L is oriented and the orientation of L is preserved by the mutation, then it is called a *positive mutant*. Many link invariants fail to distinguish links from their mutants. It is known that mutation preserves link invariants like Alexander, Jones, Kauffman and HOMFLY polynomials, and positive mutation preserves link signatures and S-equivalence classes of knots. The problem to distinguish links from positive mutants *up to concordance* is even subtler. The only known result is that Casson-Gordon invariants are effective to distinguish some knots from their positive mutants up to concordance, due to Kirk and Livingston [16, 17]. Almost nothing is known about the effect by (positive) mutation on link concordance classes beyond knot concordance. Our result says that both the set of slice links and the set of links concordant to boundary links are not closed under positive mutations. We remark that our result can also be viewed as a generalization of the result of [13] where it was shown that there is a boundary link with a mutant which is not a (homology) boundary link. However, since both the link and its mutant in [13] are ribbon links, it says nothing up to concordance.

Following techniques of the classification of boundary link concordance classes using Seifert matrices [18], the appropriate concordance classes of homology boundary links with a given pattern are classified by Cochran and Orr [8]. Thus it is now more natural to ask whether all links (with vanishing $\bar{\mu}$-invariants if $q = 1$) are concordant to homology boundary links, instead of boundary links. As a partial answer, we show the following result by investigating signatures of covering links of homology boundary links.

Theorem 1.1. *For any finite collection of patterns, there exist infinitely many homology boundary links which are never concordant to any homology boundary link admitting a pattern in the given collection.*

Note that every pattern is realized by a ribbon link [5]. Combined with our result, it can be seen that the variety of patterns which arise in a concordance class of a homology boundary link depends heavily on the choice of the concordance classes.

2. Seifert matrices of covering links

In this section we derive formulae to compute Seifert matrices and signature jump functions (of unions of parallel copies of components) of covering links of homology boundary links. It seems natural to expect such formulae. First, Seifert matrices together with patterns have enough information to classify the appropriate concordance classes of homology boundary links [8]. Since the signature jump function is invariant under link concordance [4], it is expected that signatures can be calculated from Seifert matrices and patterns. Second, since the Blanchfield form is determined by a Seifert matrix [15, 12, 8] and the Blanchfield form of a covering link of a link L is the image of the Blanchfield form of L under a transfer homomorphism [7, 8], it is expected that a Seifert matrix of a covering link of L can also be obtained from a Seifert matrix of L. In this sense, our formula for Seifert matrices is analogous to the transfer homomorphism for Blanchfield forms. We remark that no explicit formula for the latter is known.

Throughout this paper, we consider ordered and oriented links only. We use the following notation introduced in [4] for parallel copies. For a framed submanifold M in an ambient space and an n-tuple $\alpha = (s_1, \dots, s_n)$ with $s_i = \pm 1$, let $i_\alpha M$ be the union of n parallel copies of M, where the i-th copy is oriented according to the sign of s_i, and let n_α be the sum of s_i. For a nonzero integer r, let $i_r M$ be the union of $|r|$ parallel copies of M oriented according to the sign of r.

We will consider only two component links to simplify notation, although the arguments of this section can also be applied for links with more than two components. Suppose that L is a $(2q-1)$-dimensional homology boundary link with components J and K in a $\mathbf{Z}_p$-homology sphere Σ. Let $\{E, F\}$ be a system of singular Seifert surfaces properly embedded in the exterior E_L such that ∂E, ∂F are homologous to J, K in a tubular neighborhood of L, respectively. An epimorphism from $\pi_1(E_L)$ onto the free group on x and y is obtained by a Thom–Pontryagin construction on $E \cup F$. By choosing meridians ν and μ based at a fixed basepoint outside $E \cup F$, a pattern $r = (v, w)$ is determined which satisfies $v \equiv x$ and $w \equiv y$ modulo commutators.

Let p be a prime and $d = p^a$ for some positive integer a. Let $\tilde{\Sigma}$ be the d-fold cyclic cover of Σ branched along J, and t a generator of covering transformations. Fixing a basepoint of $\tilde{\Sigma}$, the lift of μ based at the basepoint is a meridian of a component $\tilde{K}$ of the pre-image of K. Then the union $\bigcup_{k=0}^{d-1} t^k\tilde{K}$ is a covering link of L. We consider the link $\tilde{L} = \bigcup i_{r_k} t^k\tilde{K}$, where the parallel copies are taken with respect to the framing induced by E and F. We will compute the signature jump function $\delta_{\tilde{L}}(\theta)$ defined in [4] from the given data r and $\{E, F\}$.

We will construct $\tilde{\Sigma}$ using well-known cut and paste arguments as in [1, 3, 12, 14], and construct a Seifert surface of $\tilde{L}$ by taking parallel copies of lifts of F. Denote the pre-image of J in $\tilde{\Sigma}$ by $\tilde{J}$. Choose a smaller tubular neighborhood V of L in $\Sigma - E_L$. We can cancel out boundary components of E with opposite orientations by attaching to E annuli properly embedded in $\Sigma - \operatorname{int}(E_L \cup V)$, and then we obtain a proper submanifold N in $\Sigma - \operatorname{int} V$ such that ∂N is a single parallel of J on ∂V. Removing from Σ the interior of the component of V containing J, we obtain an exterior E_J. Choose a bicollar $N \times [-1, 1]$ in E_J so that $N \times 1$ is a translation of N along the positive normal direction. For $k = 0, \dots, d-1$, let $t^k\tilde{X}$ be a copy of $X = E_J - N \times (-1, 1)$ and $g_\pm^k \colon N \to t^k\tilde{X}$ be a copy of the inclusions $g_\pm \colon N \to N \times \{\pm 1\} \subset X$. Then the exterior $E_{\tilde{J}}$ of $\tilde{J}$ in $\tilde{\Sigma}$ is homeomorphic to the quotient space

$$\Big(\bigcup_{k=0}^{d-1} t^k\tilde{X}\Big)\Big/\sim$$

where $g_+^k(z)$ and $g_-^{k+1}(z)$ (indices are modulo d) are identified for $z \in N$. $\tilde{\Sigma}$ is obtained by gluing $E_{\tilde{J}}$ and $S^q \times D^2$ along boundaries. We remark that we can construct $E_{\tilde{J}}$ using E instead of N. The reason why we use N is that there is a duality isomorphism between $H_q(X; \mathbf{Q}) \cong H_q(E_J - N; \mathbf{Q})$ and $H_q(N; \mathbf{Q})$ for any q. This isomorphism will be needed later and is not established for $q = 1$ if we use E instead of N.

Let $t^k\tilde{N} = g_+^k(N) \subset E_{\tilde{J}}$, and denote the lift of F in $t^k\tilde{X}$ by $t^k\tilde{F}$. Then

$$\partial(t^k\tilde{F}) = \Big(\bigcup_{l=0}^{d-1} i_{\alpha_{kl}} t^l\tilde{K}\Big) \cup i_\alpha\tilde{J}$$

for some tuples α and α_{kl}. Obviously, $n_\alpha = 0$. $n_{\alpha_{kl}}$ is determined by the pattern as follows. Since $w \equiv y$ modulo commutators, we can write $w = \prod_i x^{a_i}y^{b_i}x^{-a_i}$ where $b_i = \pm 1$ and $\sum_i b_i = 1$. Let $c_n(r)$ be the sum of b_i over all i such that $a_i = n$. If we travel along the lift of μ which is a meridian of $\tilde{K}$ in $\tilde{\Sigma}$, a $\pm$-intersection with $t^{a_i}\tilde{F}$ occurs for each $x^{a_i}y^{\pm 1}x^{-a_i}$ factor in w. From this observation, $n_{\alpha_{kl}}$ is the sum of $c_n(r)$ over all n satisfying $n \equiv k - l \bmod d$. We remark that for any pattern r, all but finitely many $c_n(r)$ vanish, and $\sum_n c_n(r) = 1$.

The following lemma implies that for any $r_0, \dots, r_{d-1}$, the system of d equations

$$\sum_{k=0}^{d-1} n_{\alpha_{kl}} x_k = r_l \quad (l = 0, \dots, d-1)$$

has a unique solution (x_k) over $\mathbf{Q}$.

Lemma 2.1. *If n is a prime power and $c_1, \dots, c_n$ are integers such that $c_1 + \cdots + c_n = 1$, then*

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{bmatrix}$$

is a nonsingular matrix.

Proof. Let $n = p^a$ and p be a prime. First we observe that the matrix has a symmetry in the sense that it is invariant under the $\mathbf{Z}_n$-action which cyclically shifts rows and columns.

We expand the determinant as a sum over all permutations of $\{1, \dots, n\}$, and investigate when a particular monomial, to say, $m = c_1^a c_{i_1}^{a_1} \cdots c_{i_k}^{a_k}$ $(1 < i_1 < \cdots < i_k)$ appears as a summand. Let X be the set of all subsets of $\{1, \dots, n\}$. The action on $\{1, \dots, n\}$ by $\mathbf{Z}_n$ induces on X in an obvious way. For any element x of X with cardinality a, let P_x be the set of permutations π such that x is the fixed point set of π and the product of the $(i, \pi(i))$-th entries is equal to m. If two elements x and y in X are in the same orbit, then the action induces a bijection between P_x and P_y which preserves the signs of permutations, by the symmetry. Hence the coefficient of m in the determinant is an integral linear combination of the cardinalities of orbits. If $0 < a < n$, the cardinality of an orbit is a multiple of p, and so is the coefficient of m. By the symmetry again, the same argument works when we replace c_1 by any c_i, and this shows that coefficients of all monomials except c_i^n are multiples of p. Therefore, the determinant is congruent to $c_1^n + \cdots + c_n^n \equiv c_1 + \cdots + c_n \equiv 1$ modulo p. $\square$

Let s be a common multiple of denominators of x_k and let

$$M = \bigcup_{k=0}^{p-1} i_{sx_k} t^k \tilde{F}.$$

Then we have

$$\partial M = \Big(\bigcup_{k,l} i_{sx_k} i_{\alpha_{kl}} t^l \tilde{K} \Big) \cup \Big(\bigcup_{k} i_{sx_k} i_{\alpha_k} \tilde{J} \Big) = \Big(\bigcup_{l} i_{\beta_l} t^l \tilde{K} \Big) \cup i_\beta \tilde{J}$$

where β_l, β are tuples such that $n_{\beta_l} = \sum_k s x_k n_{\alpha_{kl}} = s r_l$, $n_\beta = 0$. By attaching annuli to M in a tubular neighborhood of $(\bigcup t^k \tilde{K}) \cup \tilde{J}$ to cancel out unnecessary boundary components, we obtain a submanifold M' with boundary $i_s \tilde{L}$ and we can compute $\delta_{\tilde{L}}(\theta)$ from a Seifert matrix of M'. For $q > 1$, $\delta_{\tilde{L}}(\theta)$ can be computed from a Seifert matrix P of M since $H_q(M) \cong H_q(M')$.

For $q = 1$, we need additional arguments. Let $S\colon H_1(M') \times H_1(M') \to \mathbf{Q}$ be the Seifert pairing of M'. For a manifold V, denote the cokernel of $H_i(\partial V) \to H_i(V)$ by $\bar{H}_i(V)$. Then we have $H_1(M') \cong \bar{H}_1(M) \oplus \mathbf{Z}^{2n} \oplus \mathbf{Z}^m$, where the $\mathbf{Z}^{2n}$ factor is generated by cores of attached annuli and their dual loops, and the $\mathbf{Z}^m$ factor is generated by boundary parallel loops. We will show that S induces a well-defined "Seifert pairing" on $\bar{H}_1(M)$ and the $\mathbf{Z}^{2n} \oplus \mathbf{Z}^m$ factor has no contribution to the signature *jump* function of S. Thus $\delta_{\tilde{L}}(\theta)$ can be computed from a Seifert matrix P defined on $\bar{H}_1(M)$. Hence we can unify notation for any q by letting P be a Seifert matrix on $\bar{H}_q(M)$ and we have $\delta_{\tilde{L}}(\theta) = \delta_P^q(\theta/s)$.

Our assertion for $q = 1$ is shown as follows. We choose generators $\{c_i, d_i\}$ and $\{e_i\}$ of $\mathbf{Z}^{2n}$ and $\mathbf{Z}^m$ factor, respectively, where c_i is the core of an attached annulus, d_i is a curve on M' whose intersection number with c_j is δ_{ij} (Kronecker's delta symbol), and e_i is a boundary component of M. We will show the linking number of a loop on M and a boundary component c of M is zero. We may assume $c = t^j\tilde{K}$ or $\tilde{J}$ since c is homologous to one of them in $\tilde{\Sigma} - \operatorname{int} M$. For any $j = 0, \ldots, d-1$, the equation $\sum_k n_{\alpha_{kl}} x_k = \delta_{jl}$ has a solution by Lemma 2.1, and so we can construct a surface in $\tilde{\Sigma}$ whose boundary is homologous to $i_a t^j \tilde{K}$ for some $a > 0$ by taking parallel copies of $t^k\tilde{F}$ as before. By attaching annuli, we obtain a surface that is disjoint to M and bounded by $i_a t^j \tilde{K}$. Therefore, the linking number of $t^j\tilde{K}$ and any loop on M is zero. Similarly, the linking number of $\tilde{J}$ and any loop on M is zero since we can construct a surface which is bounded by $\tilde{J}$ and disjoint to M by attaching annuli to $\tilde{N}$. Since c_i and e_i are homologous to boundary components of M and M induces 0-linking framings of boundary components, the Seifert pairing S vanishes on the pairs (c_i, x), (x, c_i), (e_i, x), (x, e_i), (c_i, e_j), (e_i, c_j), (c_i, c_j), and (e_i, e_j) for any x in $H_1(M)$. By the choice of c_i and d_j, $S(c_i, d_i) - S(d_i, c_i) = \delta_{ij}$. From the observations, the usual Seifert pairing determines a well-defined "Seifert pairing" on $\bar{H}_1(M)$, and furthermore the Seifert matrix Q over the chosen basis of $H_1(M')$ is given by

$$Q = \left[\begin{array}{c|c|c|c} P & 0 & * & 0 \\ \hline 0 & 0 & R^T + I & 0 \\ \hline * & R & * & * \\ \hline 0 & 0 & * & 0 \end{array}\right]$$

where P is a Seifert matrix defined on $\bar{H}_1(M)$, and R represents the Seifert pairing between bases $\{d_i\}$ and $\{c_i\}$. In order to compute $\sigma_Q^+(\phi)$, we consider a complex hermitian matrix

$$\frac{wQ - Q^T}{w-1} = \left[\begin{array}{c|c|c|c} \dfrac{wP - P^T}{w-1} & 0 & * & 0 \\ \hline 0 & 0 & R^T + \dfrac{w}{w-1} I & 0 \\ \hline * & R - \dfrac{1}{w-1} I & * & * \\ \hline 0 & 0 & * & 0 \end{array}\right]$$

for a unimodular complex number w. The submatrix $R - (w-1)^{-1}I$ can be viewed as a matrix over the ring of polynomials in $z = (w-1)^{-1}$. Since the determinants of the upper-left square submatrices of $R - (w-1)^{-1}I$ are nonzero polynomials in z, the pivots used in the Gauss–Jordan elimination process are nonzero rational functions in z and hence $R - (w-1)^{-1}I$ can be transformed into a nonsingular diagonal matrix by row operations on $(wQ - Q^T)/(w-1)$ if w is not a zero of the denominators and the numerators of the pivots. Since $(wQ - Q^T)/(w-1)$ is hermitian, the submatrix $R^T + w(w-1)^{-1}I$ is transformed into a nonsingular diagonal matrix by corresponding column operations. Note that this can be performed all but finitely many w, and it does not alter vanishing blocks of $(wQ - Q^T)/(w-1)$. By further row and column operations on $(wQ-Q^T)/(w-1)$, all outer blocks except the top-left block are cleared and we eventually obtain the block sum of $(wP-P^T)/(w-1)$, a nonsingular null-cobordant matrix and a zero matrix. Therefore, $\sigma^+_Q(\phi)=\sigma^+_P(\phi)$ on a dense subset of $\mathbf{R}$. This shows our assertion for $q=1$.

Now we need to compute the Seifert matrix P defined on $\bar{H}_q(M;\mathbf{Q})$. (Note that if $q>1$, $\bar{H}_q(-)$ is identified with $H_q(-)$ and P is the usual Seifert matrix of M.) P is obtained by duplicating rows and columns of a Seifert matrix defined on $\bar{H}_q(\bigcup t^k\tilde{F};\mathbf{Q})\cong\bigoplus\bar{H}_q(F;\mathbf{Q})$, which we will compute.

Let x and y be elements of $\bar{H}_q(F;\mathbf{Q})$ and let a and b be q-cycles on F which represent the image of x and y under a fixed splitting map $\phi\colon\bar{H}_q(F;\mathbf{Q})\to H_q(F;\mathbf{Q})$, respectively. We will compute the linking number of lifts $t^k\tilde{a}^+$ and $\tilde{b}$ in $\tilde{\Sigma}$, where $\tilde{z}$ denotes the unique lift of z in $\tilde{X}$ for a chain z in X. A Seifert pairing on $\bar{H}_q(N\cup F;\mathbf{Q})\cong H_q(N;\mathbf{Q})\oplus\bar{H}_q(F;\mathbf{Q})$ is induced by ϕ and the usual Seifert pairing on $H_q(N\cup F)$. Fix basis of $H_q(N;\mathbf{Q})$ and $\bar{H}_q(F;\mathbf{Q})$, and let $\left[\begin{smallmatrix}A & B\\ \epsilon B^T & C\end{smallmatrix}\right]$ be a Seifert matrix defined on $H_q(N;\mathbf{Q})\oplus\bar{H}_q(F;\mathbf{Q})$ with respect to the basis as in [18, 8]. By duality, we have $H_q(N;\mathbf{Q})\cong H_q(X;\mathbf{Q})$. $(g_+)_*,(g_-)_*\colon H_q(N;\mathbf{Q})\to H_q(X;\mathbf{Q})$ and the composition of ϕ and $g_*\colon H_q(F;\mathbf{Q})\to H_q(X;\mathbf{Q})$ are represented by A, ϵA^T and B, respectively.

We will find q-cycles $z_0,\dots,z_{d-1}$ on N and $(q+1)$-chains $u_1,\dots,u_{d-1}$ in X such that

$$\begin{aligned}
g(b)+g_+(z_0)-g_-(z_1)&=\partial u_0\\
g_+(z_1)-g_-(z_2)&=\partial u_1\\
&\vdots\\
g_+(z_{d-2})-g_-(z_{d-1})&=\partial u_{d-2}\\
g_+(z_{d-1})-g_-(z_0)&=\partial u_{d-1}.
\end{aligned}$$

Once finding z_i and u_i, we obtain a chain $\tilde{u}=\bigcup t^k\tilde{u}_k$ in $\tilde{\Sigma}$ such that $\partial\tilde{u}=\tilde{b}$. Then we can compute the linking number of lifts of a and b as follows:

$$\begin{aligned}
\mathrm{lk}_{\tilde{\Sigma}}(t^k\tilde{a}^+,\tilde{b})&=t^k\tilde{a}^+\cdot\tilde{u}=a^+\cdot u_k\\
&=\mathrm{lk}_{\Sigma}(a^+,\partial u_k)\\
&=\begin{cases}\mathrm{lk}_{\Sigma}(a^+,b)+\mathrm{lk}_{\Sigma}(a^+,z_0)-\mathrm{lk}_{\Sigma}(a^+,z_1), & k=0,\\ \mathrm{lk}_{\Sigma}(a^+,z_k)-\mathrm{lk}_{\Sigma}(a^+,z_{k+1}), & 1\le k\le d-1.\end{cases}
\end{aligned}$$

Viewing x, y, z_k as column vectors representing elements of appropriate $\mathbf{Q}$-homology groups, the above system of equations becomes

$$\begin{bmatrix}A & -\epsilon A^T & & & \\ & A & -\epsilon A^T & & \\ & & \ddots & \ddots & \\ & & & A & -\epsilon A^T\\ -\epsilon A^T & & & & A\end{bmatrix}\begin{bmatrix}z_0\\ z_1\\ \vdots\\ z_{d-2}\\ z_{d-1}\end{bmatrix}=\begin{bmatrix}-By\\ 0\\ \vdots\\ 0\\ 0\end{bmatrix}$$

in $H_q(X;\mathbf{Q})$. Since N has one boundary component, $A-\epsilon A^T$ is nonsingular. By multiplying $(A-\epsilon A^T)^{-1}$ on the left of each row, it becomes

$$\begin{bmatrix} \Gamma & I-\Gamma & & & \\ & \Gamma & I-\Gamma & & \\ & & \ddots & \ddots & \\ & & & \Gamma & I-\Gamma \\ I-\Gamma & & & & \Gamma \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{d-2} \\ z_{d-1} \end{bmatrix} = \begin{bmatrix} -(A-\epsilon A^T)^{-1}By \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

where $\Gamma = (A-\epsilon A^T)^{-1}A$ and I is the identity matrix. Since $\Gamma^d - (\Gamma - I)^d$ is a presentation matrix of $H_q(\tilde\Sigma)$ (see [25]) and $\tilde\Sigma$ is a rational homology sphere, $\Gamma^d-(\Gamma-I)^d$ is invertible and a unique solution (z_k) exists. It is easy to check that

$$z_k = \begin{cases} -\dfrac{\Gamma^{d-1}}{\Gamma^d-(\Gamma-I)^d}(A-\epsilon A^T)^{-1}By, & k=0, \\ -\dfrac{\Gamma^{k-1}(\Gamma-I)^{d-k}}{\Gamma^d-(\Gamma-I)^d}(A-\epsilon A^T)^{-1}By, & 1\le k\le d-1. \end{cases}$$

Note that in the above fractional notation of matrices, denominators and numerators commute and so we have no ambiguity. By the above calculation of the linking number, we have

$$\mathrm{lk}_{\tilde\Sigma}(t^k\tilde a^+, \tilde b) = \begin{cases} x^T\Big(C-\epsilon B^T\dfrac{\Gamma^{d-1}-(\Gamma-I)^{d-1}}{\Gamma^d-(\Gamma-I)^d}(A-\epsilon A^T)^{-1}B\Big)y, & k=0, \\ x^T\Big(\epsilon B^T\dfrac{\Gamma^{k-1}(\Gamma-I)^{d-k-1}}{\Gamma^d-(\Gamma-I)^d}(A-\epsilon A^T)^{-1}B\Big)y, & 1\le k\le d-1. \end{cases}$$

From the above discussion, we obtain the following result.

Theorem 2.1. *Let* $\left[\begin{smallmatrix} A & B \\ B^T & C\end{smallmatrix}\right]$ *be a Seifert matrix of* L *defined on* $E\cup F$ *in the sense of* [8]. *Then the block matrix* $(A_{kl})_{0\le k,l<d}$ *given by*

$$A_{kl} = \begin{cases} C-\epsilon B^T\dfrac{\Gamma^{p-1}-(\Gamma-I)^{p-1}}{\Gamma^p-(\Gamma-I)^p}(A-\epsilon A^T)^{-1}B, & k=l, \\ \epsilon B^T\dfrac{\Gamma^{k-l-1}(\Gamma-I)^{p-k+l-1}}{\Gamma^p-(\Gamma-I)^p}(A-\epsilon A^T)^{-1}B, & k>l, \\ \epsilon B^T\dfrac{\Gamma^{p-k+l-1}(\Gamma-I)^{k-l-1}}{\Gamma^p-(\Gamma-I)^p}(A-\epsilon A^T)^{-1}B, & k<l, \end{cases}$$

is (cobordant to if $q=1$*) a Seifert matrix defined on* $\bar H_q(\bigcup t^k\tilde F;\mathbf{Q})$, *and the block matrix* $(P_{kl})_{0\le k,l<d}$ *given by*

$$P_{kl} = \begin{cases} i^q_{sx_k}A_{kl}, & k=l, \\ sx_k\times sx_l \text{ array of } A_{kl}, & k\ne l, \end{cases}$$

is (cobordant to if $q=1$*) a Seifert matrix defined on* $\bar H_q(M;\mathbf{Q})$.

In particular, $\delta_{\tilde L}(\theta) = \delta^q_{(P_{kl})}(\theta/s)$.

Proof. For $q>1$, we have already proved the theorem. For $q=1$, we have proved that the conclusion holds if $\left[\begin{smallmatrix} A & B \\ B^T & C\end{smallmatrix}\right]$ is a Seifert matrix defined on $H_1(N)\oplus\bar H_1(F)$. By observing that the first formula induces a well-defined homomorphism $G(2,\epsilon)\to G(d,\epsilon)$ which sends $\left[\begin{smallmatrix} A & B \\ B^T & C\end{smallmatrix}\right]$ to (A_{kl}) on the groups of cobordism classes of Seifert matrices in the sense of [18], it suffices to show that a Seifert matrix defined on

$\bar{H}_1(E) \oplus \bar{H}_1(F)$ in [8] and a Seifert matrix defined on $H_1(N) \oplus \bar{H}_1(F)$ in the previous discussion are cobordant in the sense of [18] since both Seifert matrices represent elements of $G(2, \epsilon)$.

This assertion is proved by a similar reduction argument used earlier for Seifert matrices on $H_1(M')$ and $\bar{H}_1(M)$. $H_1(N) \cong \bar{H}_1(E) \oplus \mathbf{Z}^{2n}$ where the $\mathbf{Z}^{2n}$ factor is generated by cores of annular components of $N - \operatorname{int} E$ and its dual generators, and the linking number of each core and any loop on $E \cup F$ is zero. Therefore,

$$\left[\begin{array}{c|ccccc|c} A & * & 0 & \cdots & * & 0 & B \\ \hline * & * & * & \cdots & * & * & * \\ 0 & * & 0 & \cdots & * & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * & * & * \\ 0 & * & 0 & \cdots & * & 0 & 0 \\ \hline B^T & * & 0 & \cdots & * & 0 & C \end{array}\right]$$

is a Seifert matrix defined on $H_1(N) \oplus \bar{H}_1(F)$, where $\left[\begin{smallmatrix} A & B \\ B^T & C \end{smallmatrix}\right]$ is a Seifert matrix defined on $\bar{H}_1(E) \oplus \bar{H}_1(F)$. It is easy to check that the block sum of this Seifert matrix and $-\left[\begin{smallmatrix} A & B \\ B^T & C \end{smallmatrix}\right]$ is null-cobordant. □

3. Application to link concordance

Concordance of boundary links. In this subsection we study examples of homology boundary links in S^{2q-1} which are not concordant to boundary links, whose existence was shown first in [6, 7] and subsequently in [11, 21]. A key observation in [7] is that a covering link of a boundary link L is again a boundary link. Since a boundary link is a primitive link in the sense of [4], the signature jump function of (any union of parallels of components of) a covering link of L must be of period 2π by Theorem 1.2 of [4]. Since corresponding covering links of concordant links are concordant, the same conclusion holds under an weaker assumption that L is concordant to a boundary link by the fact that signatures are invariants under link concordance [4]. We state this as a theorem.

Theorem 3.1. *If a link L is concordant to a boundary link, the signature jump function of any union of parallels (with respect to the 0-linking framing if L is 1-dimensional) of components of a covering link of L has the period 2π.*

Using Theorem 3.1, we prove

Theorem 3.2. *Suppose L is a 2-component homology boundary link in S^{2q+1} with a pattern r and a Seifert matrix $\left[\begin{smallmatrix} A & B \\ \epsilon B^T & C \end{smallmatrix}\right]$ in the sense of [8] such that $A = C = \left[\begin{smallmatrix} V & V \\ \epsilon V^T & \epsilon V^T \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} V & V \\ \epsilon V^T & V \end{smallmatrix}\right]$ for a Seifert matrix V of a knot with nontrivial signature jump function, and for some n_0, $c_n(r) = 0$ if and only if $n \neq n_0, n_0 + 1$. Then L is not concordant to any boundary links.*

Proof. In this proof, we denote components of L by J, K, and use the notation of Section 2. We consider the covering link of L obtained by taking the p-fold cyclic cover of S^{2q+1} branched along the first component J for an odd prime p. We denote the first component of the pre-image of K by $\tilde{K}_L$.

We have $\Gamma = (A - \epsilon A^T)^{-1}A = \left[\begin{smallmatrix} G & G \\ 1-G & 1-G \end{smallmatrix}\right]$ where $G = (V - \epsilon V^T)^{-1}V$. By a straightforward calculation using Theorem 2.1 and the fact that $\Gamma^2 = \Gamma$,

$$(A_{kl}) = \left[\begin{array}{cc|cc|cccc|c}
V & & & & & & & & V \\
 & \epsilon V^T & \epsilon V^T & & & & & & \\
\hline
 & V & V & & & & & & \\
 & & & \epsilon V^T & \epsilon V^T & & & & \\
\hline
 & & & V & V & & & & \\
 & & & & & \ddots & & & \\
 & & & & & & & \epsilon V^T & \epsilon V^T \\
\hline
 & & & & & & & V & V \\
\epsilon V^T & & & & & & & & \epsilon V^T
\end{array}\right]$$

is a Seifert matrix defined on $\bar{H}_q(\bigcup_k t^k \tilde{F}; \mathbf{Q})$.

Denote $c_n(r)$ by c_n for simplicity. By conjugating the pattern by x^{-n_0}, we may assume $n_0 = 0$ and $c_0 = m$, $c_1 = 1-m$ for some $m \neq 0, 1$. Moreover, by reversing orientations if necessary, we may assume that $m > 1$. Since $x_0 = m^{p-1}/(m^p - (m-1)^p)$, $x_k = m^{k-1}(m-1)^{p-k}/(m^p - (m-1)^p)$ $(k = 1, \ldots, p-1)$ form a solution of the linear system

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_{p-1} \\ c_{p-1} & c_0 & \cdots & c_{p-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{p-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We can compute the matrix (P_{kl}) in Theorem 2.1 by putting $s = m^p - (m-1)^p$. (P_{kl}) is transformed to the block sum of $i^q_{x_1 - x_0}V, \ldots, i^q_{x_{p-1}-x_{p-2}}V, i^q_{x_0 - x_{p-1}}V$ by permuting rows and columns. Therefore, by the reparametrization formula in [4] and Theorem 2.1, we have

$$\delta_{\tilde{K}_L}(\theta) = \sum_{k=0}^{p-1} \delta^q_V(y_k \theta)$$

where $y_0 = -\frac{1-a^{p-1}}{m(1-a^p)}$, $y_k = \frac{a^{k-1}}{m^2(1-a^p)}$ for $k = 1, \ldots, p-1$ and $a = (m-1)/m$.

Since δ^q_V is nontrivial, there exists $\theta_0 > 0$ such that $\delta^q_V(\theta_0) \neq 0$ and $\delta^q_V(\theta) = 0$ for all $|\theta| < \theta_0$. Let $\theta_1 = \theta_0/y_0$. Then $\delta_{\tilde{K}_L}(\theta_1) = \delta^q_V(\theta_0) \neq 0$ for sufficiently large p, since for $k > 0$ $|y_k/y_0|$ uniformly converges to $1/m$ as $p \to \infty$. We can choose large N such that $|y_k(\theta_0/y_0 - 2\pi)| < \theta_0$ for all $p, k > N$, since $y_k(\theta_0/y_0 - 2\pi)$ uniformly converges to 0 as $k \to \infty$. Therefore, if $p > N$,

$$\delta_{\tilde{K}_L}(\theta_1 - 2\pi) = \delta^q_V(\theta_0 - 2\pi y_0) + \delta^q_V(y_1(\theta_0/y_0 - 2\pi)) + \cdots + \delta^q_V(y_N(\theta_0/y_0 - 2\pi)).$$

Since $\{y_k(\theta_0/y_0 - 2\pi)\}_{p=1}^{\infty}$ is a monotone convergent sequence for each $k = 0, \ldots, N$ and the set of points at which δ^q_V is nonzero is discrete (see Lemma 2.1 of [4]), $\delta^q_V(y_k(\theta_0/y_0 - 2\pi)) = 0$ for any large p. This shows that $\delta_{\tilde{K}_L}(\theta)$ is not of period 2π for any large p. By Theorem 3.1, L is not concordant to boundary links. □

We remark that any knot that is not (algebraically if $q = 1$) torsion in the knot concordance group has a Seifert matrix V satisfying the hypothesis of the above theorem [19, 20].

In [8], it was shown that an arbitrary pair of a pattern and a Seifert matrix is always realized by a geometric construction of a homology boundary link. Hence we can obtain a large collection of links that satisfy the conditions of Theorem 3.2 and therefore are not concordant to boundary links. We remark that the main examples of links in [7], denoted by $L(T,m)$, also satisfy the conditions of Theorem 3.2 for $m \neq 0,1$. In fact, the conditions of Theorem 3.2 can be viewed as an algebraic description of $L(T,m)$. In the case of $q=1$, $L(T,m)$ is illustrated in Figure 1. In the left diagram, the obvious Seifert surface of J has Seifert matrix $\left[\begin{smallmatrix}0 & m\\ m-1 & 0\end{smallmatrix}\right]$. The surfaces in the right diagram show that $L(T,m)$ is a homology boundary link; by attaching pipes joining dotted circles and K, we obtain a system of singular Seifert surfaces. It can be seen that a Seifert matrix of this system is of the desired form, where a Seifert matrix of T plays the role of V.

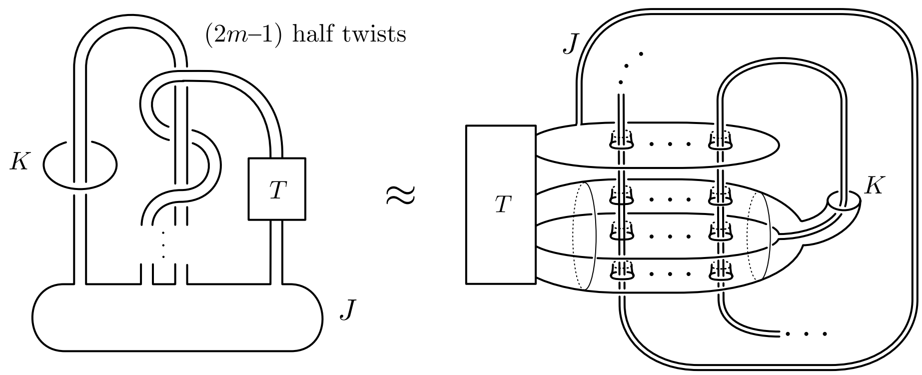

FIGURE 1.

Mutation and link concordance. In this subsection we illustrate an example of classical dimensional links which are positive mutants of ribbon links but not concordant to boundary links. Consider the 1-dimensional link L with two components shown in Figure 2. The first component K_1 of L has the same knot type as that of $L(T,m)$. The other component K_2 has the knot type of the mirror image of K_1.

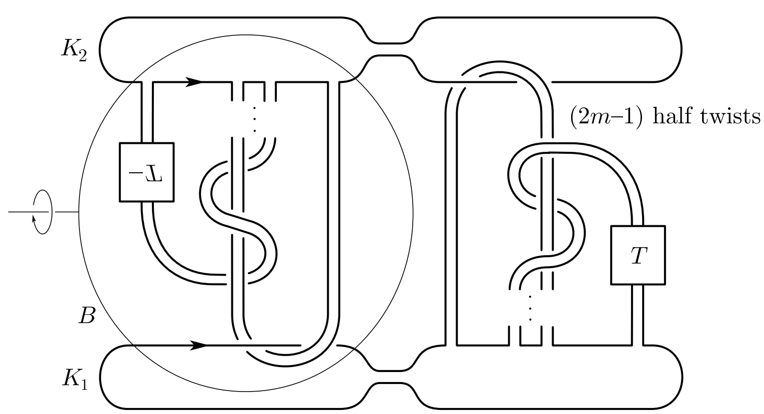

FIGURE 2.

Theorem 3.3. *L has the following properties:*

1. *L is a positive mutant of a ribbon link.*
2. *L is a fused boundary link. In particular, L has vanishing $\bar{\mu}$-invariants.*
3. *If T is not torsion in the algebraic knot concordance group and $m \neq 0, 1$, L is not concordant to boundary links.*

Proof. By the positive mutation on the 3-ball B shown in Figure 2, we obtain a mutant L^* of L. L^* is a connected sum of $L(T, m)$ and its mirror image, and in particular, L^* is a ribbon link.

L is obtained by attaching two bands joining disjoint components of the boundary link that is the split union of two parallel copies of T and $-T$ (the mirror image of T). Therefore, L is a fused boundary link and has vanishing $\bar{\mu}$-invariants.

To show the last conclusion, we consider the first component $\tilde{K}_L$ of the p-fold covering link of L as before. L is a connected sum of $L(T, m)$ and L', where L' is the link obtained by exchanging the order of the components of $L(-T, -m)$. Hence $\tilde{K}_L$ is the connected sum of $\tilde{K}_{L(T,m)}$ and $\tilde{K}_{L'}$. By the additivity of signature jump function [4], $\delta_{\tilde{K}_L}(\theta) = \delta_{\tilde{K}_{L(T,m)}}(\theta) + \delta_{\tilde{K}_{L'}}(\theta)$. Since the first component of L' (which is the second component of $L(-T, -m)$) is unknotted, the ambient space of $\tilde{K}_{L'}$ is the 3-sphere and $\delta_{\tilde{K}_{L'}}(\theta)$ is of period 2π. (In fact, $\tilde{K}_{L'}$ has the knot type of $-(T\#T)$.) Therefore, the period of $\delta_{\tilde{K}_L}(\theta)$ is equal to that of $\delta_{\tilde{K}_{L(T,m)}}(\theta)$, and is not equal to 2π for any sufficiently large prime p by the proof of Theorem 3.2. This proves that L is not concordant to boundary links. □

Concordance of homology boundary links with given patterns. In this subsection we generalize the previous arguments to show Theorem 1.1.

Proof of Theorem 1.1. Fix a pattern r. Suppose that $L = K_1 \cup \cdots \cup K_n$ is a homology boundary link admitting r as a pattern. Let $F_1 \cup \cdots \cup F_n$ be a singular Seifert surface. Consider the covering link $\bigcup_{i>1,k} t^k \tilde{K}_i$ of L obtained by taking the p-fold cover branched along K_1 as before. Attaching annuli to a lift of F_2 as done in Section 2, we obtain a submanifold in the ambient space of the covering link which is bounded by $\tilde{L} = \bigcup_{i>1,k} i_{c_{ik}} t^k \tilde{K}_i$ for some integers c_{ik}. In particular, $\tilde{L}$ is a primitive link and so $\delta_{\tilde{L}}(\theta)$ has the period 2π. Note that it was proved in Section 2 that the numbers c_{ik} are determined by r and $\sum_k c_{ik} = 1$ if $i = 2$ or $\sum_k c_{ik} = 0$ otherwise.

If L were concordant to a homology boundary link admitting pattern r, then the signature jump function of the link $\tilde{L} = \bigcup_{i>1,k} i_{c_{ik}} t^k \tilde{K}_i$ constructed as above would have the period 2π, since the signature jump function of $\tilde{L}$ is a concordance invariant of L. Let L be the distant union of $(n-2)$-component unlink and a link whose Seifert matrix and pattern are as in Theorem 3.2. We will show that for any sufficiently large m, L does not satisfy the above periodicity condition. This completes the proof of Theorem 1.1.

We assume $m > 0$ and fix $p = 3$. Let c_{ik} be the numbers determined by the pattern r and let $\tilde{L} = \bigcup_{i,k} i_{c_{ik}} t^k \tilde{K}_i$ as above. Since the split unlink part has no contribution to the signature, the signature of $\tilde{L}$ is equal to the signature of $\bigcup_k i_{c_{2k}} t^k \tilde{K}_2$. By Theorem 2.1, we have $\delta_{\tilde{L}}(\theta) = \delta_V^q(y_1\theta) + \delta_V^q(y_2\theta) + \delta_V^q(y_0\theta)$ where

$y_0 = x_0 - x_2$, $y_1 = x_1 - x_0$, $y_2 = x_2 - x_1$ and $\{x_i\}$ is a solution of

$$\begin{bmatrix} m & 1-m & 0 \\ 0 & m & 1-m \\ 1-m & 0 & m \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_{20} \\ c_{21} \\ c_{22} \end{bmatrix}$$

as in the proof of Theorem 3.2. By solving the equations, we have

$$y_i = (a_i m + b_i)/(3m^2 - 3m + 1),$$

where

$$\begin{aligned} a_0 &= 3c_{20} + 3c_{21} - 2, & b_0 &= 1 - c_{20} - 2c_{21}, \\ a_1 &= 1 - 3c_{20}, & b_1 &= 2c_{20} + c_{21} - 1, \\ a_2 &= 1 - 3c_{21}, & b_2 &= c_{20} - c_{21}. \end{aligned}$$

Since $a_0 \equiv a_1 \equiv a_2 \equiv 1 \pmod 3$ and $a_0 + a_1 + a_2 = 0$, we may assume that $|a_0| > |a_1|, |a_2|$ by permuting indices. Choose minimal $\theta_0 > 0$ such that $\delta_V^q(\theta_0) \neq 0$. Choose $\epsilon > 0$ such that $\delta_V^q(\theta) = 0$ for all $0 < |\theta - \theta_0| < \epsilon$. Since $\lim_{m\to\infty} |y_i/y_0| < 1$ for $i = 1, 2$, we can choose $\epsilon' > 0$ such that $|y_1/y_0|, |y_2/y_0| < 1 - \epsilon'$ for any large m. We remark that $0 < 2\pi|y_0| < \epsilon$ and $2\pi|y_1|, 2\pi|y_2| < \epsilon'\theta_0$ are satisfied for any large m since $y_i \to 0$ as $m \to \infty$.

Let $\theta_1 = \theta_0/y_0$. We claim that for any large m, $\delta_{\tilde{L}}(\theta_1) \neq 0$ and $\delta_{\tilde{L}}(\theta_1 + 2\pi) = 0$. Since $|y_1/y_0|, |y_2/y_0| < 1$, $\delta_{\tilde{L}}(\theta_1) = \delta_V^q(\theta_0) \neq 0$. Since $0 < |2\pi y_0| < \epsilon$ and $|\theta_0 y_i/y_0 + 2\pi y_i| < \theta_0$ for $i = 1, 2$, it follows that

$$\delta_{\tilde{L}}(\theta_1 + 2\pi) = \delta_V^q(\theta_0 + 2\pi y_0) + \delta_V^q(\theta_0 y_1/y_0 + 2\pi y_1) + \delta_V^q(\theta_0 y_2/y_0 + 2\pi y_2) = 0.$$

This proves the claim.

The claim implies that $\delta_{\tilde{L}}(\theta)$ is not of period 2π. Therefore L is not concordant to any homology boundary links admitting pattern r if m is sufficiently large. □

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