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A GEOMETRIC CHARACTERIZATION OF INTERPOLATION IN $\hat{\mathcal{E}}'(\mathbb{R})$

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ABSTRACT. We give a geometric description of the interpolating varieties for the algebra of Fourier transforms of distributions (or Beurling ultradistributions) with compact support on the real line.

1. INTRODUCTION

Let $\mathcal{E}(\mathbb{R})$ be the space of smooth functions in $\mathbb{R}$ and let $\mathcal{E}'(\mathbb{R})$ be its dual, the space of distributions with compact support on $\mathbb{R}$. It is well known that the space $\hat{\mathcal{E}}'(\mathbb{R})$ of Fourier transforms of distributions in $\mathcal{E}'(\mathbb{R})$ coincides with the algebra of entire functions f such that

$$|f(z)| \leq C(1+|z|)^A e^{B|\operatorname{Im} z|},$$

where $A, B, C > 0$ may depend on f (see [BrGa95, Theorem 1.4.15]).

A discrete sequence $\Lambda \subset \mathbb{C}$ is called $\hat{\mathcal{E}}'(\mathbb{R})$-interpolating when the interpolation problem $f(\lambda) = v_\lambda$, $\lambda \in \Lambda$, has a solution $f \in \hat{\mathcal{E}}'(\mathbb{R})$ for every sequence of complex values $\{v_\lambda\}_{\lambda\in\Lambda}$ having the characteristic growth of $\hat{\mathcal{E}}'(\mathbb{R})$ on Λ (see the precise definition in Section 2).

The origin of the interest in $\hat{\mathcal{E}}'(\mathbb{R})$-interpolation lies in its relationship with convolution equations and, in particular, with the density of exponential families $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ in the space of solutions $g \in \mathcal{E}(\mathbb{R})$ of equations of type $\mu \star g = 0$, $\mu \in \mathcal{E}'(\mathbb{R})$. Any solution g to the convolution equation is the limit of linear combinations of $\{e^{i\lambda x}\}_{\lambda\in\Lambda}$ where Λ is the zero set of $\hat{\mu}$. If, moreover, the sequence Λ is $\hat{\mathcal{E}}'(\mathbb{R})$-interpolating, then the series that represents g enjoys better convergence properties. For more on this relationship see [EhMa74] or [BrGa95, Chapter 6] (in particular, Theorem 6.1.11).

The space $\hat{\mathcal{E}}'(\mathbb{R})$ is a particular case of the algebras

$$A_p = \{f \in H(\mathbb{C}) : \log|f(z)| \leq A + Bp(z) \text{ for some } A, B > 0\}$$

associated to positive measurable weights p, obtained by taking

$$p(z) = |\operatorname{Im} z| + \log(1+|z|^2). \tag{1}$$

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There exists an analytic characterization of interpolating sequences for general A_p spaces when p is subharmonic (see Theorem A below). However, a complete geometric description exists only for subharmonic weights p which are both radial ($p(z) = p(|z|)$) and doubling ($p(2z) \le Cp(z)$ for some $C > 0$); see [BrLi95, Corollary 4.8].

For the weight (1) Ehrenpreis and Malliavin gave a necessary geometric condition which turns out to be sufficient provided that Λ is a zero sequence of a slowly decreasing function (see [EhMa74, Theorem 4] and its proof). Later Squires, probably unaware of Ehrenpreis and Malliavin's result (which was stated in terms of solutions to convolution equations), proved the same result [Sq83, Theorem 2].

In this paper we give a geometric characterization for $\hat{\mathcal{E}}'(\mathbb{R})$-interpolating sequences (Theorem 1). The characterization shows in particular that the geometric condition given by Ehrenpreis & Malliavin and Squires is also sufficient whenever the sequence is contained in the region

$$|\operatorname{Im} z| \le C \log(1 + |z|^2).$$

In general, however, their condition alone is not sufficient.

A similar characterization is obtained for the more general Beurling weights. These weights appear naturally in the context of convolution equations when one replaces distributions with compact support with Beurling-Björck ultradistributions of compact support (see [Bj66]). They are not necessarily subharmonic, but we will prove that they are equivalent to a subharmonic weight (see Lemma 8).

The paper is structured as follows. In Section 2 we give the precise definition of interpolating variety, discuss the background for the problem and state the main result. In Section 3 we prove that the geometric conditions of Theorem 1 are necessary, while in Section 4 we show that they are also sufficient.

A final remark about notation. C will always denote a positive constant and its actual value may change from one occurrence to the next. $A = O(B)$ and $A \lesssim B$ mean that $A \le cB$ for some $c > 0$, and $A \simeq B$ is $A \lesssim B \lesssim A$.

2. Preliminaries

For the following definition and general background on the problem we refer to [BrGa95, Chapter 2].

A measurable function $p : \mathbb{C} \longrightarrow \mathbb{R}_+$ is called a *weight* if for some $C, D > 0$:

(a) $\log(1 + |z|^2) \le Cp(z)$ for all $z \in \mathbb{C}$.
(b) $p(\zeta) \le Cp(z) + D$ if $|\zeta - z| \le 1$.

The importance of these properties lies in their consequences for the ring A_p defined in the introduction: (a) implies that A_p contains all polynomials, and (b) that A_p is closed under differentiation.

The algebra A_p can be thought of as the union of the Hilbert spaces

$$A^2_{p,\alpha} := \left\{ f \in H(\mathbb{C}) : \|f\|^2_{A^2_{p,\alpha}} = \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha p(z)} dm(z) < \infty \right\}$$

for $\alpha > 0$, as well as the union of the Banach spaces

$$A^\infty_{p,\alpha} := \left\{ f \in H(\mathbb{C}) : \|f\|_{A^\infty_{p,\alpha}} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\alpha p(z)} < \infty \right\} .$$

Also, $A_p = \bigcup_{\alpha > 0} A^\infty_{p,\alpha}$ has the structure of an (LF)-space with the topology of the inductive limit.

Definition. Let Λ be a discrete sequence in $\mathbb{C}$ and let $\{m_\lambda\}_{\lambda\in\Lambda}$ be a sequence of natural numbers. The pair $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is called an *interpolating variety* for the space A_p if for every sequence of values $\{v_\lambda^l\}_{\lambda,l}$, $\lambda \in \Lambda$, $l = 0, \dots, m_\lambda - 1$, with

$$\sup_{\lambda\in\Lambda} \Big(\sum_{l=0}^{m_\lambda - 1} |v_\lambda^l| \Big) e^{-\alpha p(\lambda)} < \infty \tag{2}$$

for some $\alpha > 0$, there exists $f \in A_p$ with

$$\frac{f^{(l)}(\lambda)}{l!} = v_\lambda^l, \qquad \lambda \in \Lambda; \quad l = 0, \dots, m_\lambda - 1 \ .$$

The choice of condition (2) on the values to be interpolated reflects the fact that for every $f \in A_p$ there exists $\alpha > 0$ such that

$$\sup_{z\in\mathbb{C}} \Big(\sum_{l=0}^{\infty} \Big| \frac{f^{(l)}(z)}{l!} \Big| \Big) e^{-\alpha p(z)} < \infty.$$

Thus, denoting by $A_p(X)$ the space of sequences $\{v_\lambda^l\}_{\lambda,l}$ satisfying (2) for some $\alpha > 0$, we can equivalently define interpolating varieties X as those such that the restriction operator

$$\begin{aligned} \mathcal{R}_X : A_p(\mathbb{C}) &\longrightarrow A_p(X) \\ f \quad &\mapsto \ \Big\{ \frac{f^{(l)}(\lambda)}{l!} \Big\}_{\lambda,l} \end{aligned}$$

is onto.

There exists an analytic characterization of A_p-interpolating varieties for general subharmonic weights p (see [BrLi95, Corollary 3.5]). Results for $p(z) = |z|$ and the weight (1) were previously obtained respectively by Leont'ev [Le72] and Squires [Sq81].

Given a holomorphic function f let $\mathcal{Z}(f)$ denote its zero variety, i.e., the set of pairs $(z, m_z) \in \mathbb{C} \times \mathbb{N}$ such that $f(z) = 0$ with multiplicity m_z.

Theorem A. *A variety $X = \{(\lambda, m_\lambda)\}_{\lambda\in\Lambda}$ is A_p-interpolating if and only if there exists $f \in A_p$ such that $X \subset \mathcal{Z}(f)$ and for some constants $\delta, C > 0$,*

$$\Big| \frac{f^{(m_\lambda)}(\lambda)}{m_\lambda!} \Big| \geq \delta e^{-Cp(\lambda)}, \qquad \lambda \in \Lambda.$$

We would like to give a geometric description of A_p-interpolating varieties for the non-isotropic Beurling weights

$$p(z) = |\operatorname{Im} z| + \omega(|z|),$$

where $\omega(t)$ is a subadditive increasing continuous function, normalized with $\omega(0) = 0$ and such that:

(c) $\log(1 + t) \lesssim \omega(t)$ for $t > 1$.
(d) $\int_0^\infty \frac{\omega(t)}{1+t^2} dt < \infty$.

Canonical examples of such weights are given by $\omega(t) = \log(1 + t^2)$ and $\omega(t) = t^\gamma$, $\gamma \in (0, 1)$.

Beurling weights satisfy the following additional properties:

(e) For every $c > 0$ there exists $C > 0$ such that $p(\zeta) \leq Cp(z)$ if $\zeta \in D(z, cp(z))$.

(f) For $\varepsilon > 0$ small enough, there exists $C(\varepsilon) > 0$ such that if $z \in D(\zeta, \varepsilon p(\zeta))$, then $p(\zeta) \leq C(\varepsilon)p(z)$. Also, $C(\varepsilon)$ tends to 1 as ε goes to 0.

(g) For $x \in \mathbb{R}^+$ big enough, the function $\omega(x)$ does not oscillate too much. More precisely, for fixed $C > 0$, if $y \in (x - C\omega(x), x + C\omega(x))$, then $1/2 \leq \omega(y)/\omega(x) \leq 2$ for x big enough.

Properties (e) and (f) follow easily from the subadditivity of ω. Property (g) follows from the subadditivity and the fact that $\omega(x) = o(|x|/\log|x|)$ (see [Bj66, Lemma 1.2.8]): for any $y \in (x - C\omega(x), x + C\omega(x))$,

$$\begin{aligned}\omega(x - C\omega(x)) \leq \omega(y) \leq \omega(x + C\omega(x)) &\leq \omega(x - C\omega(x)) + \omega(2C\omega(x)) \\ &\leq \omega(x - C\omega(x)) + \omega(2Cx/\log x) \leq 2\omega(x - C\omega(x)).\end{aligned}$$

In order to state the geometric conditions on a variety X as above, we consider the counting function $n(z,r) = \sum_{\lambda \in D(z,r)} m_\lambda$ and the integrated version

$$N(z,r) = \int_0^r \frac{n(z,t) - n(z,0)}{t}\, dt + n(z,0)\log r.$$

In case we want to specify the variety X to which the functions n and N refer, we will use the notation $n(z,r,X)$ and $N(z,r,X)$ respectively.

We are ready to state our main result.

Theorem 1. *A variety $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is A_p-interpolating if and only if:*

(i) *There is $C > 0$ such that*

$$N(\lambda, p(\lambda), X) \leq Cp(\lambda) \qquad \forall \lambda \in \Lambda.$$

(ii) *The following Carleson-type condition holds:*

$$\sup_{x \in \mathbb{R}} \sum_{\lambda : |\operatorname{Im}\lambda| > \omega(|\lambda|)} m_\lambda \frac{|\operatorname{Im}\lambda|}{|x - \lambda|^2} < \infty.$$

Since the Poisson kernel at λ in the corresponding half-plane (upper half-plane if $\operatorname{Im}\lambda > 0$ and lower half-plane when $\operatorname{Im}\lambda < 0$) is $P(\lambda, x) = \frac{|\operatorname{Im}\lambda|}{|x-\lambda|^2}$, a restatement of condition (ii) is that the measure $\sum_{\lambda : |\operatorname{Im}\lambda| > \omega(|\lambda|)} m_\lambda \delta_\lambda$ has bounded Poisson balayage.

Remark 2. Notice that for sequences Λ within the region $|\operatorname{Im} z| \leq \omega(|z|)$, condition (i) (shown to be necessary by Ehrenpreis & Malliavin and Squires) provides a complete characterization. However, this is not the case in general, i.e. condition (ii) does not follow from (i), as it is shown in the following example. Take the sequence Λ contained in the sector $\mathcal{A} = \{z \in \mathbb{C};\ |\mathrm{Re} z| < \mathrm{Im} z\}$ and having in each segment $\{\mathrm{Im} z = 2^n\} \cap \mathcal{A}$ exactly 2^n equispaced points. Then Λ satisfies condition (i) (basically $n(\lambda, t) \leq t$ for $t \leq p(\lambda)$), but it does not satisfy (ii) (it is not even a Blaschke sequence).

3. Necessary conditions

A standard feature of the spaces A_p is that the interpolation can be performed in a stable way. This consequence of the open mapping theorem for (LF)-spaces applied to the restriction mapping $\mathcal{R}_X$ defined in Section 2 (see [BrGa95, Lemma 2.2.6]) is stated precisely in the following lemma.

Lemma 3. *If X is an interpolating variety, there exist $C > 0$, $M \in \mathbb{N}$ such that for every $\lambda \in \Lambda$ there are functions $f_\lambda, g_\lambda \in A_p$ with bounded norms $\|f_\lambda\|_{A^\infty_{p,M}}$, $\|g_\lambda\|_{A^\infty_{p,M}} \le C$ and*

$$f_\lambda^{(l)}(\lambda')/l! = \delta_{\lambda\lambda'}\delta_{l0},$$
$$g_\lambda^{(l)}(\lambda')/l! = \delta_{\lambda\lambda'}\delta_{l(m_\lambda-1)} \qquad \forall \lambda, \lambda' \in \Lambda,\ 0 \le l \le m_\lambda.$$

An application of Jensen's Formula to the functions f_λ, g_λ in the disk $D(\lambda, p(\lambda))$ gives the following result (see [EhMa74, Theorem 4] or [Sq83, Theorem 1]).

Theorem 4. *If X is A_p-interpolating, then condition* (i) *of Theorem* 1 *holds.*

The necessity of condition (ii) is an immediate consequence of the following result. Assume that $\Lambda \cap \mathbb{R} = \emptyset$; otherwise move the horizontal line so that it does not touch any of the points in Λ. Let $\mathbb{H}$ denote the upper half-plane.

Proposition 5. *Let X be A_p-interpolating. There exist $C > 0$ such that*

$$\sum_{\substack{\lambda' \in \Lambda\cap\mathbb{H} \\ \lambda' \ne \lambda}} m_{\lambda'} \log\Big|\frac{\lambda - \lambda'}{\lambda - \bar\lambda'}\Big|^{-1} \le Cp(\lambda) \quad \textit{for all } \lambda \in \Lambda \cap \mathbb{H}.$$

Of course, an analogous result could be given for any upper ($\{z : \mathrm{Im} z > a\}$) or lower ($\{z : \mathrm{Im} z < a\}$) half-plane.

Before giving the proof of Proposition 5, we show that it implies condition (ii) of Theorem 1. Define $\Lambda_+ = \Lambda \cap \{\mathrm{Im}\, z > \omega(|z|)\}$. Given $x \in \mathbb{R}$ consider $\lambda \in \Lambda_+$ such that $|x - \lambda| = \inf_{\Lambda_+} |x - \lambda|$. Then

$$|\lambda - \bar\lambda'| \le |\lambda - x| + |x - \bar\lambda'| = |\lambda - x| + |x - \lambda'| \le 2|x - \lambda'|,$$

and therefore

$$\sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\mathrm{Im}\, \lambda'|}{|x - \bar\lambda'|^2} \le 2 \sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\mathrm{Im}\, \lambda'|}{|\lambda - \bar\lambda'|^2}.$$

The estimate $\log t^{-1} \ge 1 - t$ for $t \in (0, 1)$ shows that

$$\sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \ne \lambda}} m_{\lambda'} \frac{|\mathrm{Im}\, \lambda||\mathrm{Im}\, \lambda'|}{|\lambda - \bar\lambda'|^2} \le \sum_{\substack{\lambda' \in \Lambda_+ \\ \lambda' \ne \lambda}} m_{\lambda'} \log\Big|\frac{\lambda - \lambda'}{\lambda - \bar\lambda'}\Big|^{-1}.$$

Since $p(\lambda) \simeq |\mathrm{Im}\, \lambda|$ for $\lambda \in \Lambda_+$, it is clear that this implies condition (ii) of Theorem 1.

Remark 6. The necessary condition of Proposition 5 can be seen as a Carleson-type condition; it can be rewritten as

$$|B_\lambda(\lambda)| \ge \delta e^{-Cp(\lambda)}, \quad \lambda \in \Lambda \cap \mathbb{H},$$

where B denotes the Blaschke product in $\mathbb{H}$ of $\{(\lambda, m_\lambda)\}_{\lambda \in \Lambda\cap\mathbb{H}}$, and

$$B_\lambda(z) = B(z)\Big(\frac{z - \bar\lambda}{z - \lambda}\Big)^{m_\lambda}.$$

It can also be seen as density conditions for the counting function associated to the hyperbolic metric in the half-plane. Letting $\nu = \sum_{\lambda \in \Lambda\cap\mathbb{H}} m_\lambda \delta_\lambda$ and using the distribution function we have

$$\sum_{\lambda \in \Lambda\cap\mathbb{H}} m_\lambda \log\Big|\frac{z - \lambda}{z - \bar\lambda}\Big|^{-1} = \int_{\mathbb{H}} \log\Big|\frac{z - \zeta}{z - \bar\zeta}\Big|^{-1} d\nu(\zeta) = \int_0^1 \frac{n_{\mathbb{H}}(z, t)}{t}\, dt,$$

where
$$D_{\mathbb{H}}(z,t)=\{\zeta:\Big|\frac{z-\zeta}{z-\bar\zeta}\Big|<t\},\quad\text{and}\quad n_{\mathbb{H}}(z,t):=\nu(D_{\mathbb{H}}(z,t))$$
is the number of points of Λ in the pseudohyperbolic disk of "center" z and "radius" t (actually the true disk of center $\mathrm{Re}z+i\frac{1+t^2}{1-t^2}\mathrm{Im}z$ and radius $\frac{2t}{1-t^2}\mathrm{Im}z$).

Proof of Proposition 5. Let $z=x+iy$ and consider the Poisson transform of $\omega(|t|)$:
$$u(z):=P[\omega](z)=\int_{\mathbb{R}}\frac{y\,\omega(|t|)}{(x-t)^2+y^2}\,dt,$$
which converges by (d). Define $H=\exp(u+i\tilde u)$, where $\tilde u$ is a harmonic conjugate of u.

Given $\lambda\in\Lambda\cap\mathbb{H}$, take the function f_λ given by Lemma 3 and define
$$h_\lambda(z)=\frac{f_\lambda(z)e^{iM_1z}}{(H(z))^{M_2}},$$
with M_1,M_2 to be chosen. It is clear that h_λ is holomorphic in $\mathbb{H}$. On the other hand, for all z in the upper half-plane $|\log|H(z)|-\omega(|\mathrm{Re}z|)|\le A+B|\mathrm{Im}z|$; see [Bj66, Lemma 1.3.11]. Moreover, $|\omega(|\mathrm{Re}z|)-\omega(|z|)|\le\omega(|\mathrm{Im}z|)\le A+B|\mathrm{Im}z|$, thus $|\log|H(z)|-\omega(|z|)|\le A+B|\mathrm{Im}z|$. Therefore, if M_1 and M_2 are big enough, h_λ is bounded in $\mathbb{H}$ by a constant which does not depend on λ:
$$|h_\lambda(z)|\le Ce^{Mp(z)-M_1\mathrm{Im}\,z-M_2\log|H(z)|}\lesssim 1.$$
Also,
$$|h_\lambda(\lambda)|=e^{-M_1\mathrm{Im}\,\lambda-M_2\log|H(\lambda)|}\ge e^{-Cp(\lambda)}.$$
Now apply Jensen's formula in the half-plane to the function h_λ:
$$\log|h_\lambda(\lambda)|=\int_{\mathbb{R}}P(\lambda,x)\log|h_\lambda(x)|\,dx-\int_{\mathbb{H}}G(\lambda,\zeta)\Delta\log|h_\lambda(\zeta)|,$$
where $P(\lambda,x)$ denotes the Poisson kernel and $G(\lambda,\zeta)=\log\Big|\frac{\lambda-\zeta}{\lambda-\bar\zeta}\Big|^{-1}$ is the Green function in $\mathbb{H}$ with pole in λ.

Since h_λ vanishes on $\Lambda\setminus\{\lambda\}$, Jensen's Formula and the estimates above yield
$$\sum_{\substack{\lambda'\in\Lambda\cap\mathbb{H}\\ \lambda'\neq\lambda}}m_{\lambda'}\log\Big|\frac{\lambda-\lambda'}{\lambda-\bar\lambda'}\Big|^{-1}\le\sup_{\mathbb{R}}\log|h_\lambda|-\log|h_\lambda(\lambda)|\lesssim p(\lambda).$$
□

4. Sufficient conditions

We split the sequence into three pieces, according to the non-isotropy of the weight p. Consider the regions
$$\Omega_0=\{z\in\mathbb{C}:|\mathrm{Im}\,z|\le\omega(|z|)\},$$
$$\Omega_+=\{z\in\mathbb{C}:\mathrm{Im}\,z>\omega(|z|)\},$$
$$\Omega_-=\{z\in\mathbb{C}:\mathrm{Im}\,z<-\omega(|z|)\},$$
and define $\Lambda_0=\Lambda\cap\Omega_0$, $\Lambda_+=\Lambda\cap\Omega_+$ and $\Lambda_-=\Lambda\cap\Omega_-$. Let also $X_0=\{(\lambda,m_\lambda)\}_{\lambda\in\Lambda_0}$, $X_+=\{(\lambda,m_\lambda)\}_{\lambda\in\Lambda_+}$ and $X_-=\{(\lambda,m_\lambda)\}_{\lambda\in\Lambda_-}$.

It is enough to prove that each piece X_+,X_-,X_0 of the variety X is A_p-interpolating. This is so because X is weakly separated (see Lemma 7(i) below), and a weakly separated union of a finite number of A_p-interpolating varieties is also A_p-interpolating [Ou03, Theorem II.1]. It is also clear that the varieties X^+ and X^- can be dealt with similarly.

We start with some easy consequences of condition (i) of Theorem 1.

Lemma 7. *If condition* (i) *in Theorem* 1 *holds, then*

(i) *X is weakly separated: there exist $\delta, C > 0$ such that the disks $D_\lambda = D(\lambda, \delta e^{-C\frac{p(\lambda)}{m_\lambda}})$ are pairwise disjoint, i.e.*

$$|\lambda - \lambda'| \geq 2\delta \max[e^{-C\frac{p(\lambda)}{m_\lambda}}, e^{-C\frac{p(\lambda')}{m_{\lambda'}}}], \qquad \lambda \neq \lambda'.$$

(ii) *There exist $\varepsilon, C > 0$ such that $n(z, \varepsilon p(z), X) \leq Cp(z)$, $\forall z \in \mathbb{C}$.*

Proof. (i) If there exists λ' such that $|\lambda' - \lambda| < 1$, then

$$N(\lambda, p(\lambda), X) \geq \int_{|\lambda'-\lambda|}^{1} \frac{n(\lambda,t) - m_\lambda}{t} dt \geq \int_{|\lambda'-\lambda|}^{1} \frac{m_{\lambda'}}{t} dt = \log\Big(\frac{1}{|\lambda' - \lambda|}\Big)^{m_{\lambda'}}.$$

Using condition (i) in Theorem 1 and reversing the roles of λ and λ' we obtain the desired estimate.

(ii) When $z = \lambda \in \Lambda$, this is immediate from the estimate

$$\int_{1/2p(\lambda)}^{p(\lambda)} \frac{n(\lambda, 1/2p(\lambda)) - 1}{t} dt \leq N(\lambda, p(\lambda)).$$

When $z \notin \Lambda$, then let $\varepsilon > 0$ be such that $\zeta \in D(z, \varepsilon p(z))$ implies

$$D(z, \varepsilon p(z)) \subset D(\zeta, 1/2p(\zeta)),$$

which exists by property (f) of the weight. Take $\lambda \in D(z, \varepsilon p(z))$ (if there is no such λ the estimate is obviously true). Then, by the previous case and property (e) of the weight

$$n(z, \varepsilon p(z)) \leq n(\lambda, 1/2p(\lambda)) \lesssim p(\lambda) \lesssim p(z).$$

□

4.1. **Case Λ_0.** We would like to prove that $X_0 = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda_0}$ is A_p-interpolating using a $\bar{\partial}$-scheme. This is easier if we can regularize the weight in the following way.

Lemma 8. *There exists $\tilde{p}$ subharmonic in $\mathbb{C}$ such that $p(z) \simeq \tilde{p}(z)$ and*

$$1/\tilde{p}(z) \lesssim \Delta \tilde{p}(z) \qquad \text{if} \quad |\mathrm{Im} z| \leq 2\omega(|z|). \tag{3}$$

The fact that $p \simeq \tilde{p}$ clearly implies that $A_p = A_{\tilde{p}}$ and the interpolating varieties for A_p and $A_{\tilde{p}}$ are the same.

Proof. We will construct $\tilde{p}(z) = |\mathrm{Im} z| + r(z)$, where r satisfies the following properties:

(i) $r \geq 0$ and $\tilde{p}$ is subharmonic in $\mathbb{C}$.
(ii) $r(z) = 0$ if $|\mathrm{Im} z| \geq 10\omega(|z|)$.
(iii) $1/p(z) \lesssim \Delta \tilde{p}(z)$ and $r(z) \simeq \omega(|z|)$ if $|\mathrm{Im} z| \leq 2\omega(|z|)$.

In order to construct r, we partition the real line into intervals I_n defined in the following way.

Let $x_1 > 1$, $x_{n+1} = x_n + \omega(x_n)$ for $n \geq 1$ and $x_n = -x_{-n}$ for $n \leq -1$. Set $I_0 = [x_{-1}, x_1]$, $I_n = [x_n, x_{n+1}]$ for $n \geq 1$ and $I_n = [x_{n-1}, x_n]$ for $n \leq -1$. Denote by ω_n the length of I_n.

We consider two measures in $\mathbb{C}$. The first one is the usual length measure $d\nu$ in $\mathbb{R}$, which we split $d\nu = \sum_n d\nu_n$, with $d\nu_n = dx_{|I_n}$. The second one is defined as a sum of convolutions of the $d\nu_n$'s: let

$$d\mu_n(z) = \big(\frac{1}{100\pi\omega_n^2}\int_{I_n}\chi_{D_n}(z-x)dx\big)\,dm(z),$$

where $D_n = D(0, 10\omega_n)$, and define $d\mu = \sum_n d\mu_n$.

Notice that when z is at a distance of I_n smaller than $2\omega_n$, we can use property (g) of the Beurling weights to deduce that $d\mu(z) \simeq 1/\omega(|z|) \simeq 1/p(z)$. Hence $d\mu(z) \simeq dm(z)/p(z)$.

Define

$$r(z) = \int_{\mathbb{C}} \log|z-w|(d\mu(w) - d\nu(w)).$$

Since $\Delta|\operatorname{Im} z| = d\nu$, we have $\Delta\tilde{p} = d\mu \geq 0$.

Let S_n denote the support of μ_n. Let

$$r_n(z) := \int_{\mathbb{C}} \log|z-w|(d\mu_n(w) - d\nu_n(w)) = \int_{S_n}\log|z-w|d\mu_n(w) - \int_{I_n}\log|z-x|dx.$$

Using the definition of μ_n and reversing the order of integration we get

$$r_n(z) = \int_{I_n} M(x)dx,$$

where

$$M(x) = \frac{1}{100\pi\omega_n^2}\int_{D(x,10\omega_n)}\log|z-w|dm(w) - \log|z-x| \geq 0.$$

In particular, r is non-negative in $\mathbb{C}$.

If $z \notin S_n$ and $x \in I_n$, $\log|z-w|$ is harmonic in $D(x, 10\omega_n)$, hence $r_n(z) = 0$.

Suppose now $z \in D(x_n, 3\omega_n)$. Then, for each $x \in I_n$, $|z-x| \leq 4\omega_n$ and

$$M(x) \geq \frac{1}{100\pi\omega_n^2}\int_{9\omega_n\leq|w-x|\leq 10\omega_n}\log\frac{|z-w|}{|z-x|}dm(w) \gtrsim 1.$$

Thus, $r_n(z) \gtrsim \omega_n \gtrsim \omega(|z|)$.

If $z \in S_n$, using that μ_n and ν_n have the same mass $\omega(x_n)$, we obtain

$$\int_{\mathbb{C}}\log|z-w|(d\mu_n(w) - d\nu_n(w)) \leq \int_{\mathbb{C}}\Big|\log\frac{|z-w|}{\omega(x_n)}\Big|(d\mu_n(w) + d\nu_n(w))$$
$$\lesssim \int_{\mathbb{C}}\Big|\log\frac{|x_n-w|}{\omega(x_n)}\Big|(d\mu_n(w) + d\nu_n(w)) \lesssim \omega(|z|).$$

Since $|\mathrm{Im} z| \leq 2\omega(|z|)$, z belongs at most to a finite number of S_n's and at least to one $D(x_n, 10\omega_n)$, by property (g) of the Beurling weights, we are done. $\square$

Let us prove now that X_0 is $A_{\tilde{p}}$-interpolating. In view of Lemma 8, we assume that $1/p \lesssim \Delta p$ on $|\mathrm{Im} z| \leq 2\omega(|z|)$.

Consider the separation radii $\delta_\lambda := \delta e^{-Cp(\lambda)}$ given by Lemma 7(i).

Given a sequence of values $\{v_\lambda^\ell\}_{\lambda,\ell}$ satisfying (2), define the smooth interpolating function

$$F(z) = \sum_{\lambda\in\Lambda_0} p_\lambda(z)\mathcal{X}\Big(\frac{|z-\lambda|^2}{\delta_\lambda^2}\Big),$$

where $p_\lambda(z) = \sum_{l=0}^{m_\lambda-1} v_\lambda^l (z-\lambda)^l$ and $\mathcal{X}$ is a smooth cut-off function with $|\mathcal{X}'| \lesssim 1$, $\mathcal{X}(x) = 1$ if $|x| \le 1$ and $\mathcal{X}(x) = 0$ if $|x| \ge 2$.

It is clear that $F^{(l)}(\lambda)/l! = v_\lambda^l$, and that F has the characteristic growth of A_p functions; the support of F is contained in $\bigcup_\lambda D_\lambda$ and for $z \in D_\lambda$,

$$|F(z)| \le \sum_{l=0}^{m_\lambda-1} |v_\lambda^l| \le Ce^{\alpha p(\lambda)} \lesssim e^{Kp(z)}.$$

There is also a good estimate on $\bar\partial F$. Its support is the union of the annuli

$$C_\lambda = \{z\in\mathbb{C} : \delta_\lambda \le |z-\lambda| \le 2\delta_\lambda\},$$

and for $z \in C_\lambda$,

$$\Big|\frac{\partial F}{\partial \bar z}(z)\Big| \lesssim \sum_{l=0}^{m_\lambda-1} |v_\lambda^l||\mathcal{X}'|\frac{1}{\delta_\lambda} \lesssim e^{Cp(\lambda)} \lesssim e^{Kp(z)},$$

for K big enough.

Altogether, there exists $\gamma > 0$ such that

$$\int_{\mathbb{C}} |F(z)|^2 e^{-\gamma p(z)} < \infty, \quad \int_{\mathbb{C}} |\bar\partial F(z)|^2 e^{-\gamma p(z)} < \infty\ . \tag{4}$$

Now, when looking for a holomorphic interpolating function of the form $f = F - u$, we are led to the $\bar\partial$-problem

$$\bar\partial u = \bar\partial F,$$

which we solve using Hörmander's theorem [Ho94, Theorem 4.2.1]: given a (pluri)-subharmonic function ψ in $\mathbb{C}$, there exists a solution u to the above equation such that

$$2\int_{\mathbb{C}} |u|^2 \frac{e^{-\psi}}{(1+|z|^2)^2}\, dm \le \int_{\mathbb{C}} |\bar\partial F|^2 e^{-\psi} dm\ .$$

We apply Hörmander's theorem with

$$\psi_\beta(z) = \beta p(z) + v(z),$$

where $\beta > 0$ will be chosen later on and

$$v(z) = \sum_{\lambda\in\Lambda_0} m_\lambda \Big[\log|z-\lambda|^2 - \frac{1}{\pi\varepsilon^2 p^2(\lambda)} \int_{D(\lambda,\varepsilon p(\lambda))} \log|z-\zeta|^2 dm(\zeta)\Big]\ .$$

Here ε is a fixed small constant to be determined later on.

Integrating by parts the equality

$$\int_0^{2\pi} \log|a - re^{i\theta}|^2\ \frac{d\theta}{2\pi} = \begin{cases} \log|a|^2 & \text{if} \quad |a| > r,\\ \log r^2 & \text{if} \quad |a| \le r,\end{cases}$$

one sees that for $a \in \mathbb{C}$ and $r > 0$,

$$\log|a|^2 - \frac{1}{\pi r^2}\int_{D(a,r)} \log|\zeta|^2 dm(\zeta) = \begin{cases} \log|\frac{a}{r}|^2 + 1 - |\frac{a}{r}|^2 & \text{if} \quad |a| \le r,\\ 0 & \text{if} \quad |a| > r.\end{cases}$$

Thus

$$v(z)=\sum_{\lambda:|\lambda-z|\le\varepsilon p(\lambda)} m_\lambda\Big[\log\frac{|z-\lambda|^2}{\varepsilon^2p^2(\lambda)}+1-\frac{|z-\lambda|^2}{\varepsilon^2p^2(\lambda)}\Big]\ .$$

In particular, $v\le 0$ and $\Delta v(z)=0$ if $z\notin\bigcup_\lambda D(\lambda,\varepsilon p(\lambda))$. For $z\in\bigcup_\lambda D(\lambda,\varepsilon p(\lambda))$ we have $|\operatorname{Im} z|\le 2\omega(|z|)$ and

$$\Delta v(z)\ge\sum_{\lambda:|\lambda-z|\le\varepsilon p(\lambda)}\frac{-m_\lambda}{\varepsilon^2p^2(\lambda)}\gtrsim\sum_{\lambda:|\lambda-z|\le C(\varepsilon)p(z)}\frac{-m_\lambda}{p^2(z)}=-\ \frac{n(z,C(\varepsilon)p(z))}{p^2(z)}\ .$$

As observed in Lemma 7(ii), with ε small enough $n(z,C(\varepsilon)p(z))\lesssim p(z)$, thus $\Delta v(z)\gtrsim -1/p(z)$. This and (3) show that ψ_β is subharmonic if β is chosen big enough.

Also, we deduce from (c) that for any $\beta'>\beta$,

$$\int_{\mathbb C}|u|^2e^{-\beta' p}dm\lesssim\int_{\mathbb C}|u|^2\frac{e^{-\psi_\beta}}{(1+|z|^2)^2}\ dm\lesssim\int_{\mathbb C}|\bar\partial F|^2e^{-\psi_\beta}dm\ .$$

We need to control ψ_β on the support of $\bar\partial F$. For $z\in C_\lambda$,

$$\begin{aligned}|\psi_\beta(z)-\beta p(z)|&\le\sum_{\lambda:|\lambda-z|\le\varepsilon p(\lambda)} m_\lambda\log\frac{\varepsilon^2p^2(\lambda)}{|z-\lambda|^2}\\&\simeq m_\lambda\log\frac{\varepsilon^2p^2(\lambda)}{|z-\lambda|^2}+\sum_{\substack{\lambda':|z-\lambda'|\le\varepsilon p(\lambda')\\ \lambda'\ne\lambda}} m_{\lambda'}\log\frac{\varepsilon^2p^2(\lambda')}{|z-\lambda'|^2}\\&\lesssim p(\lambda)+\sum_{\substack{\lambda':|\lambda'-z|\le C(\varepsilon)p(z)\\ \lambda'\ne\lambda}} m_{\lambda'}\log\frac{C(\varepsilon)^2p^2(z)}{|z-\lambda'|^2}\\&\lesssim p(z)+N(z,C(\varepsilon)p(z)).\end{aligned}$$

Claim 9. *For ε small enough $N(z,C(\varepsilon)p(z))\lesssim p(z)$ for all $z\in supp(\bar\partial F)$.*

Assuming the claim we have $|\psi_\beta(z)-\beta p(z)|\le Kp(z)$ on $supp(\bar\partial F)$. Therefore, for β big enough

$$\int_{\mathbb C}|u|^2e^{-\beta' p}dm\lesssim\int_{\mathbb C}|\bar\partial F|^2e^{-\psi_\beta}dm\le\int_{\mathbb C}|\bar\partial F|^2e^{-\gamma p}dm<\infty\ .$$

This shows that $f:=F-u\in A_p$. Since $e^{-\psi_\beta}\simeq|z-\lambda|^{-2m_\lambda}$ around each λ, also $u^{(l)}(\lambda)=0$ for all $\lambda\in\Lambda$, $l=0,\dots,m_\lambda-1$, and therefore $f^{(l)}(\lambda)/l!=F^{(l)}(\lambda)/l!=v_\lambda^l$, as required.

Proof of the claim. Assume $z\in C_\lambda$ and observe that $n(z,t)=0$ for $t<\delta_\lambda$ and that $n(z,t)\le m_\lambda$ for $\delta_\lambda\le t<2\delta_\lambda$. Since $D(z,t)\subset D(\lambda,t+2\delta_\lambda)$ and $|z|<|\lambda|+2\delta_\lambda$, we have (changing into $s=t+2\delta_\lambda$)

$$\begin{aligned}N(z,C(\varepsilon)p(z))&\le\int_{\delta_\lambda}^{2\delta_\lambda}\frac{m_\lambda}{t}\ dt+\int_{2\delta_\lambda}^{C(\varepsilon)p(z)}\frac{n(z,t)-m_\lambda}{t}\ dt\\&\le p(\lambda)+\int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda}\frac{n(\lambda,s)-m_\lambda}{s-2\delta_\lambda}\ ds\\&\lesssim p(\lambda)+\int_{4\delta_\lambda}^{C(\varepsilon)p(z)+2\delta_\lambda}\frac{n(\lambda,s)-m_\lambda}{s/2}\ ds\lesssim p(\lambda)+N(\lambda,C'(\varepsilon)p(\lambda)).\end{aligned}$$

From the properties of the weight and the hypothesis we have finally that for ε small $N(z, C(\varepsilon)p(z)) \lesssim p(\lambda) \lesssim p(z)$.

4.2. **Case Λ^+.** According to Theorem A, it is enough to construct a function $G \in A_p$ such that $X_+ \subset \mathcal{Z}(G)$ and

$$\frac{|G^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq \varepsilon e^{-Kp(\lambda)}, \qquad \lambda \in \Lambda_+,$$

for some constants $\varepsilon, k > 0$. In fact, the hypotheses of Theorem A require the weight p to be subharmonic, and our weights are not necessarily so. Nevertheless, by Lemma 8, there exists a subharmonic weight $\tilde{p}$ equivalent to p, and we may apply Theorem A to $\tilde{p}$.

Take any entire function F such that $\mathcal{Z}(F) = X_+$. Since the necessary conditions imply that X_+ satisfies the Blaschke condition in $\mathbb{H}$, we can consider also the Blaschke product

$$B(z) = \prod_{\lambda \in \Lambda_+} \Big(\frac{z-\lambda}{z-\bar{\lambda}}\Big)^{m_\lambda}, \qquad z \in \mathbb{H}.$$

Define

$$\phi(z) = \begin{cases} \log\Big|\dfrac{F(z)}{B(z)}\Big|, & \operatorname{Im} z > 0, \\ \log|F(z)|, & \operatorname{Im} z \leq 0. \end{cases}$$

Lemma 10. *ϕ is harmonic outside the real axis, subharmonic on $\mathbb{C}$ and its Laplacian is uniformly bounded.*

Proof. It is clear, by definition, that ϕ is harmonic on $\mathbb{C} \setminus \mathbb{R}$. In order to prove that ϕ is subharmonic on $\mathbb{C}$, it is enough to check the mean inequality for $x \in \mathbb{R}$. We have

$$\phi(x) = \log|F(x)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log|F(x+re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(x+re^{i\theta})d\theta.$$

Since $\Delta \log|F| \equiv 0$ around $\mathbb{R}$, it is enough to compute the Laplacian of

$$\psi(z) = \begin{cases} \log\dfrac{1}{|B(z)|}, & \operatorname{Im} z > 0, \\ 0, & \operatorname{Im} z \leq 0. \end{cases}$$

Being

$$\log\frac{1}{|B(z)|} = \frac{1}{2}\sum_{\lambda \in \Lambda^+} m_\lambda \log\Big|\frac{z-\bar{\lambda}}{z-\lambda}\Big|^2,$$

it will be enough to compute the Laplacian of each term

$$\psi_\lambda(z) = \begin{cases} \log\Big|\dfrac{z-\bar{\lambda}}{z-\lambda}\Big|^2, & \operatorname{Im} z > 0, \\ 0, & \operatorname{Im} z \leq 0. \end{cases}$$

It is clear that $\partial\psi_\lambda/\partial x = 0$ on $\mathbb{R}$, hence $\Delta\psi_\lambda = \partial^2\psi_\lambda/\partial y^2$. Since ψ_λ is continuous around $\mathbb{R}$, this Laplacian has a magnitude equivalent to the jump of the first derivative of ψ_λ. The derivative of the Green function on the half-plane with respect to the normal direction y is the Poisson kernel:

$$\frac{\partial}{\partial y}\log\Big|\frac{z-\bar{\lambda}}{z-\lambda}\Big|^2_{|y=0} = \frac{4\operatorname{Im}\lambda}{|x-\lambda|^2}.$$

Therefore,

$$\Delta\phi(x) = 4\sum_{\lambda\in\Lambda^+} m_\lambda \frac{\operatorname{Im}\lambda}{|x-\lambda|^2}\,dx,$$

which is bounded by the hypothesis. □

Define

$$\Psi(z) = N|\operatorname{Im} z| - \phi(z).$$

Observe that $\Delta\Psi(z) = N\,dx - \Delta\phi(x)\,dx$, thus according to the previous lemma $\Delta\Psi \simeq dx$ when $N\in\mathbb{N}$ is big enough. In this situation, according to [OrSe99, Lemma 3], there exists a multiplier associated to Ψ, i.e., an entire function h such that:

(a) $\mathcal{Z}(h)$ is a separated sequence contained in $\mathbb{R}$.
(b) Given any $\varepsilon > 0$, $|h(z)| \simeq \exp(\Psi(z))$ for all points z such that $d(z,\mathcal{Z}(h)) > \varepsilon$.

Define now $G = hF$. It is clear that $G\in A_p$:

$$|G(z)| \lesssim e^{\Psi(z)+\log|F(z)|} \le e^{\Psi(z)+\phi(z)} \le e^{Np(z)}, \qquad z\in\mathbb{C}.$$

It is also clear that $X_+\subset\mathcal{Z}(G)$, since $X_+\subset\mathcal{Z}(F)$.

In order to prove that there exist $\varepsilon, C>0$ such that

$$\Big|\frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!}\Big| \ge \varepsilon e^{-Cp(\lambda)} \tag{5}$$

consider then the disjoint disks $D_\lambda = D(\lambda,\delta_\lambda)$, $\delta_\lambda = \delta e^{-C\frac{p(\lambda)}{m_\lambda}}$ given by Lemma 7(i). Since Λ_+ is far from $\mathcal{Z}(h)$, the estimate

$$|G(z)| = |h(z)|e^{\phi(z)}|B(z)| \simeq e^{N|\operatorname{Im} z|}|B(z)|, \qquad z\in\partial D_\lambda,$$

holds.

Claim 11. *There exists $C>0$ such that $|B(z)| \ge \epsilon e^{-Cp(z)}$, $z\in\partial D_\lambda$.*

Assuming this we have $|G(z)| \gtrsim e^{-Cp(z)}$ for all $z\in\partial D_\lambda$. Define then $g(z) = G(z)/(z-\lambda)^{m_\lambda}$. It is clear that g is holomorphic, non-vanishing in D_λ, and $|g(z)| \gtrsim e^{-cp(\lambda)}$ for $z\in\partial D_\lambda$. By the minimum principle

$$\Big|\frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!}\Big| = |g(0)| \gtrsim e^{-cp(\lambda)},$$

as desired.

Proof of the claim. As observed in Remark 6(b), the estimate we want to prove is equivalent to

$$\int_0^1 \frac{n_{\mathbb{H}}(z,t)}{t}\,dt \lesssim p(z), \qquad z\in\partial D_\lambda.$$

This is proved like Claim 9, except we replace the Euclidean disks by the hyperbolic ones. We have

$$\int_0^1 \frac{n_{\mathbb{H}}(z,t)}{t}\,dt \lesssim \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t}dt + \int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z,t)-m_\lambda}{t}\,dt.$$

The first term is controlled by $p(\lambda)$. In order to control the second term observe that $D_{\mathbb{H}}(z,t) \subset D_{\mathbb{H}}(\lambda, \frac{t+\delta_\lambda}{1+t\delta_\lambda})$; hence changing the variable into $s = \frac{t+\delta_\lambda}{1+t\delta_\lambda}$ we get

$$\int_{2\delta_\lambda}^1 \frac{n_{\mathbb{H}}(z,t) - m_\lambda}{t}\, dt \le \int_{\frac{3\delta_\lambda}{1+2\delta_\lambda^2}}^1 \frac{n_{\mathbb{H}}(\lambda,s) - m_\lambda}{s - \delta_\lambda} \frac{1-\delta_\lambda^2}{(1-\delta_\lambda)^2}\, ds.$$

There is no restriction in assuming that $\delta_\lambda < 1/2$. Then $\frac{3\delta_\lambda}{1+2\delta_\lambda^2} > 2\delta_\lambda$ and therefore $s - \delta_\lambda > s/2$. With this and condition (ii) in Theorem 1 we obtain

$$\int_0^1 \frac{n_{\mathbb{H}}(z,t)}{t}\, dt \lesssim p(\lambda) + \int_0^1 \frac{n_{\mathbb{H}}(\lambda,s) - m_\lambda}{s}\, ds\ .$$

Since $p(\lambda) \lesssim p(z)$, we will be done as soon as we prove that

$$\int_0^1 \frac{n_{\mathbb{H}}(\lambda,s) - m_\lambda}{s}\, ds \lesssim p(\lambda).$$

There exists $\delta > 0$ (independent of λ) such that $D_{\mathbb{H}}(\lambda,\delta) \subset D(\lambda, p(\lambda))$. Then

$$\begin{aligned}\int_0^\delta \frac{n_{\mathbb{H}}(\lambda,s) - m_\lambda}{s}\, ds &= \sum_{0<|\frac{\lambda-\lambda'}{\lambda-\bar\lambda'}|<\delta} m_{\lambda'} \log \frac{\delta}{|\frac{\lambda-\lambda'}{\lambda-\bar\lambda'}|} \le \sum_{0<|\frac{\lambda-\lambda'}{\lambda-\bar\lambda'}|<\delta} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda-\lambda'|} \\ &\lesssim \sum_{0<|\lambda-\lambda'|<p(\lambda)} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda-\lambda'|} \le N(\lambda, p(\lambda)) \lesssim p(\lambda).\end{aligned}$$

For the remaining part we use condition (ii) in Theorem 1 and the estimate $\log t^{-1} \simeq 1 - t$ for $\delta < t < 1$. Taking $x = \operatorname{Re}\lambda$ we have

$$\begin{aligned}\int_\delta^1 \frac{n_{\mathbb{H}}(\lambda,s) - m_\lambda}{s}\, ds &\lesssim \sum_{\lambda\neq\lambda'} m_{\lambda'} \frac{|\operatorname{Im}\lambda||\operatorname{Im}\lambda'|}{|\lambda - \bar\lambda'|^2} \\ &\lesssim \sum_{\lambda\neq\lambda'} m_{\lambda'} \frac{|\operatorname{Im}\lambda||\operatorname{Im}\lambda'|}{|x - \lambda'|^2} \lesssim |\operatorname{Im}\lambda| \simeq p(\lambda).\end{aligned}$$

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