SPECTRA OF QUANTIZED HYPERALGEBRAS

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ABSTRACT. We describe the prime and primitive spectra for quantized enveloping algebras at roots of 1 in characteristic zero in terms of the prime spectrum of the underlying enveloping algebra. Our methods come from the theory of Hopf algebra crossed products. For primitive ideals we obtain an analogue of Duflo's Theorem, which says that every primitive ideal is the annihilator of a simple highest weight module. This depends on an extension of Lusztig's tensor product theorem.

1. INTRODUCTION

We describe the prime and primitive spectra of the restricted specialization of the quantized enveloping algebra U_ζ, where ζ is a root of unity of odd order ℓ (prime to 3 in type G_2) associated to a semisimple complex Lie algebra $\mathfrak{g}$ of rank n as in e.g. [Lu1, Lu2, Lu4, Li, An], working over a field k of characteristic zero. We refer to U_ζ as the hyperalgebra because it is defined using a divided power form in a way which mimics the usual hyperalgebra construction for an affine algebraic group.

There is a Hopf algebra map Fr : $U_\zeta \to U$, where U is the ordinary enveloping algebra, known as the quantum Frobenius map. It is known [Li, An] that the Hopf kernel of Fr is the quantized restricted enveloping algebra u_ζ, which is a normal sub-Hopf algebra of U_ζ. Furthermore U_ζ is isomorphic as an algebra to the crossed product $u_\zeta \#_t U$ as k-algebras for some 2-cocycle $t : U \otimes U \to u_\zeta$. This is explicitly shown in [An], follows from the general results in [Sch], and was earlier pointed out informally by Takeuchi. See [Mo, Do] for details about crossed products. The crossed product decomposition is an important tool for our study of prime and primitive ideals.

The crossed product structure allows us to describe the prime spectrum of U_ζ in terms of the prime spectra of u_ζ and U. The structure of the latter as a topological space is only partially understood [So], though it is explicitly described in rank one [NG] and in type A_2 [Ca]. We think of the crossed product structure as being a ring-theoretic explanation of the Lusztig twisted tensor product theorem [Lu1, Li]. Indeed, we use it to give a different proof of the tensor product theorem which moreover applies to all simple modules. This is then applied to prove a version of Duflo's Theorem, which states that every primitive ideal is the annihilator of a simple highest weight module. This result invites the study of primitive spectra for U_ζ in connection with Weyl group orbit geometry, as has been carried

out extensively for U; see [Ja] (and [Jo] 8.5.4 for more references). We note that the generic quantized version of Duflo's Theorem was proved by Joseph and Letzter; see [Jo].

In the second section we discuss U_ζ and its images modulo annihilators of a simple u_ζ-module. The crossed product in the image is shown to be a tensor product with trivial action and cocycle, facts which depend on the Skolem-Noether Theorem and the vanishing of the Sweedler two-cocycle. We use this fact to show that the prime spectrum of U_ζ is homeomorphic to $\mathrm{Spec}(U) \times \mathrm{Spec}(u_{\zeta.})$ in section 3. We turn to primitive ideals in section 4, first developing the theory of weights and simple highest weight modules. Here we allow for possibly infinite-dimensional simple modules, which are labeled by algebra maps from the Cartan part U_ζ^0 to the base field k. The resulting group of weights has a very different structure from the generic case. In passing, we give an expression for the primitive elements in U_ζ, which is not used in what follows, but may be of independent interest since they are (in a Hopf-algebraic sense) a natural choice as generators. In section 4.2, assuming k algebraically closed, we prove an extension of Lusztig's tensor product theorem [Lu1, CP, Li] to all simple modules for U_ζ. We provide the version of Duflo's Theorem in the last subsection, 4.3.

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1.1. Notation and preliminaries.

We adopt the notation and conventions for the quantized enveloping algebra U_ζ of rank n at the primitive root of unity ζ of order ℓ as in [Lu1, 2,4] (also [An, Li]), working over a field k of characteristic zero. ℓ is odd and not divisible by 3 if there is a component of type G_2. The algebra U_ζ has generators $E_i, F_i, E_i^{(\ell)}, F_i^{(\ell)}, K_i, \binom{K_i}{\ell}$, $i = 1, ..., n$. Let (a_{ij}) be the symmetrizable Cartan matrix of finite type with integers $d_i \in \{1, 2, 3\}$ such that $d_i a_{ij} = d_j a_{ji}$ for all i, j. Let $\zeta_i = \zeta^{d_i}$. We invoke the relations $K_i^\ell = 1$ (as done in e.g. [Li]), and thus only consider type 1 modules. We shall use Δ and ε for comultiplication and counit, using subscription to indicate the Hopf algebra in question.

Let $L(m)$ denote the finite-dimensional simple highest weight U_ζ-module with restricted integral weight $m \in [0, \ell - 1]^n$. Then the restriction of this module to u_ζ is simple, and every simple u_ζ-module is of this form. The proof of these facts can be found in [Lu1, Lu2] (cf. [CP, 11.2.10]) in the simply-laced case, and it is well known that they hold in general.

Working over $\mathbb{Z}[q, q^{-1}]$, let $q_i = q^{d_i}$. For $c, m \in \mathbb{Z}$ and $r, t \in \mathbb{Z}^+$ set

$$\begin{bmatrix} m \\ t \end{bmatrix}_{q_i} = \prod_{s=1}^{t} \frac{q_i^{m-s+1} - q_i^{-(m-s+1)}}{q_i^s - q_i^{-s}},$$

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix}_{q_i} = \prod_{s=1}^{t} \frac{K_i q_i^{c-s+1} - K_i^{-1} q_i^{-(c-s+1)}}{q_i^s - q_i^{-s}}.$$

Specializing q to ζ, we have the corresponding elements $\begin{bmatrix} m \\ t \end{bmatrix}_{\zeta_i} \in$ k and $\begin{bmatrix} K_i; c \\ t \end{bmatrix}_{\zeta_i}$ $\in U_\zeta$. Thus $\binom{K_i}{\ell} = \begin{bmatrix} K_i; 0 \\ \ell \end{bmatrix}_{\zeta_i} \in U_\zeta$.

We shall need several formulas for later use that we summarize here. Fix $1 \leq i, j \leq n$. Then we have the defining conjugation relations

$$K_i E_j K_i^{-1} = \zeta_i^{a_{ij}} E_j,$$
$$K_i F_j K_i^{-1} = \zeta_i^{-a_{ij}} F_j.$$

These relations imply

$$(1.1) \qquad \binom{K_i}{\ell} F_j^{(t)} = F_j^{(t)} \binom{K_i; -t a_{ij}}{\ell},$$

$$(1.2) \qquad \binom{K_i}{\ell} E_j^{(t)} = E_j^{(t)} \binom{K_i; t a_{ij}}{\ell}.$$

The next equation is a simple verification:

$$(1.3) \qquad \begin{bmatrix} m \\ t \end{bmatrix}_{q_i} = (-1)^t \begin{bmatrix} -m + t - 1 \\ t \end{bmatrix}_{q_i}.$$

For $m \in \mathbb{Z}$, we write $m = m_0 + m_1 \ell \in \mathbb{Z}$ with unique $m_0, m_1 \in \mathbb{Z}$ with $0 \leq m_0 \leq \ell - 1$ and refer to this as the "short ℓ-adic decomposition of m". We mention the well-known fact that

$$(1.4) \qquad \begin{bmatrix} m \\ \ell \end{bmatrix}_{\zeta_i} = m_1;$$

see [Lu1], also [CP, 9.3.6].

Let m be an integer with $0 \leq m \leq \ell - 1$, and let $c = c_0 + c_1 \ell$ be the short ℓ-adic decomposition of c. Then we have

$$(1.5) \qquad \begin{bmatrix} m - c \\ \ell \end{bmatrix}_{\zeta_i} = \begin{cases} -c_1 \text{ if } m \geq c_0, \\ -(c_1 + 1) \text{ if } m < c_0, \end{cases}$$

$$(1.6) \qquad \begin{bmatrix} m + c \\ \ell \end{bmatrix}_{\zeta_i} = \begin{cases} c_1 \text{ if } m + c_0 < \ell, \\ c_1 + 1 \text{ if } m + c_0 \geq \ell. \end{cases}$$

The first equation, (1.5), of this last pair of equations can seen by applying equation (1.3) to obtain $\begin{bmatrix} m - c \\ \ell \end{bmatrix}_{\zeta_i} = (-1)^\ell \begin{bmatrix} c_1 \ell - (m - c_0) + \ell - 1 \\ \ell \end{bmatrix}_{\zeta_i}$. If $m - c_0 \geq 0$, we get $(-1)^\ell c_1 = -c_1$ using (1.4). If $m - c_0 < 0$, then $c_1 \ell - (m - c_0) + \ell - 1 = (c_1 + 1)\ell + c_0 - m - 1$ and we get $-(c_1 + 1)$ for the second case. The second equation, (1.6), is similar.

2. Crossed products

We begin by briefly reviewing some results concerning cleft extensions and crossed products. Readers may wish to refer to [Mo, Do] for more detailed discussions. Fix a subalgebra A of a Hopf algebra U and a right U-comodule algebra T. An extension of algebras $A \subset T$ is said to be *cleft*, or more precisely U-*cleft,* if there exists a convolution-invertible U-comodule map $\chi \in \text{Hom}(U, T)$ and the subalgebra of coinvariants $T^{\text{co}U}$ for U equals A. Such a map χ is called a *section*. A result from [DT] is that T is isomorphic to a crossed product $A \#_t U$ for some action of U on A and invertible cocycle t if and only if T is U-cleft.

In [Do], it is shown that for a Hopf algebra U and any U-cleft extension T of the algebra A, any two sections $\chi', \chi \in \text{Hom}(U, T)$ are related by $\chi' = v* \chi$ for

some convolution invertible $v \in \mathrm{Hom}(U, A)$. Recall also that corresponding to each section χ, there is a k-isomorphism

$$\tilde{\chi} : A \otimes U \to T$$

given by $\tilde{\chi}(a \otimes y) = a\chi(y)$, $a \in A$, $y \in U$, which is an algebra isomorphism when the tensor product is given the crossed product structure. The crossed product is denoted by $A\#_t U$ (with action and twisting t depending on χ) and the elements $a \otimes y$ are written $a\#y$ or $a\#_t y$. Using the second section, we affix $'$ and obtain an equivalent crossed product and isomorphism $\tilde{\chi}' : A\#_{t'} U \to T$ with new twisting t' and

$$\tilde{\chi}'(a\#_{t'}y) = a\chi'(y) = \sum av(y_1)\chi(y_2)$$

with $a\#_{t'}y \in A\#_{t'}U$. The corresponding algebra isomorphism $A\#_{t'}U \to A\#_t U$ sends $a\#_{t'}y$ to $\sum av(y_1)\#_t y_2$ [Do, p. 3064].

We now look at U_ζ and fix notation. The Hopf algebra extension (see [An])

$$\mathrm{k} \to u_\zeta \to U_\zeta \xrightarrow{\mathrm{Fr}} U \to \mathrm{k}$$

is cleft. Here $U = U(\mathfrak{g})$, U_ζ is a right U-comodule using $(\mathrm{id}\otimes \mathrm{Fr})\Delta$ and u_ζ is the coinvariant subalgebra. The fact that u_ζ is the set of right coinvariants follows directly from the PBW theorem. We note for later use that U_ζ is also a left U-comodule using $(\mathrm{Fr}\otimes id)\Delta$ and u_ζ is the left coinvariant subalgebra. An explicit section $\gamma : U \to U_\zeta$ is given by first specifying

$$\gamma(e_i) = E_i^{(\ell)}, \gamma(f_i) = F_i^{(\ell)}, \gamma(h_i) = \binom{K_i}{\ell},$$

where the e's, f's and h's are the usual generators for the enveloping algebra [An]; we then extend to obtain three algebra maps $U^+ \to U_\zeta^+$, $U^- \to U_\zeta^-$, and $U^0 \to U_\zeta^0$. The definition of γ is completed by tensoring, using the triangular decompositions

$$\begin{aligned} U &= U^- \otimes U^0 \otimes U^+, \\ U_\zeta &= U_\zeta^- \otimes U_\zeta^0 \otimes U_\zeta^+, \end{aligned}$$

i.e., $\gamma(x^-x^0x^+) = \gamma(x^-)\gamma(x^0)\gamma(x^+)$ for $x^+ \in U^+$, $x^0 \in U^0$, $x^- \in U^-$. It is very straightforward check that γ is both a right and left U-comodule map. The section γ corresponds to the crossed product $u_\zeta\#_t U \cong U_\zeta$ for some action of U on u_ζ and some 2-cocycle $t : U \otimes U \to u_\zeta$.

We remark that it follows from results in [Sch] that a section $U \to U_\zeta$ exists. Also, our section γ differs slightly from the one in [An], since there the relations $K_i^\ell = 1$ are not assumed.

Let A be an algebra with crossed product action by a Hopf algebra U. Then $T = A\#_t U$ is a associative algebra. We briefly review some ring-theoretic facts and definitions. If I is a U-invariant ideal of A, then it is standard and easy to check that $TJ \subset JT = J\#T$ is an ideal of T. For an ideal I of A, as usual (e.g. [Ch]), we put $(I : U) = \{a \in A | x.a \in I, \text{ all } x \in U\}$, the largest U-invariant ideal of A contained in I. A U-invariant ideal J of A is said to be U-prime if $KL \subset J$ implies $K \subset J$ or $L \subset J$, for all U-invariant ideals $L, K \subset A$.

The action of U on A is said to be *inner* (see [Mo] or [Do]) if there exists a convolution-invertible $u \in \mathrm{Hom}_\mathrm{k}(U, A)$ such that $y.a = \sum u(y_1)au^{-1}(y)$ for all $a \in A$ and $y \in U$. Here the action of U is said to be *implemented* by u. Note that u is not, in general, an algebra map.

Lemma 1. *Consider the crossed product $U_\zeta = u_\zeta \#_t U$ as above. Then the U-prime ideals and the prime ideals of u_ζ coincide.*

Proof. Let $\mathfrak{m}$ be a U-prime ideal of u_ζ. Then by [Ch], $\mathfrak{m}$ is a prime ideal. Conversely, let $\mathfrak{m}$ be a prime ideal of u_ζ. Then $(\mathfrak{m} : U)$ is a U-prime ideal. By [Ch] again, $(\mathfrak{m} : U)$ is actually a prime and hence maximal ideal. So $\mathfrak{m} = (\mathfrak{m} : U)$. This shows that every prime ideal of u_ζ is U-prime. □

Proposition 1. *Consider the crossed product $U_\zeta = u_\zeta \#_t U$ arising from the section γ as above. Let $\mathfrak{m}$ be a prime ideal of u_ζ and set $R = u_\zeta / \mathfrak{m}$. Let $T = U_\zeta / \mathfrak{m} U_\zeta$ and let $\pi : U_\zeta \to T$ denote the natural map. Then*
  (a) $\chi = \pi\gamma \in Hom(U, T)$ is a section which yields the crossed product

$$T \cong R \#_{\bar{t}} U$$

with cocycle $\bar{t} = \pi t : U \otimes U \to R$, via an isomorphism $\tilde{\chi} : R \#_t U \to T$ arising from the section χ.
  (b) There is another section $\chi' \in Hom_k(U, T)$ giving rise to the trivial crossed product

$$T \cong R \otimes U$$

via an isomorphism $\tilde{\chi}' : R \otimes U \to T$ of cleft extensions of R.
  (c) The corresponding isomorphism $\theta' : R \#_t U \tilde{\to} R \otimes U$ is given by

$$\theta'(a \# y) = \sum av(y_{(1)}) \otimes y_{(2)}$$

for all $a \in R$, $y \in U$, for some $v \in Hom_k(U, R)$.

Proof. By the lemma $\mathfrak{m}$ is a U-invariant ideal. Therefore, as pointed out before Lemma 1, $U_\zeta \mathfrak{m} \subseteq \mathfrak{m} U_\zeta = \mathfrak{m} \#_t U$, so $\mathfrak{m} U_\zeta$ is an ideal of U_ζ. Since $\mathfrak{m}$ is a trivial right U-comodule, $\mathfrak{m} U_\zeta$ is also a right U-subcomodule of U_ζ. Hence T inherits a U-comodule algebra structure.

(a): Plainly, χ is a section of T. Thus we have $T \cong R \#_{\bar{t}} U$ with cocycle $\bar{t} = \pi t : U \otimes U \to R$.

(b) and (c): Obviously R is a simple algebra; in fact $\mathfrak{m}$ is the annihilator of a finite-dimensional simple u_ζ-module of dimension, say n. Since the simple modules are absolutely simple, R is isomorphic to the algebra of $n \times n$ matrices over k.

The Skolem-Noether Theorem ([Sw], see also [Mo, 6.2]) asserts that the action of U on R is inner. By [Mo, 7.3] we can perform a change of basis to obtain $R \#_t U \cong R_\tau[U]$, a twisted product, with trivial action and a new cocycle τ. The isomorphism is given explicitly by a map that is the identity on $R = R \# 1$ as follows:

$$a \#_t y \longmapsto \sum a u^{-1}(y_{(1)}) \#_\tau y_{(2)}$$

for all $y \in U$, $a \in R$, where the inner action is implemented by $u, u^{-1} \in \text{Hom}(U, R)$ as in [Mo, 7.3].

Furthermore, the action of U is now trivial, so R is a (trivial) U-module. By [Mo, 7.1] or [Do], since U is cocommutative, the cocycle τ has image in the center of R, namely k. The reader may also check as an exercise that the twisted module condition quickly implies that, for a trivial action of any Hopf algebra, any cocycle has central values. Thus the Sweedler cocycle τ corresponds to a Lie cocycle ([Sw, 4.1 or 4.3]) with coefficients in the trivial module. Therefore τ is equivalent to a Sweedler coboundary by Whitehead's Second Lemma. Here we have used the equivalence of the Lie (and Hochschild) and Sweedler cohomologies in degree 2 [Sw,

section 4]. We conclude by [Do, 2.4] that $R_\tau[U]$ is isomorphic to a smash product with the same trivial action of U on R; i.e, $R_\tau[U] \cong R \otimes U$. Thus $R\#_t U \cong R \otimes U$ by an isomorphism that is the identity on $R = R\#1 = R \otimes 1$.

The isomorphism $R\#_t U \cong R \otimes U$ is given by

$$a\#y \to \sum av(y_{(1)}) \otimes y_{(2)},$$

$a \in R$, $y \in U$, where $v = u * w \in \mathrm{Hom}(U, R)$ and w is the map whose image is the coboundary τ, by [Do, 2.4]. The corresponding section χ' is given by $v * \chi$. $\square$

3. PRIME SPECTRUM

Theorem 1. $\mathrm{Spec}(U_\zeta)$ *is homeomorphic to* $\mathrm{Spec}(U) \times \mathrm{Spec}(u_\zeta)$.

Proof Fix a prime ideal $P \lhd U_\zeta$ and let $\mathfrak{m} = P \cap u_\zeta$, and put $R = u_\zeta/\mathfrak{m}$. Then $\mathfrak{m}$ is a U-prime, and hence maximal, ideal of R by the lemma. As in [Ch], we define

$$\mathrm{Spec}_\mathfrak{m}(U_\zeta) = \{Q \in \mathrm{Spec}(U_\zeta) : Q \cap u_\zeta = \mathfrak{m}\}.$$

Note that the sets $\mathrm{Spec}_\mathfrak{m}(U_\zeta)$ are the connected components of $\mathrm{Spec}(U_\zeta)$. By e.g. [Ch] we have $\mathrm{Spec}_\mathfrak{m}(U_\zeta) \approx \mathrm{Spec}(\frac{U_\zeta}{\mathfrak{m}U_\zeta}) \approx \mathrm{Spec}(R\#_t U)$. By the proposition above we see that $\mathrm{Spec}_\mathfrak{m}(U_\zeta) \approx \mathrm{Spec}(R\#U) \approx \mathrm{Spec}(U)$. Thus we obtain a mapping $\pi : \mathrm{Spec}(U_\zeta) \to \mathrm{Spec}(U)$ that restricts to a homeomorphism on each $\mathrm{Spec}_\mathfrak{m}(U_\zeta)$.

To complete the proof, define the mapping $h : \mathrm{Spec}(U_\zeta) \to \mathrm{Spec}(u_\zeta) \times \mathrm{Spec}(U)$ by $h(P) = (P \cap u_\zeta) \times \pi(P)$. Clearly, h maps $\mathrm{Spec}_\mathfrak{m}(U_\zeta)$ homeomorphically onto the connected component $\mathfrak{m} \times \mathrm{Spec}(U)$.

4. PRIMITIVE SPECTRUM

4.1. **Simple highest weight modules.**

4.1.1. *Weights.* Let U_ζ^0 denote the Hopf subalgebra of U_ζ generated by the K_i and the $\binom{K_i}{\ell}$, $i = 1, ..., n$, which may also be constructed by specializing the corresponding subalgebra of the Lusztig form to ζ. Also let G denote the multiplicative group of group-likes generated by the K_i's. Since U_ζ^0 is a pointed commutative and cocommutative Hopf algebra, by the Kostant structure theorem (cf. [Mo]) it is the tensor product of the (Hopf) algebras kG and the commutative enveloping algebra $U(P(U_\zeta^0))$, with primitive generators $P(U_\zeta^0)$). On the other hand, restricting the mapping Fr to U_ζ^0 and the section γ to the subalgebra $U^0 = k[h_1, ..., h_n]$, we see that U_ζ^0 is a crossed product $kG\#k[\binom{K_1}{\ell}, ..., \binom{K_n}{\ell}]$ with trivial action and trivial cocycle. Thus

Proposition 2. U_ζ^0 *is the polynomial algebra generated over* kG *by the algebraically independent elements* $\binom{K_i}{\ell}$, $i = 1, ..., n$.

The $\binom{K_i}{\ell}$ are not the primitive generators of U_ζ^0; cf. Remark 1 following Proposition 4 below.

We consider *weights* to be algebra homomorphisms $\lambda : U_\zeta^0 \to k$. We let

$$X = Alg_k(U_\zeta^0, k)$$

denote the set of weights. Weights λ are specified by a pair of algebra maps $\lambda = (\lambda_0, \lambda_1)$ with $\lambda_0 : kG \to k$ and

$$\lambda_1 : k[\binom{K_1}{\ell}, ..., \binom{K_n}{\ell}] \to k,$$

and thus are given by a pair of n-tuples

$$(\lambda_{0,1}, ..., \lambda_{0,n}) \in \{0, 1, ..., \ell - 1\}^n,$$
$$(\lambda_{1,1}, ..., \lambda_{1,n}) \in k^n,$$

with $\lambda(K_i) = \zeta_i^{\lambda_{0i}}$ and $\lambda(\binom{K_i}{\ell}) = \lambda_{1i}$.

We say that λ is a *restricted* weight if $\lambda_1 = 0$. By slight notational abuse we write $\lambda = \lambda_0$ in this case, and similarly $\lambda = \lambda_1$ if $\lambda_0 = \varepsilon_{kG}$.

Integral weights. Let m denote an n-tuple of integers $(m_1, ..., m_n) \in \mathbb{Z}^n$. We define an embedding $\tilde{} : \mathbb{Z}^n \to X$ of sets by setting

$$\tilde{m}(K_j) = \zeta_j^{m_{0j}},$$
$$\tilde{m}(\binom{K_j}{\ell}) = m_{1j},$$

where m_{0j} and m_{1j} are the unique integers in the short ℓ-adic decompositions $m_{0j} + m_{1j}\ell = m_j$, $j = 1, ..., n$. In our above labeling of weights as pairs of n-tuples, this is equivalent to

$$\tilde{m}_{0j} = m_{0j},$$
$$\tilde{m}_{1j} = m_{1j},$$

$j = 1, ..., n$.

We need to specify the convolution group structure on the set of weights which arises from the cocommutative Hopf algebra structure on U_ζ^0. It is given by "carrying" from the finite abelian factor to the additive factor as we next state. We use $*$ for the group operation on X.

Proposition 3. *The group structure on $X = Alg(U_\zeta^0, k)$ is given by*

$$((\lambda_0, \lambda_1) * (\mu_0, \mu_1))_j = \begin{cases} (\lambda_{0j} + \mu_{0j}, \lambda_{1j} + \mu_{1j}) & \text{if } \lambda_{0j} + \mu_{0j} < \ell, \\ (\lambda_{0j} + \mu_{0j} - \ell, \lambda_{1j} + \mu_{1j} + 1) & \text{if } \lambda_{0j} + \mu_{0j} \geqslant \ell \end{cases}$$

for $\lambda, \mu \in X$, $j = 1, ..., n$.

Proof. Let $\lambda, \mu \in X$. Since each K_i is group-like,

$$\lambda * \mu(K_i) = \lambda(K_i)\mu(K_i)$$

$$= \zeta^{\lambda_{0i} + \mu_{0i}} = \begin{cases} \zeta^{\lambda_{0i} + \mu_{0i}} & \text{if } \lambda_{0i} + \mu_{0i} < \ell, \\ \zeta^{\lambda_{0i} + \mu_{0i} - \ell} & \text{otherwise.} \end{cases}$$

Further, recall (e.g. [CP]) that $\Delta(\binom{K_i}{\ell}) = \sum_{j=0}^{\ell} \binom{K_i}{\ell-j} K_i^{-j} \otimes \binom{K_i}{j} K_i^{\ell-j}$, and observe that $\lambda(\binom{K_i}{j}) = \binom{\lambda_{0i}}{j}_{\zeta_i}$ for all $j < \ell$. We infer that

$$\lambda * \mu(\binom{K_i}{\ell}) = \lambda_{1i} + w + \mu_{1i},$$

where the cross-term is $w = \sum_{j=1}^{\ell-1} \zeta_i^{-(\lambda_{0i}+\mu_{0i})j} \binom{\lambda_{0i}}{\ell-j}_{\zeta_i} \binom{\mu_{0i}}{j}_{\zeta_i}$. The proof is completed by applying equation (4.4) below to deduce that $w = \binom{\lambda_{0i}+\mu_{0i}}{\ell}_{\zeta_i}$. By (1.4) we immediately obtain the desired result that $w = 0$ or $w = 1$ in cases as in the statement. $\qquad\square$

The following consequence is now straightforward. It explains the relationship between our weights and the integral weights as considered in [Lu1, Lu2], which were given as n-tuples of integers.

Corollary 1. *The map* $\tilde{\ } : \mathbb{Z}^n \rightarrow X$ *defined above is an embedding of abelian groups.*

From now on we will use additive notation for the group operation on X. For future use we record the following easy facts.

Proposition 4. *Let* $\lambda = (\lambda_0, \lambda_1)$ *and* $\mu = (\mu_0, \mu_1)$ *be weights and write* $\lambda_j = (\lambda_{0j}, \lambda_{1j})$, $j = 1, ..., n$. *The following equations hold:*

$$(-\lambda)_j = \begin{cases} (\ell - \lambda_{0j}, -\lambda_{1j} - 1) \text{ if } \lambda_{0j} > 0, \\ (0, -\lambda_{1j}) \text{ if } \lambda_{0j} = 0, \end{cases}$$

$$(\lambda - \mu)_j = \begin{cases} (\ell + \lambda_{0j} - \mu_{0j}, \lambda_{1j} - \mu_{1j} - 1) \text{ if } \lambda_{0j} < \mu_{0j}, \\ (\lambda_{0j} - \mu_{0j}, \lambda_{1j} - \mu_{1j}) \text{ if } \lambda_{0j} \geq \mu_{0j}. \end{cases}$$

Remark 1. We describe the primitive elements of U_ζ^0. It suffices to work in rank 1, so let $U = U_\zeta^0(sl(2))$. Then up to a scalar multiple, U has a unique primitive element

$$\binom{K}{\ell} + \sum_{i=0}^{\ell-1} a_i K^i,$$

where

$$a_i = \frac{1}{\ell^2} \sum_{j=0}^{\ell-1} j\zeta^{-ij}, i = 0, 1, ..., \ell - 1.$$

Proof. The weight group $X = \text{Alg}(U, k)$ can be written as the set of pairs (ζ^i, a), $a \in k$, $i = 0, 1, ..., \ell - 1$. Let X' be the group with the same underlying set but with group structure $\langle \zeta \rangle \times \alpha_k$, where α_k denotes the additive group of k. Note that the coordinate Hopf algebra of X' is $U' = k[K, d]$, where d is primitive and K is a group-like of order ℓ. Thus $\text{Alg}(U', k)$ is isomorphic to X', where the entries in the pairs correspond to evaluations at K and d, respectively. It is easy to see that X and X' are isomorphic via $\eta : X \rightarrow X'$, where

$$\eta((\zeta^i, a)) = (\zeta^i, a + \frac{i}{\ell}).$$

Therefore we have an isomorphism of Hopf algebras $\eta^* : U' \rightarrow U$.

We proceed to calculate the primitive element $\eta^*(d)$. Let $(\zeta^i, a) \in X$. Evaluating, we have

$$\langle (\zeta^i, a), \eta^*(d) \rangle$$
$$= \langle \eta(\zeta^i, a), d \rangle$$
$$= \langle (\zeta^i, a + \frac{i}{\ell}), d \rangle$$
$$= a + \frac{i}{\ell}.$$

On the other hand, since $\binom{K}{\ell}$ is primitive modulo the coradical $k[K]$, we may write

$$\eta^*(d) = b\binom{K}{\ell} + \sum_{i=0}^{\ell-1} a_j K^j$$

for some $b, a_j \in k$. Evaluating at (ζ^i, a) yields the system of ℓ equations

$$ba + \sum_{j=0}^{\ell-1} a_i \zeta^{ij} = a + \frac{i}{\ell},$$

$i = 0, ..., \ell - 1$. Setting $a = 0, 1$ immediately yields $b = 1$. So we have the system of equations

$$\sum_{j=0}^{\ell-1} a_i \zeta^{ij} = \frac{i}{\ell},$$

which can be solved for the a_i by inverting the Vandermonde matrix (ζ^{ij}). Its inverse is (fortunately, as pointed out to us by J. Angelos) $\frac{1}{\ell}(\zeta^{-ij})$, as can easily be checked. This yields the asserted expressions for the coefficients a_i.

Since U_ζ^0 has Krull dimension one and the coradical is finite, the k-space of primitives of U_ζ^0 must be of dimension one. The uniqueness follows from the Kostant structure theorem; see [Mo]. $\square$

4.1.2. *Highest weight modules.* We proceed to develop the theory of highest weight modules, following the standard strategy. The main difference is that we allow nonintegral weights, and the weights are an extension of the additive group of k^n by the finite abelian group $G = \langle K_1, ..., K_n \rangle$.

Definition 1. Let V denote a U_ζ-module. We say that $v \in V$ is a *primitive vector* of weight $\lambda \in X$ if

$$K_i.v = \lambda(K_i)v,$$
$$\binom{K_i}{\ell}v = \lambda(\binom{K_i}{\ell})v,$$

and $E_i v = E_i^{(\ell)} v = 0$ for all j. We say that V is a *highest weight module* if V is generated by a primitive vector v.

In this case, V is the sum of its weight spaces

$$V_\mu = \{v \in V | x.v = \mu(x)v, \text{ all } x \in U_\zeta^0\}$$

for weights μ.

We define a partial ordering on X next. Let (a_{ij}) be the symmetrizable Cartan matrix with entries $d_i a_{ij} = d_j a_{ji}$ in the standard notation. *Simple roots* are defined

to be the restricted weights $\rho_i = (a_{1i}, a_{2i}, ..., a_{ni})$, the ith column of (a_{ij}), $i = 1, ..., n$. We say that $\lambda \preceq \mu$ if $\mu - \lambda$ is a nonegative integral linear combination of simple roots. This ordering is sensible because of the following proposition. Recall that we are using additive notation for the convolution group operation on X.

Proposition 5. *Let V be a U_ζ-module and let $v \in V$ be a vector of weight $\lambda \in X$. Then $F_i^{(t)} v$ has weight $\lambda - t\rho_i$ and $E_i^{(t)} v$ has weight $\lambda + t\rho_i$ for all $t \in \mathbb{N}$.*

Proof. Observe that

$$K_j F_i^{(t)} v = (K_j F_i^{(t)} K_j^{-1})(K_j v)$$
$$= (\zeta_j^{-t a_{ji}} F_i^{(t)})(\zeta_j^{\lambda_{0j}} v)$$
$$= \zeta_j^{\lambda_{0j} - t a_{ji}} F_i^{(t)} v.$$

This establishes the first equation to be checked. We leave the analogous formula for $K_j E_i^{(t)} v$ to the reader.

The second pair of equations to be checked are

$$\binom{K_j}{\ell} F_i^{(t)} v = (\lambda - t\rho_i) F_i^{(t)} v,$$

$$\binom{K_j}{\ell} E_i^{(t)} v = (\lambda + t\rho_i) E_i^{(t)} v.$$

We begin by recalling a pair of identities (e.g. [Lu2, g9-g10, p. 270]). Let $m, c \in \mathbb{N}$. Then

$$(4.1) \qquad \begin{bmatrix} K_j; -c \\ \ell \end{bmatrix}_{q_j} = \sum_{s=0}^{\ell} (-1)^s q_j^{c(\ell-s)} \begin{bmatrix} c+s-1 \\ s \end{bmatrix}_{q_j} K_j^s \begin{bmatrix} K_j \\ \ell - s \end{bmatrix}_{q_j},$$

$$(4.2) \qquad \begin{bmatrix} K_j; c \\ \ell \end{bmatrix}_{q_j} = \sum_{s=0}^{\ell} q_j^{c(\ell-s)} \begin{bmatrix} c \\ s \end{bmatrix}_{q_j} K_j^{-s} \begin{bmatrix} K_j \\ \ell - s \end{bmatrix}_{q_j}.$$

The next pair of formulas are obtained from these identities by specializing K_j to q_j^m :

$$(4.3) \qquad \begin{bmatrix} m - c \\ \ell \end{bmatrix}_{q_j} = \sum_{s=0}^{\ell} (-1)^s q_j^{\ell c + s(m-c)} \begin{bmatrix} c+s-1 \\ s \end{bmatrix}_{q_j} \begin{bmatrix} m \\ \ell - s \end{bmatrix}_{q_j},$$

$$(4.4) \qquad \begin{bmatrix} m + c \\ \ell \end{bmatrix}_{q_j} = \sum_{s=0}^{\ell} q_j^{\ell c - s(m+c)} \begin{bmatrix} c \\ s \end{bmatrix}_{q_j} \begin{bmatrix} m \\ \ell - s \end{bmatrix}_{q_j}.$$

If we put $m = \lambda_{0j}$, the $s = 0$ term in equations (4.3) and (4.4) is zero. Let S denote the sum of the terms in equations (4.1) or (4.2) with $s > 0$. Now using the obvious fact that $\binom{K_j}{\ell - s} v = \begin{bmatrix} \lambda_{0j} \\ \ell - s \end{bmatrix}_{\zeta_j} v$ for all $s > 0$, we see that

$$S.v = \begin{bmatrix} \lambda_{0j} - c \\ \ell \end{bmatrix}_{\zeta_j} v.$$

Therefore

$$\begin{bmatrix} K_j; -c \\ \ell \end{bmatrix}_{\zeta_j} v = \begin{pmatrix} K_j \\ \ell \end{pmatrix} v + \begin{bmatrix} \lambda_{0j} - c \\ \ell \end{bmatrix}_{\zeta_j} v$$

$$= (\lambda_{1j} + \begin{bmatrix} \lambda_{0j} - c \\ \ell \end{bmatrix}_{\zeta_j}) v.$$

Set $c = ta_{ji}$. Now by equation (1.1),

$$\begin{pmatrix} K_j \\ \ell \end{pmatrix} F^{(t)}.v = (\lambda_{1j} + \begin{bmatrix} \lambda_{0j} - c \\ \ell \end{bmatrix}_{\zeta_j}) F^{(t)}.v.$$

Writing the short ℓ-adic decomposition $c = c_0 + c_1\ell$, one has by (1.5)

$$\begin{bmatrix} \lambda_{0j} - c \\ \ell \end{bmatrix}_{\zeta_j} = \begin{cases} -c_1 \text{ if } \lambda_{0j} \geq c_0, \\ -(c_1 + 1) \text{ otherwise.} \end{cases}$$

This is the desired result for the action of the $F_i^{(t)}$. The analogous result for the action of $E_i^{(t)}$ is obtained similarly using equations (1.2), (1.6) and (4.4) in place of (1.1), (1.5) and (4.3). $\qquad\square$

Corollary 2. *Let v be a primitive vector of weight $\lambda \in X$. Then λ is the highest weight in $U_\zeta v$ in the ordering $\preceq$ defined above.*

4.1.3. *Simple highest weight modules.* Given a weight λ, we can construct a highest weight module for U_ζ by the usual Verma module construction. Using the standard argument, we find that there is a unique simple factor module which we denote by $L(\lambda)$ with primitive vector of weight λ, and every simple highest weight module is obtained in this way. Let

$$T_\lambda = U_\zeta/\text{ann}_{u_\zeta}(L(\lambda))U_\zeta.$$

Clearly the representation $U_\zeta \to \text{End}_k(L(\lambda))$ factors through T_λ. We write

$$\phi_\lambda : T_\lambda \to \text{End}_k(L(\lambda))$$

for the representation of T_λ.

Consider a restricted weight λ; then $L(\lambda)$ is simple as a u_ζ-module. Let

$$R_\lambda = u_\zeta/\text{ann}_{u_\zeta}(L(\lambda)) \cong \text{End}_k(L(\lambda)).$$

On the other hand, when $\lambda = \lambda_1$, then λ factors through $U^0 = k[h_1, ..., h_n]$ and thus λ corresponds in the obvious way to a weight $\bar\lambda$ for U. (This makes sense since λ vanishes on $\ker \text{Fr} \cap U_\zeta^0 = \sum(K_i - 1)U_\zeta^0$.) The corresponding simple U-module with highest weight $\bar\lambda = \bar\lambda_1$ shall be denoted by $L_U(\bar\lambda)$, so that $L_U(\bar\lambda)^{\text{Fr}} \cong L(\lambda)$.

We will often drop the subscript U.

4.2. **Tensor product theorem for simple modules.** We recall the algebra map $\chi' : U \to T$ from Proposition 1.

Lemma 2. *Let λ be a restricted weight. Then U acts trivially on $L(\lambda)$ via $\phi_\lambda\chi' : U \to \text{End}_k(L(\lambda))$, i.e., $\phi_\lambda\chi' = \varepsilon_U$.*

Proof. By Proposition 1, the elements of $\phi_\lambda\chi'(U)$ commute with R (as endomorphisms of $L(\lambda)$). Therefore $\phi_\lambda\chi'$ sends U to $k = \text{End}_R(L(\lambda)) \subset \text{End}_k(L(\lambda))$. $\qquad\square$

Let R and U be rings, and let L and V be R and U-modules, respectively. We give $L \otimes V$ (resp. $V \otimes L$) the structure of an $R \otimes U$-module (resp. $U \otimes R$), where the action is along tensor factors. If L is a U_ζ-module and V is a U-module, we give $L \otimes V^{\mathrm{Fr}}$ and $V^{\mathrm{Fr}} \otimes L$ the usual module structure via the comultiplication of U_ζ, with V^{Fr} considered as a U_ζ-module along Fr. When $R = u_\zeta/\mathrm{ann}_{u_\zeta}(L)$ we have both module structures on the vector space $L \otimes V$, because L can be viewed as a U_ζ-module.

We shall assume k is algebraically closed for the remainder of the paper.

Theorem 2. (a) *Every simple U_ζ-module is isomorphic to $L \otimes V^{\mathrm{Fr}}$ for some simple highest weight module $L \cong L(\lambda_0)$ with restricted weight λ_0 and a simple U-module V whose U_ζ-module structure is via* Fr.

(b) *Conversely, every module of this form, with L and V simple, is a simple U_ζ-module.*

(c) *In addition, $V^{\mathrm{Fr}} \otimes L \cong L \otimes V^{\mathrm{Fr}} \cong L \otimes V \cong V \otimes L$.*

Proof. Let M be a simple U_ζ-module. Then, as the annihilator $\mathfrak{m} = \mathrm{ann}_{u_\zeta}(M)$ is a prime ideal, M is a semisimple u_ζ-module with unique isotypic component, say $L = L(\lambda_0)$.

Let e be a primitive idempotent in $R = u_\zeta/\mathfrak{m}$. Then $L \cong Re$ and $eRe = $ k. As in Proposition 1 we have $T = U_\zeta/\mathfrak{m}U_\zeta \cong R \otimes U \cong M_n(U)$. Accordingly, $eTe \cong U$ and Te is a projective indecomposable module for T. Clearly L is a generator for $\mathrm{mod}(R)$, so Te is a generator for module category $\mathrm{Mod}(T)$.

By Morita equivalence, as left T-modules,

$$M \cong Te \otimes_U eM$$
$$\cong (L \otimes U) \otimes_U eM$$
$$\cong L \otimes eM,$$

where, in the last expression, $T = R \otimes U$ acts along tensor factors, and $eM = V$ is the corresponding simple $eTe \cong U$-module.

Consider $L \otimes V$ as a U_ζ-module via $\pi : U_\zeta \to T = R \otimes U$, and $L \otimes V^{\mathrm{Fr}}$ as a U_ζ-module by comultiplying. Note that $L \otimes V^{\mathrm{Fr}}$ inherits a T-module structure since $\mathfrak{m}U_\zeta$ is a right U-subcomodule. We finish the proof of (a) by showing that $L \otimes V \cong L \otimes V^{\mathrm{Fr}}$. To do this it suffices to show that $\chi'(U)$ only acts on the right factor V^{Fr}, i.e.,

$$\chi'(a)(l \otimes v^{\mathrm{Fr}}) = l \otimes av$$

for all $a \in U$, $l \in L$ and $v \in V$. To show this, recall that U_ζ is a right U-comodule via $\rho_\zeta = (1 \otimes \mathrm{Fr})\Delta_{U_\zeta}$. Thus T inherits a right U-comodule with structure map denoted by $\rho : T \to T \otimes U$. We summarize these maps in the following diagram:

$$
\begin{array}{ccc}
U_\zeta & \xrightarrow{\Delta} & U_\zeta \otimes U_\zeta \\
\downarrow{id} & & \downarrow{id \otimes \mathrm{Fr}} \\
U_\zeta & \xrightarrow{\rho_\zeta} & U_\zeta \otimes U \\
\downarrow{\pi} & & \downarrow{\pi \otimes id} \\
T & \xrightarrow{\rho} & T \otimes U
\end{array}
$$

Similarly T is a left U-comodule with structure map $\lambda : T \to U \otimes T$.

We now compute

$$\chi'(a)(l \otimes v^{\mathrm{Fr}}) = (\rho\chi'(a))(l \otimes v)$$
$$= (\chi' \otimes id_T)\Delta_U(a)(l \otimes v)$$
$$= (\varepsilon_U \otimes id_T)\Delta_U(a)(l \otimes v)$$
$$= (id_U \otimes a)(l \otimes v),$$

where the second equality holds because χ' is a right U-comodule map, and the third equality holds by the preceding lemma. This completes the proof of (a).

To prove (b), let V be a simple U-module and L a simple u_ζ-module. By a well-known result [Qu] (since k is algebraically closed), L and V have trivial endomorphism rings k. This fact, with a straightforward density theorem argument, shows that $L \otimes V$ is a simple T-module, with R and U acting along factors. Thus, as a U_ζ-module via $\pi : U_\zeta \to T$, we see that $L \otimes V$ is a simple module.

To prove (c) we first claim that the right section χ' is also a left U-comodule map. As in section 2, we have $U_\zeta = u_\zeta\gamma(U)$ and u_ζ is both left and right U-coinvariant for the respective left and right coactions $\lambda_\zeta = (\mathrm{Fr} \otimes id_{U_\zeta})\Delta_{U_\zeta}$ and $\rho_\zeta = (id_{U_\zeta} \otimes \mathrm{Fr})\Delta_{U_\zeta}$. Let $a \in u_\zeta$ and $y \in U$. Then in Sweedler notation,

$$\rho_\zeta(a\gamma(y)) = \sum a\gamma(y_1) \otimes y_2,$$

since a is right coinvariant and γ is a right U-comodule map. On the other hand,

$$\lambda_\zeta(a\gamma(y)) = \sum y_1 \otimes a\gamma(y_2),$$

since a is left coinvariant and γ is a right U-comodule map. Since U is cocommutative, we see that $\lambda_\zeta = \tau\rho_\zeta$, where τ denotes the twist map. The same fact pushes down to T, i.e., $\lambda = \tau\rho$. Thus for any right comodule map $\varkappa : U \to T$,

$$\lambda\varkappa = \tau\rho\varkappa$$
$$= \tau(\varkappa \otimes id_U)\rho$$
$$= (id_U \otimes \varkappa)\lambda.$$

This establishes the claim that $\varkappa$ is a left U-comodule map.

By the left-comodule version of the argument just used in the last paragraph of the proof of part (a), we see that $V \otimes L \cong V^{\mathrm{Fr}} \otimes L$. Obviously, $V \otimes L \cong L \otimes V$ as U_ζ-modules. Thus $V^{\mathrm{Fr}} \otimes L \cong V \otimes L \cong L \otimes V \cong L \otimes V^{\mathrm{Fr}}$. This completes the proof of (c) and the theorem. $\square$

Remark 2. The tensor symmetry in (c) follows from the quasi-tensor structure on U_ζ (see [Lu3]) when the modules are finite dimensional. We do not know this to be the case in general.

4.3. **An analog of Duflo's Theorem.**

Theorem 3. *Every primitive ideal of U_ζ is the annihilator of a simple highest weight module.*

Proof. Let M be a simple left U_ζ-module. Let P denote the annihilator in U_ζ and set $m = u_\zeta \cap P$, which is a prime ideal of u_ζ by Lemma 1. As a U_ζ-module $M \cong L \otimes V^{\mathrm{Fr}}$, where L and V are simple u_ζ- and U-modules, respectively. By Duflo's Theorem [Duf], $\mathrm{ann}_U(V)$ is the annihilator of a highest weight module for U, say $L_U(\bar\lambda_1)$. Therefore $\mathrm{ann}_{U_\zeta}(V^{\mathrm{Fr}}) = \mathrm{ann}_{U_\zeta}(L(\lambda_1))$ for some weight λ_1 (notation as in section 4.1).

Note also that L is a simple highest weight module for u_ζ, say of highest weight λ_0. Thus if $x \in L = L(\lambda_0)$ and $y \in L_U(\bar{\lambda}_1)$ are primitive vectors for the respective u_ζ- and U-module structures, then adopting the argument in [Lu1],

$$E_i.(x \otimes y) = (1 \otimes E_i + E_i \otimes K).(x \otimes y) = E_i.x \otimes y = 0,$$

$$E_i^{(\ell)}.(x \otimes y) = (1 \otimes E_i^{(\ell)} + E_i^{(\ell)} \otimes 1).(x \otimes y) = E_i^{(\ell)}.x \otimes y + 1 + e_i.y = 0$$

$$\text{and}$$

$$K_i.(x \otimes y) = \lambda_0(K_i)(x \otimes y) = \lambda_{0,i}(x \otimes y),$$

$$\binom{K_i}{\ell}.(x \otimes y) = \lambda_1(\binom{K_i}{\ell})(x \otimes y) = \lambda_{1,i}(x \otimes y)$$

for all i. Hence $x \otimes y \in L \otimes L(\lambda_1)^{\mathrm{Fr}}$ is a primitive vector, and thus $L \otimes L_U(\bar{\lambda}_1)^{\mathrm{Fr}}$ is a simple highest weight module. Finally $\mathrm{ann}_{U_\zeta}(M) = \mathrm{ann}(L(\lambda_0) \otimes V^{\mathrm{Fr}})$ can be computed using the Sweedler wedge in the coalgebra U_ζ (see [Mo]) to be

$$\mathrm{ann}_{U_\zeta}(L(\lambda_0)) \wedge \mathrm{ann}_{U_\zeta}(V^{\mathrm{Fr}})$$
$$= \mathrm{ann}_{U_\zeta}(L(\lambda_0)) \wedge \mathrm{ann}_{U_\zeta}(L_U(\bar{\lambda}_1)^{\mathrm{Fr}})$$
$$= \mathrm{ann}_{U_\zeta}(L(\lambda_0) \otimes L_U(\bar{\lambda}_1)^{\mathrm{Fr}}).$$

This completes the proof of the theorem. $\square$

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